Evaluate $\int_0^{1/6} y \cdot \tan^{-1}(6y)\, dy$.

Solution: Remember that

$$\tan^{-1}(x) = \arctan(x).$$

$$\tan^{-1}(x) \neq \frac{1}{\tan(x)} = (\tan(x))^{-1}.$$

(In other words, $\arctan(x) \neq \cot(x)$)

So, $\int_0^{1/6} y \cdot \tan^{-1}(6y)\, dy$

$$= \int_0^{1/6} x \cdot \arctan(6x)\, dx.$$

Let $u = 6x$. Then, $x = \frac{1}{6} u$ and

$dx = \frac{1}{6} du$. So, $\int_0^{1/6} x \cdot \arctan(6x)\, dx$

$$= \int_0^1 \left(\tfrac{1}{6} u\right) \cdot \arctan(u) \cdot \tfrac{1}{6}\, du$$

$$= \frac{1}{36} \int_0^1 u \cdot \arctan(u)\, du = \frac{1}{36} \int_0^1 t \cdot \arctan(t)\, dt.$$

Integration by Parts : $u = \arctan(t)$ $\quad dv = t \cdot dt$

$$\downarrow \qquad\qquad \int$$

$$du = \frac{1}{1+t^2} \qquad v = \frac{1}{2}t^2$$

$$\Rightarrow \int_0^1 t \cdot \arctan(t)\, dt =$$

$$\frac{1}{2}t^2 \cdot \arctan(t)\Big|_0^1 - \int_0^1 \frac{1}{2}t^2 \cdot \frac{1}{1+t^2}\, dt$$

$$= \frac{\pi}{8} - \frac{1}{2}\int_0^1 \frac{t^2}{1+t^2}\, dt.$$

long Division :

$$\begin{array}{r} 1 \\ t^2+1 \,\big)\, \overline{t^2} \\ \underline{t^2+1} \\ -1 \end{array} \Rightarrow \frac{t^2}{1+t^2} = 1 - \frac{1}{1+t^2}$$

$$\Rightarrow \int_0^1 \frac{t^2}{1+t^2}\, dt = \int_0^1 1 - \frac{1}{1+t^2}\, dt$$

$$= t - \arctan(t)\Big|_0^1$$

$$= 1 - \frac{\pi}{4}.$$

So, $\int_0^1 t\cdot\arctan(t)\,dt = \frac{\pi}{8} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right)$

$$= \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{2}$$

and

$$\int_0^{1/6} x\cdot\arctan(6x)\,dx = \frac{1}{36}\int_0^1 t\cdot\arctan(t)\,dt.$$

$$= \frac{1}{36}\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi}{144} - \frac{1}{72}.$$

Kind of Hard for a quiz problem...