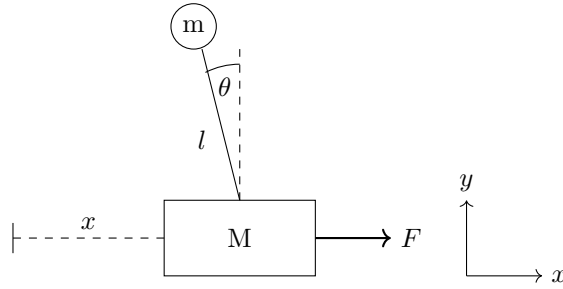


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1 Equations of Motion



- A cart of mass M is constrained to move along the x axis with its distance from an arbitrary point on the x axis denoted x . A driving force of magnitude F is applied to the cart in the x direction. A simple pendulum consisting of a mass m and a massless rod of length l is connected to the cart with its angle from the positive y axis denoted θ .
- The kinetic energy of the cart is

$$T_{\text{cart}} = \frac{1}{2}M\dot{x}^2.$$

- The x and y coordinates of the pendulum are

$$X = x - l \sin \theta$$

$$Y = l \cos \theta,$$

thus its x and y velocities are

$$\dot{X} = \dot{x} - l\dot{\theta} \cos \theta$$

$$\dot{Y} = -l\dot{\theta} \sin \theta$$

and its kinetic energy is

$$\begin{aligned}
T_{\text{pendulum}} &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}(\dot{X}^2 + \dot{Y}^2) \\
&= \frac{1}{2}m[(\dot{x} - l\dot{\theta}\cos\theta)^2 + (-l\dot{\theta}\sin\theta)^2] \\
&= \frac{1}{2}m(\dot{x}^2 - 2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2).
\end{aligned}$$

- The total kinetic energy of the system is

$$\begin{aligned}
T &= T_{\text{cart}} + T_{\text{pendulum}} \\
&= \frac{1}{2}(m + M)\dot{x}^2 + \frac{1}{2}m(l^2\dot{\theta}^2 - 2l\dot{x}\dot{\theta}\cos\theta).
\end{aligned}$$

- The potential energy of the system is equal to the gravitational potential energy of the pendulum. If its potential energy is 0 when $\theta = \frac{\pi}{2}$ then

$$U = mgl\cos\theta.$$

- The Lagrangian of the system is

$$\begin{aligned}
\mathcal{L} &= T - U \\
&= \frac{1}{2}(m + M)\dot{x}^2 + \frac{1}{2}m(l^2\dot{\theta}^2 - 2l\dot{x}\dot{\theta}\cos\theta) - mgl\cos\theta.
\end{aligned}$$

- By d'Alembert's principle the generalized forces associated with the θ and x coordinates are 0 and F , respectively.
- The Euler-Lagrange equation for the θ coordinate is

$$\begin{aligned}
\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} &= 0 \\
\frac{d}{dt}(ml^2\dot{\theta} - ml\dot{x}\cos\theta) - ml\dot{x}\sin\theta - mgl\sin\theta &= 0 \\
l\ddot{\theta} - \ddot{x}\cos\theta - g\sin\theta &= 0.
\end{aligned}$$

- The Euler-Lagrange equation for the x coordinate is

$$\begin{aligned}
\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} - \frac{\partial\mathcal{L}}{\partial x} &= F \\
\frac{d}{dt}[(m + M)\dot{x} - ml\dot{\theta}\cos\theta] &= F \\
(m + M)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta &= F.
\end{aligned}$$

- Solving these equations for $\ddot{\theta}$ and \ddot{x} gives

$$\ddot{\theta} = \frac{(m + M)g \sin \theta + F \cos \theta - ml\dot{\theta}^2 \cos \theta \sin \theta}{l(m + M) - ml \cos^2 \theta}$$

and

$$\ddot{x} = \frac{2F + mg \sin 2\theta - 2ml\dot{\theta}^2 \sin \theta}{m + 2M - m \cos 2\theta}.$$

2 Linearization, Stability, and Controllability

- The state vector for this system is

$$\begin{pmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{pmatrix}.$$

- The fixed point about which the system will be linearized is

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

- The \mathbf{A} matrix is equal to the Jacobian matrix evaluated at the fixed point

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{g(m+M)}{lM} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{gm}{M} & 0 & 0 & 0 \end{pmatrix}.$$

- The non-zero eigenvalues of \mathbf{A} are

$$\pm \sqrt{\frac{g(m + M)}{lM}}.$$

Because one of these has a positive real part the system is unstable.

- Rearranging the equations of motion to find the coefficients of F gives

$$\ddot{\theta} = f(\theta, \dot{\theta}) + \frac{\cos \theta}{l(m + M) - ml \cos^2 \theta} F$$

and

$$\ddot{x} = g(\theta, \dot{\theta}) + \frac{2}{m + 2M - m \cos 2\theta} F.$$

Using the small angle approximation for \cos gives

$$\ddot{\theta} = f(\theta, \dot{\theta}) + \frac{1}{lM}F$$

and

$$\ddot{x} = g(\theta, \dot{\theta}) + \frac{1}{M}F$$

resulting in the \mathbf{B} matrix

$$\begin{pmatrix} 0 \\ \frac{1}{lM} \\ 0 \\ \frac{1}{M} \end{pmatrix}.$$

- The controllability matrix

$$C = (\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \mathbf{A}^3\mathbf{B})$$

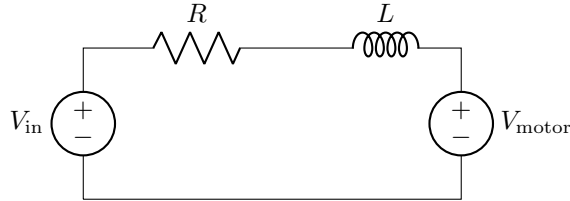
has full rank (4) so the system is controllable via the force F on the cart.

- The ideal state feedback gains matrix \mathbf{K} can be determined using Mathematica's `LQRegulatorGains` function.
- Thus, the force to apply to the cart when the system is in state \mathbf{x} is

$$F = u = -\mathbf{K}\mathbf{x}.$$

3 Motor Control

- Using the equations above we can calculate the force to apply to the cart when the system is in a given state. However, the motor (which applies a force to the cart) is controlled via pulse-width modulation (PWM) and it's not clear what force results from a particular duty cycle.
- The motor can be electrically modelled by the circuit



where V_{in} is the (effective) voltage delivered to the motor via PWM, R is the resistance of the motor's windings, L is the inductance of the motor's windings, and V_{motor} is the back EMF generated as the motor is spinning.

- If we assume that the back EMF generated by the motor is proportional to its angular velocity, i.e. $V_{\text{motor}} = K_b\omega$, then Kirchoff's voltage law gives

$$V_{\text{in}} - IR - \frac{dI}{dt}L - V_{\text{motor}} = 0$$

$$V_{\text{in}} - IR - \frac{dI}{dt}L - K_b\omega = 0.$$

- The motor can be mechanically modelled as a rotating cylinder with equation of motion

$$\tau_m + \tau_r = J\alpha$$

$$K_t I - b\omega = J\alpha$$

where τ_m is the torque due to the motor which is proportional to the current, τ_r is the resistive torque due to air resistance, friction, etc. which is proportional to the angular velocity ω , J is the moment of inertia, and α is the angular acceleration.

- Rearranging the last equation for I we find

$$I = \frac{1}{K_t}(b\omega + J\alpha)$$

and

$$\frac{dI}{dt} = \frac{1}{K_t}(b\alpha + J\dot{\alpha}).$$

- Substituting this into the electrical equation above gives

$$V_{\text{in}} - \frac{R}{K_t}(b\omega + J\alpha) - \frac{L}{K_t}(b\alpha + J\dot{\alpha}) - K_b\omega = 0$$

$$V_{\text{in}} - \left(\frac{bR}{K_t} + K_b\right)\omega - \left(\frac{JR}{K_t} + \frac{bL}{K_t}\right)\alpha - \frac{JL}{K_t}\dot{\alpha} = 0$$

or

$$A\ddot{\omega} + B\dot{\omega} + C\omega + V_{\text{in}} = 0$$

where A , B , and C are constants.

- The general solution to this equation is

$$\omega = AV_{\text{in}} + Be^{Ct} + De^{Et}$$

where A , B , C , D , and E are different constants from above.

- The equation can be simplified by assuming $D = 0$

$$\omega = AV_{\text{in}} + Be^{Ct}.$$

- In reality ω quickly changes from 0 to some equilibrium value that has the same sign as V_{in} . That being the case, we know:
 - C must be negative otherwise ω would increase without bound,
 - A must be positive as the equilibrium value has the same sign as V_{in} , and
 - $B = -AV_{\text{in}}$ because $\omega(0) = 0$.

This gives a final equation

$$\omega = AV_{\text{in}}(1 - e^{-Bt})$$

where $A, B > 0$ are constants that can be determined from experimental data.