1) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^n}$

Ratio Test: $\begin{cases} a_n = \dfrac{1}{(\ln(n))^n} \\ a_{n+1} = \dfrac{1}{(\ln(n+1))^{n+1}} \end{cases}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \underbrace{\frac{(\ln(n))^n}{(\ln(n+1))^{n+1}}}$$

$$= \lim_{n\to\infty} \left(\frac{\ln(n)}{\ln(n+1)}\right)^n \cdot \frac{1}{\ln(n+1)}.$$

$$= 1 \cdot 0 = 0.$$

2) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\ln(n)}}$.

Limit Comparison Test:

We know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

[p-series w/ $p=2$].

So, let $a_n = \frac{1}{n^{\ln(n)}}$ and $b_n = \frac{1}{n^2}$.

Then, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^{\ln(n)}} = 0$,

because $\ln(n) > 2$ for $n > e^2$. So, by the limit comparison test, $\sum_{n=1}^{\infty} a_n$ converges.

3) $\sum_{n=3}^{\infty} \frac{1}{10^n} = \sum_{n=3}^{\infty} \left(\frac{1}{10}\right)^n$

$$= \underbrace{\sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n}_{} - \sum_{n=0}^{2} \left(\frac{1}{10}\right)^n$$

[geometric series, $r = 1/10$.]

$$= \frac{1}{1-\frac{1}{10}} - \left[1 + \left(\frac{1}{10}\right) + \left(\frac{1}{10}\right)^2\right]$$

$$= \frac{10}{9} - 1 - \frac{1}{10} - \frac{1}{100} = \frac{1}{900}.$$

$\Rightarrow$ converges.

4) $\sum_{n=1}^{\infty} \frac{1}{\ln(10^n)}$ Integral test.

$$\Rightarrow \int_1^{\infty} \frac{1}{\ln(10^x)}\, dx$$

Let $u = \ln(10^x)$. $\Rightarrow$

$$du = \frac{1}{10^x} \cdot 10^x \cdot \ln(10)\, dx$$

$$= \ln(10)\, dx.$$

$$\Rightarrow \int_1^{\infty} \frac{dx}{\ln(10^x)} = \int_{\ln(10)}^{\infty} \frac{1}{\ln(10)} \cdot \frac{1}{u}\, du.$$

$$= \frac{1}{\ln(10)} \underbrace{\int_{\ln(10)}^{\infty} \frac{du}{u}}$$

$$\ln|u| \Big|_{\ln(10)}^{\infty} = \infty.$$

$\Rightarrow$ Diverges.

5) $$\sum_{n=0}^{\infty} \ln\left(\frac{n+2}{n+1}\right).$$

$$= \sum_{n=0}^{\infty} \left[\ln(n+2) - \ln(n+1)\right].$$

Let's look at the partial sum,

$$S_k = \sum_{n=0}^{k} \left[\ln(n+2) - \ln(n+1)\right].$$

Telescoping series

$$= \left[\cancel{\ln(2)} - \ln(1)\right] + \left[\cancel{\ln(3)} - \cancel{\ln(2)}\right]$$

$$+ \left[\cancel{\ln(4)} - \cancel{\ln(3)}\right] + \cancel{\cdots} +$$

$$\left[\cancel{\ln(k+1)} - \cancel{\ln(k)}\right] + \left[\ln(k+2) - \cancel{\ln(k+1)}\right].$$

$$= \ln(k+2) - \ln(1) = \ln(k+2).$$

So, $\lim_{k\to\infty} S_k = \lim_{k\to\infty} \ln(k+2) = \infty$

$\Longrightarrow$ Divergence.

6) $\sum_{n=1}^{\infty} n^{-\frac{1}{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{n}}}$.

Let's compare this w/ $\sum_{n=1}^{\infty} \frac{1}{n}$

[Harmonic Series], which diverges.

Notice that ~~for~~ all $n \in \mathbb{N}^+$ [natural numbers]

$\frac{1}{n^{\frac{1}{n}}} \geq \frac{1}{n}$. Then,

$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{n}}} \geq \sum_{n=1}^{\infty} \frac{1}{n} = \infty$,

so $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{n}}}$ diverges.

7) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^p}$ for some $p \in \mathbb{R}$.

If $p \le 0$, then the series diverges.

So, let $p > 0$.

case i) $0 < p \le 1$. $\Rightarrow \displaystyle\sum_{n=2}^{\infty} \frac{1}{n^p}$ diverges.

$n > \ln(n)$ for $n \ge 2$.

$\Rightarrow \dfrac{1}{n} < \dfrac{1}{\ln(n)} \Rightarrow \dfrac{1}{n^p} < \dfrac{1}{\ln(n)^p}$

$\Rightarrow \underbrace{\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^p}}_{+\infty} < \displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)^p}$

$\Rightarrow$ diverges.

case ii) suppose that $p > 1$.

We know that $\sum_{n=2}^{\infty} \frac{1}{n^p}$ converges.

Limit comparison Test:

$a_n = \frac{1}{n^p}$.

$b_n = \frac{1}{(\ln(n))^p}$.

$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\ln(n)}{n}\right)^p$

$= 0$.

$\Rightarrow$ I think we have a convergence by Limit comparison Test...?

8) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^p}$. If $p<0$, then series diverges.

case i) suppose $p \leq 1$.

We know $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

Notice that

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^p}}_{\infty} \leq \sum_{n=1}^{\infty} \frac{\ln(n)}{n^p}.$$

∞, we have a divergence.

case ii) Now, with $p > 1$, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

Let $b_n = \frac{\ln(n)}{n^p}$ and $a_n = \frac{1}{n^p}$.

L.C.T: $\lim_{n \to \infty} \frac{1}{\cancel{n^p}} \cdot \frac{\cancel{n^p}}{\ln(n)} = \lim_{n \to \infty} \frac{1}{\ln(n)}$

$= 0$

$\Rightarrow$ convergence.

9) $\sum_{n=1}^{\infty} \frac{3^n}{4^n-2^n}$.

Ratio Test: $a_n = \frac{3^n}{4^n-2^n}$.

$$a_{n+1} = \frac{3^{n+1}}{4^{n+1}-2^{n+1}} = \frac{3 \cdot 3^n}{4 \cdot 4^n - 2 \cdot 2^n}$$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{3 \cdot 3^n}{4 \cdot 4^n - 2 \cdot 2^n} \cdot \frac{4^n-2^n}{3^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{3 \cdot (4^n-2^n)}{4 \cdot 4^n - 2 \cdot 2^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{3-3(\frac{1}{2})^n}{4-2(\frac{1}{2})^n} \right| = \frac{3}{4}.$$

< 1

$\Rightarrow$ convergence

10) $\sum_{n=0}^{\infty} \frac{n^p}{(n!)^q}$. Ratio Test :

$a_n = \frac{n^p}{(n!)^q}$, $a_{n+1} = \frac{(n+1)^p}{((n+1)!)^q}$

$\Rightarrow \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^p}{(n+1)^q (n!)^q} \cdot \frac{(n!)^q}{n^p} \right|$

$= \lim_{n\to\infty} \left| \underbrace{\left(\frac{n+1}{n}\right)^p}_{1} \cdot \underbrace{\frac{1}{(n+1)^q}}_{0} \right|$

$1 \cdot 0 = 0 < 1$

$\Rightarrow$ Convergence .