INFINITE DIMENSIONAL DYNAMICAL SYSTEMS

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Abstract. In this paper we introduce the theory of infinite-dimensional dynamical systems. This theory may be applied to the study of the long-term dynamics of partial differential equations. We briefly introduce and review the theory of Sobolev spaces as the underlying space for this theory. We then define dissipative systems and their global attractors, and characterize these attractors in a special case. Finally, we show that the fractal dimension of the attractor may be bounded as a finite value, and describe some results which follow from this work on particular PDEs.

1. Introduction

In the study of dynamical systems, we are often interested in the long-term behaviour of some system of interest. Plenty of powerful tools have been developed in the finite dimensional case to quantify what we mean by the "long term behaviour" of a system and how much information can be gathered about it by estimating its long term behaviour. However, many systems of physical interest evolve in a function space on some underlying manifold and thus form infinite-dimensional systems. Such systems are often described by partial differential equations on the underlying manifold; these equations may be viewed as infinite-dimensional ordinary differential equations by letting the independent variable be the function u on the manifold, rather than the point x in the manifold. For instance, whereas the evolution of a small passive swimmer with a prescribed flow field can be seen as the evolution of a point in a finite dimensional space (namely the space which the swimmer is confined to), the evolution of the fluid field itself is an infinite dimensional problem.

The transition to infinite dimensional systems leads to two primary obstructions to our study: choosing the correct norm and topology on our space, as this choice is non-arbitrary in infinite dimensions, and characterizing the existence and behaviour of our evolution in such a space. The goal of this paper will be to introduce the major tools and techniques used to analyze the long-term behaviour of a certain class of partial differential equations. We will see that, given certain technical assumptions, these systems can be cast as infinite dimensional dynamical systems and studied using relatively standard topological methods. In particular, we will see that such systems "settle down" into a finite dimensional space which essentially contains the steady state solutions and perturbation thereof to the given partial differential equation. We note that the m-dimensional Reaction-Diffusion equation with Dirichlet boundary conditions and the $2D$ Navier-Stokes equation with periodic boundary conditions are both special cases of our system.

We begin with a brief note on the relevant spaces and choice of topology which we will be using to cast the problem. We then go on to prove the existence of a so-called "strange attractor" which will help determine the long term dynamics of our system. After this we will show how to bound the dimension of the attractor based on spectral properties of an appropriate linearized form of the problem. Like we saw in Floquet theory, the integral of the trace will be the determining factor in the system's long term behaviour.

2. Introduction to Sobolev Spaces

In short, Sobolev spaces are spaces of functions u such that the function and all its derivatives up to some point are integrable. To make this concept more useful, we will introduce a broader definition of the derivative, which may be defined irrespective of the continuity properties of u.

Definition 2.1. *Let Ω be an open, bounded subset of $\mathbb{R}^n$, and let $u : \Omega \to \mathbb{R}^n$. We say that u has a weak derivative with respect to x_j if there is a function $v : \Omega \to \mathbb{R}^n$ such that for any $\phi \in C_c^\infty(\Omega)$ we have*

$$\int_\Omega v\phi dx = -\int_\Omega u\frac{d\phi}{dx_j}dx. \tag{1}$$

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We then say that v is the weak derivative of u with respect to x_j.

Remark 2.2. More generally, given a multi-index $\alpha \in \mathbb{N}^n$, a weak derivative of u with respect to α is a function v such that

$$\int_\Omega v\phi dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi dx. \tag{2}$$

Note that a weak derivative need not exist even if a function is differentiable almost everywhere. For instance, let $u \in L^1(0,2)$ be given by

$$u(x) = \begin{matrix} x, & 0 < x \leq 1 \\ 2, & 1 < x < 2. \end{matrix} \tag{3}$$

Then if $\phi \in C_c^\infty(0,2)$, we see that

$$-\int_0^2 u\phi' dx = -\int_0^1 x\phi' dx - \int_1^2 2\phi' dx = -[x\phi]_0^1 + \int_0^1 \phi dx - 2\phi(2) + 2\phi(1) = \int_0^1 \phi dx + \phi(1). \tag{4}$$

Clearly there is no function v such that $\int_0^2 v\phi dx$ is equal to (4) for every $\phi \in C_c^\infty(0,2)$. Hence a weak derivative for u does not exist. To find an object which works as a derivative for u in this case we would need to explore the larger class of distributional derivatives; we will not do so here.[1]

Definition 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. For $1 \leq p < \infty$, $k \in \mathbb{N}$, the space*

$$W^{k,p}(\Omega) := \{u : D^\alpha u \in L^p(\Omega)\, \forall \alpha \in \mathbb{N}^n, 0 \leq |\alpha| \leq k\}, \tag{5}$$

equipped with the norm

$$\|u\|_{W^{k,p}} := \left(\sum_{0 \leq |\alpha| \leq k} \|D^\alpha u\|_{L^p}^p \right)^{1/p} \tag{6}$$

is a Sobolev space *on Ω. For $p = \infty$, the norm for the space $W^{k,\infty}$ is $\|u\|_{W^{k,\infty}} := \max_{0 \leq |\alpha \leq k} \operatorname{esssup}_{x \in \Omega} |u(x)|$. In particular, we define $H^k(\Omega) = W^{k,2}$.*

It can be shown that $W^{k,p}(\Omega)$ is a separable Banach space for each k, p. Furthermore, $H^k(\Omega)$ is a Hilbert space with the inner product

$$((u,v))_{H^k} = \sum_{0 \leq |\alpha| \leq k} (D^\alpha u, D^\alpha v). \tag{7}$$

Observe that $\|u\|_{H^k}^2 := ((u,u))_{H^k} = \|u\|_{W^{k,2}}^2$, so that the Sobolev space norm is the same as the norm induced by the inner product.
From this point on, we restrict our attention to the spaces H^k. Our goal is to describe approximation properties for these spaces. We begin by noting that $C_c^\infty(\Omega)$ is not dense in $H^k(\Omega)$. Instead, we define $H_0^k(\Omega)$ to be the completion of $C_c^\infty(\Omega)$ in $H^k(\Omega)$. This allows for the following definition:

Definition 2.4. *Let Ω be an open bounded subset of $\mathbb{R}^n$. We define the Sobolev spaces of negative integer values $H^{-k}(\Omega)$ to be the spaces of bounded linear functionals on the spaces $H_0^k(\Omega)$.*

Hence we have defined the Sobolev Hilbert spaces H^k for all $k \in \mathbb{Z}$. We note in passing that these definitions may be extended in the same way to the case $p \neq 2$.

Remark 2.5. Observe that $H^k(\Omega) \subseteq H^l(\Omega)$ for $0 \leq l \leq k$. This implies that $H_0^k(\Omega) \subseteq H_0^l(\Omega)$, and hence that $H^{-k}(\Omega) \supseteq H^{-l}(\Omega)$ for $0 \leq l \leq k$ (note that we are identifying $H^k(\Omega)$ with its dual). Hence we have that $H^k(\Omega) \subseteq H^l(\Omega)$ for every $l \leq k \in \mathbb{Z}$.

Although $C_c^\infty(\Omega)$ is not dense in $H^k(\Omega)$, it may be shown that $C^\infty(\Omega)$ is. Indeed, it is the compact support requirement of $C_c^\infty(\Omega)$ which prevents its density in $H^k(\Omega)$, as the fact that Ω is open implies that any element of $C_c^\infty(\Omega)$ is necessarily zero on $\partial\Omega$. This is visualized in the case of $k = 1$ by the following theorem:

[1]It is, however, true that $v = D_j u$ if u is continuous everywhere *and* differentiable almost everywhere. This follows by integration by parts, since $\lim u\phi$ exists at each point and for each ϕ.

Theorem 2.6. *If Ω is open and bounded, let $T : H^1(\Omega) \to L^2(\partial\Omega)$ be the bounded linear operator given by $Tu = u|_{\partial\Omega}$. Then $u \in H^1_0(\Omega)$ if and only if $u \in H^1$ and $Tu = 0$.*

Two more theorems complete the characterization of the relationship between $H^k(\Omega)$ and spaces of infinitely differentiable functions. Proofs for each of these theorems and the preceding one involve several technical lemmas. The full details may be found in [1].

Theorem 2.7. *$C^\infty(\Omega) \cap H^k(\Omega)$ is dense in $H^k(\Omega)$.*

Theorem 2.8. *If Ω is an open bounded subset of $\mathbb{R}^n$ such that $\overline{\Omega}$ is a compact C^k-manifold of dimension n with boundary, then $C^\infty(\overline{\Omega})$ is dense in $H^k(\Omega)$.*

Finally, we state the Rellich-Kondrachov compactness theorem, which will be useful in proving several of our results later. Again, a proof may be found in [1].

Theorem 2.9. *Let $\Omega \subset \mathbb{R}^n$ be such that $\overline{\Omega}$ a bounded C^1-submanifold of $\mathbb{R}^n$ of dimension n with boundary. Then $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$.*

3. Existence of Strange Attractors

After establishing the existence and uniqueness of a solution $u(t, x_0)$ to the Cauchy problem for a given partial differential equation for all positive times, we associate to each partial or ordinary differential equation a solution operator $S(t)$ given by $S(t)x_0 = u(t, x_0)$. Note that this operator is continuous in both t and x_0 and has the following properties:

$$\begin{aligned} &S(0) = \mathrm{Id}, \\ &S(t)S(s) = S(t+s) = S(s)S(t). \end{aligned} \tag{8}$$

We say that any one parameter family of functions that satisfies (8) is a *semi-group*. As a bit of terminology, by continuous-time topological semi-dynamical system (which we will refer to as a semi-dynamical system) we simply mean a pair (X, f^t) where X is an underlying topological space and $f^t : X \to X$ is a one parameter family of maps continuously parameterized by R_0^+ such that the family forms a semi-group. Thus we can associate to each well-posed differential equation a semi-dynamical system $(H, S(t))$ where H is the underlying phase space.

Many systems of physical importance have some form of energy dissipation which constrain the dynamics over time. We would like to connect this form of energy dissipation to some topological properties of our dynamics. As such, we introduce the following notions of dissipation.

Definition 3.1. *We say that a semi-dynamical system $(H, S(t))$ is **point dissipative** if there exists a bounded set $B \subset H$ such that for every $x_0 \in H$ there exists a $t_0(x_0)$ such that $S(t)x_0 \in B$ for all $t \geq t_0(x_0)$.*

This definition can be interpreted as saying that for each point in our phase space there exists a time after which it has lost some critical amount of energy. Note however that the exact time required to cross that threshold could vary dramatically from point to point, even in a small neighborhood. To accommodate this we introduce a stronger notion of dissipation which allows for some error in initial position measurement.

Definition 3.2. *We say that a semi-dynamical system is **bounded dissipative** if there exists a bounded set $B \subset H$ such that for each bounded set $X \subset H$ there is a time $t_0(X)$ such that $S(t)X \subset B$ for all $t \geq t_0(X)$.*

Note that bounded dissipative is certainly stronger than point dissipative, as definition 3.2 implies definition 3.1 by taking $X = \{x_0\}$.[2] When restricting to $\mathbb{R}^m$, there is no major distinction between a bounded set and a compact set (consider taking the closure of your bounded set). Compact sets form a rich space for convergence and existence, and thus we give the most powerful notion of dissipative as:

Definition 3.3. *We say that a semi-dynamical system is **dissipative** if it is bounded dissipative and its corresponding B is a compact set.*

We wish to characterize the long-term behaviour of our dynamics, and thus we introduce an associated set which tries to capture what points remain near our trajectory for all times:

[2] Indeed, if H is finite-dimensional, it can be shown that the two definitions are equivalent.

Definition 3.4. *We define the ω-limit set of a set X as the set of all limit points of the orbit of X,*

$$\omega(X) = \{y | \exists\, t_n \to \infty, x_n \in X, S(t_n)x_n \to y\}.$$

Equivalently, we have the following characterization:

$$\omega(X) = \cap_{t\geq 0}\overline{\cup_{s\geq t}S(s)X}$$

.

Note that $\omega(X)$ is in general a positively invariant set. However, if the image of our bounded set lies in a compact set, we can say much more than this.

Proposition 3.5. *If for some $t_0 > 0$ the set*

$$\overline{\cup_{t\geq t_0}S(t)X}$$

is compact, then $\omega(X)$ is both non-empty and compact.

Proof. Since for all $t \geq t_0$,

$$\cup_{s\geq t}S(s)X = S(t-t_0)\cup_{s\geq t_0} S(s)X, \tag{9}$$

we have

$$\overline{\cup_{s\geq t}S(s)X} = \overline{S(t-t_0)\cup_{s\geq t_0} S(s)X} \subseteq S(t-t_0)\overline{\cup_{s\geq t_0}S(s)X}, \tag{10}$$

where the last inclusion follows by continuity. Thus $\overline{\cup_{s\geq t}S(s)X}$ is a closed subset of a compact set which implies it is compact for all $t \geq t_0$. Note that $t \geq t' \geq t_0 \implies \overline{\cup_{s\geq t}S(s)X} \subset \overline{\cup_{s\geq t}S(s)X}$. We therefore have a decreasing sequence of compact sets, therefore their intersection $\omega(X)$ is compact and non-empty. ☐

In fact our proof goes further. If B is compact and $f(B) \subset B$, then $\omega(X)$ is nonempty, compact, and forward invariant for all closed subsets of B (including B itself). In particular it is nonempty for all singleton subsets of X (i.e. $\omega(\{x\}) \neq \emptyset\ \forall\ x \in X$). We might expect that therefore the ω-limit set of our compact absorbing set B would capture all of our dynamics. To further clarify this notion we introduce another structure:

Definition 3.6. *The global attractor $\mathcal{A}$ of a dynamical system (H, f) is the maximal compact invariant set and the minimal set that attracts all bounded sets, i.e., the smallest set such that*

$$\lim_{t\to\infty} \operatorname{dist}(S(t)X, \mathcal{A}) \to 0$$

for all bounded X.

Note that such an attractor must be both a maximal set by inclusion and a minimal set by attraction of bounded sets. Thus we cannot use a construction akin to Zorn's Lemma to prove its existence. However, we have already developed an appropriate candidate.

Theorem 3.7. *If $S(t)$ is dissipative and B is any associated compact absorbing set then there exists a global attractor $\mathcal{A} = \omega(B)$. If H is connected then so is $\mathcal{A}$.*

Proof. By Proposition 3.5, $\omega(B)$ is non-empty, forward invariant, and compact. It is also the maximal compact invariant set; to see this, suppose Y is another compact invariant set. Then there exists a $t_0(Y)$ such that $Y = S(t_0(Y))Y \subset B$ and therefore $\omega(Y) \subset \omega(B)$. (Note in particular that by 3.5, this implies that $\mathcal{A}$ is independent of the choice of B and hence is well-defined.) Hence $\mathcal{A}$ is the maximal compact absorbing set for $S(t)$.

To show that the set attracts all bounded sets, assume not. Then there exists some bounded set X, a $\delta > 0$, and a sequence $t_n \to \infty$ such that $\operatorname{dist}(S(t_n)X, \mathcal{A}) \geq \delta$. It therefore follows that there are $x_n \in X$ with $\operatorname{dist}(S(t_n)x_n, \mathcal{A}) \geq \delta/2$. Note that X is bounded and therefore for all $t_n \geq t_0(X)$ we have $S(t_n)x_n \in B$. By compactness, there exists a convergent subsequence n_k. such that $S(t_{n_k})x_{n_k} \to \beta \in B$ and $\operatorname{dist}(\beta, \mathcal{A}) = \lim_{k\to\infty} \operatorname{dist}(S(t_{n_k})x_{n_k}, \mathcal{A}) = \delta/2$. But

$$\beta = \lim_{k\to\infty} S(t_{n_k})x_{n_k} = \lim_{k\to\infty} S(t_{n_k} - t_0(X))S(t_0(X))x_{n_k} \in \omega(B) = \mathcal{A},$$

which gives a contradiction. Since $\mathcal{A}$ is invariant and compact itself, it must be the minimal such set that has this property, as it attracts itself. ☐

While the dynamics on the whole space may not be reversible, the dynamics restricted to the attractor may in fact be. This allows us to employ further tools and is another reason why the attractor is a good place to extract dynamics from our system. To make this precise we have

Theorem 3.8. *If the semigroup is injective on $\mathcal{A}$, i.e.,*

$$S(t)u_0 = S(t)v_0 \in \mathcal{A} \textit{ for some } t > 0 \implies u_0 = v_0$$

then every trajectory on $\mathcal{A}$ is defined for all $t \in \mathbb{R}$. In other words,

$$(\mathcal{A}, \{S(t)\}_{t \geq 0})$$

is a dynamical system.

Proof. For each $u \in \mathcal{A}$, consider $S(t)\mathcal{A}$. By forward invariance, $u \in S(t)\mathcal{A}$ and by injectivity there exists a unique v such that $S(t)v = u$. We then define the negative time maps as $S(-t)u = v$ to extend $S(t)$ to be defined for all times (i.e. $S(-t) = S(t)^{-1}$). □

4. Characterization of Strange Attractors

4.1. **Structure of the Attractor.** We begin with a few theorems which help us understand the general structure of these attractors. First we introduce a definition which will help in our statement of the theorem.

Definition 4.1. *A complete orbit is an element of H for which $S(t)$ is defined for all times. Equivalently, it is a solution $u(t)$ to our underlying PDE or ODE which can be defined for all $t \in \mathbb{R}$.*

Theorem 4.2. *All complete bounded orbits lie in $\mathcal{A}$. If the semi-group is injective on $\mathcal{A}$ then $\mathcal{A}$ is the union of all complete bounded orbits.*

Proof. In order for the orbit $\mathcal{O}$ to not fully lie within the attractor$\mathcal{A}$, there needs to be some point $x_0 \in \mathcal{O}$ such that $\mathrm{dist}(x_0, \mathcal{A}) > \varepsilon$ for some $\varepsilon > 0$. However, the orbit is also a bounded set and so by dissipation we have that for some t large enough:

$$\mathrm{dist}(S(t)z, \mathcal{A}) < \varepsilon \ \forall \ z \in \mathcal{O}.$$

Because the orbit is complete, there exists an $x_0 = S(t)\tilde{x}$ for the same t used above. Combining these two, we obtain a contradiction. □

Now we have to determine what type of points we may expect to be on the attractor. Fixed points must obviously be on the attractor as they trivially form complete bounded orbits. The same is true for periodic orbits if they exist. However, these structures also bring in related points. To see this, we need to quickly review the notions of a stable and unstable manifold.

Definition 4.3. *The unstable manifold of a fixed point p is the set:*

$$W^u(p) = \{x \mid S(t)x \textit{ exists for all times } t \in \mathbb{R} \textit{ and } \lim_{t\to\infty} S(-t)x = p\}.$$

The stable manifold of a fixed point p is defined as the set

$$W^s(p) = \{x \mid S(t)x \textit{ exists for all times } t \in \mathbb{R} \textit{ and } \lim_{t\to\infty} S(t)x = p\}.$$

The unstable and stable manifolds of any invariant set X are defined similarly by replacing $\lim_{t\to\infty} S(\pm t)x \to p$ with $\lim_{t\to\pm\infty} \mathrm{dist}(S(t)x, X) = 0$.

Now we show that the unstable manifolds of all compact invariant sets (in particular the fixed points and periodic orbits we mentioned earlier) are contained in the global attractor.

Theorem 4.4. *If X is a compact invariant set, then $W^u(X) \subset \mathcal{A}$.*

Proof. Let $x \in W^u(X)$. Then by definition 4.3, x lies on a complete orbit, and we have $\lim_{t\to-\infty} \mathrm{dist}(S(t)x, X) = 0$. Furthermore from the attraction of bounded sets in definition 3.6, we have $\lim_{t\to\infty} \mathrm{dist}(S(t)x, \mathcal{A}) = 0$. Thus $\{S(t)x\}_{t\in\mathbb{R}}$ forms a complete bounded orbit and therefore $x \in \mathcal{A}$ by theorem 4.2. Since x was arbitrary, we thus have $W^u(X) \subset \mathcal{A}$. □

We now have some sense of what types of points will be included in the attractor for a specific equation, but often there is some uncertainty in the underlying equation. This uncertainty may be imagined as a weak forcing term or a weak process which should be included in the full description but which is removed for the analysis. Another kind would be small uncertainties in the measurement. Ideally, we would hope that small perturbations would maintain the structure of the attractor and that it would vary continuously as a function of the perturbation size. We will not go this far, but we are able to make considerable progress in this direction.

Theorem 4.5. *Let S_η be a family of perturbed semi-groups where $\eta \in [0, \eta_0)$ such that each S_η has a global attractor $\mathcal{A}_\eta$ and there exists a bounded set X such that*

$$\bigcup_{0 \le \eta \le \eta_0} A_\eta \subset X.$$

If, in addition the semi-groups S_η converge to S_0 in that, for each $t > 0$, $S_\eta(t)x \to S_0(t)x_0$ uniformly on bounded subsets Y of H, i.e.,

$$\lim_{\eta \to 0} \sup_{u_0 \in Y} \left|(S_\eta(t) - S_0(t))u_0\right| = 0.$$

then

$$\lim_{\eta \to 0} dist(\mathcal{A}_0, \mathcal{A}_\eta) = 0.$$

Proof. This proof essentially relies on a combination of the attraction of bounded sets and the convergence of the semi-groups. To see this, note that each attractor is invariant under its corresponding semi-flow by construction. Note that $A_\eta \subset X$ which is bounded, thus there exists a $\tau > 0$ such that

$$\sup_{x \in X} \operatorname{dist}(S_0(\tau)x, A_0) \le \varepsilon/2.$$

By convergence of the semi-groups, we can then find a $\eta(\tau, \varepsilon)$ such that

$$\sup_{x \in X} \left|S_\eta(t)x - S_0(t)x\right| < \varepsilon/2.$$

Thus for $\eta \le \eta(\varepsilon)$ we have

$$\mathcal{A}_\eta = S_\eta \mathcal{A}_\eta \subset S_\eta(t)X \tag{11}$$

$$\operatorname{dist}(\mathcal{A}_\eta, \mathcal{A}_0) \le \operatorname{dist}(S_\eta(t)X, \mathcal{A}_0) \le \varepsilon \tag{12}$$

Since ε can be made arbitrarily small by taking small η we have that

$$\lim_{\eta \to 0} \operatorname{dist}(\mathcal{A}_\eta, \mathcal{A}_0) = 0.$$

□

A word of warning: this is the one directional distance, not the full Hausdorff metric. This statement prevents our attractor from exploding out beyond the original attractor as we vary η, but it does not prevent implosion of the attractor into itself. To have that we would need

$$\lim_{\eta \to 0} \operatorname{dist}(A_0, A_\eta) \to 0$$

which would give us convergence in the Hausdorff metric ($d_H(X, y) = \max(\operatorname{dist}(X, Y), \operatorname{dist}(Y, X))$). In general, this is unknown. We can make progress on this point, but first we need to restrict ourselves to a smaller class.

4.1.1. *Lyapunov Systems.* When a system has a strictly decreasing quantity, we can get a much stronger grasp on the asymptotics and attractor of our system. We give a particularly strong definition.

Definition 4.6. *A Lyapunov function for a semigroup $S(t)$ on a positively invariant set X is a continuous function $\Phi : X \to \mathbb{R}$ such that*

(1) for each $u_0 \in X$, the function $t \mapsto \Phi(S(t)u_0)$ is non-increasing, and
(2) if $\Phi(S(\tau)u) = \Phi(u)$ for any $\tau > 0$, then u is a fixed point of $S(t)$.

Note this is a strong definition of a Lyapunov function and is in some sense tailored to our exact situation. Note that Lyapunov functions usually require the function to be positive; here this is not as relevant because we will be considering continuous functions on compact invariant sets. It also fully excludes periodic orbits by point (2). Regardless, it is a useful definition as it allows us to state the following:

Proposition 4.7. *Suppose that $S(t) : H \to H$ has a Lyapunov function L on $X \subset H$ where X is a forward invariant absorbing set. Denote the fixed points of $S(t)$ by* $\mathrm{Fix}(S)$. *Then $\omega(u_0) \subset Fix(S)$ for every $u_0 \in H$. Furthermore, if H is connected and* $\mathrm{Fix}(S)$ *is discrete, then $\omega(u_0) \in \mathrm{Fix}(S)$.*

Proof. For each $u_0 \in H$, there exists a time t such that $S(t)u_0 \in X$ by definition of absorbing sets. Since clearly $\omega(u_0) = \omega(S(t)u_0)$, we may without loss assume that $u_0 \in X$. Since X is compact and connected, $\omega(u_0)$ is non-empty and connected as well as forward invariant. The Lyapunov functional must be constant on $\omega(u_0)$ as we have:

$$\Phi|_{\omega(u_0)} = \lim_{t\to\infty} \Phi(u(t)) = \inf_{t\in\mathbb{R}} \Phi(u(t))$$

where we use continuity and non-increasing nature of Φ. Because $\{u(t)\}_{t\in\mathbb{R}^+}$ is a subset of X by forward invariance, Φ must achieve its maximum on the set and is therefore bounded from below and the limit exists and is achieved by at least one point. Thus by our strict defintion of a Lyapunov function, $\omega(u_0)$ consists of only stationary points. If H is connected then so is $\omega(u_0)$ in the same way as before and therefore if the fixed points are discrete then $\omega(u_0)$ must be precisely one of them i.e. $\omega(u_0) \in \mathrm{Fix}(S)$. □

We can leverage this same type of argument to give a strong condition for our attractor:

Theorem 4.8. *Suppose that $S(t)$ has a Lyapunov function Φ on $\mathcal{A}$. Then*

$$\mathcal{A} = W^u(\mathrm{Fix}(S)). \tag{13}$$

Also, if H is connected and $\mathrm{Fix}(S)$ *is discrete then*

$$\mathcal{A} = \cup_{z\in \mathrm{Fix}(S)} W^u(z) \tag{14}$$

and we also have

$$\mathcal{A} = \cup_{z\in \mathrm{Fix}(S)} W^s(z). \tag{15}$$

Proof. Clearly $\mathrm{Fix}(S) \subset \mathcal{A}$, hence it is bounded. Since the set of fixed points of a continuous function are closed, we thus have that $\mathrm{Fix}(S)$ is compact. It is also clearly invariant; hence by Theorem 4.4, we have that $W^u(S) \subset \mathcal{A}$. Now, let $u_0 \in \mathcal{A}$, and set

$$\gamma = \cap_{s<0} \overline{\{u(t) : t < s\}}, \tag{16}$$

where $u(t) = S(t)u_0$. We want to show that $\gamma \subset \mathrm{Fix}(S)$; in this case we have that $u_0 \in W^u(\gamma) \subset W^u(S)$, hence $\mathcal{A} \subset W^u(S)$. To show that $\gamma \subset \mathrm{Fix}(S)$, note that by Theorem 4.3, $W^u(\gamma) \subset \mathcal{A}$. But

$$\Phi|_\gamma = \lim_{t\to-\infty} \Phi(u(t)) = \sup_{t\in\mathbb{R}} \Phi(u(t)). \tag{17}$$

Φ is bounded above because $\mathcal{A}$ is compact, and since Φ is non-decreasing on any backwards trajectory in $W^u(\mathcal{A})$, we have that the limit in (17) exists. This then implies as above that the set γ consists only of stationary points, that is, $\gamma \in \mathrm{Fix}(S)$. This completes the proof of the first claim.
Now, if H is connected and $\mathrm{Fix}(S)$ is discrete, then $\gamma = z$ for some $z \in \mathrm{Fix}(S)$ by Proposition 4.7. Hence any element of $\mathcal{A}$ is an element of $W^u(\gamma)$ for some γ, and the claim (14) holds. The claim (15) holds by a similar argument. □

As an interesting side note, we observe that this implies that if $\mathrm{Fix}(S)$ is discrete, then every point on the attractor satisfies

$$\lim_{t\to-\infty} S(t)u_0 = z_1, \lim_{t\to\infty} S(t)u_0 = z_2, z_1, z_2 \in \mathrm{Fix}(S).$$

Therefore the structure of such attractors can be fully specified by a finite directed graph without cycles. Given a specific problem of this type, lots of techniques have been developed to infer which fixed points should be connected to which (called the *connection problem*) for certain systems.
On this very special subset we are able to achieve the continuity properties we are looking for.

Theorem 4.9. *Let the assumptions of Theorem 4.5 hold and in addition assume that $\mathcal{A}_0$ is given by the closure of the unstable manifolds of a finite number of fixed points. Provided that the unstable manifolds vary continuously with η near $\eta = 0$ in some neighborhood of the each fixed point, then the attractor is lower semi-contiuous i.e.*

$$\lim_{\eta \to 0} dist(A_0, A_\eta) = 0.$$

Combining this with 4.5 gives us continuity in the Hausdorff Metric i.e.

$$\lim_{\eta \to 0} dist_{\mathcal{H}}(A_0, A_\eta) = 0.$$

Proof. By compactness of A_0, there exists a finite sequence of points $\{x_n^0\}_{n=1}^N \subset A_0$ such that

$$\min_{1 \le n \le N} \left| x^0 - x_n^0 \right| < \frac{\varepsilon}{2}.$$

If we can then show that there exists a sequence $\{x_n^\eta\}_{n=1}^N \in A_\eta$ such that $\min_{1\le n \le N} \left| x_n^\eta - x_n^0 \right| < \frac{\varepsilon}{2}$, then it follows that for every $x^0 \in A^0$ there exists an $x^\eta \in A^\eta$ such that $\left| x^0 - x^\eta \right| < \varepsilon$ i.e. $\mathrm{dist}(A_0, A_\eta) < \varepsilon$ as desired. Since A_0 is the closure of unstable manifolds, there exists a sequence $\{y_n\}_{n=1}^N$ such that every y_n is on the unstable manifold of some fixed point z_n and $\left| x_n^0 - y_n \right| < \frac{\varepsilon}{4}$. Since each y_n is on the unstable manifold of some z_n we can write $y_n = S(t_n)\tilde{z_n}$ such that $\tilde{z_n}$ is in a small neighborhood of z_n. By continuity of S_0, there exists a $\delta > 0$ such that

$$|\tilde{z_n} - u| \implies \left| S_0(t_n)\tilde{z_n} - S_0(t_n)u \right| < \frac{\varepsilon}{8} \text{ for all } \tilde{z_n}.$$

Since A_0 is bounded, then so is $N(A_0, \delta) = \{x \in H | \mathrm{dist}(x, A_0) \le \delta\}$ is a bounded set and therefore there we can pick an η^* small enough such that for all $\eta < \eta^*$ we have by continuity of the semi-group under the perturbation we have

$$\left| S_0(t_n)u - S_\eta(t_n)u \right| \le \frac{\varepsilon}{8} \text{ for all } u \in N(A_0, \delta), 1 \le n \le N.$$

Since the local unstable manifolds perturb smoothly, we can pick a possibly smaller η such that there are points within A_η such that $\left| \tilde{z_n^\eta} - \tilde{z_n} \right| < \delta$. We thus have that

$$\left| S_\eta(t_n)\tilde{z_n^\eta} - y_n \right| = \left| S_\eta(t_n)\tilde{z_n^\eta} - S_0(t_n)\tilde{z_n} \right| \tag{18}$$

$$\le \left| S_\eta(t_n)\tilde{z_n^\eta} - S_0(t_n)\tilde{z_n^\eta} \right| + \left| S_0(t_n)\tilde{z_n^\eta} - S_0(t_n)\tilde{z_n} \right| \le \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{4}. \tag{19}$$

Since $S_\eta(t_n)\tilde{z_n^\eta} \in A_\eta$ by forward invariance, we thus have that

$$\left| S_\eta(t_n)\tilde{z_n^\eta} - x_n \right| \le \left| x_n^0 - y_n \right| + \left| S_\eta(t_n)\tilde{z_n^\eta} - y_n \right| \le \frac{\varepsilon}{4} + \frac{\varepsilon}{4}$$

as was desired. Thus we have that $\lim_{\eta \to 0} \mathrm{dist}(A_0, A_\eta) = 0$ and therefore the perturbation is continuous in the Hausdorff metric i.e.

$$\lim_{\eta \to 0} \mathrm{dist}_{\mathcal{H}}(A_0, A_\eta) = 0.$$

□

4.2. **How Attractor Dynamics Approximate Underlying Dynamics.** We have given some indication that the dynamics on the attractor have something to do with the long-term behaviour of our system, but we have yet to make the connection exact. The connection is not necessarily obvious or what one might expect. For instance, it is not true that for every forward orbit of our system that there exists an orbit on the attractor which stays arbitrarily close after a certain amount of time. As an example, consider the following set of ordinary differential equations:[3]

$$\dot{x} = z(x + y) \tag{20}$$

$$\dot{y} = z(y - x) \tag{21}$$

$$\dot{z} = -z|z|. \tag{22}$$

[3]Note: The system given in Robinson appears to have a typo, as it does not give the proper equation for $\dot{\theta}$. We have written the corrected system here.

The set has $z = 0$ as an attractor. Note the system is not dissipative, so this is not a global attractor; instead, by attractor here we just mean the second component of the definition of the global attractor in that it is the minimal set which attracts all bounded sets. However, if we consider the dynamics in cylindrical coordinates we have

$$z(t) = \frac{z_0}{1 + |z_0|\, t} \tag{23}$$

$$\dot{r}(t) = rz \tag{24}$$

$$\dot{\theta} = -z \tag{25}$$

$$r(t) = r_0(1 + |z_0|\, t)^{\operatorname{sgn}(z_0)} \tag{26}$$

$$\theta(t) = \theta_0 - \operatorname{sgn}(z_0)\ln(1 + |z_0|\, t) \tag{27}$$

Note that the dynamics on the attractor are trivial: $z = 0 \implies \dot{x}, \dot{y}, \dot{z} = 0$. However, by our calculation the r, θ coordinates of each point off of the attractor evolve monotonically. Thus, no points on the attractor capture the dynamics for all time. However, note that ln becomes "approximately constant" over time, and can be well approximated by a series of test functions where our constant approximation stays approximately correct for longer and longer intervals. We make this precise in the following proposition.

Proposition 4.10. *Given a trajectory $u(t) = S(t)u_0,\ \varepsilon > 0,\ T > 0$, there exists a time $\tau(\varepsilon, T) > 0$ and a point $v_0 \in \mathcal{A}$ such that*

$$\big|u(t+\tau) - S(t)v_0\big| \le \varepsilon \text{ for all } 0 \le t \le T.$$

Proof. Since the semigroup is continuous in both initial position and time, for any given $\varepsilon > 0,\ T > 0$ we have $\delta(\varepsilon, T)$ such that

$$|u_0 - v_0| \le \delta(\varepsilon, T) \implies \big|u(t) - v(t)\big| \le \varepsilon \text{ for all } t \in [0, T].$$

Since any point is clearly a bounded set, the attraction of bounded sets gives us that there exists some τ and a point $v_0 \in \mathcal{A}$ such that $\operatorname{dist}(S(\tau)x, v_0) \le \delta$. Combining these gives the desired result. □

Note that the v_0 may change after each time step.

4.3. **Measures of Dimensionality.** In our construction of the attractor, we did not make any statements about its regularity as a space. We would hope the fact that the attractors are compact would help bound their dimensionality as the unit ball in an infinite-dimensional space is not in general compact. In fact we can show these global attractors are finite-dimensional, but we will need to expand our notions of dimensionality before arriving there. To this end, we will give two distinct definitions of dimensionality, both of which will be useful. We will then obtain techniques to bound the dimension of our attractors by seeing how the flow acts on n-dimensional test objects. If the flow contracts these objects, then we know that $\dim(\mathcal{A}) < n$, since the attractor is invariant and therefore not contracted.

First, we itemize some key properties we would hope any definition of dimensionality would hold:

(1) $d(X) \le d(Y)$ whenever $X \subseteq Y$;
(2) $d(X) = m$ if X is an m-manifold;
(3) $d(X \cup Y) = \max(d(X), d(Y))$.

The intuition for these can be seen as follows. First, we clearly want smaller spaces to have smaller dimensionality. Second, we want any definition of dimensionality to agree with our standard definition for well-behaved sets of $\mathbb{R}^n$. Third, the union of any two sets should be realizable as the "dominant" or "largest" dimension of the set; hence attaching a two-dimensional sheet to a three-dimensional shape defines a set which is "essentially" three-dimensional.

4.3.1. *Fractal Dimension.*

Definition 4.11. *If $\overline{X}$ is compact, then we denote the fractal dimension of X to be $d_f(X)$ which is given by*

$$d_f(X) = \limsup_{\varepsilon \to 0} \frac{\ln(cov(X, \varepsilon))}{\ln(1/\varepsilon)}$$

where $cov(X, \varepsilon)$ is the minimum number of closed balls of size ε needed to cover X (such a number is guaranteed to be finite by compactness of $\overline{X}$).

Before going further, let us give a few properties of this fractal dimension. In addition to the first three properties that we wanted for any notion of dimensionality, the following are properties of the fractal dimension.

Proposition 4.12 (Properties of the Fractal Dimension)**.**

(1) The fractal dimension is stable under finite unions i.e.

$$d_f(\cup_{k=1}^N X_k) \leq \max_k d_f(X_k)$$

(2) If $g : H \to H$ is Hölder continuous with exponent α i.e.

$$\big|g(x) - g(y)\big| \leq L|x - y|^\alpha$$

then $d_f(g(X)) \leq d_f(X)/\alpha$,

(3) $d_f(X \times Y) \leq d_f(X) + d_f(Y)$

(4) $d_f(\overline{X}) = d_f(X)$.

Proof.

(1) This follows rather quickly from

$$N(\cup_{k=1}^n X_k, \varepsilon) \leq \sum_{k=1}^n N(x, \varepsilon)$$

which follows by simply taking the minimal covering of each X_k and using their combination as a cover of $\cup_{k=1}^n X_k$.

(2) If X is covered by $N(X, \varepsilon)$ ε-balls, then (by Hölder continuity) $g(X)$ is covered by the same number of balls of radius $L\varepsilon^\alpha$. Thus we have

$$N(g(X), L\varepsilon^\alpha) \leq N(X"\varepsilon)$$

$$d_f(g(X)) \leq \limsup_{\varepsilon\to 0} \frac{\ln(g(X), L\varepsilon^\alpha)}{-\ln(L\varepsilon^\alpha)} \leq \limsup_{\varepsilon\to 0} \frac{N(X, \varepsilon)}{-\ln(L) + \alpha\ln(\varepsilon^{-1})} = d_f(X)/\alpha$$

(3) This follows in a similar vein to (1) by noting that if X and Y admit a minimal cover, then in particular the product of all possible pairs of balls. Thus $\ln(N(X \times Y, 2\varepsilon) \leq \ln(N(X, \varepsilon) + \ln(Y, \varepsilon)$.

(4) Note that since the cover is a finite cover of closed balls, a cover will cover X if and only if it covers $\overline{X}$.

□

4.4. **Bounding the Attractor Dimension Dynamically.** We now seek a way to bound the fractal dimension of our attractor using the evolution of our partial differential equation. The idea is that we know that the attractor itself is invariant under the flow. Thus if the equation causes a contraction on all test volumes above a certain dimension d, then we know that $d_f(\mathcal{A}) < d$. To get a better handle of how equation causes test volumes to evolve, we introduce an appropriate linearization to our system.

Definition 4.13. *We say that a semi-group $S(t)$ is called uniformly differentiable on $\mathcal{A}$ if for every $u \in \mathcal{A}$ there exists a linear operator $\Lambda(t, u)$ such that for all $t \geq 0$*

$$\lim_{\varepsilon\to 0} \sup_{u,v\in\mathcal{A}; 0<|u-v|\leq\varepsilon} \frac{\big|S(t)v - S(t)u - \Lambda(t, u)(v - u)\big|}{|v - u|} = 0 \tag{28}$$

$$\text{and } \sup_{u\in\mathcal{A}} \|\Lambda(t, u)\|_{op} < \infty. \tag{29}$$

The major tool used to prove whether an equation satisfies this involves estimating the partial differential equation by a series of finite dimensional modes in what is known as a Galerkin expansion. The main idea is to project the solution onto its dominant eigenmodes. We will simply take that our system is uniformly differentiable as a given.

For this we will consider the abstract system

$$\frac{du}{dt} = F(u(t)), \quad u(0) = u_0.$$

Suppose this system is uniformly differentiable and $\Lambda(t,u)$ is the flow of the linearized differential equation

$$\frac{dU}{dt} = F'(S(t)u_0)U(t) = L(t;u_0)U(t), \quad U(0) = \xi.$$

If we then consider the evolution of the volume formed by a sequence of displacement vectors ξ_n each centered at u_0 then we have that the volume is related by

$$\frac{d}{dt}V_n(t) = \big(\det M(\xi_1, \xi_2, ..., \xi_n)\big)^{1/2} \tag{30}$$

$$M(\xi_1, \xi_2, ..., \xi_n)_{ij} = \langle \xi_i, \xi_j \rangle. \tag{31}$$

As a connection to the fractal dimension, it will be easier to study the evolution of the logarithm of the volume rather than the volume in and of itself. Noting that $\ln(\det(M)) = \mathrm{Tr}(\ln(M))$ and $\frac{d}{dt}\ln(M) = \mathrm{Tr}\left[M^{-1}\frac{dM}{dt}\right]$ we have that

$$\frac{d}{dt}\ln(V_n(t)) = \frac{1}{2}\mathrm{Tr}\left[M^{-1}\frac{dM}{dt}\right].$$

By the fact that the displacement vectors follow the linearized equations, we have that

$$\frac{d}{dt}\ln(V_n(t)) = \mathrm{Tr}(LP^{(n)})$$

where $P^{(n)}$ is the projection onto the n-dimensional space spanned by ξ_n. Thus, integrating forward in time we get that

$$V_n(t) = V_n(0)\exp\int_0^t \mathrm{Tr}(L(s;u_0)P^{(n)}(s))ds.$$

We therefore have that our n-dimensional volume has an asymptotic growth rate as if it were of base

$$\lim_{t\to\infty}\exp\frac{1}{t}\int_0^t \mathrm{Tr}(L(s;u_0)P^{(n)}(s))ds.$$

Note that this is dependent on both the choice of displacement vectors and the base point. To get a maximal growth rate, we define the following

$$\mathcal{TR}_n(\mathcal{A}) = \sup_{x_0\in\mathcal{A}}\sup_{P^{(n)}(0)}\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathrm{Tr}(L(s;u_0)P^{(n)}(s))ds.$$

We now state the result as our intuition would expect. Note that actually proving the following is not at all straightforward.

Theorem 4.14. *Suppose that S(t) is uniformly differentiable on $\mathcal{A}$ and that there exists a t_0 such that $\Lambda(t,u_0)$ is a compact operator for all $t \geq t_0$. If $\mathcal{TR}_n(\mathcal{A}) < 0$ then $d_f(\mathcal{A}) \leq n$.*

As a few examples we cite the following facts which can be shown with the method we have outlined:

Theorem 4.15. *The Chaffee-Infante equation*

$$u_t - \Delta u = \lambda(u - u^3), u|_{\partial\Omega} = 0, \Omega \subset \mathbb{R}^m$$

is a well-posed forward in time partial differential equation. It has a compact absorbing set in $H_0^1(\Omega)$ and therefore a global attractor. The system is uniformly differentiable and the fractal dimension of the attractor is bounded above

$$d_f(\mathcal{A}) \leq c\lambda^{m/2}$$

for some $c > 0$.

Theorem 4.16. *The 2D periodic Navier-Stokes equations*

$$u_t + u \cdot \nabla u = -\nabla p + \nu\Delta u, \nabla \cdot u = 0, \Omega = \mathbb{T}^2$$

is a well-posed partial differential equation forward in time and is uniformly differentiable. It has a well defined global attractor bounded in norm by some constant p_ν and the fractal dimension of the attractor is bounded above by $d_f(\mathcal{A}) \leq \alpha\left(\frac{p_\nu}{\nu}\right)^2$ for some $\alpha > 0$.

We make no claims that the above estimates are sharp.

5. Conclusion

In this paper we have given a brief introduction to the field of infinite dimensional dynamical systems. Many further questions remain in the field. For a particular problem where dissipative behaviour is expected, finding the correct norm to establish a compact absorbing set is in general an open problem. Another open problem is to determine which fixed points are connected in Lyapunov systems and how can the attractor be reconstructed from long time data of the system. The theory may also be extended for non-autonomous systems as well as systems where uniqueness of the solution is not guaranteed (via the concept of a "trajectory attractor") [?][3].

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