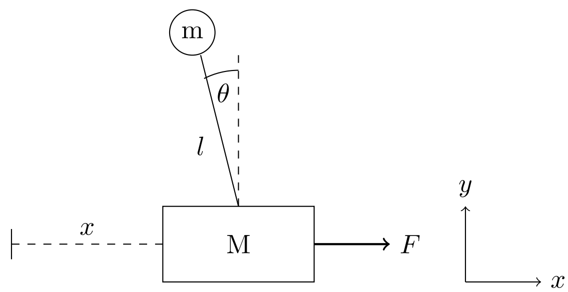
Contents

1 Equations of Motion



- A cart of mass M is constrained to move along the x axis with its distance from an arbitrary point on the x axis denoted x. A driving force of magnitude F is applied to the cart in the x direction. A simple pendulum consisting of a mass m and a massless rod of length l is connected to the cart with its angle from the positive y axis denoted θ.

- The kinetic energy of the cart is
$$T_{\text{cart}} = \frac{1}{2}M\dot{x}^2.$$

- The x and y coordinates of the pendulum are
$$\begin{aligned} X &= x - l\sin\theta \\ Y &= l\cos\theta, \end{aligned}$$
thus its x and y velocities are
$$\begin{aligned} \dot{X} &= \dot{x} - l\dot{\theta}\cos\theta \\ \dot{Y} &= -l\dot{\theta}\sin\theta \end{aligned}$$
and its kinetic energy is
$$\begin{aligned} T_{\text{pendulum}} &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}(\dot{X}^2 + \dot{Y}^2) \\ &= \frac{1}{2}m[(\dot{x} - l\dot{\theta}\cos\theta)^2 + (-l\dot{\theta}\sin\theta)^2] \\ &= \frac{1}{2}m(\dot{x}^2 - 2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2). \end{aligned}$$

- The total kinetic energy of the system is

$$\begin{aligned} T &= T_{\text{cart}} + T_{\text{pendulum}} \\ &= \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}m(l^2\dot{\theta}^2 - 2l\dot{x}\dot{\theta}\cos\theta). \end{aligned}$$

- The potential energy of the system is equal to the gravitational potential energy of the pendulum. If its potential energy is 0 when $\theta = \frac{\pi}{2}$ then

$$U = mgl\cos\theta.$$

- The Lagrangian of the system is

$$\begin{aligned} \mathcal{L} &= T - U \\ &= \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}m(l^2\dot{\theta}^2 - 2l\dot{x}\dot{\theta}\cos\theta) - mgl\cos\theta. \end{aligned}$$

- By d'Alembert's principle the generalized forces associated with the θ and x coordinates are 0 and F, respectively.

- The Euler-Lagrange equation for the θ coordinate is

$$\begin{aligned} \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} &= 0 \\ \frac{d}{dt}(ml^2\dot{\theta} - ml\dot{x}\cos\theta) - ml\dot{x}\dot{\theta}\sin\theta - mgl\sin\theta &= 0 \\ l\ddot{\theta} - \ddot{x}\cos\theta - g\sin\theta &= 0. \end{aligned}$$

  As the goal is to keep θ close to 0 this can be linearized by using the approximations $\cos\theta \approx 1$ and $\sin\theta \approx \theta$ giving

$$l\ddot{\theta} - \ddot{x} - g\theta = 0.$$

- The Euler-Lagrange equation for the x coordinate is

$$\begin{aligned} \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} - \frac{\partial\mathcal{L}}{\partial x} &= F \\ \frac{d}{dt}[(m+M)\dot{x} - ml\dot{\theta}\cos\theta] &= F \\ (m+M)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta &= F. \end{aligned}$$

  Using the same approximations as above gives

$$(m+M)\ddot{x} - ml\ddot{\theta} + ml\dot{\theta}^2\theta = F.$$