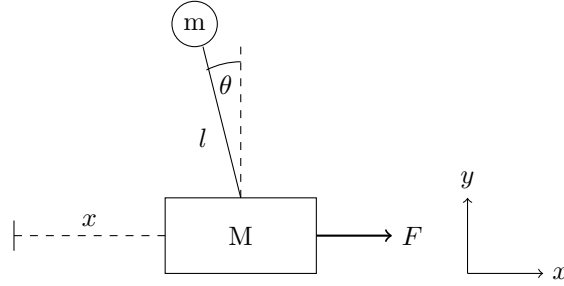


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1 Equations of Motion

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1 Equations of Motion



- A cart of mass M is constrained to move along the x axis with its distance from an arbitrary point on the x axis denoted x . A driving force of magnitude F is applied to the cart in the x direction. A simple pendulum consisting of a mass m and a massless rod of length l is connected to the cart with its angle from the positive y axis denoted θ .
- The kinetic energy of the cart is

$$T_{\text{cart}} = \frac{1}{2}M\dot{x}^2.$$

- The x and y coordinates of the pendulum are

$$\begin{aligned} X &= x - l \sin \theta \\ Y &= l \cos \theta, \end{aligned}$$

thus its x and y velocities are

$$\begin{aligned} \dot{X} &= \dot{x} - l\dot{\theta} \cos \theta \\ \dot{Y} &= -l\dot{\theta} \sin \theta \end{aligned}$$

and its kinetic energy is

$$\begin{aligned} T_{\text{pendulum}} &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}(\dot{X}^2 + \dot{Y}^2) \\ &= \frac{1}{2}m[(\dot{x} - l\dot{\theta} \cos \theta)^2 + (-l\dot{\theta} \sin \theta)^2] \\ &= \frac{1}{2}m(\dot{x}^2 - 2l\dot{x}\dot{\theta} \cos \theta + l^2\dot{\theta}^2). \end{aligned}$$

- The total kinetic energy of the system is

$$\begin{aligned} T &= T_{\text{cart}} + T_{\text{pendulum}} \\ &= \frac{1}{2}(m + M)\dot{x}^2 + \frac{1}{2}m(l^2\dot{\theta}^2 - 2l\dot{x}\dot{\theta}\cos\theta). \end{aligned}$$

- The potential energy of the system is equal to the gravitational potential energy of the pendulum. If its potential energy is 0 when $\theta = \frac{\pi}{2}$ then

$$U = mgl \cos \theta.$$

- The Lagrangian of the system is

$$\begin{aligned} \mathcal{L} &= T - U \\ &= \frac{1}{2}(m + M)\dot{x}^2 + \frac{1}{2}m(l^2\dot{\theta}^2 - 2l\dot{x}\dot{\theta}\cos\theta) - mgl \cos \theta. \end{aligned}$$

- By d'Alembert's principle the generalized forces associated with the θ and x coordinates are 0 and F , respectively.
- The Euler-Lagrange equation for the θ coordinate is

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} &= 0 \\ \frac{d}{dt}(ml^2\dot{\theta} - ml\dot{x}\cos\theta) - ml\dot{x}\sin\theta - mgl \sin \theta &= 0 \\ l\ddot{\theta} - \ddot{x}\cos\theta - g \sin \theta &= 0. \end{aligned}$$

- The Euler-Lagrange equation for the x coordinate is

$$\begin{aligned} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} &= F \\ \frac{d}{dt}[(m + M)\dot{x} - ml\dot{\theta}\cos\theta] &= F \\ (m + M)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2 \sin \theta &= F. \end{aligned}$$

- Solving these equations for $\ddot{\theta}$ and \ddot{x} gives

$$\ddot{\theta} = \frac{(m + M)g \sin \theta + F \cos \theta - ml\dot{\theta}^2 \cos \theta \sin \theta}{l(m + M) - ml \cos^2 \theta}$$

and

$$\ddot{x} = \frac{2F + mg \sin 2\theta - 2ml\dot{\theta}^2 \sin \theta}{m + 2M - m \cos 2\theta}.$$