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1 Introduction

I recently built this balancing pendulum on a cart. If you've ever tried to keep something like a broom or a stick upright on your hand you'll know it can be quite tricky! Gravity's constantly pulling it down, so if it's not perfectly vertical it'll start to fall. The same thing happens to the pendulum but the cart's programmed to move left and right in a way that keeps it upright.

In this video I'll explain how I built it. I'll start by deriving the system's equations of motion, I'll apply some control theory to those equations to analyse the stability of the system and determine how to control it, I'll talk about how I physically built it, and finally I'll talk about the code that controls it.

The math will make the most sense if you're familiar with calculus and a little linear algebra, but if not, don't worry, you should still be able to follow along.

2 Equations of Motion

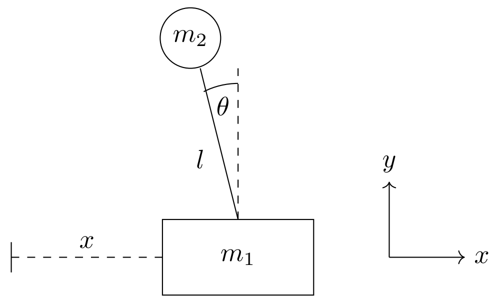
If we want to control this system we need to know how it behaves when the cart's free to move, so let's derive its equations of motion. First we need to define our coordinate system and variables in a diagram:

We have a cart of mass m_1 that can only move along the x-axis and its position x is measured from some origin point. The cart is connected to a simple pendulum of length l and mass m_2 which is at an angle θ from vertical.

With that out of the way, now we can calculate the Lagrangian of the system which is defined as the difference between its kinetic and potential energies: $\mathcal{L} = T - U$. Well, what are those energies?

Starting with the cart, its kinetic energy is

$$T_{\text{cart}} = \frac{1}{2}m_1 v^2$$



but because it can only move along the x-axis its velocity has no y component this is equivalent to

$$T_{\text{cart}} = \frac{1}{2} m_1 \dot{x}^2$$

I'm going to use Newton's notation for derivatives so when you see a dot or a double dot over a variable that's what that means. As for the cart's potential energy, it can't move up or down so its gravitational potential energy can't change and there's no springs or anything involved so we might as well set it to

$$U_{\text{cart}} = 0.$$

Next, the pendulum. Because it's joined to the cart its x coordinate changes as cart moves, and both of its coordinates change as it rotates. We can write its coordinates as

$$\begin{aligned} X &= x - l \sin\theta \\ Y &= l \cos\theta. \end{aligned}$$

Differentiating these equations with respect to time gives the x and y components of the pendulum's velocity

$$\begin{aligned} \dot{X} &= \dot{x} - l\dot{\theta} \cos\theta \\ \dot{Y} &= -l\dot{\theta} \sin\theta. \end{aligned}$$

Using the Pythagorean theorem we can combine these to find the squared magnitude of the pendulum's velocity

$$\begin{aligned} V^2 &= \dot{X}^2 + \dot{Y}^2 \\ &= (\dot{x} - l\dot{\theta}\cos\theta)^2 + (-l\dot{\theta}\sin\theta)^2 \\ &= \dot{x}^2 - 2l\dot{\theta}\dot{x}\cos\theta + l^2\dot{\theta}^2 \end{aligned}$$

and we can use this to find its kinetic energy

$$\begin{aligned} T_{\text{pendulum}} &= \frac{1}{2} m_2 V^2 \\ &= \frac{1}{2} m_2 (\dot{x}^2 - 2l\dot{\theta}\dot{x}\cos\theta + l^2\dot{\theta}^2). \end{aligned}$$

Unlike the cart, the pendulum's potential energy can change — as we saw, when it rotates it moves up and down so its gravitational potential energy changes. If we say its potential energy is 0 when its y coordinate is 0, then we can define its potential energy to be

$$\begin{aligned} U_{\text{pendulum}} &= m_2 g y \\ &= m_2 g l \cos\theta. \end{aligned}$$

Combining all of those energies gives a Lagrangian of

$$\begin{aligned} \mathcal{L} &= T - U \\ &= T_{\text{cart}} + T_{\text{pendulum}} - U_{\text{cart}} - U_{\text{pendulum}} \\ &= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (\dot{x}^2 - 2l\dot{\theta}\dot{x}\cos\theta + l^2\dot{\theta}^2) - m_2 g l \cos\theta \\ &= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_2 (l^2\dot{\theta}^2 - 2l\dot{\theta}\dot{x}\cos\theta) - m_2 g l \cos\theta. \end{aligned}$$

Now that we have the Lagrangian we can apply the Euler-Lagrange equation to each of the system's two coordinates θ and x. For θ we get

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} \\ &= \frac{d}{dt}(m_2 l^2 \dot{\theta} - m_2 l \dot{x}\cos\theta) - m_2 l \dot{\theta}\dot{x}\sin\theta - m_2 g l \sin\theta \\ &= l\ddot{\theta} - \ddot{x}\cos\theta - g\sin\theta \end{aligned}$$

and for x we get

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} \\ &= \frac{d}{dt}[(m_1 + m_2)\dot{x} - m_2 l \dot{\theta}\cos\theta] \\ &= (m_1 + m_2)\ddot{x} - m_2 l \ddot{\theta}\cos\theta + m_2 l \dot{\theta}^2 \sin\theta. \end{aligned}$$

Rearranging these two equations and solving for $\ddot{\theta}$ and $\ddot{x}$ gives us our equations of motion

$$\begin{aligned} \ddot{\theta} &= \frac{(m_1 + m_2) g \sin\theta - m_2 l \dot{\theta}^2 \cos\theta \sin\theta}{l(m_1 + m_2) - m_2 l \cos^2\theta} \\ \ddot{x} &= \frac{m_2 \sin 2\theta - 2 m_2 l \dot{\theta}^2 \sin\theta}{2m_1 + m_2 - m_2 \cos 2\theta}. \end{aligned}$$

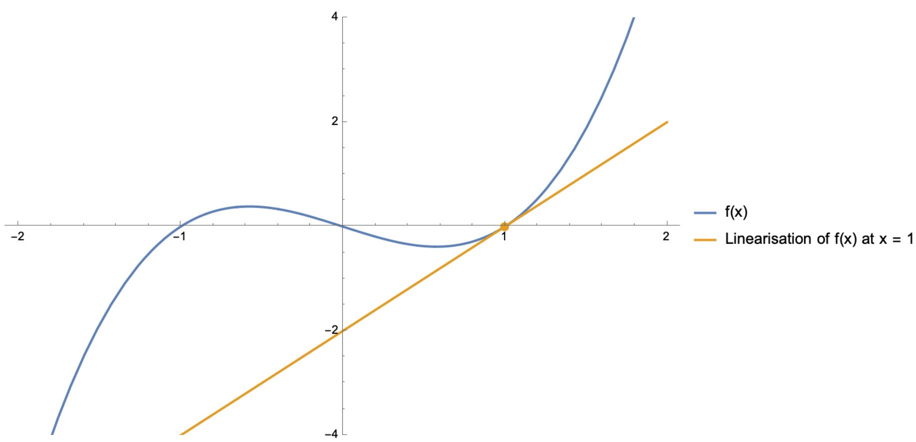
Now that we have them how do we know they're correct? One way would be to solve them for θ and x and see if their predictions are reasonable. I certainly don't know how to solve them analytically, but we can solve them numerically. This is a Mathematica notebook I created to do just that. You can see we define some constants like gravity, the length of the pendulum, etc. We define the initial conditions for the simulation — here the pendulum is starting 30° from vertical. We solve the equations of motion numerically, and we generate an animation of the solution which you can see when I evaluate the notebook.

The pendulum swings back and forth as we expect, but interestingly it also causes the cart to move. This model doesn't include air resistance or friction so it'll keep swinging forever, but we can tweak the variables and see how that affects the solution. Let's try making the cart much heavier. It still moves, but much less than before, which makes sense.

3 Linearisation

These equations look good, but they're quite complicated so it might be difficult to analyse the stability of the system and determine how to control it. Can we simplify them at all? Well, one approach is to linearise them.

Linearising a function means finding a linear approximation of it at a particular point. For example, if we graph a function $y = f(x)$ and we want to linearise it at $x = 1$, we evaluate it at the point to find its y coordinate, and we evaluate its derivative at the point to find its slope. Using this information we can plot a new function that passes through the point and extends the slope in both directions. It's a line. If you zoom in the line does a pretty good job of approximating the function. As you move away it doesn't do as good of a job, but if you only care about the region around the point that's fine.



In the case of the pendulum, our goal is to keep the cart in the middle of the track, the pendulum upright, and neither the cart nor the pendulum moving. In other words, we want all four variables θ, $\dot{\theta}$, x, and $\dot{x}$ to be 0. The further they are from 0 the less likely it is we'll be able to recover and we might have to accept that the pendulum's going to fall over or hit the end of the track. If they're always going to be near 0 then it sounds like we can tolerate the approximation error and linearisation might work for us!

So how do we linearise our equations of motion? First, let's combine them into one vector-valued function

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} = \mathbf{f}(\theta, \dot{\theta}) = \begin{bmatrix} \frac{(m_1+m_2)g\sin\theta - m_2 l\dot{\theta}^2\cos\theta\sin\theta}{l(m_1+m_2)-m_2 l\cos^2\theta} \\ \frac{m_2\sin 2\theta - 2m_2 l\dot{\theta}^2\sin\theta}{2m_1+m_2-m_2\cos 2\theta} \end{bmatrix}.$$

Let's also introduce the state vector $\mathbf{x}$ which describes the state of the system

$$\mathbf{x} = \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix}.$$

Now we can change our function to just accept the state vector $\mathbf{x}$

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} = \mathbf{f}(\mathbf{x}) = \begin{bmatrix} \frac{(m_1+m_2)g\sin\theta - m_2 l\dot{\theta}^2\cos\theta\sin\theta}{l(m_1+m_2)-m_2 l\cos^2\theta} \\ \frac{m_2\sin 2\theta - 2m_2 l\dot{\theta}^2\sin\theta}{2m_1+m_2-m_2\cos 2\theta} \end{bmatrix}$$

and if we include $\dot{\theta}$ and $\dot{x}$ in the output you can see that the function now returns the derivative of $\mathbf{x}$

$$\begin{aligned} \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \mathbf{f}(\mathbf{x}) &= \begin{bmatrix} \dot{\theta} \\ \frac{(m_1+m_2)g\sin\theta - m_2 l\dot{\theta}^2\cos\theta\sin\theta}{l(m_1+m_2)-m_2 l\cos^2\theta} \\ \dot{x} \\ \frac{m_2\sin 2\theta - 2m_2 l\dot{\theta}^2\sin\theta}{2m_1+m_2-m_2\cos 2\theta} \end{bmatrix} \\ \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) &= \begin{bmatrix} \dot{\theta} \\ \frac{(m_1+m_2)g\sin\theta - m_2 l\dot{\theta}^2\cos\theta\sin\theta}{l(m_1+m_2)-m_2 l\cos^2\theta} \\ \dot{x} \\ \frac{m_2\sin 2\theta - 2m_2 l\dot{\theta}^2\sin\theta}{2m_1+m_2-m_2\cos 2\theta} \end{bmatrix}. \end{aligned}$$

The general equation to linearise a function like this is at a particular point $\mathbf{p}$ is

$$\mathbf{f}(\mathbf{x}) \approx f(\mathbf{p}) + D\mathbf{f}|_{\mathbf{p}}(\mathbf{x} - \mathbf{p}).$$

As I mentioned before we want to linearise our function at the point where all the elements of $\mathbf{x}$ are 0. That means $\mathbf{p}$ is the zero vector

$$\mathbf{f}(\mathbf{x}) \approx f\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) + D\mathbf{f}|_{\mathbf{p}}\left(\mathbf{x} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right).$$

Looking at the equations within $\mathbf{f}(\mathbf{x})$ you can see that at the point $\mathbf{p}$, $\dot{\theta}$ is 0, the numerator of $\ddot{\theta}$ has a common term of $\sin\theta$ which is also 0, $\dot{x}$ is 0, and the numerator of $\ddot{x}$ contains the terms $\sin 2\theta$ and $\sin\theta$, both of which are 0. So $\mathbf{f}(\mathbf{p})$ is also the zero vector and we can remove it from our linearised function

$$\mathbf{f}(\mathbf{x}) \approx D\mathbf{f}|_{\mathbf{p}}\mathbf{x}.$$

$D\mathbf{f}$ is called the Jacobian matrix of our function and $D\mathbf{f}|_{\mathbf{p}}$ is it evaluated at the point $\mathbf{p}$. It plays the same role the derivative did in the previous two-dimensional example — when you multiply it by a displacement vector $\mathbf{x}$ it tells you how much each component of the function differs from the point at which it was linearised. It's like extending the slope in each dimension. If we actually evaluate this at the point $\mathbf{p}$ we get

$$D\mathbf{f}|_{\mathbf{p}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{g(m_1+m_2)}{lm_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{gm_2}{m_1} & 0 & 0 & 0 \end{bmatrix}.$$

I'm going to call this matrix $\mathbf{A}$ which means the linearised version of our function is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{g(m_1+m_2)}{lm_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{gm_2}{m_1} & 0 & 0 & 0 \end{bmatrix} \mathbf{x}.$$

I've intentionally skimmed over some details in this section in the interests of time. If you'd like to learn more about linearising nonlinear systems and when it's actually valid to do that, these two videos from Steve Brunton are really great: Linearizing Nonlinear Differential Equations Near a Fixed Point, and The Hartman-Grobman Theorem, Structural Stability of Linearization, and Stable/Unstable Manifolds.

4 Stability

Now that we have a linearised version of our equations of motion we can apply some tools from linear control theory. One of those tools is stability analysis. This tells us that if any of the eigenvalues of our $\mathbf{A}$ matrix have a positive real component, the system is unstable. Our four eigenvalues are

$$0,\ 0,\ -\sqrt{\frac{g(m_1+m_2)}{lm_1}},\ \text{and}\ \sqrt{\frac{g(m_1+m_2)}{lm_1}}.$$

All of the values under that radical are positive, so the second two eigenvalues are real and the last eigenvalue is positive which tells us what we already know: the pendulum's going to fall over.

5 Control

I mentioned earlier that the cart moves left and right to keep the pendulum upright. A more formal way to say that is: we can say we apply a force F on the cart in the x direction. We can recalculate our equations of motion to include this force, the only difference is we apply d'Alembert's principle to the Euler-Lagrange equation for the x coordinate and equate it to F. This results in the following updated equations of motion where I've hidden some terms so we can focus on the coefficients of F

$$\begin{aligned}\ddot{\theta} &= f(\theta, \dot{\theta}) + \frac{\cos\theta}{l(m_1+m_2) - lm_2\cos^2\theta}F \\ \ddot{x} &= g(\theta, \dot{\theta}) + \frac{2}{2m_1+m_2-m_2\cos 2\theta}F.\end{aligned}$$

Again, our goal is for θ to be close to 0 so we can use the small angle approximation where $\cos\theta = 1$ and that gives the simplified equations

$$\begin{aligned}\ddot{\theta} &= f(\theta, \dot{\theta}) + \frac{1}{lm_1}F \\ \ddot{x} &= g(\theta, \dot{\theta}) + \frac{1}{m_1}F.\end{aligned}$$

If we use the coefficients of F to create a new matrix $\mathbf{B}$ and substitute this into our linearised equations of motion we get

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}F \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{g(m_1+m_2)}{lm_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{gm_2}{m_1} & 0 & 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ \frac{1}{lm_1} \\ 0 \\ \frac{1}{m_1}\end{bmatrix}F.\end{aligned}$$

Another useful tool from control theory is the concept of controllability. A system is said to be controllable if it's possible to move it into any state you want using only its inputs. In our case, the only input is F — is that enough to control the system? It is if the rank of the controllability matrix equals the dimension of the state space

$$\text{rank}(\mathbf{C}) = \text{rank}(\begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}) = n.$$

If we perform this calculation using our $\mathbf{A}$ and $\mathbf{B}$ matrices we get the value 4. This is the dimension of our state vector and so we can control the system using the force F.

Now we have a way to control the system, but how do we choose F? Obviously we want it to be a function of the state of the system — for example, if the pendulum is close to vertical we want to apply a small force but if it's far from vertical we want to apply a larger force. This suggests we could define it as

$$F = \mathbf{Kx}$$

where $\mathbf{K}$ is a row matrix. Conventionally this is written as

$$F = -\mathbf{Kx}.$$

If we substitute this into our linearised equation of motion we get

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} \\ &= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}.\end{aligned}$$

This looks very similar to the original equation but $\mathbf{A}$ has been replaced by $\mathbf{A} - \mathbf{BK}$. If we can choose $\mathbf{K}$ in such a way that the eigenvalues of this new matrix don't have positive real components, the system will be stable.

So how do we choose $\mathbf{K}$? One approach is to use the Linear Quadratic Regulator algorithm. Now I don't completely understand how this works, but thankfully there's a Mathematica function that does. It accepts our matrices $\mathbf{A}$ and $\mathbf{B}$ plus two additional matrices that define how much it "costs" to apply a force to the cart and for each variable to differ from 0. For example, if we don't want to use much electricity running the motor we could set the force cost high and the resulting $\mathbf{K}$ matrix would minimise its use. Or if we don't want the cart to move far from the centre of the track we could set the x cost high and the matrix would try to minimise it but possibly at the expense of another variable like the pendulum angle.

This Mathematica notebook calculates the $\mathbf{K}$ matrix and generates an animation of a pendulum controlled by it. Like the last notebook, we define some constants, initial conditions, our matrices, we calculate $\mathbf{K}$, and generate an animation. If I evaluate the notebook we get our $\mathbf{K}$ matrix, we can see its eigenvalues don't have positive real parts so the system is stable, and we can play our animation. It looks good! Let's see what happens if we tweak one of the cost matrices. I'll increase the cost of applying a force to the cart. Regenerating the animation, we can see the cart moves much further from the centre of the track because it's trying to minimise the applied force.

6 Building

That's enough math for now. Let's talk about how I built it!

This is a model of the system that I designed in Fusion 360. At the core are two steel rods that the cart slides along on linear bearings. The cart itself and the two ends are 3D printed in PLA and on the bottom of the cart are two extensions that the timing belt connects to, but it's not visible in this model. The timing belt comes from one side of the cart, over the ilder pulley, all the way over to the timing pulley, and back to the other side of the cart. This means that when the timing pulley rotates, the cart moves.

The timing pulley is on an axle that's connected to an incremental rotary encoder so we can measure the position of the cart. On the other side of the axle is a 3D printed 5:1 gear train that's connected to a DC motor. I had to add the gear train because I bought a 24V motor which ended up being overkill for this project and I needed a way to reduce the RPM. On top of the housing is a limit switch which is primarily used to zero the cart position on startup, but it's also used to kill the motor if the cart gets too close to the end.

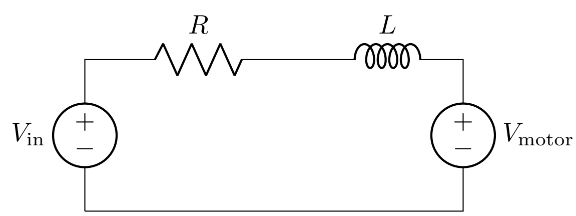
The pendulum itself is a long steel rod connected to a shorter steel rod with a 90° clamp. The shorter rod runs through some bearings and to another incremental rotary encoder that lets us measure the angle of the pendulum.

The system is all controlled by an Arduino Uno, the motor is controlled by a motor driver which is connected to a 24V power supply, and everything is connected via a breadboard.

After printing and assembling everything this is what it looks like. You can see that there are some long wires connected to the cart so it can move around.

7 Motor Control

Great! We have our control algorithm and we've built the physical system, but there's still one missing piece. The control algorithm tells us the force F to apply to the cart, but the Arduino controls the cart via a motor, and it controls the motor via a voltage. How do we convert a force into a voltage? We'll need an equation for that.



We can model the electrical properties of a DC motor using this simple circuit. V_{in} is the voltage applied to the motor, R is the resistance of its windings, L is the inductance of its windings, and V_{motor} is the back emf generated as the motor is spinning. Applying Kirchoff's voltage law to this circuit gives

$$V_{\text{in}} - IR - \frac{dI}{dt}L - V_{\text{motor}} = 0.$$

If we assume the back emf is proportional to the motor's angular velocity ω this becomes

$$V_{\text{in}} - IR - \frac{dI}{dt} - a_1\omega = 0$$

where a_1 is a proportionality constant.

We can model the mechanical properties of a DC motor by treating it as a rotating cylinder. It experiences a torque from the motor τ_m and a resistive torque τ_r from back emf, friction, etc. This gives an equation of motion

$$\tau_m - \tau_r = J\dot{\omega}$$

where J is the cylinder's moment of inertia. If we assume the torque from the motor is proportional to the current flowing through it and the resistive torque is proportional to the cylinder's angular velocity the equation of motion becomes

$$a_2 I - a_3\omega = J\dot{\omega}.$$

Solving this equation of motion for I gives

$$I = \frac{1}{a_2}(a_3\omega + J\dot{\omega})$$

and differentiating this with respect to time gives

$$\frac{dI}{dt} = \frac{1}{a_2}(a_3\dot{\omega} + J\ddot{\omega}).$$

Substituting these into the electrical equation gives

$$V_{\text{in}} - \frac{R}{a_2}(a_3\omega + J\dot{\omega}) - \frac{1}{a_2}(a_3\dot{\omega} + J\ddot{\omega}) - a_1\omega = 0.$$

This looks a little complicated, but if we rearrange and collect constants we get

$$b_1\ddot{\omega} + b_2\dot{\omega} + b_3\omega + V_{\text{in}} = 0$$

which has the general solution

$$\omega = c_1 V_{\text{in}} + c_2 e^{c_3 t} + c_4 e^{c_5 t}.$$

Taking a step back, what does this equation represent? It tells us how the motor's angular velocity changes over time if we apply a constant voltage V_{in}. Can we use our intuition about how the motor behaves to give us any clues about these constants? First of all, we probably don't need two exponential terms, so let's assume c_4 is 0

$$\omega = c_1 V_{\text{in}} + c_2 e^{c_3 t}.$$

Next, we know that if we apply a constant voltage the motor's angular velocity increases quite quickly initially but then resistive forces cause it to stabilise around a steady state value. If that's the case c_3 must be negative, otherwise the angular velocity would increase exponentially forever. The motor's angular velocity is 0 at $t = 0$ so $c_2 = -c_1 V_{\text{in}}$

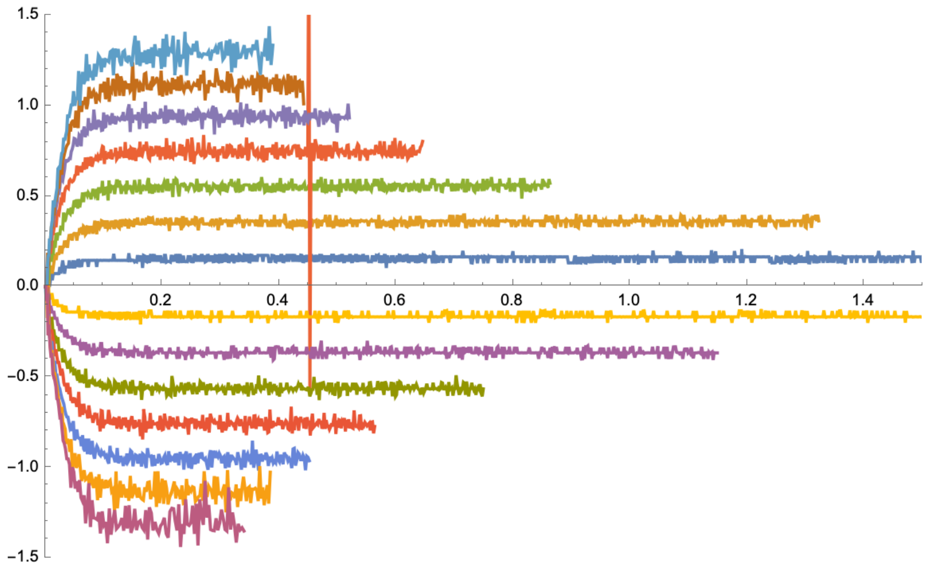
$$\omega = c_1 V_{\text{in}}(1 - e^{c_3 t}),\ c_3 < 0.$$

Finally, if we multiply both sides of the equation by the radius of the timing pulley we get the velocity of the cart instead of the angular velocity of the motor which will make things a little easier for us in an upcoming step

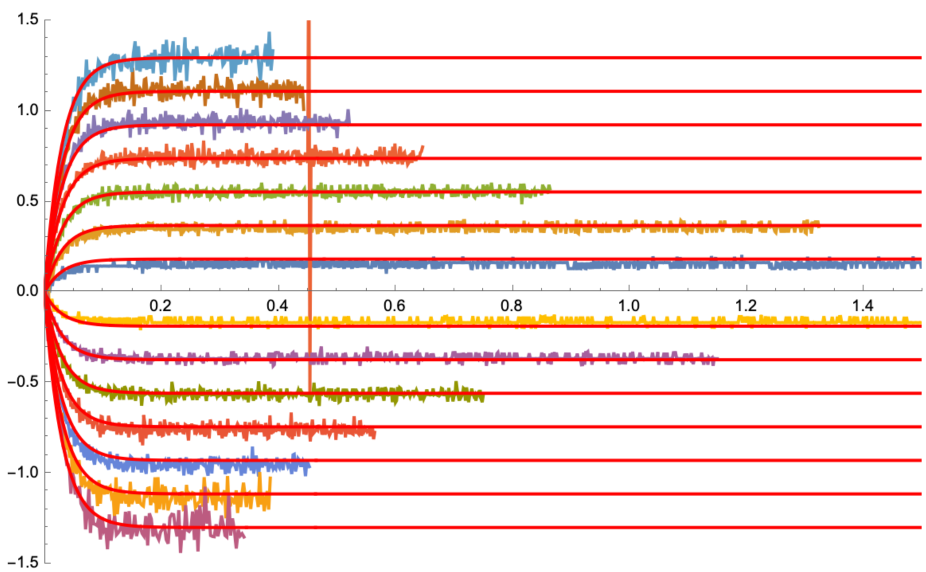
$$v = d_1 V_{\text{in}}(1 - e^{-d_2 t}).$$

This is the general equation for the cart's velocity over time at particular input voltage. In order to use it for our pendulum we need to find the values of the constants d_1 and d_2 for our motor. We can do this by collecting data on the motor's behaviour at different voltages and fitting our model to that data.

This is a Mathematica notebook I created to do just that. Here you can see the cart's velocity over time for 14 different voltages, some forwards, some backwards. Other than this blip around 0.5 s it looks as we would expect.



Mathematica's `FindFit` function can be used to find the best parameters to fit an expression to a dataset. If we use it to fit our velocity equation this dataset and plot it on top, it looks like this



A pretty good fit! One last step and we're done.

We have an equation for the cart's velocity over time but it's not very useful. Firstly because it assumes a constant voltage and we're going to be changing the voltage to move the cart back and forth. Secondly, remember, we're trying to find an equation to convert a force into voltage and this doesn't involve force at all!

How can we fix this? Newton's second law tells us that $F = ma$. If we can find an equation for the acceleration of the cart based on the input voltage, we could rearrange Newton's second law to get the voltage in terms of the force which is what we're looking for!

Let's take the derivative of our velocity equation with respect to time

$$a = d_1 d_2 V_{\text{in}} e^{-d_2 t}.$$

This looks promising, but it still assumes a constant voltage and what value would we use for t? Let's keep looking.

Earlier we found a time-indepdendent equation for the angular velocity of the motor

$$b_1 \ddot{\omega} + b_2 \dot{\omega} + b_3 \omega + V_{\text{in}} = 0.$$

If we assume that the first term is small and can be ignored

$$b_2 \dot{\omega} + b_3 \omega + V_{\text{in}} = 0,$$

multiply by the radius of the timing pulley

$$b_2 \dot{v} + b_3 v + R V_{\text{in}} = 0,$$

rearrange, and collect constants we get an equation for the cart's acceleration

$$\dot{v} = a = k_1 v + k_2 V_{\text{in}}.$$

Aha! What if we equate this with our other equation for the cart's acceleration

$$d_1 d_2 V_{\text{in}} e^{-d_2 t} = k_1 v + k_2 V_{\text{in}}.$$

We can also substitute in our equation for the cart's velocity

$$d_1 d_2 V_{\text{in}} e^{-d_2 t} = k_1 d_1 V_{\text{in}} (1 - e^{-d_2 t}) + k_2 V_{\text{in}}.$$

Rearranging we find

$$d_1 d_2 V_{\text{in}} e^{-d_2 t} = -k_1 d_1 V_{\text{in}} e^{-d_2 t} + (k_1 d_1 + k_2) V_{\text{in}}.$$

The coefficients on both sides must be equal, so $k_1 = -d_2$ and $k_2 = d_1 d_2$. This gives us an equation for the cart's acceleration based on its velocity and the input voltage

$$a = d_1 d_2 V_{\text{in}} - d_2 v.$$

Is this equation correct? One way to check is to integrate it numerically and check if the result matches the data we collected. Back in the Mathematica notebook we do that, and can see it matches pretty well.

Finally we can multiply the equation by the cart's mass to find the applied force

$$F = m_2(d_1 d_2 V_{\text{in}} - d_2 v)$$

and rearrange to find the voltage required to apply that force

$$V_{\text{in}} = \frac{1}{d_1}\left(v + \frac{F}{d_2 m_2}\right).$$