9.4: Cross Product Exercises

1) Let $\vec{a} = \langle 1, 1, 4 \rangle$, $\vec{b} = \langle 1, 1, 4 \rangle$.

What is $\vec{a} \times \vec{b}$?

solution: Intuitively, there's no "parallelogram" formed with just one vector ($\vec{a} = \vec{b} = \langle 1, 1, 4 \rangle$), so the area of the parallelogram is zero, hence we should expect the cross product to yield the zero vector.

Let us verify.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 4 \\ 1 & 1 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 4 \\ 1 & 4 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 4 \\ 1 & 4 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \vec{k} = \langle 0, 0, 0 \rangle.$$

2. Suppose that $\vec{v} \circ \vec{w} = 7$ and

$$\|\vec{v} \times \vec{w}\| = 2.$$

Find $\tan(\theta)$ and θ, where θ is the angle between $\vec{v}, \vec{w}$.

Solution:

- From $\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos(\theta)$, we have $7 = \|\vec{v}\| \cdot \|\vec{w}\| \cos(\theta)$.

- From $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \sin(\theta)$, we have $2 = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \sin(\theta)$. Hence,

$$\frac{2}{7} = \frac{\|\vec{v}\| \cdot \|\vec{w}\| \cdot \sin(\theta)}{\|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos(\theta)} = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta),$$

and $\theta = \arctan(2/7)$.

3. Let $\vec{u} = \langle 0, 9, 0 \rangle$ and

$\vec{v} = \langle 9 \cdot \cos(\pi/4), 9 \cdot \sin(\pi/4), 0 \rangle$.

Then, $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 9 & 0 \\ \frac{9\sqrt{2}}{2} & \frac{9\sqrt{2}}{2} & 0 \end{vmatrix}$

$$= \begin{vmatrix} 9 & 0 \\ \frac{9\sqrt{2}}{2} & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 0 & 0 \\ \frac{9\sqrt{2}}{2} & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 0 & 9 \\ \frac{9\sqrt{2}}{2} & \frac{9\sqrt{2}}{2} \end{vmatrix} \vec{k}$$

$$= 0\vec{i} - 0\vec{j} + \left(\frac{-81\sqrt{2}}{2}\right)\vec{k}$$

$$= \left\langle 0, 0, \frac{-81\sqrt{2}}{2} \right\rangle.$$

$\Rightarrow$ Downward facing vector in $\mathbb{R}^3$.

4) Let $\vec{a} = \langle 1, 10, 1 \rangle$, $\vec{b} = \langle 1, 15, 1 \rangle$,

what is a unit vector orthogonal to $\vec{a}$ and $\vec{b}$?

Solution: $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$ and $\vec{b}$.

$\Rightarrow \frac{1}{\|\vec{a} \times \vec{b}\|} (\vec{a} \times \vec{b})$ should be the unit vector that happens to be orthogonal to $\vec{a}$ and $\vec{b}$.

$$\Rightarrow \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 10 & 1 \\ 1 & 15 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 10 & 1 \\ 15 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 10 \\ 1 & 15 \end{vmatrix} \vec{k}$$

$= -5\vec{i} - 0\vec{j} + 5\vec{k} = \langle -5, 0, 5 \rangle$.

Now, $\|\langle -5, 0, 5 \rangle\| = \sqrt{25 + 0 + 25} = 5\sqrt{2}$.

$$\Rightarrow \frac{1}{5\sqrt{2}} \langle -5, 0, 5 \rangle$$

$= \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$ is a unit vector.

$\left[\left\langle \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \right\rangle \right.$ is also fine,

as unit vector in the opposite direction is still a unit vector,

and still orthogonal to

$\vec{a}$ and $\vec{b}$. $\left. \right]$

5) Let $\vec{p} = \langle 0,2,2\rangle$, $\vec{q} = \langle 4,0,7\rangle$

Find the area of the triangle formed with $\vec{p}$ and $\vec{q}$.

Solution: Area of the triangle is simply half of the area of the parallelogram: $\frac{1}{2}\|\vec{p}\times\vec{q}\|$. So,

$$\vec{p}\times\vec{q} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 2 \\ 4 & 0 & 7 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 2 \\ 0 & 7 \end{vmatrix}\vec{i} - \begin{vmatrix} 0 & 2 \\ 4 & 7 \end{vmatrix}\vec{j} + \begin{vmatrix} 0 & 2 \\ 4 & 0 \end{vmatrix}\vec{k}$$

$$= 14\vec{i} + 8\vec{j} - 8\vec{k} = \langle 14, 8, -8\rangle.$$

$$\Rightarrow \frac{1}{2}\|\langle 14,8,-8\rangle\| = \frac{1}{2}\sqrt{14^2+64+64} = \frac{1}{2}\sqrt{324}.$$

$$= \frac{1}{2}\cdot 18 = 9.$$

6) Let $\vec{a} = \langle -3, 3, -2 \rangle$,

$\vec{b} = \langle -4, -2, -3 \rangle$,

$\vec{p} = (k, k, k)$ 😡

For what $k \in \mathbb{R}$ would make vector from $\vec{a}$ to $\vec{b}$ perpendicular to a vector from $\vec{a}$ to $\vec{p}$?

Solution:

- vector from $\vec{a}$ to $\vec{b}$:

$$\vec{v} = \vec{b} - \vec{a} = \langle -4+3, -2-3, -3+2 \rangle$$

$$= \langle -1, -5, -1 \rangle.$$

- vector from $\vec{a}$ to $\vec{p}$:

$$\vec{w} = \vec{p} - \vec{a} = \langle k+3, k-3, k+2 \rangle.$$

Remember that $\vec{v}$ and $\vec{w}$ are orthogonal if $\vec{v} \circ \vec{w} = 0$.

$$\Rightarrow \vec{v} \circ \vec{w} = -(k+3) - 5(k-3) - (k+2) = 0 .$$

$$\Rightarrow -k-3-5k+15-k-2=0$$

$$\Rightarrow 10 = 7k \quad \Rightarrow k = 10/7 .$$