This document contains my notes on Steve Brunton's Control Bootcamp video series. Each section corresponds to the video of the same title.

Contents

1 Overview

- **Passive controls** attempt to control a system passively, i.e. they are built into the system and don't vary based on observations of the system.
- **Active controls** attempt to control a system actively, i.e. they change their behaviour based on observations of the system.
- **Open-loop controllers** don't observe the output of the system — their inputs are predetermined. One downside of this approach is that you may unnecessarily input energy into the system when it already has the desired output.
- **Closed-loop controllers** observe the output of the system to determine their inputs. They have several benefits over open-loop controllers:
  - They handle uncertainty in the system, e.g. if you don't always know how the system will respond, or if your model isn't completely accurate.
  - They handle disturbances in the system, e.g. if someone pushes a self-balancing inverted pendulum. It may not be possible to account for these in the model.
  - They can be more energy efficient than open-loop controllers, i.e. they don't have the downside mentioned above.
- The mathematical model used in this course is a state-space based system of linear differential equations

  $$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

  where $\mathbf{x}$ is the **state vector** — a vector containing all the system's values of interest — and $\dot{\mathbf{x}}$ are their rates of change at a time t.
- The solution to the above equation is

  $$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$$

  and the eigenvalues of $\mathbf{A}$ can be used to determine the stability of the system, e.g. if they all have negative real components the system is stable.

- Control is introduced to the system by modifying the equation to

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where $\mathbf{B}$ is a coefficient matrix and $\mathbf{u}$ is the input to the system.

- If we make the input to the system

$$\mathbf{u} = -\mathbf{K}\mathbf{x}$$

then

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} \\ &= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}\end{aligned}$$

and now it is the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ that determine the stability of the system, i.e. we can make an unstable system stable by appropriate choice of inputs.

2 Linear Systems

- The solution to the linear system of differential equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$$

where

$$\begin{aligned}e^{\mathbf{A}t} &= \sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!} \\ &= \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \ldots\end{aligned}$$

- If a matrix $\mathbf{A}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ and eigenvectors $\mathbf{k}_1, \ldots, \mathbf{k}_n$ then

$$\mathbf{A}\mathbf{T} = \mathbf{T}\mathbf{D}$$

or

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$$

where

$$\mathbf{T} = \begin{pmatrix} \mathbf{k}_1 & \cdots & \mathbf{k}_n \end{pmatrix}$$

and

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

- Using the above gives
$$e^{\mathbf{A}t} = \mathbf{T}e^{\mathbf{D}t}\mathbf{T}^{-1}$$
which is simpler because $e^{\mathbf{D}t}$ is easy to calculate.

- If we define $\mathbf{z}$ to be $\mathbf{x}$ transformed to the eigenvector basis, i.e.
$$\mathbf{x} = \mathbf{T}\mathbf{z}$$
and
$$\dot{\mathbf{x}} = \mathbf{T}\dot{\mathbf{z}}$$
then
$$\begin{aligned}
\mathbf{z} &= \mathbf{T}^{-1}\mathbf{x} \\
\dot{\mathbf{z}} &= \mathbf{T}^{-1}\dot{\mathbf{x}} \\
&= \mathbf{T}^{-1}\mathbf{A}\mathbf{x} \\
&= \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{z} \\
&= \mathbf{D}\mathbf{z}
\end{aligned}$$
where $\mathbf{D}$ is the diagonal matrix consisting of the eigenvalues of $\mathbf{A}$. The solution to this equation is
$$\mathbf{z}(t) = e^{\mathbf{D}t}\mathbf{z}_0$$
where
$$e^{\mathbf{D}t} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix},$$
i.e. the equations are uncoupled such that
$$\begin{aligned}
z_1(t) &= z_{0,1}e^{\lambda_1 t} \\
&\vdots \\
z_n(t) &= z_{0,n}e^{\lambda_n t}.
\end{aligned}$$

- The equivalence between the original and eigenvector coordinates can be seen by manipulating the solution
$$\begin{aligned}
\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}_0 \\
&= \mathbf{T}e^{\mathbf{D}t}\mathbf{T}^{-1}\mathbf{x}_0 \\
&= \mathbf{T}e^{\mathbf{D}t}\mathbf{z}_0 \\
&= \mathbf{T}\mathbf{z}(t).
\end{aligned}$$