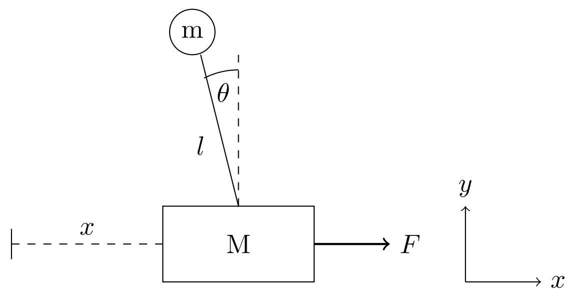
Contents

1 Equations of Motion



- A cart of mass M is constrained to move along the x axis with its distance from an arbitrary point on the x axis denoted x. A driving force of magnitude F is applied to the cart in the x direction. A simple pendulum consisting of a mass m and a massless rod of length l is connected to the cart with its angle from the positive y axis denoted θ.

- The kinetic energy of the cart is

$$T_{\text{cart}} = \frac{1}{2}M\dot{x}^2.$$

- The x and y coordinates of the pendulum are

$$X = x - l\sin\theta$$
$$Y = l\cos\theta,$$

thus its x and y velocities are

$$\dot{X} = \dot{x} - l\dot{\theta}\cos\theta$$
$$\dot{Y} = -l\dot{\theta}\sin\theta$$

and its kinetic energy is

$$\begin{aligned}
T_{\text{pendulum}} &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}(\dot{X}^2 + \dot{Y}^2) \\
&= \frac{1}{2}m[(\dot{x} - l\dot{\theta}\cos\theta)^2 + (-l\dot{\theta}\sin\theta)^2] \\
&= \frac{1}{2}m(\dot{x}^2 - 2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2).
\end{aligned}$$

- The total kinetic energy of the system is

$$\begin{aligned}
T &= T_{\text{cart}} + T_{\text{pendulum}} \\
&= \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}m(l^2\dot{\theta}^2 - 2l\dot{x}\dot{\theta}\cos\theta).
\end{aligned}$$

- The potential energy of the system is equal to the gravitational potential energy of the pendulum. If its potential energy is 0 when $\theta = \frac{\pi}{2}$ then

$$U = mgl\cos\theta.$$

- The Lagrangian of the system is

$$\begin{aligned}
\mathcal{L} &= T - U \\
&= \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}m(l^2\dot{\theta}^2 - 2l\dot{x}\dot{\theta}\cos\theta) - mgl\cos\theta.
\end{aligned}$$

- By d'Alembert's principle the generalized forces associated with the θ and x coordinates are 0 and F, respectively.
- The Euler-Lagrange equation for the θ coordinate is

$$\begin{aligned}
\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} &= 0 \\
\frac{d}{dt}(ml^2\dot{\theta} - ml\dot{x}\cos\theta) - ml\dot{x}\dot{\theta}\sin\theta - mgl\sin\theta &= 0 \\
l\ddot{\theta} - \ddot{x}\cos\theta - g\sin\theta &= 0.
\end{aligned}$$

- The Euler-Lagrange equation for the x coordinate is

$$\begin{aligned}
\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} - \frac{\partial\mathcal{L}}{\partial x} &= F \\
\frac{d}{dt}[(m+M)\dot{x} - ml\dot{\theta}\cos\theta] &= F \\
(m+M)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta &= F.
\end{aligned}$$

- Solving these equations for $\ddot{\theta}$ and $\ddot{x}$ gives

$$\ddot{\theta} = \frac{(m+M)g\sin\theta + F\cos\theta - ml\dot{\theta}^2\cos\theta\sin\theta}{l(m+M) - ml\cos^2\theta}$$

  and

$$\ddot{x} = \frac{2F + mg\sin 2\theta - 2ml\dot{\theta}^2\sin\theta}{m + 2M - m\cos 2\theta}.$$

2 Linearization, Stability, and Controllability

- The state vector for this system is

$$\begin{pmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{pmatrix}.$$

- The fixed point about which the system will be linearized is

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

- The $\mathbf{A}$ matrix is equal to the Jacobian matrix evaluated at the fixed point

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{g(m+M)}{lM} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{gm}{M} & 0 & 0 & 0 \end{pmatrix}.$$

- The non-zero eigenvalues of $\mathbf{A}$ are

$$\pm\sqrt{\frac{g(m+M)}{lM}}.$$

  Because one of these has a positive real part the system is unstable.

- Rearranging the equations of motion to find the coefficients of F gives

$$\ddot{\theta} = f(\theta, \dot{\theta}) + \frac{\cos\theta}{l(m+M) - m\cos^2\theta}F$$

  and

$$\ddot{x} = g(\theta, \dot{\theta}) + \frac{2}{m + 2M - m\cos 2\theta}F.$$

Using the small angle approximation for cos gives

$$\ddot{\theta} = f(\theta, \dot{\theta}) + \frac{1}{lM} F$$

and

$$\ddot{x} = g(\theta, \dot{\theta}) + \frac{1}{M} F$$

resulting in the **B** matrix

$$\begin{pmatrix} 0 \\ \frac{1}{lM} \\ 0 \\ \frac{1}{M} \end{pmatrix}.$$

- The controllability matrix

$$C = \begin{pmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \mathbf{A}^3\mathbf{B} \end{pmatrix}$$

has full rank (4) so the system is controllable via the force F on the cart.

- The ideal state feedback gains matrix **K** can be determined using Mathematica's `LQRegulatorGains` function.