**Exercise 4**: Imagine a set of N measurements $t_i$, with uncertainty variances $\sigma_{t_i}^2$, all of the same (unknown) quantity T. Assuming the generative model that each $t_i$ differs from T by a Gaussian-distributed offset, taken from a Gaussian with zero mean and variance $\sigma_{t_i}^2$, write down an expression for the log likelihood ln L for the data given the model parameter T. Take a derivate to show that the maximum likelihood value for T is the usual weighted mean.

**Solution:** The frequency distribution for the data $y_i$ in this model is

$$p(t_i|\sigma_{ti}, T) = \frac{1}{\sqrt{2\pi\sigma_{t_i}^2}} \exp\left( - \frac{(t_i - T)^2}{2\sigma_{t_i}^2} \right), \tag{1}$$

The objective function that maximizes the probability of the observed data given the model is the likelihood, defined as

$$\mathfrak{L} = \prod_{i=1}^{N} p(t_i|\sigma_{ti}, T) \tag{2}$$

since the data are assumed to be independent. Taking the logarithm,

$$\ln \mathfrak{L} = C - \sum_{i=1}^{N} \frac{(t_i - T)^2}{2\sigma_{t_i}^2} \tag{3}$$

for some constant C. Maximizing $\ln L$ by taking the derivative with respect to the model parameter $T$ gives

$$0 = \frac{\partial ln\mathfrak{L}}{\partial T} = + \sum_{i=1}^{N} \frac{t_i - T}{\sigma_{t_i}^2} \tag{4}$$

$$\implies NT = \sum_{i=1}^{N} t_i \tag{5}$$

$$\implies T = \frac{1}{N} \sum_{i=1}^{N} t_i \tag{6}$$

This is the usual weighted mean, $\bar{t} = \frac{\sum_{i=1}^{N} w_i t_i}{\sum_{i=1}^{N} w_i}$ with weights $w_i = 1$.