This document contains my notes on . Each section corresponds to the video of the same title.

Contents

1   Overview

- **Passive controls** attempt to control a system passively, i.e. they are built into the system and don't vary based on observations of the system.

- **Active controls** attempt to control a system actively, i.e. they change their behaviour based on observations of the system.

- **Open-loop controllers** don't observe the output of the system — their inputs are predetermined. One downside of this approach is that you may unnecessarily input energy into the system when it already has the desired output.

- **Closed-loop controllers** observe the output of the system to determine their inputs. They have several benefits over open-loop controllers:

  - They handle uncertainty in the system, e.g. if you don't always know how the system will respond, or if your model isn't completely accurate.

– They handle disturbances in the system, e.g. if someone pushes a self-balancing inverted pendulum. It may not be possible to account for these in the model.

– They can be more energy efficient than open-loop controllers, i.e. they don't have the downside mentioned above.

- The mathematical model used in this course is a state-space based system of linear differential equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

where $\mathbf{x}$ is the **state vector** — a vector containing all the system's values of interest — and $\dot{\mathbf{x}}$ are their rates of change at a time t.

- The solution to the above equation is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$$

and the eigenvalues of $\mathbf{A}$ can be used to determine the stability of the system, e.g. if they all have negative real components the system is stable.

- Control is introduced to the system by modifying the equation to

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where $\mathbf{B}$ is a coefficient matrix and $\mathbf{u}$ is the input to the system.

- If we make the input to the system

$$\mathbf{u} = -\mathbf{K}\mathbf{x}$$

then

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x}$$
$$= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$$

and now it is the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ that determine the stability of the system, i.e. we can make an unstable system stable by appropriate choice of inputs.

2 Linear Systems

- The solution to the linear system of differential equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$$

where

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!}$$

$$= \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \dots$$

- If a matrix $\mathbf{A}$ has eigenvalues $\lambda_1, \dots, \lambda_n$ and eigenvectors $\mathbf{k}_1, \dots, \mathbf{k}_n$ then

$$\mathbf{A}\mathbf{T} = \mathbf{T}\mathbf{D}$$

or

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$$

where

$$\mathbf{T} = \begin{pmatrix} \mathbf{k}_1 & \cdots & \mathbf{k}_n \end{pmatrix}$$

and

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

- Using the above gives

$$e^{\mathbf{A}t} = \mathbf{T} e^{\mathbf{D}t} \mathbf{T}^{-1}$$

which is simpler because $e^{\mathbf{D}t}$ is easy to calculate.

- If we define $\mathbf{z}$ to be $\mathbf{x}$ transformed to the eigenvector basis, i.e.

$$\mathbf{x} = \mathbf{T}\mathbf{z}$$

and

$$\dot{\mathbf{x}} = \mathbf{T}\dot{\mathbf{z}}$$

then

$$\mathbf{z} = \mathbf{T}^{-1}\mathbf{x}$$
$$\dot{\mathbf{z}} = \mathbf{T}^{-1}\dot{\mathbf{x}}$$
$$= \mathbf{T}^{-1}\mathbf{A}\mathbf{x}$$
$$= \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{z}$$
$$= \mathbf{D}\mathbf{z}$$

where $\mathbf{D}$ is the diagonal matrix consisting of the eigenvalues of $\mathbf{A}$. The solution to this equation is

$$\mathbf{z}(t) = e^{\mathbf{D}t}\mathbf{z}_0$$

where

$$e^{\mathbf{D}t} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix},$$

i.e. the equations are uncoupled such that

$$z_1(t) = z_{0,1}e^{\lambda_1 t}$$

$$\vdots$$

$$z_n(t) = z_{0,n}e^{\lambda_n t}.$$

- The equivalence between the original and eigenvector coordinates can be seen by manipulating the solution

$$\begin{aligned} \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}_0 \\ &= \mathbf{T}e^{\mathbf{D}t}\mathbf{T}^{-1}\mathbf{x}_0 \\ &= \mathbf{T}e^{\mathbf{D}t}\mathbf{z}_0 \\ &= \mathbf{T}\mathbf{z}(t). \end{aligned}$$

3   Stability and Eigenvalues

- A system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is stable if and only if all of the eigenvalues of $\mathbf{A}$ have negative real parts. This is because the solution is of the form[1]

$$\mathbf{x} = \mathbf{T}e^{\mathbf{D}t}\mathbf{T}^{-1}\mathbf{x}_0$$

where

$$e^{\mathbf{D}t} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix}$$

and

$$e^{\lambda_1 t} = e^{(a+ib)t} = e^{at}e^{ibt}$$

which is only stable if $a < 0$.

- Sometimes you can make a system stable (i.e. make all the eigenvalues have negative real parts) via the control term $\mathbf{B}\mathbf{u}$.

- In a real-world system, control signals will be sent and observations received in discrete steps rather than continuously. A variation of the equation that captures this is

$$\mathbf{x}_{k+1} = \tilde{\mathbf{A}}\mathbf{x}_k$$

---

[1]See here for a proof of why $e^{\mathbf{A}t} = \mathbf{T}e^{\mathbf{D}t}\mathbf{T}^{-1}$.

where

$$\tilde{\mathbf{A}} = e^{\mathbf{A}\Delta t}.$$

In other words,

$$\begin{aligned}
\mathbf{x}_k &= \tilde{\mathbf{A}}^k \mathbf{x}_0 \\
&= (\tilde{\mathbf{T}}\tilde{\mathbf{D}}\tilde{\mathbf{T}}^{-1})^k \mathbf{x_0} \\
&= \tilde{\mathbf{T}}\tilde{\mathbf{D}}^k \tilde{\mathbf{T}}^{-1} \mathbf{x}_0
\end{aligned}$$

where

$$\tilde{\mathbf{D}}^k = \begin{pmatrix} \tilde{\lambda}_1^k & & 0 \\ & \ddots & \\ 0 & & \tilde{\lambda}_n^k \end{pmatrix}$$

so the discrete system is stable if the moduli of all the eigenvalues of $\tilde{\mathbf{A}}$ are less than or equal to 1.

- Note that the stability criteria of a discrete system (the moduli of of all eigenvalues must be less than 1) differs from that of a continuous system (the real parts of all eigenvalues must be negative) because we're raising $\tilde{\mathbf{A}} = e^{\mathbf{A}\Delta t}$ to a power directly rather than exponentiating it so both the real and imaginary components of its eigenvalues contribute to the "radius"

$$\begin{aligned}
\lambda_n^k &= (a+ib)^k \\
&= (\sqrt{a^2+b^2}e^{i\arctan\frac{b}{a}})^k \\
&= \sqrt{a^2+b^2}^k e^{ik\arctan\frac{b}{a}}.
\end{aligned}$$

4 Linearizing Around a Fixed Point

- The **Jacobian matrix** of a vector-valued function of several variables $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is defined as

$$D\mathbf{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

- If $\mathbf{x}$ is a point and $\Delta\mathbf{x}$ is a displacement, both in $\mathbb{R}^n$, the Jacobian matrix can be used to linearize $\mathbf{f}$ around $\mathbf{x}$

$$\mathbf{f}(\mathbf{x}+\Delta\mathbf{x}) \approx \mathbf{f}(\mathbf{x}) + D\mathbf{f}|_{\mathbf{x}}\Delta\mathbf{x}.$$

- To convert a nonlinear system $\dot{\mathbf{x}} = f(\mathbf{x})$ to a linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$:

1. Find fixed points $\overline{\mathbf{x}}$ such that $f(\overline{\mathbf{x}}) = \mathbf{0}$.
2. Use the Jacobian matrix to linearize $f(\mathbf{x})$ about $\overline{\mathbf{x}}$, i.e.

$$\dot{\mathbf{x}} = f(\mathbf{x})$$

is approximated by

$$\begin{aligned} \dot{\mathbf{x}} &= f(\overline{\mathbf{x}}) + Df|_{\overline{\mathbf{x}}}\Delta\mathbf{x} \\ &= Df|_{\overline{\mathbf{x}}}\Delta\mathbf{x} \\ &= \mathbf{A}\Delta\mathbf{x} \end{aligned}$$

where $\mathbf{A} = Df|_{\overline{\mathbf{x}}}$. However this is only valid if all eigenvalues of $\mathbf{A}$ have a nonzero real part.

5   Controllability

- A system with control can be modelled using the equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where $\mathbf{u}$ is the input to the system.

- If we can completely observe the state of the system, i.e. we have access to all state variables, we can let $\mathbf{u} = -\mathbf{K}\mathbf{x}$ and thus

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ &= \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} \\ &= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} \end{aligned}$$

and we can control the eigenvalues/stability of the system via careful choice of $\mathbf{K}$.

- A system is said to be **controllable** if it's possible to choose $\mathbf{u} = -\mathbf{K}\mathbf{x}$ such that the system has any arbitrary eigenvalues and thus the state can be manipulated in any desired way.

- In most cases $\mathbf{A}$ (the linearized equations of motion) and $\mathbf{B}$ (the effects of various inputs) are predetermined and you only get to control $\mathbf{u}$.

- Some combinations of $\mathbf{A}$ and $\mathbf{B}$ are controllable and some aren't. For example,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathbf{u}$$

isn't controllable because $\dot{x}_1 = x_1$ isn't affected by $\mathbf{u}$. However if $\mathbf{B}$ were instead

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

the system would be controllable because $\mathbf{u}$ can affect both $\dot{x}_1$ and $\dot{x}_2$.

- A system with state space in $\mathbb{R}^n$ is controllable if the controllability matrix

$$C = \begin{pmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{pmatrix}$$

has rank n.

6 Controllability, Reachability, and Eigenvalue Placement

- The following are all equivalent:
    - The system is controllable.
    - It's possible to arbitrarily place the eigenvalues of the system by careful choice of $\mathbf{K}$.
    - The state space is fully reachable, i.e. for any state in $\mathbb{R}^n$ there exist inputs $\mathbf{u}(t)$ that result in the system being in that state.

7 Controllability and the Discrete-Time Impulse Response

- The first column in the controllability matrix can be thought of as an initial "kick" to the system via the input vector $\mathbf{u}$ and each subsequent column can be thought of as the state of the system one time step later. This lets us see which dimensions of the state space can be controlled via input.

8 Degrees of Controllability and Gramians

- The **Gramian** of a system is defined as

$$W_t = \int_0^t e^{\mathbf{A}t} \mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T \tau} \, d\tau.$$

It is real-valued and symmetric so it has positive, real eigenvalues. If you sort the eigenvalues from largest to smallest, the same ordering of the associated eigenvectors is also ordering them from most to least controllable.

- The Gramian gives a finer grained answer to the question "Is the system controllable?" than the rank of the controllability matrix — it tells you how controllable it is in different dimsensions of the state space.

- A system is said to be **stabilisable** if all unstable eigenvectors of $\mathbf{A}$ are in the controllable subspace, i.e. it's OK if a subset of the state vector isn't controllable providing it's stable — it will stabilise by itself.

10 Cayley-Hamilton Theorem

- The **Cayley-Hamilton theorem** states that every square matrix satisfies its own characteristic equation, i.e. if you take the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$
$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

and substitute λ with $\mathbf{A}$, the equation is satisfied.

14 Linear Quadratic Regulator (LQR) Control for the Inverted Pendulum on a Cart

- The **linear quadratic regulator** is a feedback controller that minimises the cost of operating the system.

- The cost is defined as

$$J = \int_0^\infty (\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})\,dt$$

where $\mathbf{x}$ is an $n \times 1$ state vector, $\mathbf{Q}$ is an $n \times n$ matrix that determines the cost of a given state vector, $\mathbf{u}$ is an $q \times 1$ input vector, and $\mathbf{R}$ is a $q \times q$ matrix that determines the cost of a given input vector.

- $\mathbf{Q}$ can be used to associate a cost with given states, e.g. in the case of an inverted pendulum we could say it is 1 unit cost for every 1 unit distance x is away from its target, or it is 100 unit cost for every radian θ is away from its target.

- Similarly, $\mathbf{R}$ can be used to associate a cost with given inputs, e.g. if moving the cart uses a lot of electricity we could define some cost for that.

- The linear quadratic regulator finds a matrix $\mathbf{K}$ that adjusts the eigenvalues of the system to minimise the above cost.