DEGREE THEORY: AN INTRODUCTION AND APPLICATIONS

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1. Introduction

Let us begin with a question of fundamental mathematical importance: given a function f, a domain Ω, and an image point y, for how many $x \in \Omega$ do we have $f(x) = y$? By definition, injective functions would always have either one or zero and surjective functions would always have at least one. Of course, such an answer in general is complicated. However, we might have more luck by instead of demanding that we know the **exact** number of such x, we can find a function which would indicate to us that at least some x exists. It would also be great if such a function is topologically stable, as then we can connect possibly more interesting and complicated scenarios with much simpler and well understood spaces. There are many ways to go about such a construction; but, with the goal of moving onto a general differentiable manifold, let us start with coordinate patches.

2. Subsets of $\mathbb{R}^n$

2.1. **Degree defined on $\mathbb{R}^n$.** Let us consider a bounded, open set Ω in $\mathbb{R}^n$ and let $f : \overline{\Omega} \to \mathbb{R}^n$ be a continuous function. Note that even inclusion in the image **at all** is not stable if the point we are considering is on the boundary of our region. To avoid this, let $y \in \mathbb{R}^n \backslash \partial\Omega$. With all of this in place, we search for a function $\deg : (f, \Omega, y) \to \mathbb{Z}$ which satisfies the following properties:

(D1) : $\deg(id_\Omega, \Omega, y) = 1$ if $y \in \Omega$,

(D2) : $\deg(f, \Omega_1, y) + \deg(f, \Omega_2, y) = \deg(f, \Omega, y)$ where $\Omega_1, \Omega_2 \subseteq \Omega$, $y \notin f(\overline{\Omega}\backslash(\Omega_1 \cup \Omega_2))$,

(D3) : $\deg(h(t, \cdot), \Omega, y(t))$ is independent of t as long as h: $[0, 1] \times \overline{\Omega} \to \mathbb{R}^n$ is a continuous homotopy, $y(t)$ is continuous and $y(t) \notin h(t, \partial\Omega)$ for all $t \in [0, 1]$.

The first requirement is a kind of normalization, but matches our intuition as we know the identity map is bijective onto itself and thus should have one pre-image for every point in its image. The second requirement lets us break our domain into different spaces assuming we know that our image point is not outside of our decomposition. The third requirement enforces topological stability, allowing us to deform our domain and the image point as long as we do not reach the boundrary at any point.

Such a function not only exists, but is known to be unique. If y is a regular value of f, it takes the following form:

$$\deg(f, \Omega, y) = \sum_{x \in f^{-1}(y)} \operatorname{sgn}(\det(D_x f)) \tag{1}$$

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Note that the sum is finite as we are considering a continuous map from a compact set ($\overline{\Omega}$), thus the pre-image is compact. Since y is a regular value, the pre-image must also be a discrete space (at most countably many isolated points). A discrete, compact space in a Hausdorff space must be finite, thus the sum is a finite sum of ± 1 and thus is finite.

We will refer to $\deg(f, \Omega, y)$ as the **degree of f**. We will refer any interested reader to the details/proof in [1]. The key point of the proof is the transition from general continuous functions to linear functions defined at appropriate points. We also recover a few properties of the degree function, namely:

(D4) : $\deg(f, \Omega, y) \neq 0 \implies f^{-1}(y) \neq \emptyset$,

(D5) : $\deg(f, \Omega, y)$ is constant on every connected component of $\mathbb{R}^n \backslash f(\partial\Omega)$,

(D6) : $\deg(g, \Omega, y) = \deg(f, \Omega, y)$ if $g|_{\partial\Omega} = f|_{\partial\Omega}$,

(D7) : $\deg(f, \Omega, y) = \deg(f, \Omega_1, y)$ if $y \notin f(\overline{\Omega}\backslash\Omega_1)$.

Intuitively, our degree function will not necessarily tell us how many x exist such that $f(x) = y$; but we are at least able to tell that if the degree is not equal to zero, we have at least one solution! Just to underscore the power of this tool, we prove a familiar result in mod 2 intersection theory (we will see this connection many times):

Theorem 2.1 (Brower's Fixed Point Theorem for the Compact Ball)**.** *Let $f : \overline{B_r(0)} \to \overline{B_r(0)}$ be continuous, then f has a fixed point*

Proof. Suppose not. Then if we consider the homotopy $h : [0, 1] \times \overline{B_r(0)} \to \overline{B_r(0)}$: $h(t, x) = x - tf(x)$, note that

$$\big|h(t, \partial B_r(0))\big| \geq |x| - t\big|f(x)\big| = r - t\big|f(x)\big| \geq r(1 - t) > 0$$

on $t \in [0, 1)$ and by assumption we have that $h(1, \partial B_r(0)) = x - f(x) \neq 0$. Thus using **(D3)** we have that

$$\deg(h(t, \cdot), B_r(0), 0) = deg(x - f(x), B_r(0), 0) = \deg(x, B_r(0), 0) = 1$$

by **(D1)**. Using **(D4)**, there then exists a y s.t.

$$f(y) - y = 0 \implies f(y) = y.$$

□

This also allows a direct proof of another important result.

Theorem 2.2 (Hairy Ball/ Hedgehog Theorem)**.** *Let n be odd. Let Ω be an open bounded subset of $\mathbb{R}^n$ with $0 \in \Omega$. Let $f : \partial\Omega \to \mathbb{R}^n\backslash\{0\}$ be continuous. Then there exists an $x \in \partial\Omega$ such that $f(x) = \lambda x$.*

Proof. The proof primarily lies on the fact that if two maps have different degree, then there cannot exist a homotopy between them which keeps the value off of the boundary. This would require a negation of **(D3)** which would cause a contradiction. We also note that since the identity map is clearly a smooth submersion and $0 \in \Omega$ and n is odd, we have from (1) that $\deg(-id_\Omega, \Omega, 0) = -1$ ($\mathrm{sgn}(\det(D_0(-id_\Omega)) = -1^n = -1$). Since $\partial\Omega$ is compact, without loss we assume there exists an extension $\tilde{f} \in C(\overline{\Omega})$ such that $\tilde{f}|_{\partial\Omega} = f|_{\partial\Omega}$. There is a concern about whether or not the degree of such an extension is unique. **(D6)** implies that even if the extension of the function is not unique, the degree of the extension will

be unique.
If $\deg(\tilde{f}, \Omega, 0) \neq -1$, then consider the map

$$h : [0,1] \times \overline{\Omega} : h(t,x) = t(-x) + (1-t)\tilde{f}(x).$$

This map clearly defines a homotopy between the two functions, but $\deg(-id_\Omega, \Omega, 0) \neq \deg(\tilde{f}, \Omega, 0)$ so **(D3)** implies there is some (t_0, x_0) such that $x_0 \in \partial\Omega$ and $h(t_0, x_0) = 0$. This implies that

$$h(t_0, x_0) = 0 \implies f(x_0) = \tilde{f}(x_0) = \frac{-t_0}{1-t_0} x_0$$

and we have found a point $x_0 \in \partial\Omega$ such that $f(x_0) = \lambda x_0$. If $\deg(\tilde{f}, \Omega, 0) = -1$ we apply the same argument above to the map

$$h : [0,1] \times \overline{\Omega} : h(t,x) = t(x) + (1-t)\tilde{f}(x)$$

and obtain λ of the form

$$h(t_0, x_0) = 0 \implies f(x_0) = \tilde{f}(x_0) = \frac{t_0}{1-t_0} x_0.$$

□

In particular, the Hairy Ball/ Hedgehog theorem implies that if n is even, we cannot have a nowhere vanishing vector field on S^n. This is because S^n is always the boundary of a ball in $\mathbb{R}^{n+1}$. Hairy Ball would then imply that there would be one point where the vector field points directly out of the ball. Since the vector field is a section of the tangent space, this then forces the vector field to be zero at that point.

Note that **(D5)** implies that $\deg(f, \Omega, K_i)$ is well defined for all connected components K_i of $\Omega \backslash f(\partial\Omega)$ by setting $\deg(f, \Omega, K_i) = \deg(f, \Omega, y),\ y \in K_i$. Due to Sard's Theorem, we can choose y to be a regular value of f. This leads us to an important relationship

Theorem 2.3 (Product Formula for Degree)**.** *Let Ω be open and bounded, $f : \Omega \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$ with both continuous. Let K_i denote an enumeration of the connected components of $\mathbb{R}^n \backslash f(\partial\Omega)$ and suppose that $y \notin gf(\partial\Omega)$. Then we have:*

$$\deg(gf, \Omega, y) = \sum_i \deg(f, \Omega, K_i) \deg(g, K_i, y)$$

with only finitely many terms nonzero

Proof. First, note that $y \notin gf(\partial\Omega) \implies g^{-1}(y) \cap f(\partial\Omega) = \emptyset$. Thus $f(\overline{\Omega}) \cap g^{-1}(y) \subseteq \mathbb{R}^n \backslash f(\partial\Omega) = \cup_i K_i$. However, because f and g are continuous $f(\overline{\Omega}) \cap g^{-1}(y)$ is compact and $\cup_i K_i$ is a cover, therefore $\cup_i K_i$ admits a finite subcover $\cup_{i=1}^n K_i \supseteq f(\overline{\Omega}) \cap g^{-1}(y)$. Thus there are only finitely many i s.t. $\deg(g, K_i, y) \neq 0$ and $\deg(f, \Omega, K_i) \neq 0$ both by **(D4)**. Thus only finitely many terms may be nonzero.
Note that because of **(D5)** (as well as the general construction of the degree), it is enough to check the formula for regular values of gf. By the chain rule we thus have

$$\begin{aligned}
\deg(gf, \Omega, y) &= \sum_{x \in (gf)^{-1}(y)} \operatorname{sgn}(\det(D_x(gf))), \\
&= \sum_{x \in (gf)^{-1}(y)} \operatorname{sgn}(\det(D_{f(x)}(g)))\operatorname{sgn}(\det(D_x(f))),
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{x\in f^{-1}(z)\\ z\in g^{-1}(y)}} \operatorname{sgn}(\det(D_z(g)))\operatorname{sgn}(\det(D_x(f))),\\
&= \sum_{\substack{x\in f^{-1}(z)\\ z\in f(\Omega)}} \operatorname{sgn}(\det(D_z(g)))\left[\sum_{x\in f^{-1}(z)} \operatorname{sgn}(\det(D_x(f)))\right],\\
&= \sum_{\substack{z\in f(\Omega)\\ g(z)=y}} \operatorname{sgn}(\det(D_z(g)))\deg(f,\Omega,z),\\
&= \sum_{i=1}^{n}\sum_{\substack{z\in K_i\\ z\in g^{-1}(y)}} \operatorname{sgn}(\det(D_z(g)))\deg(f,\Omega,z),\\
&= \sum_{i=1}^{n}\deg(f,\Omega,K_i)\sum_{\substack{z\in K_i\\ z\in g^{-1}(y)}} \operatorname{sgn}(\det(D_z(g))),\\
&= \sum_{i}\deg(f,\Omega,K_i)\deg(g,K_i,y).
\end{aligned}
$$

□

We will use the product formula to show a "hard to prove but intuitive" theorem.

Theorem 2.4 (Jordan/Brouwer Separation Theorem)**.** *Let $C_1, C_2 \subset \mathbb{R}^n$ be compact homeomorphic sets. Then $\mathbb{R}^n\backslash C_1$ and $\mathbb{R}^n\backslash C_2$ have the same number of finite connected components, or both have infinite connected components.*

Proof. First note that by Heine-Borel, C_1, C_2 compact implies they are bounded. $\mathbb{R}^n\backslash C_1$ and $\mathbb{R}^n\backslash C_2$ then both have exactly one unbounded connected component. It is sufficient to check then that $\mathbb{R}^n\backslash C_1$ and $\mathbb{R}^n\backslash C_2$ have the same number of bounded connected components.

Let K_j and L_i denote bounded connected components of $\mathbb{R}^n\backslash C_1$ and $\mathbb{R}^n\backslash C_2$ respectively. Let K_0 and L_0 denote the unique unbounded connected components of $\mathbb{R}^n\backslash C_1$ and $\mathbb{R}^n\backslash C_2$ respectively. Note that for all K_j and L_i we have $\partial K_j \subseteq C_1$ and $\partial L_i \subseteq C_2$ as both C_1 and C_2 are closed. Let us fix a j. Then we have

$$(\cup_i L_i) \cup L_0 = \mathbb{R}^n\backslash C_2 \subseteq \mathbb{R}^n\backslash h(\partial K_j) = (\cup_q G_q) \cup G_0. \tag{2}$$

Note that $h(\partial K_j) \subseteq h(C_1) = C_2$ implies $h(\partial K_j)$ is bounded, and again we can break it into bounded connected components $(\cup_q G_q)$ and a unique unbounded connected component (G_0). Thus for every L_i we have a unique q(i) s.t. $L_i = G_q(i)$ as L_i and G_q are maximal, connected sets. In particular, $q(0) = 0$ as these are the only unbounded sets.

Let $h : C_1 \to C_2$ denote a homeomorphism onto C_2. Let $\hat{h}, \widetilde{h^{-1}}$ be a continuous extensions of h and h^{-1} respectively to all of $\mathbb{R}^n$. Pick a $y \in K_j$. Note that $\partial K_j \subseteq C_1$ gives us that $\widetilde{h^{-1}}\hat{h}|_{\partial K_J} = hh^{-1}|_{\partial K_j} = id_{K_j}|_{\partial K_j}$. Thus by **(D1)**, **(D6)**, and Theorem 2.3 we have

$$1 = \deg(id_{K_j}, K_j, y) = \deg(\widetilde{h^{-1}}\hat{h}, K_j, y) = \sum_q \deg(\hat{h}, K_j, G(q))\deg(\widetilde{h^{-1}}, G_q, y) \tag{3}$$

(4)
$$= \sum_i \deg(\hat{h}, K_j, G(q(i))) \deg(\widetilde{h^{-1}}, G_q(i), y) + \sum_{\substack{q \ s.t. \\ q(i) \neq q \forall \ i}} \deg(\hat{h}, K_j, G(q)) \deg(\widetilde{h^{-1}}, G_q, y).$$

We now focus on the second term. Note that $q \neq q(i)$ for all i implies that $G_q \subseteq \mathbb{R}^n \backslash (\partial K_j) \backslash (\mathbb{R}^n \backslash C_2) = C_2 \backslash h(\partial K_j)$. Then by **(D4)**, $\deg(\widetilde{h^{-1}}, G_q, y) \neq 0$ for q such that $q \neq q(i)$ for all i implies there exists an $x \in C_2$ such that $h^{-1}(x) = \widetilde{h^{-1}}(x) = y$ which is impossible as $h^{-1}(x) \in C_1, y \in K_j \subseteq \mathbb{R}^n \backslash C_1$. Thus the second term in our sum is zero and we have

(5)
$$1 = \sum_i \deg(\hat{h}, K_j, G(q(i))) \deg(\widetilde{h^{-1}}, G_q(i), y) = \sum_i \deg(\hat{h}, K_j, L_i) \deg(\widetilde{h^{-1}}, L_i, K_j).$$

The exact same chain of reasoning (now fixing an L_i) leads us to

$$1 = \sum_j \deg(\widetilde{h^{-1}}, L_i, y) \deg(\hat{h}, K_j, L_i). \tag{6}$$

Combining these we see that

$$\sum_i \sum_j \deg(\widetilde{h^{-1}}, L_i, y) \deg(\hat{h}, K_j, L_i) = \sum_j 1 = \sum_i 1 \Leftrightarrow i = j. \tag{7}$$

□

We would now like to expand this tool to general differentiable manifolds. However, we encounter a few problems we will need to correct for. For instance, how do we know the expression in (1) is well defined? What if we could pick two different coordinate charts at each pre-image which changes the sign of the determinant, thereby changing my sum? To prevent this, we need some global notion to orient our manifolds so that we can remove this ambiguity.

2.2. **Orientation on Vector Spaces.** A strong understanding of orientation is necessary to continue in degree theory. We refer the reader to Chapter 15 in [4] for an introduction. A useful construction is the *direct sum orientation.* Let $W = V_1 \oplus V_2$. If we introduce an orientation on any two of these vector spaces, we naturally induce an orientation on the third in the following way. If $\beta = (\beta_1, \beta_2)$ where β is an ordered basis for W and β_1, β_2 are ordered bases for V_1, V_2 respectively, assign orientations in such a way that $sgn(\beta) = sgn(\beta_1)sgn(\beta_2)$.

3. Manifolds

3.1. **Degree on Manifolds.** An orientation for a given manifold is a smooth choice of orientations for the manifold. A manifold is called *orientable* if it can be given an orientation. Unlike vector spaces, not all manifolds are orientable. A classic example of a non-orientable manifold is the Möbius Strip. However, if our manifold is connected and orientable, we have the following proposition

Proposition 3.1. *A connected, orientable manifold has exactly two orientations*

See a proof for the above proposition in [3]. Thus if we have a smooth map $f : M \to N$ between two oriented, connected manifolds, df_x can only be either orientation preserving or reversing, so its orientation number is well-defined. As a quick review, the orientation number of a smooth map is $+1$ if its differential is orientation preserving and -1 if it is orientation reversing at a given point. We are now ready to introduce our notion of degree

Definition 3.2. Let X be a compact,smooth manifold and $f : X \to X$ be a smooth map with regular value y. Then we define

$$\deg(f) = \sum_{x \in f^{-1}(y)} \text{orientation number}(df_x) \tag{8}$$

This inherits many of the properties from the finite dimensional degree in Euclidean space, most notability its stability and nonzero implies the pre-image is non-empty (to see this, we simply reduce to coordinate charts). However, we can generalize this notion even further.

3.2. **Intersection Theory with Functions.** We generalize degree theory to not only include solutions to $f(x) = y$ but instead to capture equations like $f(x) \in Z$ where Z is a submanifold of some ambient space. When introducing orientations and orientation numbers, we needed both the image and pre-image to be manifolds so that we could effectively talk about the differential map (i.e. we want any map $f : X \to Z$ we consider to be transversal). We also want our pre-image space to be compact so that we can construct finite sums like those in (1). All of this justifies the following definition

Definition 3.3. Let X, Y, and Z be connected, orientated manifolds with $X, Z \subseteq Y$ with dim X + dim Z = dim Y. Let X be compact. Let $f : X \to Y$ be transversal to Z. Denote the intersection number as $I(f, Z)$ to be the sum of orientation numbers at each of the points in $f^{-1}(Z)$

Let us unpack the above definition. First, why did we impose dim X + dim Z = dim Y? This is because (since f is transversal to Z), codim $f^{-1}(Z)$ = codim Z but this then gives

$$\dim X - \dim f^{-1}(z) = \text{codim}(f^{-1}(Z)) = \text{codim} Z = \dim Y - \dim Z \tag{9}$$

$$\implies \dim f^{-1}(Z) = \dim X + \dim Z - \dim Y \tag{10}$$

We thus require dim X + dim Z = dim Y so that our pre-image is a zero dimensional submanifold of a compact manifold, i.e. a finite number of discrete points. Second, how do we define an orientation number at these points? X and Z may not even have the same dimension in general. The key aspect again is that $f^{-1}(Z)$ has the same codimension as Z. To see why, pick an arbitrary complimentary subspace H to $T_x(f^{-1}(Z))$ i.e. an H such that:

$$H \oplus T_x(f^{-1}(Z)) = T_x(X)$$

Let z = f(x). Transversality then gives us

$$df_x(T_x(X)) + T_z(Z) = T_z(Y) \tag{11}$$

$$df_x(H \oplus T_x(f^{-1}(Z))) + T_z(Z) = T_z(Y) \tag{12}$$

$$df_x(H) \oplus T_z(Z) = T_z(Y) \tag{13}$$

Since $df_x^{-1}(T_z(Z)) = T_x(f^{-1}(Z))$. By dimension counting, $df_x|_H$ is an isomorphism onto its image. We obtain orientations on H and $df_x(H)$ using the direct sum orientations from X, Z, and Y. Set the orientation number at $x \in f^{-1}(Z)$ to be the orientation number of $df_x|_H$. It is left as an exercise that such a choice is independent of the complimentary subspace H (details in [3]).

Proposition 3.4. *If f and g are homotopic , then $I(f, Z) = I(g, Z)$*

Proof. This follows from stability of degree. Note that in local coordinates this is just a map from a compact subset of $\mathbb{R}^n$ to $\mathbb{R}^n$. See details in [3] ☐

This allows us to even further extend our definition and remove the need for transversality altogether:

Definition 3.5. Let X, Y, Z be smooth, oriented manifolds with dim X + dim Z = dim Y. Let X be compact. Then for any $f : X \to Y$, there exists a function $\hat{f} : X \to Y$ such that $\hat{f}$ is homotopic to f and $\hat{f}$ is transversal to Z. Define $I(f, Z) = I(\hat{f}, Z)$

3.3. **Intersection Theory with Manifolds.** We may then take this one step further by noting that every sub-manifold of a given space is identified with its inclusion map. With this in mind, lets extend our definition of transversality to cover this possibility:

Definition 3.6. $f : X \to Y$, $g : Z \to Y$ are called transversal (denoted $f \pitchfork g$) if

$$df_x(T_x(X)) + dg_z(T_z(Z)) = T_y(Y)$$

whenever $f(x) = g(y) = z$

Further, we have another characterization of the above:

Proposition 3.7. *Let $f : X \to Y$, $g : Z \to Y$ be smooth functions with X, Y, Z connected and oriented. Let dim X + dim Z = dim Y. Let Z and X be compact. Let $\Delta = \{(y, y)|y \in Y\}$. Then $f \pitchfork g \Leftrightarrow f \times g \pitchfork \Delta$*

Proof. This directly follows from the defintion in the following way:

$$f \pitchfork g \Leftrightarrow df_x(T_x(X)) \oplus dg_z(T_z(Z)) = T_y(Y) \tag{14}$$
$$\Leftrightarrow df_x(T_x(X)) \cap dg_z(T_z(Z)) = \emptyset \tag{15}$$
$$\Leftrightarrow df_x(T_x(X)) \times dg_z(T_z(Z)) \cap T_y(Y) \times T_y(Y) = \emptyset \tag{16}$$
$$\Leftrightarrow df_x(T_x(X)) \times dg_z(T_z(Z)) \oplus T_y(Y) \times T_y(Y) \tag{17}$$
$$\Leftrightarrow f \times g \pitchfork \Delta \tag{18}$$

☐

This allows us to further our definition of intersection theory to account for transversal maps in general:

Definition 3.8. Let X, Z, Y be oriented, connected manifolds with X, Z compact and dim Y = dim X + dim Z. Let $f : X \to Y$, $g : Y \to Z$ be smooth maps such that $f \pitchfork g$. Then in particular we have

$$df_x(T_x(X)) \oplus dg_z(T_z(Z)) = T_y(Y)$$

for f(x) = y = g(z). Set the orientation number of a point (x,z) to +1 if $df_x(T_x(X)) \oplus dg_z(T_z(Z))$ has the same orientation as $T_y(Y)$ and -1 otherwise. Then we have

$$I(f, g) = \sum_{\substack{(x,z), y \in Y \\ g(z) = y, f(x) = y}} \text{orientation number}((x, z))$$

Again, homotopy invariance follows from homotopy invariance of Intersection, just exchanging the roles of X and Z in terms of which we consider "the manifold" and which we consider "the transverse function". This then gives the immediate consequence that

Proposition 3.9. $I(f,g) = (-1)^{(dim\ X)(dim\ Z)} I(g,f)$

To see this, simply note that we require exactly $(\dim X)(\dim Z)$ transpositions to transform $df_x(T_x(X)) \oplus dg_z(T_z(Z))$ into $dg_z(T_z(Z)) \oplus df_x(T_x(X))$. Again, identifying manifolds with their inclusion mappings we have

Corollary 3.10. *If X, Z are oriented, compact submanifolds of Y, then*

$$I(X,Z) = (-1)^{(\dim X)(\dim Z)} I(Z,X) \tag{19}$$

We are now interested in a special case of this, namely the intersection of a space with itself.

Definition 3.11. Let $\Delta = \{(x,x)|x \in X\}$ be the diagonal of $X \times X$. If X is compact and oriented then we define the Euler characteristic of X (denoted $\chi(X)$) as

$$\chi(X) = I(\Delta, \Delta)$$

The Euler characteristic is an important topological invariant which we will use in just a moment. For now, we will consider the above definition as the sole definition of the Euler characteristic. We will show that that this notion agrees with other notions we may have seen for the Euler characteristic for some given surfaces.

As a helpful immediate consequence, we have the following

Corollary 3.12. *Let X be an odd dimensional, compact, oriented manifold. Then $\chi(X) = 0$*

Proof. Apply Corollary 3.10 with $\dim \Delta = \dim X$ odd. □

3.4. **Lefschetz Fixed Point Theory.** When we are seeking fixed points of a map, we are seeking x in our space such that for a given function $f : X \to X$ we have $f(x) = x$. This plays an incredibly important role in a wide variety of application from finding zeros in algebraic equations to characterizing behaviour in dynamical systems. With the tools we have built thus far, we are able to create a tool which helps us find such fixed points.

Definition 3.13. Let X be a compact, oriented manifold. Let $f : X \to X$ be a smooth map. Recall that $\Delta = \{(x,x)|x \in X\}$ and $\text{graph}(f) = \{(x, f(x))|x \in X\}$. We define the *global Lefschetz number* (denoted L(f)) as

$$L(f) = I(\Delta, \text{graph}(f))$$

As f is arbitrary, in general there may be infinitely many such points. Much like the degree, this will not tell us exactly how many fixed points there are, but we do have the powerful result that:

Theorem 3.14. *Let $f : X \to X$ be a smooth map with X compact and orientable. Then if $L(f) \neq 0$, there exists at least one fixed point of f*

Proof. Suppose not. Then as $L(f)$ is defined as the intersection number between two manifolds, we only need to sum up the orientation numbers in their intersection. f has no fixed points implies that $\Delta \cap \text{graph}(f) = \emptyset$ and thus $L(f) = 0$. □

Proposition 3.15. *If $f : X \to X$ is homotopic to the identity with X compact and orientable, then $L(f) = \chi(X)$*

Proof. This is a direct consequence of identifying graph(f) and Δ with the inclusion maps $i_f : X \to X \times X : x \to (x, f(x))$ and $\imath_\Delta : X \to X \times X : x \to (x, x)$. If f(x) is homotopic to the identity then i_f is homotopic to i_Δ trivially. We then have

$$L(f) = I(\Delta, \text{graph}(f)) = (-1)^{(\dim X)^2} I(\text{graph}(f), \Delta) = (-1)^{(\dim X)^2} I(i_f, \Delta) \tag{20}$$

$$= (-1)^{(\dim X)^2} I(i_\Delta, \Delta) = (-1)^{(\dim X)^2} I(\Delta, \Delta) = (-1)^{(\dim X)^2} \chi(X) \tag{21}$$

If dim X is even, the equality is clear. If dim X is odd, then by Corollary 3.12 then $\chi(X) = 0$ and the equality still holds. □

We hope the following is not hard to believe based on the results we have built thus far

Theorem 3.16. *Let $f : X \to X$ be smooth with X compact and connected. Let Δ denote the diagonal of X. If df_x has no eigenvalue equal to 1, then graph$(f) \pitchfork \Delta$ and we can calculate $L(f)$ as*

$$L(f) = \sum_{x s.t. f(x)=x} \textit{orientation number } (Id - df_x) \tag{22}$$

We will need one other variation of this theorem.

Proposition 3.17. *If f satisfies the conditions in Theorem 3.16, then around each fixed point x_i for f we can find coordinate maps Ψ_i such that each fixed point is isolated. Moreover, we have that:*

$$L(f) = \sum_i \deg\left(\frac{\psi_i f \psi_i^{-1} - x}{\|\psi_i f \psi_i^{-1} - x\|}\right)$$

See a proof for the above theorem, proposition, and further extensions in [3].

3.5. The Poincaré-Hopf Theorem.

We now introduce a consequence of Corollary 3.15 of profound importance.

Theorem 3.18 (The Poincaré-Hopf Theorem)**.** *Let Φ_t be the flow generated by a vector field v on a compact,oriented, connected manifold X. Let Φ_t have only finitely many fixed points. Then*

$$L(\Phi_t) = \chi(X) = \sum_{x s.t. \Phi_t(x)=x} \deg\left(\frac{v}{\|v\|}\right)$$

Proof. First, note all flows are homotopic to the identity map under the homotopy $h : [0, 1] \times X : h(s, x) = \Phi_{t(1-s)}(x)$. Thus $L(f) = \chi(X)$ follows from Proposition 3.15. Due to Theorem 21, we only need to check the expression in a coordinate patch around each fixed point. Let x then be a fixed point and let $\psi : U \to B_1(0)$ be a coordinate function which sends x to the origin. Without loss, let U contain no other fixed points (only finitely many). Since the flow is smooth in the t-parameter, we thus have

$$\psi\Phi_t\psi^{-1}(y) = \psi\Phi_0\psi^{-1}(y) + \frac{d}{dt}\left(\psi\Phi_t\psi^{-1}(y)\right)|_{t=0} t + r(y, t)t^2 \tag{23}$$

$$= y + \mathcal{L}_v(\psi)\psi^{-1}(y)t + r(y,t)t^2 = y + tv\psi\psi^{-1}(y) + r(y,t)t^2 \tag{24}$$

$$= y + vt + r(x,t)t^2 \tag{25}$$

Without loss, let y not be a fixed point of the flow. Then we have

$$\psi\Phi_t\psi^{-1}(y) - y = vt + r(y,t)t^2 \tag{26}$$

$$\implies \frac{\psi\Phi_t\psi^{-1}(y) - y}{\|\psi\Phi_t\psi^{-1}(y) - y\|} = \frac{v + r(y,t)t}{\|v + r(y,t)t\|} \tag{27}$$

The degree of the term on the right is exactly the contribution to the Lefschetz number by Proposition 3.17. The term on the right is homtopic to $\frac{v}{\|v\|}$ and thus must have the same degree. Summing up over all fixed points, we achieve the statement of the theorem. □

Returning to our definition of the Euler-characteristic, we can use Poincaré-Hopf to show that this abstract defintion agrees with our definition for some common manifolds.

Theorem 3.19. $\chi(S^2) = 2,\ \chi(S^n) = (-1)^n + 1,\ \chi(T^2) = 0,\ \chi(D^2) = 1$

Proof. Note by Poincaré-Hopf it is enough to construct a vector field on each of these manifolds, and then simply check the degree of the vector field in some neighborhood at each of the zero points of the vector field. For S^2, let ϕ denote the azimuthal angle and let $v = \sin(\phi)\ \hat{\phi}$. Through direct computation (using stereographic projection at the poles)

$$\chi(S^2) = \deg(Id, \Phi_S, 0) + \deg(-Id, \Phi_N, 0) = 1 + 1 = 2.$$

For a general n, we can let ϕ be the last angle for a general hyper-sphere and define the same vector field. Then we again have by direct computation:

$$\chi(S^n) = \deg(Id, \Phi_S, 0) + \deg(-Id, \Phi_N, 0) = 1 + (-1)^n.$$

For T^2, consider the standard representation with two angles (θ, ϕ) and define the vector field $\hat{\phi}$. Note this vector field is perfectly well defined and has zero fixed points, thus

$$\chi(T^2) = 0.$$

For D^2, define the vector field $v = -\hat{r}$. By direct computation we have:

$$\chi(D^2) = \deg(-Id, D^2, 0) = 1.$$

□

The last calculation is particularly useful in the study of two dimensional differential equations. It states that any region enclosed by a periodic curve must contain a set of fixed points with degrees that sum to 1. This says that if we know all fixed points in a given region are saddles (degree is equal to -1), then no periodic orbits can exist in this region. Details of degree of vector fields for common examples are found in [3] and [5]. For an example of how this can be used to show the non-existence of cycling behaviour in competing species, see [5].

4. Conclusion

Degree Theory and Intersection Theories are powerful tools which allow us to transfer between analytic (the degree of a vector field at its fixed points) and topological (Euler characteristic, transversality, connected regions) notions. The Poincaré-Hopf Theorem itself has spawned many other techniques which pass topological and analytic information back and forth, including the celebrated Atiyah-Singer Index Theorem. We have seen how such a notion is not only natural to search for when trying to find solutions to an equation, but that there is in fact a unique correspondence which connects these questions to the topology of the space itself.

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