9.8 : arc length & curvature.

2) Let $\vec{r} = \langle e^{-2t}\cos(6t), e^{-2t}\sin(6t), e^{-2t} \rangle$

$$\Rightarrow \quad x(t) = e^{-2t} \cdot \cos(6t)$$

$$y(t) = e^{-2t} \cdot \sin(6t)$$

$$z(t) = e^{-2t}.$$

$$\Rightarrow \quad x'(t) = -2e^{-2t}(3 \cdot \sin(6t) + \cos(6t))$$

$$y'(t) = e^{-2t}(6 \cdot \cos(6t) - 2 \cdot \sin(6t))$$

$$z'(t) = -2e^{-2t}$$

$$\Rightarrow \quad [x'(t)]^2 = 36e^{-4t}\sin^2(6t)$$

$$+ 4e^{-4t}\cos^2(6t)$$

$$+ 24e^{-4t} \cdot \sin(6t)\cos(6t)$$

$$[y'(t)]^2 = 4e^{-4t}\sin^2(6t) + 36e^{-4t}\cos^2(6t)$$
$$-24e^{-4t}\sin(6t)\cos(6t)$$

$$[z'(t)]^2 = 4e^{-4t}$$

$$\Rightarrow \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} =$$

$$\sqrt{40e^{-4t}\sin^2(6t) + 40e^{-4t}\cos^2(6t) + 4e^{-4t}}$$

$$= \sqrt{40e^{-4t}\underbrace{(\sin^2(6t)+\cos^2(6t))}_{1} + 4e^{-4t}}$$

$$= \sqrt{40\cdot e^{-4t}(1) + 4e^{-4t}}$$

$$= \sqrt{44e^{-4t}} = \sqrt{4(11)(e^{-2t})^2}$$

$$= \sqrt{4}\cdot\sqrt{11}\cdot\sqrt{(e^{-2t})^2}$$

$$= 2\sqrt{11}e^{-2t}$$

So, $\quad s(t) = \int_0^t \sqrt{(x'(w))^2 + (y'(w))^2 + (z'(w))^2}\, dw$

$$= \int_0^t 2\sqrt{11} \cdot e^{-2w}\, dw$$

$$= 2\sqrt{11} \int_0^t e^{-2w}\, dw$$

$$= 2\sqrt{11} \cdot \left[-\frac{1}{2} e^{-2w} \right]_0^t$$

$$= 2\sqrt{11} \left[-\frac{1}{2} e^{-2t} - \left(-\frac{1}{2} e^{0} \right) \right]$$

$$= 2\sqrt{11} \left(-\frac{1}{2} e^{-2t} + \frac{1}{2} \right)$$

$$= \sqrt{11} \left(1 - e^{-2t} \right).$$

That took a while!!!

4) Let $y = \sin(-3x)$.

What is the curvature at $x = \pi/4$?

Parametrize via $x = t$:

$$\vec{r}(t) = \langle t, \sin(-3t) \rangle.$$

$$\vec{r}\,'(t) = \langle 1, -3\cdot\cos(-3t) \rangle.$$

$$|\vec{r}\,'(t)| = \sqrt{1 + 9\cdot\cos^2(-3t)}.$$

$$\Rightarrow \vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

$$= \left\langle \frac{1}{\sqrt{1+9\cdot\cos^2(-3t)}}, \frac{-3\cdot\cos(-3t)}{\sqrt{1+9\cdot\cos^2(-3t)}} \right\rangle$$

$$\Rightarrow \vec{T}\,'(t) =$$

3^3 $\qquad\qquad$ 3^2

$$\left\langle \frac{27\cdot\sin(3t)\cdot\cos(3t)}{(1+9\cdot\cos^2(-3t))^{3/2}}, \frac{9\cdot\sin(3t)}{(1+9\cdot\cos^2(-3t))^{3/2}} \right\rangle$$

$$\Rightarrow \left|\vec{T}'(t)\right| = \sqrt{\frac{3^6 \cdot \sin^2(3t)\cos^2(3t) + 3^4 \cdot \sin^2(3t)}{(1 + 9\cdot\cos^2(-3t))^3}}$$

$$= \sqrt{\frac{3^4 \cdot \sin^2(3t)\cancel{(1 + 3^2 \cdot \cos^2(3t))}}{(1 + 9\cdot\cos^2(-3t))^{\cancel{3}\,2}}}$$

$$= \frac{9 \cdot \sin(3t)}{1 + 9\cdot\cos^2(-3t)}.$$

$$\Rightarrow K = \frac{|T'(t)|}{|\vec{r}'(t)|}$$

$$= \frac{1}{\sqrt{1 + 9\cdot\cos^2(-3t)}} \cdot \frac{9\cdot\sin(3t)}{(1 + 9\cdot\cos^2(-3t))}$$

$$= \frac{9\cdot\sin(3t)}{(1 + 9\cdot\cos^2(-3t))^{3/2}}.$$

So, at $x = t = \pi/4$,

$$K(\pi/4) = \frac{9\cdot\sin(3\pi/4)}{(1 + 9\cdot\cos^2(-\frac{3\pi}{4}))^{3/2}}$$

$$= \frac{9\cdot\frac{\sqrt{2}}{2}}{\left(1+9\cdot\left(-\frac{\sqrt{2}}{2}\right)^2\right)^{3/2}}$$

$$= \frac{9\sqrt{2}}{2}\cdot\frac{1}{\left(1+\frac{9}{2}\right)^{3/2}}$$

$$= \frac{9\sqrt{2}}{2}\cdot\frac{2^{3/2}}{11^{3/2}}$$

$$= \frac{9\cdot 2^2}{2\cdot(11)^{3/2}} = \frac{36}{2\cdot(11)^{3/2}} = \frac{18}{(11)^{3/2}}.$$

💀 This also took a while.