This document contains my notes on Steve Brunton's Control Bootcamp video series. Each section corresponds to the video of the same title.

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1 Overview

- **Passive controls** attempt to control a system passively, i.e. they are built into the system and don't vary based on observations of the system.
- **Active controls** attempt to control a system actively, i.e. they change their behaviour based on observations of the system.
- **Open-loop controllers** don't observe the output of the system — their inputs are predetermined. One downside of this approach is that you may unnecessarily input energy into the system when it already has the desired output.
- **Closed-loop controllers** observe the output of the system to determine their inputs. They have several benefits over open-loop controllers:
  - They handle uncertainty in the system, e.g. if you don't always know how the system will respond, or if your model isn't completely accurate.
  - They handle disturbances in the system, e.g. if someone pushes a self-balancing inverted pendulum. It may not be possible to account for these in the model.
  - They can be more energy efficient than open-loop controllers, i.e. they don't have the downside mentioned above.
- The mathematical model used in this course is a state-space based system of linear differential equations

  $$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

  where $\mathbf{x}$ is the **state vector** — a vector containing all the system's values of interest — and $\dot{\mathbf{x}}$ are their rates of change at a time t.

- The solution to the above equation is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$$

and the eigenvalues of $\mathbf{A}$ can be used to determine the stability of the system, e.g. if they all have negative real components the system is stable.

- Control is introduced to the system by modifying the equation to

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where $\mathbf{B}$ is a coefficient matrix and $\mathbf{u}$ is the input to the system.

- If we make the input to the system

$$\mathbf{u} = -\mathbf{K}\mathbf{x}$$

then

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} \\ &= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} \end{aligned}$$

and now it is the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$ that determine the stability of the system, i.e. we can make an unstable system stable by appropriate choice of inputs.

2 Linear Systems

- The solution to the linear system of differential equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$$

where

$$\begin{aligned} e^{\mathbf{A}t} &= \sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!} \\ &= \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \dots \end{aligned}$$

- If a matrix $\mathbf{A}$ has eigenvalues $\lambda_1, \dots, \lambda_n$ and eigenvectors $\mathbf{k}_1, \dots, \mathbf{k}_n$ then

$$\mathbf{A}\mathbf{T} = \mathbf{T}\mathbf{D}$$

or

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$$

where

$$\mathbf{T} = \begin{pmatrix} \mathbf{k}_1 & \cdots & \mathbf{k}_n \end{pmatrix}$$

and

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

- Using the above gives
$$e^{\mathbf{A}t} = \mathbf{T}e^{\mathbf{D}t}\mathbf{T}^{-1}$$
which is simpler because $e^{\mathbf{D}t}$ is easy to calculate.

- If we define $\mathbf{z}$ to be $\mathbf{x}$ transformed to the eigenvector basis, i.e.
$$\mathbf{x} = \mathbf{T}\mathbf{z}$$
and
$$\dot{\mathbf{x}} = \mathbf{T}\dot{\mathbf{z}}$$
then
$$\begin{aligned} \mathbf{z} &= \mathbf{T}^{-1}\mathbf{x} \\ \dot{\mathbf{z}} &= \mathbf{T}^{-1}\dot{\mathbf{x}} \\ &= \mathbf{T}^{-1}\mathbf{A}\mathbf{x} \\ &= \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{z} \\ &= \mathbf{D}\mathbf{z} \end{aligned}$$
where $\mathbf{D}$ is the diagonal matrix consisting of the eigenvalues of $\mathbf{A}$. The solution to this equation is
$$\mathbf{z}(t) = e^{\mathbf{D}t}\mathbf{z}_0$$
where
$$e^{\mathbf{D}t} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix},$$
i.e. the equations are uncoupled such that
$$\begin{aligned} z_1(t) &= z_{0,1}e^{\lambda_1 t} \\ &\vdots \\ z_n(t) &= z_{0,n}e^{\lambda_n t}. \end{aligned}$$

- The equivalence between the original and eigenvector coordinates can be seen by manipulating the solution

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}_0 \\ &= \mathbf{T}e^{\mathbf{D}t}\mathbf{T}^{-1}\mathbf{x}_0 \\ &= \mathbf{T}e^{\mathbf{D}t}\mathbf{z}_0 \\ &= \mathbf{T}\mathbf{z}(t).\end{aligned}$$

3 Stability and Eigenvalues

- A system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is stable if and only if all of the eigenvalues of $\mathbf{A}$ have negative real parts. This is because the solution is of the form[1]

  $$\mathbf{x} = \mathbf{T}e^{\mathbf{D}t}\mathbf{T}^{-1}\mathbf{x}_0$$

  where

  $$e^{\mathbf{D}t} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix}$$

  and

  $$e^{\lambda_1 t} = e^{(a+ib)t} = e^{at}e^{ibt}$$

  which is only stable if $a < 0$.

- Sometimes you can make a system stable (i.e. make all the eigenvalues have negative real parts) via the control term $\mathbf{Bu}$.

- In a real-world system, control signals will be sent and observations received in discrete steps rather than continuously. A variation of the equation that captures this is

  $$\mathbf{x}_{k+1} = \tilde{\mathbf{A}}\mathbf{x}_k$$

  where

  $$\tilde{\mathbf{A}} = e^{\mathbf{A}\Delta t}.$$

  In other words,

  $$\begin{aligned}\mathbf{x}_k &= \tilde{\mathbf{A}}^k\mathbf{x}_0 \\ &= (\tilde{\mathbf{T}}\tilde{\mathbf{D}}\tilde{\mathbf{T}}^{-1})^k\mathbf{x_0} \\ &= \tilde{\mathbf{T}}\tilde{\mathbf{D}}^k\tilde{\mathbf{T}}^{-1}\mathbf{x}_0\end{aligned}$$

  where

  $$\tilde{\mathbf{D}}^k = \begin{pmatrix} \tilde{\lambda}_1^k & & 0 \\ & \ddots & \\ 0 & & \tilde{\lambda}_n^k \end{pmatrix}$$

[1] See here for a proof of why $e^{\mathbf{A}t} = \mathbf{T}e^{\mathbf{D}t}\mathbf{T}^{-1}$.

so the discrete system is stable if the moduli of all the eigenvalues of $\tilde{\mathbf{A}}$ are less than or equal to 1.

- Note that the stability criteria of a discrete system (the moduli of of all eigenvalues must be less than 1) differs from that of a continuous system (the real parts of all eigenvalues must be negative) because we're raising $\tilde{\mathbf{A}} = e^{\mathbf{A}\Delta t}$ to a power directly rather than exponentiating it so both the real and imaginary components of its eigenvalues contribute to the "radius"

$$\begin{aligned}\lambda_n^k &= (a+ib)^k \\ &= (\sqrt{a^2+b^2}e^{i\arctan\frac{b}{a}})^k \\ &= \sqrt{a^2+b^2}^k e^{ik\arctan\frac{b}{a}}.\end{aligned}$$

4 Linearizing Around a Fixed Point

- The **Jacobian matrix** of a vector-valued function of several variables $\mathbf{f}: \mathbb{R}^n \to \mathbb{R}^m$ is defined as

$$D\mathbf{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

- If $\mathbf{x}$ is a point and $\Delta\mathbf{x}$ is a displacement, both in $\mathbb{R}^n$, the Jacobian matrix can be used to linearize $\mathbf{f}$ around $\mathbf{x}$

$$\mathbf{f}(\mathbf{x}+\Delta\mathbf{x}) \approx \mathbf{f}(\mathbf{x}) + D\mathbf{f}|_{\mathbf{x}}\Delta\mathbf{x}.$$

- To convert a nonlinear system $\dot{\mathbf{x}} = f(\mathbf{x})$ to a linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$:

  1. Find fixed points $\overline{\mathbf{x}}$ such that $f(\overline{\mathbf{x}}) = \mathbf{0}$.
  2. Use the Jacobian matrix to linearize $f(\mathbf{x})$ about $\overline{\mathbf{x}}$, i.e.

  $$\dot{\mathbf{x}} = f(\mathbf{x})$$

  is approximated by

  $$\begin{aligned}\dot{\mathbf{x}} &= f(\overline{\mathbf{x}}) + Df|_{\overline{\mathbf{x}}}\Delta\mathbf{x} \\ &= Df|_{\overline{\mathbf{x}}}\Delta\mathbf{x} \\ &= \mathbf{A}\Delta\mathbf{x}\end{aligned}$$

  where $\mathbf{A} = Df|_{\overline{\mathbf{x}}}$. However this is only valid if all eigenvalues of $\mathbf{A}$ have a nonzero real part.