SF2522 Final Project:

Verification of an approximation model error for Lévy processes

based on: *"Diffusion approximation of Lévy processes with a view towards finance." [1]*

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Abstract

In this Project a diffusion base model for approximating Lévy processes is presented. Simulations with Monte Carlo method are carried out on an example that has a financial application. The whole project is based on the paper cited in the title. The research paper gives an error estimate for the diffusion model. The practical purpose of the Project is to verify the accuracy of this estimation by comparing the simulation results.

Contents

1 Background theory and model

1.1 Lévy processes

A Lévy process is a stochastic process which satisfies the properties:

- $X_0 = 0$
- for any time t increments X_t are independent
- increments $X_t - X_s$ are equal in distribution to X_{t-s} for any $t > s$
- $\lim_{h\to 0} P(|X_{t+h} - X_t| > \epsilon) = 0$ holds for any $\epsilon > 0$ and $t \geq 0$

The distribution of a Lévy process X_t has the characteristic function:

$$E[e^{i\Theta X_t}] = exp(t(i\gamma\Theta - \frac{1}{2}\sigma^2\Theta^2 + \int_R (e^{i\Theta x} - 1 - i\Theta x \mathbf{1}_{|x|<1})\nu(dx))) \quad (1)$$

where ν is the Lévy measure and the set of parameters (γ, σ^2, ν) is called the characteristic triple.
If the measure is finite, $\nu < \infty$, X_t is said to have finite jump activity, and can be rewritten as the sum of three different terms, each linked to one of the parameters in the characteristic triple:

$$X_t = \gamma' t + \sigma W_t + \sum_{i=1}^{N_t} Y_i \quad (2)$$

The first term is characterized by the relation

$$\gamma' = \gamma - \int_{|x|<1} x\nu(dx) \quad (3)$$

and can be seen as the drift term of the scaled Wiener process W_t with diffusion σ^2, the second term. Finally the summation term represents a compound Poisson process. A compound Poisson process is a stochastic process which incorporates jumps, hence it's presence in the breakdown of a Lèvy process. The jumps have distribution Y_i and occur at random moments, the number of them N_t is distributed according to a Poisson process. [1, 2]

1.2 Model and its error

1.2.1 Diffusion approximation model

The model over which most of the work is performed is a toy example based on Example 4.6 in paper [1]. It is of financial nature and models a European digital option with maturity $T = 1$ and payoff funcition

$$g(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \quad (4)$$

The underlying process is a pure jump Lévy process with measure

$$\nu(dx) = \frac{1}{x^2}\mathbf{1}_{0<x<1}dx \quad (5)$$

The objective of a financial model as such is to compute $E[g(X_t)]$. This can be done efficiently by approximating the process X_t with $\bar{X}_t$ and then computing $E[g(\bar{X}_t)]$.
$\bar{X}_t$ is defined in Definition 3.1 of [1] as the diffusion approximation, which has characteristic triple $(\nu^\epsilon, \sigma^2 + \sigma^2(\epsilon), \gamma^\epsilon)$, where

$$\nu^\epsilon(x) = \mathbf{1}_{|x|>\epsilon}\nu(x) = \begin{cases} \nu(x) & |x| > \epsilon \\ 0 & |x| < \epsilon \end{cases} \quad (6)$$

and

$$\gamma^\epsilon = \begin{cases} \gamma & \epsilon < 1 \\ \gamma + \int_{1<|x|<\epsilon} x\nu(dx) & \epsilon \geq 1 \end{cases} \quad (7)$$

This approximation is obtained by breaking down the original process into a sum as follows:

$$X_t = X_t^\epsilon + R_t^\epsilon \ (8)$$

In this way we separate the sum of the jumps which do not exceed a small constant ϵ. This sum is represented by R_t^ϵ, a process with characteristic triple $(\mathbf{1}_{|x|<\epsilon}\nu, 0, 0)$, and can be approximated by a diffusion term, giving:

$$\bar{X}_t = X_t^\epsilon + \sigma(\epsilon)W_t \ (9)$$

As we know X_t^ϵ, being a finite jump activity process, can be written as the sum of a compound Poisson term, and a Brownian term with drift, as in equation (2), with characteristic triple $(\nu^\epsilon, \sigma^2, \gamma^\epsilon)$.

We now put all this information together and turn to our specific case, picking up from equation (5). Combining equation (5) with equation (6) we obtain that our specific measure is given by

$$\nu^\epsilon(dx) = \frac{1}{x^2}\mathbf{1}_{\epsilon<x<1}dx \ (10)$$

and the volatility is $\sigma(\epsilon) = \sqrt{\epsilon}$. For simplicity purposes, especially in the implementation, we will be using $\epsilon \leq 1$, so the drift is:

$$\gamma^\epsilon = \gamma \ (11)$$

For computational purposes we need to explicitly compute $\gamma^{\epsilon\prime}$ using equation (3) and (11):

$$\gamma^{\epsilon\prime} = \gamma - \int_{|x|<1} x\frac{1}{x^2}\mathbf{1}_{\epsilon<x<1}dx = \gamma - \int_{\epsilon<x<1}\frac{1}{x}dx = \gamma - ln\left(\frac{1}{\epsilon}\right) \quad (12)$$

Our final approximation model of the process is written then in a form which is easy to interpret for simulation:

$$\bar{X}_t = \left(\gamma - ln\left(\frac{1}{\epsilon}\right)\right)t + (\sigma + \sqrt{\epsilon})W_t + \sum_{i=1}^{N_t} Y_i \ (13)$$

Y_i are $i.i.d$ Normal distributions and N_t is the output of a Poisson random variable.

1.2.2 Model error

Using the diffusion approximation $\bar{X}_t$ clearly produces a certain amount of error, this is calculated in section 4 of [1] and is given as:

$$err\_model = E[g(X_T)] - E[g(\bar{X}_T)] = \frac{Tf(\epsilon)}{6} E[g^{(3)}(\bar{X}_T)] + \mathcal{O}(\epsilon^{2+\beta}) \quad (14)$$

where

$$f(\epsilon) = \int_{|y|<\epsilon} y\nu(dy) \quad (15)$$

in our particular case

$$f(\epsilon) = \frac{\epsilon^2}{6} \quad (16)$$

from the example in [1].

For the computation of the expected value we will use Monte Carlo simulations. From an error point of view this introduces an extra statistical error term because only a finite number of simulations can be carried out and convergence is not perfect. The statistical error, which can be added to the model one obtaining $err\_total = err\_model + err\_statistical$ is:

$$err\_statistical = E[g(\bar{X_T})] - \frac{1}{N}\sum_{i=1}^{N} g(\bar{X_T}(\omega_i)) \quad (17)$$

2 Simulation and results

The model in absence of market data could not be calibrated to resemble "real life" behaviour, so a toy example was constructed. The criteria for choosing the parameters of the example for simulations fell upon that of being able to produce significant results even with limited computational power: in fact maybe the biggest challenge in the whole work was to be able to carry out a sufficient amount of Monte Carlo simulations. The parameters chosen for the simulations, as well as the values already mentioned above, are $\lambda = 0.4$, $\sigma = 0.4$, and $dt = 0.01$ for subdivisions of time between 0 and $T = 1$. Further parameters will be set within the construction of the algorithm.

2.1 Algorithm and implementation

The algorithm is constructed taking into account the purpose of the simulations: to compute $E[g(X_T)]$ and verify the error estimate for the diffusion approximation model. With this in mind the method used to simulate the process is a Monte Carlo method. Using equation (13) the iterations of a process are computed with random number generators (which we will address further on) for a large number of times, and then the average is taken to obtain a final result of the expected value. This as we know from Monte Carlo methods, converges better the more simulations are carried out.

The Monte Carlo simulations are carried out for different values of ϵ ranging in between 0.01 and 1. The results are then plotted next to a theoretical error line given from the estimate in equation (14).
The model error in equation (14) is computed on a large amount of values of ϵ to give a seemingly continuous function. The 3rd derivative of $g(\bar{X}_T)$ is computed by using values from a simulation with $\epsilon = 1$, so that it is a pure diffusion process, and a discretization of the derivative as such:

$$g^{(3)}(x) = \frac{1}{8h^3}(g(x-3h)-8g(x-2h)+13g(x-h)-13g(x+h)+8g(x+2h)-g(x+3h))+\mathcal{O}(h^4) \quad (18)$$

the value chosen for the discretization is $h = 0.2$.
The actual error of the model should be computed comparing it to true or ideal data, this was not possible for the lack of data, the simulation with $\epsilon = 0.01$ was taken as a benchmark since the approximation can be considered fairly accurate. This is also the techique adopted in [1].

Descending into more detail the simulation was implemented by breaking down the stochastic process into its different parts as in equation (13). So for each Monte Carlo iteration we simulated a Brownian term, with drift computed accordingly, and a compound Poisson process.
The Brownian term required only a random Gaussian number generator and the appropriate scaling. Instead the compound Poisson process was slightly more complex and was simulated as follows:

- a random number was generated from a Poisson distribution, this represents the number of jumps
- each jumps' value was generated from a random Gaussian generator
- the time of each jump was taken from a Uniform distribution on the time interval $t[(0, T)]$

Finally after simulations were carried out for each Monte Carlo step, the expected value was computed for each of the values of ϵ.

2.2 Results

The highest number of Monte Carlo simulations managed without incurring in computation time issues was 10^5, not too satisfactory in theory but allowed to obtain some significant results, as we can see in figure 1. The red line in the figure represents the model error computed from equation (14) on many values of ϵ, showing what the behaviour should be like. The blue stars are the computed errors obtained from the Monte Carlo simulations at certain values of ϵ. It is easy to see that the theoretical prediction is maintained. In figure 2 we have the same values which though are on logarithmic scale only on the y-axis. The same kind of information can be inferred.

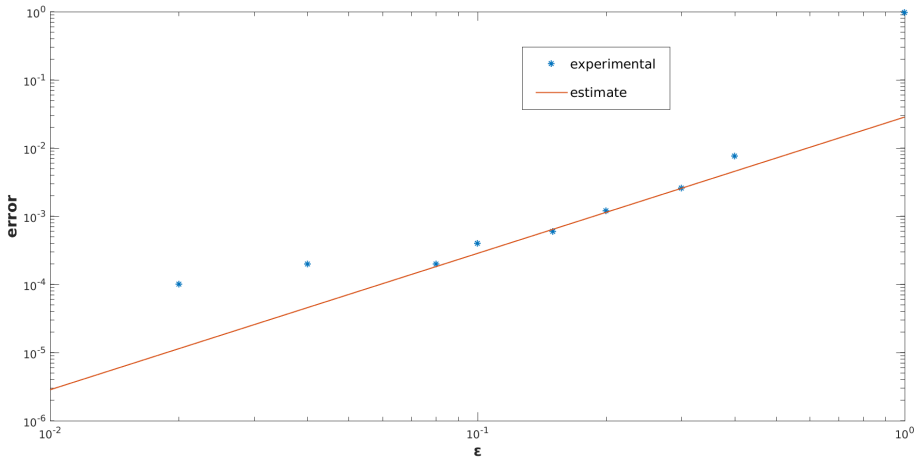

Figure 1: Comparison of the theoretical error model with simulation results on logarithmic scale

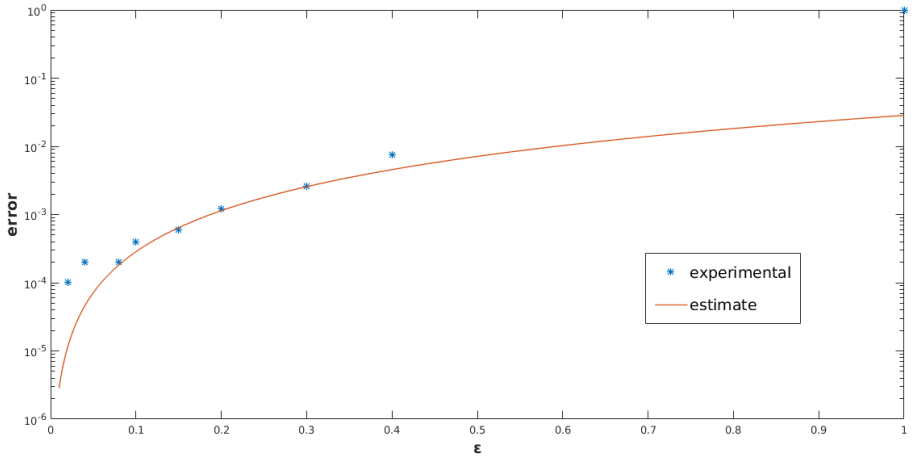

Figure 2: Comparison of the theoretical error model with simulation results on semi-logarithmic scale

References

[1] Kiessling Jonas and Tempone Raúl. "Diffusion approximation of Lévy processes with a view towards finance". In: *Monte Carlo Methods and Applications* 17.1 (2011), pp. 11–45. ISSN: 0010-4655. DOI: https://doi.org/10.1515/mcma.2011.003.

[2] Wikipedia. *Lévy process*. URL: https://en.wikipedia.org/wiki/L%C3%A9vy_process (visited on 10/05/2018).