

This document contains my notes on [Steve Brunton's Control Bootcamp video series](#). Each section corresponds to the video of the same title.

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1 Overview

- **Passive controls** attempt to control a system passively, i.e. they are built into the system and don't vary based on observations of the system.
- **Active controls** attempt to control a system actively, i.e. they change their behaviour based on observations of the system.
- **Open-loop controllers** don't observe the output of the system — their inputs are predetermined. One downside of this approach is that you may unnecessarily input energy into the system when it already has the desired output.
- **Closed-loop controllers** observe the output of the system to determine their inputs. They have several benefits over open-loop controllers:
 - They handle uncertainty in the system, e.g. if you don't always know how the system will respond, or if your model isn't completely accurate.
 - They handle disturbances in the system, e.g. if someone pushes a self-balancing inverted pendulum. It may not be possible to account for these in the model.
 - They can be more energy efficient than open-loop controllers, i.e. they don't have the downside mentioned above.
- The mathematical model used in this course is a state-space based system of linear differential equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

where \mathbf{x} is the **state vector** — a vector containing all the system's values of interest — and $\dot{\mathbf{x}}$ are their rates of change at a time t .

- The solution to the above equation is

$$\mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{x}(0)$$

and the eigenvalues of \mathbf{A} can be used to determine the stability of the system, e.g. if they all have negative real components the system is stable.

- Control is introduced to the system by modifying the equation to

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where \mathbf{B} is a coefficient matrix and \mathbf{u} is the input to the system.

- If we make the input to the system

$$\mathbf{u} = -\mathbf{K}\mathbf{x}$$

then

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} \\ &= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} \end{aligned}$$

and now it is the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ that determine the stability of the system, i.e. we can make an unstable system stable by appropriate choice of inputs.

2 Linear Systems

- The solution to the linear system of differential equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

is

$$\mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{x}_0$$

where

$$\begin{aligned} e^{\mathbf{A}t} &= \sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!} \\ &= \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \dots \end{aligned}$$

- If a matrix \mathbf{A} has eigenvalues $\lambda_1, \dots, \lambda_n$ and eigenvectors $\mathbf{k}_1, \dots, \mathbf{k}_n$ then

$$\mathbf{A}\mathbf{T} = \mathbf{T}\mathbf{D}$$

or

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$$

where

$$\mathbf{T} = (\mathbf{k}_1 \quad \cdots \quad \mathbf{k}_n)$$

and

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

- Using the above gives

$$e^{\mathbf{A}t} = \mathbf{T}e^{\mathbf{D}t}\mathbf{T}^{-1}$$

which is simpler because $e^{\mathbf{D}t}$ is easy to calculate.

- If we define \mathbf{z} to be \mathbf{x} transformed to the eigenvector basis, i.e.

$$\mathbf{x} = \mathbf{T}\mathbf{z}$$

and

$$\dot{\mathbf{x}} = \mathbf{T}\dot{\mathbf{z}}$$

then

$$\begin{aligned} \mathbf{z} &= \mathbf{T}^{-1}\mathbf{x} \\ \dot{\mathbf{z}} &= \mathbf{T}^{-1}\dot{\mathbf{x}} \\ &= \mathbf{T}^{-1}\mathbf{A}\mathbf{x} \\ &= \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{z} \\ &= \mathbf{D}\mathbf{z} \end{aligned}$$

where \mathbf{D} is the diagonal matrix consisting of the eigenvalues of \mathbf{A} . The solution to this equation is

$$\mathbf{z}(t) = e^{\mathbf{D}t}\mathbf{z}_0$$

where

$$e^{\mathbf{D}t} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix},$$

i.e. the equations are uncoupled such that

$$\begin{aligned} z_1(t) &= z_{0,1}e^{\lambda_1 t} \\ &\vdots \\ z_n(t) &= z_{0,n}e^{\lambda_n t}. \end{aligned}$$

- The equivalence between the original and eigenvector coordinates can be seen by manipulating the solution

$$\begin{aligned}
\mathbf{x}(t) &= e^{\mathbf{A}t} \mathbf{x}_0 \\
&= \mathbf{T} e^{\mathbf{D}t} \mathbf{T}^{-1} \mathbf{x}_0 \\
&= \mathbf{T} e^{\mathbf{D}t} \mathbf{z}_0 \\
&= \mathbf{T} \mathbf{z}(t).
\end{aligned}$$

3 Stability and Eigenvalues

- A system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is stable if and only if all of the eigenvalues of \mathbf{A} have negative real parts. This is because the solution is of the form¹

$$\mathbf{x} = \mathbf{T} e^{\mathbf{D}t} \mathbf{T}^{-1} \mathbf{x}_0$$

where

$$e^{\mathbf{D}t} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix}$$

and

$$e^{\lambda_1 t} = e^{(a+ib)t} = e^{at} e^{ibt}$$

which is only stable if $a < 0$.

- Sometimes you can make a system stable (i.e. make all the eigenvalues have negative real parts) via the control term $\mathbf{B}\mathbf{u}$.
- In a real-world system, control signals will be sent and observations received in discrete steps rather than continuously. A variation of the equation that captures this is

$$\mathbf{x}_{k+1} = \tilde{\mathbf{A}} \mathbf{x}_k$$

where

$$\tilde{\mathbf{A}} = e^{\mathbf{A}\Delta t}.$$

In other words,

$$\begin{aligned}
\mathbf{x}_k &= \tilde{\mathbf{A}}^k \mathbf{x}_0 \\
&= (\tilde{\mathbf{T}} \tilde{\mathbf{D}} \tilde{\mathbf{T}}^{-1})^k \mathbf{x}_0 \\
&= \tilde{\mathbf{T}} \tilde{\mathbf{D}}^k \tilde{\mathbf{T}}^{-1} \mathbf{x}_0
\end{aligned}$$

where

$$\tilde{\mathbf{D}}^k = \begin{pmatrix} \tilde{\lambda}_1^k & & 0 \\ & \ddots & \\ 0 & & \tilde{\lambda}_n^k \end{pmatrix}$$

¹See [here](#) for a proof of why $e^{\mathbf{A}t} = \mathbf{T} e^{\mathbf{D}t} \mathbf{T}^{-1}$.

so the discrete system is stable if the moduli of all the eigenvalues of $\tilde{\mathbf{A}}$ are less than or equal to 1.

- Note that the stability criteria of a discrete system (the moduli of all eigenvalues must be less than 1) differs from that of a continuous system (the real parts of all eigenvalues must be negative) because we're raising $\tilde{\mathbf{A}} = e^{\mathbf{A}\Delta t}$ to a power directly rather than exponentiating it so both the real and imaginary components of its eigenvalues contribute to the “radius”

$$\begin{aligned}\lambda_n^k &= (a + ib)^k \\ &= (\sqrt{a^2 + b^2} e^{i \arctan \frac{b}{a}})^k \\ &= \sqrt{a^2 + b^2}^k e^{ik \arctan \frac{b}{a}}.\end{aligned}$$

4 Linearizing Around a Fixed Point

- The **Jacobian matrix** of a vector-valued function of several variables $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is defined as

$$D\mathbf{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

- If \mathbf{x} is a point and $\Delta\mathbf{x}$ is a displacement, both in \mathbb{R}^n , the Jacobian matrix can be used to linearize \mathbf{f} around \mathbf{x}

$$\mathbf{f}(\mathbf{x} + \Delta\mathbf{x}) \approx \mathbf{f}(\mathbf{x}) + D\mathbf{f}|_{\mathbf{x}}\Delta\mathbf{x}.$$

- To convert a nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ to a linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$:
 1. Find fixed points $\bar{\mathbf{x}}$ such that $\mathbf{f}(\bar{\mathbf{x}}) = \mathbf{0}$.
 2. Use the Jacobian matrix to linearize $\mathbf{f}(\mathbf{x})$ about $\bar{\mathbf{x}}$, i.e.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

is approximated by

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\bar{\mathbf{x}}) + D\mathbf{f}|_{\bar{\mathbf{x}}}\Delta\mathbf{x} \\ &= D\mathbf{f}|_{\bar{\mathbf{x}}}\Delta\mathbf{x} \\ &= \mathbf{A}\Delta\mathbf{x}\end{aligned}$$

where $\mathbf{A} = D\mathbf{f}|_{\bar{\mathbf{x}}}$. However this is only valid if all eigenvalues of \mathbf{A} have a nonzero real part.

5 Controllability

- A system with control can be modelled using the equation

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

where \mathbf{u} is the input to the system.

- If we can completely observe the state of the system, i.e. we have access to all state variables, we can let $\mathbf{u} = -\mathbf{Kx}$ and thus

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} \\ &= \mathbf{Ax} - \mathbf{BKx} \\ &= (\mathbf{A} - \mathbf{BK})\mathbf{x}\end{aligned}$$

and we can control the eigenvalues/stability of the system via careful choice of \mathbf{K} .

- A system is said to be **controllable** if it's possible to choose $\mathbf{u} = -\mathbf{Kx}$ such that the system has any arbitrary eigenvalues and thus the state can be manipulated in any desired way.
- In most cases \mathbf{A} (the linearized equations of motion) and \mathbf{B} (the effects of various inputs) are predetermined and you only get to control \mathbf{u} .
- Some combinations of \mathbf{A} and \mathbf{B} are controllable and some aren't. For example,

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} \\ \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathbf{u}\end{aligned}$$

isn't controllable because $\dot{x}_1 = x_1$ isn't affected by \mathbf{u} . However if \mathbf{B} were instead

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

the system would be controllable because \mathbf{u} can affect both \dot{x}_1 and \dot{x}_2 .

- A system with state space in \mathbb{R}^n is controllable if the controllability matrix

$$C = (\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B})$$

has rank n .