Contents

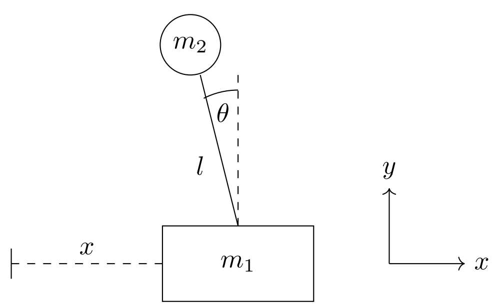
1 Introduction

I recently built this self-balancing inverted pendulum on a cart. If you've ever tried to keep something like a broom or a stick upright on your hand you'll know it can be quite tricky! Gravity's constantly pulling it down, so if it's not perfectly vertical it'll start to fall. The same thing happens to the pendulum but the cart's programmed to move left and right in a way that keeps it upright.

In this video I'll explain how I built this, starting with the math and physics, and then the construction and code. The math will make the most sense if you're familiar with calculus and a little linear algebra but if not, don't worry, you should still be able to follow along.

2 Equations of Motion

If we want to control this system we need to know how it behaves when the cart's free to move, so let's derive its equations of motion. First we need to define our coordinate system and variables in a diagram:



We have a cart of mass m_1 that can only move along the x-axis and its position x is measured from some origin point. The cart is connected to a simple pendulum of length l and mass m_2 which is at an angle θ from vertical.

With that out of the way, now we can determine the Lagrangian of the system which is defined as the difference between its kinetic and potential energies: $\mathcal{L} = T - U$. Well, what are those energies?

Starting with the cart, its kinetic energy is

$$T_{\text{cart}} = \frac{1}{2} m_1 v^2$$

but because it can only move along the x-axis its velocity has no y component this is equivalent to

$$T_{\text{cart}} = \frac{1}{2} m_1 \dot{x}^2.$$

Again, it can't move up or down, so its potential energy can't change and we might as well set it to 0

$$U_{\text{cart}} = 0.$$

Next, the pendulum. Because it's joined to the cart its x coordinate changes as cart moves, and both of its coordinates change as it rotates. We can write its coordinates as

$$\begin{aligned} X &= x - l \sin\theta \\ Y &= l \cos\theta. \end{aligned}$$

Differentiating these equations with respect to time gives the x and y components of the pendulum's velocity

$$\begin{aligned} \dot{X} &= \dot{x} - l\dot{\theta}\cos\theta \\ \dot{Y} &= -l\dot{\theta}\sin\theta. \end{aligned}$$

Using the Pythagorean theorem we can combine these to find the squared magnitude of the pendulum's velocity

$$\begin{aligned} V^2 &= \dot{X}^2 + \dot{Y}^2 \\ &= (\dot{x} - l\dot{\theta}\cos\theta)^2 + (-l\dot{\theta}\sin\theta)^2 \\ &= \dot{x}^2 - 2l\dot{\theta}\dot{x}\cos\theta + l^2\dot{\theta}^2 \end{aligned}$$

and we can use this to find its kinetic energy

$$\begin{aligned} T_{\text{pendulum}} &= \frac{1}{2} m_2 V^2 \\ &= \frac{1}{2} m_2 (\dot{x}^2 - 2l\dot{\theta}\dot{x}\cos\theta + l^2\dot{\theta}^2). \end{aligned}$$

Unlike the cart, the pendulum's potential energy can change — as we saw, when it rotates it moves up and down so its gravitational potential energy changes. If we say its potential energy is 0 when its y coordinate is 0, then we can define its potential energy to be

$$\begin{aligned} U_{\text{pendulum}} &= m_2 g y \\ &= m_2 g l \cos\theta. \end{aligned}$$

Combining all of those energies gives a Lagrangian of

$$\begin{aligned}
\mathcal{L} &= T - U \\
&= T_{\text{cart}} + T_{\text{pendulum}} - U_{\text{cart}} - U_{\text{pendulum}} \\
&= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x}^2 - 2l\dot{\theta}\dot{x}\cos\theta + l^2\dot{\theta}^2) - m_2 gl\cos\theta \\
&= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2(l^2\dot{\theta}^2 - 2l\dot{\theta}\dot{x}\cos\theta) - m_2 gl\cos\theta.
\end{aligned}$$

Now that we have the Lagrangian we can apply the Euler-Lagrange equation to each of the system's two coordinates θ and x. For θ we get

$$\begin{aligned}
0 &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} \\
&= \frac{d}{dt}(m_2 l^2\dot{\theta} - m_2 l\dot{x}\cos\theta) - m_2 l\dot{\theta}\dot{x}\sin\theta - m_2 gl\sin\theta \\
&= l\ddot{\theta} - \ddot{x}\cos\theta - g\sin\theta
\end{aligned}$$

and for x we get

$$\begin{aligned}
0 &= \frac{d}{dt}\frac{\partial L}{\partial\dot{x}} - \frac{\partial\mathcal{L}}{\partial x} \\
&= \frac{d}{dt}[(m_1 + m_2)\dot{x} - m_2 l\dot{\theta}\cos\theta] \\
&= (m_1 + m_2)\ddot{x} - m_2 l\ddot{\theta}\cos\theta + m_2 l\dot{\theta}^2\sin\theta.
\end{aligned}$$

Rearranging these two equations and solving for $\ddot{\theta}$ and $\ddot{x}$ gives us our equations of motion

$$\begin{aligned}
\ddot{\theta} &= \frac{(m_1 + m_2)g\sin\theta - m_2 l\dot{\theta}^2\cos\theta\sin\theta}{l(m_1 + m_2) - m_2 l\cos^2\theta} \\
\ddot{x} &= \frac{m_2\sin 2\theta - 2m_2 l\dot{\theta}^2\sin\theta}{2m_1 + m_2 - m_2\cos 2\theta}.
\end{aligned}$$

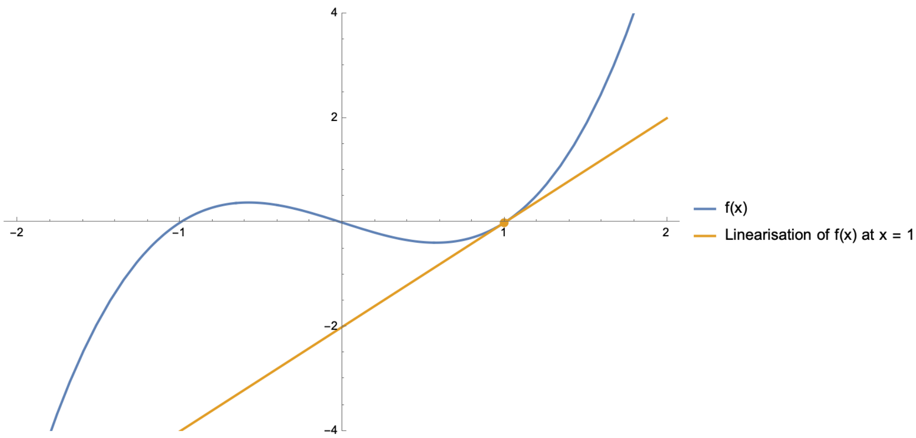
Now that we've got them how do we know they're correct? One way would be to solve them for θ and x and see if their predictions are reasonable. I certainly don't know how to solve them analytically, but we can solve them numerically. This is a Mathematica notebook I created to do just that. You can see we define some constants like gravity, the length of the pendulum, etc. We define the initial conditions for the simulation — here the pendulum is starting 30° from vertical. We solve the equations of motion numerically, and generate an animation of the solution which you can see if I evaluate the notebook.

The pendulum swings back and forth as we expect, but interestingly it also causes the cart to move. This model doesn't include air resistance or friction so it'll keep swinging forever, but we can tweak the variables and see how that affects the solution. Let's make the cart much heavier. It still moves, but much less than before, which makes sense.

3 Linearisation

These equations look good, but they're quite complicated so it might be difficult to analyse the stability of the system and determine how to control it. Can we simplify them at all? Well, one approach is to linearise them.

Linearising a function means finding a linear approximation of it at a particular point. For example, if we graph a function $y = f(x)$ and we want to linearise it at $x = 1$, we evaluate it at the point to find its y coordinate, and we evaluate its derivative at the point to find its slope. Using this information we can plot a new function that passes through the point and extends its slope in both directions. It's a line. If you zoom in the line does a pretty good job of approximating the function. As you move away it doesn't do as good of a job, but if you only care about the area around the point that's fine.



In the case of the pendulum, our goal is to keep the cart in the middle of the track, the pendulum upright, and neither of them moving. In other words, we want all four variables θ, $\dot{\theta}$, x, and $\dot{x}$ to be 0. The further they are from 0 the less likely it is we'll be able to recover and we might have to accept that the pendulum's going to fall over. It sounds like we can tolerate the approximation error and linearisation might work for us!

So how do we linearise our equations of motion? First, let's rewrite them in vector form. We'll combine our two real-valued functions into one vector-valued function

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{x} \end{bmatrix} = \mathbf{f}(\theta, \dot{\theta}) = \begin{bmatrix} \frac{(m_1+m_2)g\sin\theta - m_2 l\dot{\theta}^2\cos\theta\sin\theta}{l(m_1+m_2) - m_2 l\cos^2\theta} \\ \frac{m_2\sin 2\theta - 2m_2 l\dot{\theta}^2\sin\theta}{2m_1+m_2-m_2\cos 2\theta} \end{bmatrix}.$$

Let's also introduce the state vector $\mathbf{x}$ which can be used to describe the state of the system

$$\mathbf{x} = \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix}.$$

Now we can change our function to accept a single variable $\mathbf{x}$

$$\begin{bmatrix}\ddot{\theta}\\ \ddot{x}\end{bmatrix} = \mathbf{f}(\mathbf{x}) = \begin{bmatrix}\frac{(m_1+m_2)g\sin\theta - m_2 l\dot{\theta}^2\cos\theta\sin\theta}{l(m_1+m_2)-m_2 l\cos^2\theta}\\ \frac{m_2\sin 2\theta - 2m_2 l\dot{\theta}^2\sin\theta}{2m_1+m_2-m_2\cos 2\theta}\end{bmatrix}$$

and if we include $\dot{\theta}$ and $\dot{x}$ in the output vector you can see that the function now returns the derivative of $\mathbf{x}$

$$\begin{bmatrix}\dot{\theta}\\ \ddot{\theta}\\ \dot{x}\\ \ddot{x}\end{bmatrix} = \mathbf{f}(\mathbf{x}) = \begin{bmatrix}\dot{\theta}\\ \frac{(m_1+m_2)g\sin\theta - m_2 l\dot{\theta}^2\cos\theta\sin\theta}{l(m_1+m_2)-m_2 l\cos^2\theta}\\ \dot{x}\\ \frac{m_2\sin 2\theta - 2m_2 l\dot{\theta}^2\sin\theta}{2m_1+m_2-m_2\cos 2\theta}\end{bmatrix}$$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \begin{bmatrix}\dot{\theta}\\ \frac{(m_1+m_2)g\sin\theta - m_2 l\dot{\theta}^2\cos\theta\sin\theta}{l(m_1+m_2)-m_2 l\cos^2\theta}\\ \dot{x}\\ \frac{m_2\sin 2\theta - 2m_2 l\dot{\theta}^2\sin\theta}{2m_1+m_2-m_2\cos 2\theta}\end{bmatrix}.$$

The general equation to linearise a vector-valued multivariable function like this is at a particular point $\mathbf{p}$ is

$$\mathbf{f}(\mathbf{x}) \approx f(\mathbf{p}) + D\mathbf{f}|_{\mathbf{p}}(\mathbf{x}-\mathbf{p}).$$

As I mentioned before we want to linearise our function at the point where all the elements of $\mathbf{x}$ are 0. That means $\mathbf{p}$ is the zero vector

$$\mathbf{f}(\mathbf{x}) \approx f\left(\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\right) + D\mathbf{f}|_{\mathbf{p}}\left(\mathbf{x} - \begin{bmatrix}0\\0\\0\\0\end{bmatrix}\right).$$

Looking at the equations within $\mathbf{f}(\mathbf{x})$ you can see that at this point $\dot{\theta}$ is 0, the numerator of $\ddot{\theta}$ has a common term of $\sin\theta$ which is also 0, $\dot{x}$ is 0, and the numerator of $\ddot{x}$ contains the terms $\sin 2\theta$ and $\sin\theta$, both of which are 0. So $\mathbf{f}(\mathbf{p})$ is also the zero vector and we can remove it from our linearised function

$$\mathbf{f}(\mathbf{x}) \approx D\mathbf{f}|_{\mathbf{p}}\mathbf{x}.$$

The term $D\mathbf{f}|_{\mathbf{p}}$ is the Jacobian matrix of our function evaluated at the point $\mathbf{p}$. It plays the same role the derivative did in the previous two-dimensional example — namely when multiplied by a vector $\mathbf{x}$ it tells you how much each component of the function differs from the point at which it was linearised. If we actually evaluate this at the point $\mathbf{p}$ we get

$$D\mathbf{f}|_{\mathbf{p}} = \begin{bmatrix}0 & 1 & 0 & 0\\ \frac{g(m_1+m_2)}{lm_1} & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ \frac{gm_2}{m_1} & 0 & 0 & 0\end{bmatrix}.$$

I'm going to call this matric $\mathbf{A}$ which means the linearised version of our function is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{g(m_1+m_2)}{lm_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{gm_2}{m_1} & 0 & 0 & 0 \end{bmatrix} \mathbf{x}.$$

I've intentionally skimmed over some details in this section because it's quite a complicated topic. If you'd like to learn more about linearising nonlinear systems and when it's actually valid to do that, these two videos from Steve Brunton are great: Linearizing Nonlinear Differential Equations Near a Fixed Point, and The Hartman-Grobman Theorem, Structural Stability of Linearization, and Stable/Unstable Manifolds.

4 Stability

Now that we have a linearised version of our equations of motion we can apply the tools of linear control theory. One such tool is stability analysis. This tells us that if any of the eigenvalues of our $\mathbf{A}$ matrix have a positive real component, the system is unstable or it'll move away from the point at which it was linearised. Our four eigenvalues are

$$0, \ 0, \ -\sqrt{\frac{g(m_1+m_2)}{lm_1}}, \text{ and } \sqrt{\frac{g(m_1+m_2)}{lm_1}}.$$

The last of these has a positive real component which tells us what we already know: the system is unstable.