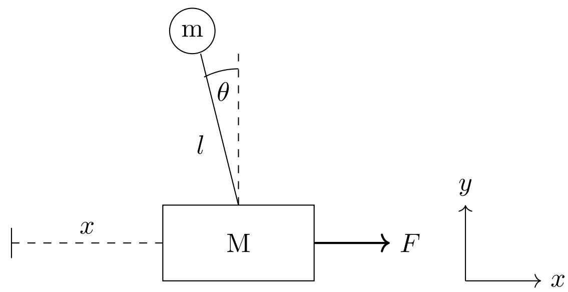
Contents

1 Equations of Motion



- A cart of mass M is constrained to move along the x axis with its distance from an arbitrary point on the x axis denoted x. A driving force of magnitude F is applied to the cart in the x direction. A simple pendulum consisting of a mass m and a massless rod of length l is connected to the cart with its angle from the positive y axis denoted θ.
- The kinetic energy of the cart is

$$T_{\text{cart}} = \frac{1}{2}M\dot{x}^2.$$

- The x and y coordinates of the pendulum are

$$\begin{aligned} X &= x - l\sin\theta \\ Y &= l\cos\theta, \end{aligned}$$

thus its x and y velocities are

$$\begin{aligned} \dot{X} &= \dot{x} - l\dot{\theta}\cos\theta \\ \dot{Y} &= -l\dot{\theta}\sin\theta \end{aligned}$$

and its kinetic energy is

$$\begin{aligned}
T_{\text{pendulum}} &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}(\dot{X}^2 + \dot{Y}^2) \\
&= \frac{1}{2}m[(\dot{x} - l\dot{\theta}\cos\theta)^2 + (-l\dot{\theta}\sin\theta)^2] \\
&= \frac{1}{2}m(\dot{x}^2 - 2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2).
\end{aligned}$$

- The total kinetic energy of the system is

$$\begin{aligned}
T &= T_{\text{cart}} + T_{\text{pendulum}} \\
&= \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}m(l^2\dot{\theta}^2 - 2l\dot{x}\dot{\theta}\cos\theta).
\end{aligned}$$

- The potential energy of the system is equal to the gravitational potential energy of the pendulum. If its potential energy is 0 when $\theta = \frac{\pi}{2}$ then

$$U = mgl\cos\theta.$$

- The Lagrangian of the system is

$$\begin{aligned}
\mathcal{L} &= T - U \\
&= \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}m(l^2\dot{\theta}^2 - 2l\dot{x}\dot{\theta}\cos\theta) - mgl\cos\theta.
\end{aligned}$$

- By d'Alembert's principle the generalized forces associated with the θ and x coordinates are 0 and F, respectively.
- The Euler-Lagrange equation for the θ coordinate is

$$\begin{aligned}
\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} &= 0 \\
\frac{d}{dt}(ml^2\dot{\theta} - ml\dot{x}\cos\theta) - ml\dot{x}\dot{\theta}\sin\theta - mgl\sin\theta &= 0 \\
l\ddot{\theta} - \ddot{x}\cos\theta - g\sin\theta &= 0.
\end{aligned}$$

- The Euler-Lagrange equation for the x coordinate is

$$\begin{aligned}
\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} - \frac{\partial\mathcal{L}}{\partial x} &= F \\
\frac{d}{dt}[(m+M)\dot{x} - ml\dot{\theta}\cos\theta] &= F \\
(m+M)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta &= F.
\end{aligned}$$

- Solving these equations for $\ddot{\theta}$ and $\ddot{x}$ gives

$$\ddot{\theta} = \frac{(m+M)g\sin\theta + F\cos\theta - ml\dot{\theta}^2\cos\theta\sin\theta}{l(m+M) - ml\cos^2\theta}$$

and

$$\ddot{x} = \frac{2F + mg\sin 2\theta - 2ml\dot{\theta}^2\sin\theta}{m + 2M - m\cos 2\theta}.$$

2 Linearization, Stability, and Controllability

- The state vector for this system is

$$\begin{pmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{pmatrix}.$$

- The fixed point about which the system will be linearized is

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

- The **A** matrix is equal to the Jacobian matrix evaluated at the fixed point

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{g(m+M)}{lM} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{gm}{M} & 0 & 0 & 0 \end{pmatrix}.$$

- The non-zero eigenvalues of **A** are

$$\pm\sqrt{\frac{g(m+M)}{lM}}.$$

Because one of these has a positive real part the system is unstable.

- Rearranging the equations of motion to find the coefficients of F gives

$$\ddot{\theta} = f(\theta, \dot{\theta}) + \frac{\cos\theta}{l(m+M) - ml\cos^2\theta}F$$

and

$$\ddot{x} = g(\theta, \dot{\theta}) + \frac{2}{m + 2M - m\cos 2\theta}F.$$

Using the small angle approximation for cos gives

$$\ddot{\theta} = f(\theta, \dot{\theta}) + \frac{1}{lM}F$$

and

$$\ddot{x} = g(\theta, \dot{\theta}) + \frac{1}{M}F$$

resulting in the **B** matrix

$$\begin{pmatrix} 0 \\ \frac{1}{lM} \\ 0 \\ \frac{1}{M} \end{pmatrix}.$$

- The controllability matrix

$$C = \begin{pmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \mathbf{A}^3\mathbf{B} \end{pmatrix}$$

has full rank (4) so the system is controllable via the force F on the cart.

- The ideal state feedback gains matrix **K** can be determined using Mathematica's `LQRegulatorGains` function.

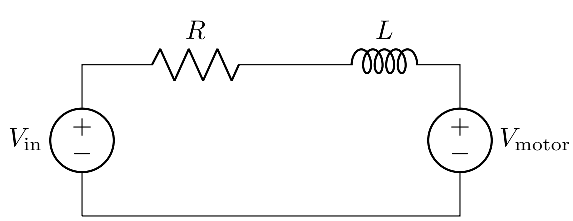
- Thus, the force to apply to the cart when the system is in state $\mathbf{x}$ is

$$F = u = -\mathbf{K}\mathbf{x}.$$

3 Motor Control

- Using the equations above we can calculate the force to apply to the cart when the system is in a given state. However, the motor (which applies a force to the cart) is controlled via pulse-width modulation (PWM) and it's not clear what force results from a particular duty cycle.

- The motor can be electrically modelled by the circuit



where V_{in} is the (effective) voltage delivered to the motor via PWM, R is the resistance of the motor's windings, L is the inductance of the motor's windings, and V_{motor} is the back EMF generated as the motor is spinning.

- If we assume that the back EMF generated by the motor is proportional to its angular velocity, i.e. $V_{\text{motor}} = K_{\text{b}}\omega$, then Kirchoff's voltage law gives

$$V_{\text{in}} - IR - \frac{dI}{dt}L - V_{\text{motor}} = 0$$
$$V_{\text{in}} - IR - \frac{dI}{dt}L - K_{\text{b}}\omega = 0.$$

- The motor can be mechanically modelled as a rotating cylinder with equation of motion

$$\tau_{\text{m}} + \tau_{\text{r}} = J\alpha$$
$$K_{\text{t}}I - b\omega = J\alpha$$

where τ_m is the torque due to the motor which is proportional to the current, τ_{r} is the resistive torque due to air resistance, friction, etc. which is proportional to the angular velocity ω, J is the moment of inertia, and α is the angular acceleration.

- Rearranging the last equation for I we find

$$I = \frac{1}{K_{\text{t}}}(b\omega + J\alpha)$$

and

$$\frac{dI}{dt} = \frac{1}{K_{\text{t}}}(b\alpha + J\dot{\alpha}).$$

- Substituting this into the electrical equation above gives

$$V_{\text{in}} - \frac{R}{K_{\text{t}}}(b\omega + J\alpha) - \frac{L}{K_{\text{t}}}(b\alpha + J\dot{\alpha}) - K_{\text{b}}\omega = 0$$
$$V_{\text{in}} - \left(\frac{bR}{K_{\text{t}}} + K_{\text{b}}\right)\omega - \left(\frac{JR}{K_{\text{t}}} + \frac{bL}{K_{\text{t}}}\right)\alpha - \frac{JL}{K_{\text{t}}}\dot{\alpha} = 0$$

or

$$A\ddot{\omega} + B\dot{\omega} + C\omega + V_{\text{in}} = 0$$

where A, B, and C are constants.

- The general solution to this equation is

$$\omega = AV_{\text{in}} + Be^{Ct} + De^{Et}$$

where A, B, C, D, and E are different constants from above.

- The equation can be simplified by assuming $D = 0$

$$\omega = AV_{\text{in}} + Be^{Ct}.$$

- In reality ω quickly changes from 0 to some equilibrium value that has the same sign as V_{in}. That being the case, we know:
  - C must be negative otherwise ω would increase without bound,
  - A must be positive as the equilibrium value has the same sign as V_{in}, and
  - $B = -AV_{\text{in}}$ because $\omega(0) = 0$.

  Finally, the equation can be multiplied by the radius of the timing pulley R to give the velocity of the cart v. This gives

  $$v = AV_{\text{in}}(1 - e^{-Bt})$$

  where $A, B > 0$ are constants that can be determined from experimental data.

- Differentiating this equation with respect to time gives

  $$a = ABV_{\text{in}}e^{-Bt}$$

  which can be equated with the time-independent equation for $a = R\dot{\omega}$

  $$\begin{aligned} ABV_{\text{in}}e^{-Bt} &= CV_{\text{in}} + Dv \\ &= CV_{\text{in}} + D[AV_{\text{in}}(1 - e^{-Bt})] \\ &= (AD + C)V_{\text{in}} - ADV_{\text{in}}e^{-Bt}. \end{aligned}$$

  From this we can see $AB = -AD$ or $D = -B$ and $AD+C = 0$ or $C = AB$. Once A and B are determined from experimental data we can determine C and D and thus the time-independent equation for a.

- The time-independent equation for a can be multiplied by the mass of the cart M to find the force on the cart F. This equation can then be rearranged to find the voltage V_{in} required to exert a force F on the cart when it has velocity v

  $$\begin{aligned} a &= CV_{\text{in}} + Dv \\ F &= M(CV_{\text{in}} + Dv) \\ V_{\text{in}} &= \frac{1}{C}\left(\frac{F}{M} - Dv\right). \end{aligned}$$

  This equation bridges the control equations of section 2 and the motor equations of this section.