MODELING AXION-GRAVITON INTERACTIONS FROM AN EFFECTIVE ACTION APPROACH

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ABSTRACT

Axions or axion-like particles are hypothetical elementary particles that serve as promising dark matter candidates and address some challenges in the Standard Model of particle physics. An indirect method to detect axions is by studying the effect of axion exchange between two neutron stars that are emitting gravitational waves prior to a binary merger. This method is based on the coupling between axions, neutron stars and the linearized weak gravitational field. In this thesis, we outline and derive the key steps behind the recent development of an effective action approach. We derive the effective Lagrangian for axion-graviton interactions starting from the action for Einstein-Hilbert gravity, a scalar field and the point particle approximation for neutron stars. We present the power counting scheme for the effective Lagrangian approach and present results from the computation of Feynman diagrams to first order in the post-Newtonian expansion of the binary's orbital dynamics. These theoretical calculations, while only part of the full perturbative series, provide a useful schema to achieve the ultimate goal of calculating the radiated gravitational signal in presence of axion-like interactions between two neutron stars.

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CHAPTER 1

INTRODUCTION

1.1 The Standard Model and the Axion

The Standard Model (SM) of Particle Physics [1] is currently the most complete unified theory of three of the four known fundamental forces in nature. The SM has, however, only been tested up to an energy scale of a few Teraelectron Volts (TeV or 10^{12} eV) thanks to CERN's Large Hadron Collider (LHC) [2]. Formally, the SM is a renormalizable quantum field theory of the electromagnetic, weak, and strong force based on the gauge principle where interactions are described by the exchange of gauge bosons (spin-1 particles) and the elementary particles are quarks and leptons (spin-1/2 particles). The Higgs boson is the only scalar (spin-0) particle detected so far, and it plays a special role by giving mass to the elementary particles (and itself) [3]. The SM is regarded as an effective field theory or EFT because new physics can emerge at higher energy scales up to the Planck scale ($l_P = \sqrt{\hbar G/c^3} \approx 10^{-35}$ m in natural units). The SM has been extremely successful in predicting and describing the results of particle production and interactions at colliders up to LHC energies. However, it has several issues that cannot be addressed without extending its current framework. One of them, which is related to this thesis, is the unnaturalness of certain parameters in the theory.

The Standard Model has 19 free parameters that must be determined by experiment. The principle of "naturalness" states that, absent some symmetry reason, these parameters should be, in some appropriate system of units, of order 1. One of the 19 parameters, the vacuum angle of Quantum Chromodynamics $\theta_{\mathrm{Q}CD}$, is experimentally determined to be consistent with zero, because current experimental bounds from the neutron's electric dipole moment imply that $\theta_{\mathrm{Q}CD} \leq 10^{-10}$ [4]. This implies some sort of fine-tuning from physics beyond the Standard Model, as there is no fundamental symmetry reason for this parameter to vanish. The θ_{QCD} term in the QCD Lagrangian violates charge-parity conservation (CP symmetry). In effect, the vanishing of θ_{QCD} means that QCD is CP-symmetric. Ordinarily, one might expect CP symmetry is reason enough, but we know that the weak force violates CP, so why not QCD? Furthermore, QCD admits a class of gauge transformations called "large" gauge transformations that connect topologically different but equivalent vacua, and $\theta_{QCD} \neq 0$ is a parameter that builds the true vacuum from the superposition of these many vacua. In short, there is no reason to expect that θ_{QCD}=0, but experimentally it is consistent with a vanishing value. How to explain this unnatural value of θ_{QCD} without fine-tuning (or other unscientific assumptions such as the Anthropic principle) is termed the Strong CP problem in particle physics [5]. One popular solution to this problem is motivating an additional light scalar particle called the *a*xion.

Another reason to study the axion, besides the strong CP problem of QCD, is as a candidate dark matter particle. It is believed that up to 21% of the total matter-energy density of our Universe must come from non-baryonic dark matter [6]. The ordinary baryonic matter that interacts with photons which is also known as "visible" matter is less than 5% of the total energy density of the universe. To explain this dark matter component, among the most common ideas are WIMPS (weakly-interacting massive particles) with a mass of 100 GeV or more, and light scalar particles that may or may not self-interact [7]. The WIMP scenario, which is supported by supersymmetric theories, would likely imply production and detection at LHC, but this has not been seen thus far. Attention has consequently turned to light scalars in the wide allowed mass range 10^{-22} e$V \leq m_a \leq 1$ GeV. The QCD axion, which is a particular kind of light scalar has a smaller allowed mass range of 25 μeV $\leq m_a \leq 5$ meV [8], which follows from the scale of the symmetry breaking of the Peccei-Quinn symmetry $U(1)_{\text{PQ}}$ [1]

[1]The Peccei-Quinn symmetry is the global symmetry introduced by R. Peccei and H. Quinn [9] to promote θ_{QCD} to a Nambu-Goldstone mode and thereby resolve the strong CP problem.

Even with this small mass, QCD axions are non-relativistic in the early universe and can be produced out of equilibrium, comprising a candidate for cold dark matter (CDM). Therefore, the axion serves as an attractive solution to some outstanding problems in particle physics and cosmology. If confirmed, it would be a dramatic advance in our understanding of Physics beyond the SM.

1.2 Axion Detection Methods

Currently, there are several ways in which experiments try to find axions or axion-like particles (ALP). We summarize a few of them before discussing the proposal to detect them indirectly using the Laser Interferometric Gravitational Wave Observatory (LIGO). Any detection method needs to be extremely sensitive due to the tiny magnitude of the ALP coupling, be it to photons or matter.

The first method is the “haloscope,” which aims to detect cold axions streaming from the galactic halo. The detection principle is based on the inverse Primakoff effect, which looks for the decay photons as a distinctive signature of dark matter axions. The decay rate is enhanced by a magnetic field of a microwave resonator. The Axion Dark Matter Experiment (ADMX) has yet to find such a signature, but has successfully ruled out ALPs from the galactic halo in the 2-4 μeV mass range. A related class of detection methods involving the Primakoff effect include looking for changes in the polarization of light propagating in a strong magnetic field.

The second method is a “helioscope”, using electron recoil events in a shielded underground vat of liquid Xenon as a signature of ALPs from the Sun. Such particles would not be dark matter candidates, but could be axions nevertheless. Recently, the XENON1T experiment in Gran Sasso, Italy reported a 3.5σ detection of a candidate axion. While this is an exciting data point, confirmation at the 5σ or standard discovery level will have to wait for the planned upgrade to the even larger XENONnT facility.

Finally, there are astrophysical searches that aim to detect the radio frequency photons produced by axions as they pass through neutron star magnetospheres. Axions may also be produced by Bremsstrahlung in neutron-neutron collisions in a neutron star, and constraints on the cooling of neutron stars limit the axion mass to be less than about 0.08 eV. Several other experiments based on similar principles exist (CAST, PVLAS, HAYSTAC, ALPS I and II), but none have as yet detected axions/ALPs.

1.3 Axion Detection in LIGO

It has been proposed [10] that gravitational waves emitted by merging neutron stars can carry the imprint of particle physics beyond the standard model, including the presence of new scalar particles. This "fifth" force mediated by scalar particles would alter the binary neutron star's inspiral rate, hence changing the amplitude and phase of the gravitational waveform, which can be detected by LIGO. Such forces have been constrained to be very weak in low-gravity environments, but in the strong gravity environment of neutron star mergers, there are no constraints. This intriguing idea was applied by Huang et al. [11] to an estimation of the prospects for axion detection in neutron star mergers. In this thesis, we follow the derivations laid out in that paper in some detail and arrive at the basic tools that allow us to compute the effect of axions on gravitational waves from a binary neutron star merger.

1.4 Organization of this Thesis

In Chapter 2, we provide some theoretical details of the theory behind the axion and the $U(1)_{\mathrm{PQ}}$ Peccei-Quinn symmetry. This chapter also presents the Einstein-Hilbert theory of gravitation and the procedure to linearize it to obtain weak-field solutions that describe gravitational waves.

In Chapter 3, we review the effective action approach and identify the light (radiation) and heavy (potential) modes for the axion and graviton fields. We identify the relevant propagators and vertex factor for axion-graviton coupling. We explain the relation of the power counting scheme in the effective Lagrangian to the post-Newtonian (PN) approximation for the binary neutron star system.

In Chapter 4, we present our main results, which are the Feynman diagrams computed using the Feynman rules that follow from the effective Lagrangian. The amplitudes for neutron stars interacting via axions and gravitons at leading order and 1PN order are calculated in momentum space. These amplitudes together give the leading correction from the axion sector to the binary's gravitational interaction.

In Chapter 5, we conclude with an outline of the steps to integrate out the heavy modes and obtain the effective theory for radiation power. This calculation is left for the future.

CHAPTER 2

THEORETICAL BACKGROUND FOR AXIONS AND GRAVITONS

2.1 Introduction

In this chapter, we introduce more formally the axion and the graviton, before moving on to discuss their interaction in the subsequent chapter. The mathematical treatment of the axion and graviton is standard, but by no means unique. There are alternate descriptions of axions and ALPs as well as extensions of the Einstein-Hilbert action for gravity, but we follow the standard prescription in this thesis.

2.2 The Strong CP Problem

To motivate the axion, we begin with the QCD Lagrangian (see glossary for the meaning of mathematical symbols in the equations)

$$\mathcal{L}_{\mathrm{QCD}} = \bar{\psi}_i \left(i(\gamma^\mu D_\mu)_{ij} - m_{ij} \right) \psi_j - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a - \theta \frac{g_s^2}{32\pi^2} G^{\mu\nu a} \tilde{G}^a_{\mu\nu} \,. \tag{2.1}$$

It is well known that that the θ-term ($\theta = \theta_{QCD}$ of Chapter 1) violates CP symmetry:

$$\mathcal{L}_\theta = -\theta \frac{g_s^2}{32\pi^2} G^{\mu\nu a} \tilde{G}^a_{a\mu\nu} \,. \tag{2.2}$$

This can be shown by using the definition of the hodge dual of the gluonic field strength tensor:

$$\tilde{G}^a_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} G^{\alpha\beta a} \tag{2.3}$$

$$\Rightarrow G^{\mu\nu a} \tilde{G}^a_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} G^{\mu\nu a} G^{\alpha\beta a} \,. \tag{2.4}$$

The quantity $G\tilde{G}$ in Eq.(2.4) is even under charge conjugation but odd under parity conjugation because the pseudotensor $\epsilon_{\mu\nu\alpha\beta}$ changes sign under parity conjugation. The neutron's electric dipole moment (nEDM) is directly proportional to this CP-violating term, which is the lowest-dimensional operator (and hence least suppressed) that can contribute to a non-zero nEDM. The induced nEDM is given by (e is the electron charge in cgs units)

$$d_n \sim e\theta \frac{m_q}{m_N^2} \sim 10^{-16} e\,cm\,, \tag{2.5}$$

if $\theta \sim \mathcal{O}(1)$, which is numerically 10 orders of magnitude above the experimental constraint of $d_n \sim 10^{-26} e\,cm$. The lack of experimental evidence for a non-vanishing nEDM indicates that QCD is CP-preserving (and hence also T-preserving since CPT is assumed to hold). The fine-tuning of this result (nEDM $\leq 10^{-26} e\,\mathrm{cm} \Rightarrow \theta \leq 10^{-10}$) is called the strong CP problem. In fact, the strong CP problem is not just an issue within QCD, but within the full standard model, since additional CP-violating terms arise from diagonalizing the quark mass matrix in the electroweak sector.

2.3 The Peccei-Quinn Solution to the Strong CP Problem - QCD Axions

In 1977, Roberto Peccei and Helen Quinn suggested a possible resolution to this problem by making θ a dynamical field that carries a $U(1)_{\mathrm{PQ}}$ charge (unrelated to the $U(1)$ of electromagnetism). A potential, for this dynamic field, is induced as a result of its coupling to the $G\tilde{G}$ term. The dynamical field is resolved into an equilibrium value (minimum of the potential which occurs at θ=0) and its low-energy fluctuations about this value (the axion). Therefore, the axion emerges as a Goldstone boson of the spontaneously broken $U(1)_{\mathrm{PQ}}$ symmetry. Ideally, it should be massless, but QCD's non-perturbative nature introduces an effective axion mass m_a, whose order of magnitude is set by the symmetry-breaking scale f_{PQ} as

$$m_a \sim \frac{f_\pi m_\pi}{f_{\mathrm{PQ}}} \tag{2.6}$$

if the axion mass is too large ($m_a > 10^{-2}$ eV), it upsets conventional stellar evolution and supernova theory since they would carry away too much energy. If they are too light

($m_a < 1\,\mu$eV), they would be produced in high abundance in the early universe and overclose the universe (standard cosmology requires a high degree of flatness). To ensure a light axion and weak axion coupling to SM particles, we must have $f_\pi \ll f_{\mathrm{PQ}}$. Bounds from experiments suggest that $f_{\mathrm{PQ}} \gg v$, where the electroweak scale $v \sim 100$ GeV. In this thesis, we assume f_{PQ} large enough that the axion mass is somewhere within this allowed range of $\mu eV - meV$.

2.4 The Einstein-Hilbert Action for Gravity

Besides the axion, the other component of this thesis is the graviton, the quantum of gravity. Since there is no accepted or fully consistent quantum theory of gravity yet, we will simply refer to the wave-like solution of linearized classical gravity as the "graviton." In the context of this thesis, it represents the perturbations of the classical gravitational field in Einstein's theory of general relativity. When two neutron stars or black holes merge, the generated perturbations travel far and can be detected on Earth with gravitational wave detectors. We will describe the linearization procedure of the gravitational field in classical general relativity. The starting point is the Einstein-Hilbert action which yields the standard Einstein equations for gravity from an application of the variational principle.

The action is invariant under diffeomorphisms, that is, smooth transformations from one differentiable manifold to another. This is similar to the invariance of the electromagnetic action under $U(1)$ gauge invariance. For perturbative treatments of gravity, as we require here, it is necessary to fix the gauge so we can extract the physical degrees of freedom. The Einstein-Hilbert action for pure space-time (no mass-energy sources) is:

$$S_{\mathrm{GR}} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}R\,. \tag{2.7}$$

The integral is usually taken over some restricted patch of space-time to ensure convergence. It is convenient (and conventional in this field) to introduce the Planck mass

$$m_{\rm Pl} = \sqrt{\frac{1}{32\pi G}} \approx 1.2 \times 10^{19}\ {\rm Ge}V\,. \tag{2.8}$$

To give an idea, the Planck mass, which can be thought of (in length units) as the Compton wavelength of a graviton, is about 22 μg, about the mass of a flea! However, it is an enormous amount of energy for a quantum system. In terms of $m_{\rm Pl}$ then,

$$S_{\rm GR} = -2m_{\rm Pl}^2 \int d^4x\sqrt{-g}\ R\,. \tag{2.9}$$

As motivated above, we choose the harmonic gauge which naturally yields wave equations for weak perturbations to the gravitational background field. These waves can be interpreted as gravitational waves in the weak-field limit. The choice of harmonic gauge modifies the action to

$$S_{\rm GR} = -2m_{\rm Pl}^2 \int d^4x\sqrt{-g}\left[R - \frac{1}{2}\Gamma^\mu\Gamma^\nu g_{\mu\nu}\right]\,, \tag{2.10}$$

where $\Gamma^\mu = \Gamma^\mu_{\alpha\beta}g^{\alpha\beta}$ and $\Gamma^\mu_{\alpha\beta}$ is the gauge given in Appendix C. We now linearize this action around the Minkowski metric $\eta_{\mu\nu}$.

2.5 Linearized Gravity and the Equations of Motion for Gravitational Waves

The Einstein Field Equations (**E**FE) following from the action $S_{{\rm G}R}$ in Eq.(2.10) are:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}\,. \tag{2.11}$$

These are 10 equations that describe the connection of the curvature tensor of space-time to the stress-energy tensor. The geometry of space-time is governed by fields that induce stresses on its fabric; this means that the gravitational field is actually a characteristic of space-time's geometry. Using the EFE we can relate the distributions of matter and radiation to the geometry of space-time. It should be noted that a vanishing Ricci tensor correlates to space-time free of any matter distribution. However, this does necessarily mean that the Riemann tensor

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma} \,, \tag{2.12}$$

will vanish as well. Since we can always choose a locally flat coordinate system, an observer far away from a matter distribution will have the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,. \tag{2.13}$$

$h_{\mu\nu}$ describes the perturbation of flat space caused by a matter distribution very far away, or equivalently, describes the ripples in space-time known as gravitational waves.

We can calculate the equations obeyed by $h_{\mu\nu}$ by linearizing the EFE. Plugging Eq.(2.13) into the Riemann tensor Eq.(2.12) yields terms that are nonlinear in $h_{\mu\nu}$. For calculating numerical solutions this is fine but impractical if the goal is to calculate exact solutions for weak gravity where perturbations of the metric are small, i.e, $|h_{\mu\nu}| \ll 1$. The nonlinear terms can then be neglected, and we obtain the *linearized* theory of gravity. The linearized Riemann tensor becomes:

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \eta_{\mu\lambda}\partial_\rho \Gamma^{\lambda}_{\nu\sigma} - \eta_{\mu\lambda}\partial_\sigma \Gamma^{\lambda}_{\nu\rho} \\ &= \frac{1}{2}\left(\partial_\nu \partial_\rho h_{\mu\sigma} + \partial_\mu \partial_\sigma h_{\nu\rho} - \partial_\mu \partial_\rho h_{\nu\sigma} - \partial_\nu \partial_\sigma h_{\mu\rho}\right) \,. \end{aligned} \tag{2.14}$$

Contracting this expression over μ and ρ then gives the linearized Ricci tensor (after relabeling $\nu \to \mu$ and $\sigma \to \nu$):

$$R_{\mu\nu} = \frac{1}{2}\left(\partial_\sigma \partial_\nu h^\sigma{}_\mu + \partial_\sigma \partial_\mu h^\sigma_\nu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu}\right) , \tag{2.15}$$

where h=$\eta^{\mu\nu} h_{\mu\nu}$. The Einstein field tensor in linearized theory now becomes:

$$G_{\mu\nu} = \frac{1}{2}[\partial_\sigma \partial_\nu h^\sigma_\mu + \partial_\sigma \partial_\mu h^\sigma_\nu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho \partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu}\Box h] . \tag{2.16}$$

There is in fact a more convenient notation that compacts the linearized equations of motion. By defining the tensor:

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h , \tag{2.17}$$

and noting that $\bar{h} \equiv \eta^{\mu\nu}\bar{h}_{\mu\nu} = -h$, inverting (2.17) for $h_{\mu\nu}$ returns the "trace-reversed" $h_{\mu\nu}$:

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} . \tag{2.18}$$

Substituting (2.18) into (2.15) and then the EFE yields

$$\Box \bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\rho \partial^\sigma \bar{h}_{\rho\sigma} - \partial^\rho \partial_\nu \bar{h}_{\mu\rho} - \partial^\rho \partial_\mu \bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4} T_{\mu\nu} . \tag{2.19}$$

the EFE in the weak field limit given by (2.19) are a system of 10 linear PDE's for 10 independent components of $\bar{h}_{\mu\nu}$ in terms of the sources given by $T_{\mu\nu}$.

There is also further gauge freedom within the $h_{\mu\nu}$ as the decomposition Eq.(2.13) does not determine $h_{\mu\nu}$ uniquely. Because of the presence of slowly-varying diffeomorphisms, the linearized equations still contain residual symmetries. Perturbing the coordinates used to generate the metric by an infinitesimal transformation

$$x^{\mu\prime} = x^{\mu} + \xi^{\mu} \, , \tag{2.20}$$

we see that the space time metric now transforms as

$$\begin{aligned} g_{\mu'\nu'}\left(x'\right) &= \frac{\partial x^{\alpha}}{\partial x^{\mu'}} \frac{\partial x^{\beta}}{\partial x^{\nu'}} g_{\alpha\beta}(x) \\ &= \left(\delta^{\alpha}_{\mu'} - \frac{\partial \xi^{\alpha}}{\partial x^{\mu\prime}}\right) \left(\delta^{\beta}_{\nu'} - \frac{\partial \xi^{\beta}}{\partial x^{\nu\prime}}\right) \left[g_{\alpha\beta}\left(x'\right) - \frac{\partial g_{\alpha\beta}\left(x'\right)}{\partial x^{\sigma}} \xi^{\sigma}\right] \\ &= g_{\mu\nu}(x) - \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} g_{\alpha\nu} - \frac{\partial \xi^{\beta}}{\partial x^{\nu}} g_{\mu\beta} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \xi^{\sigma} \, . \end{aligned} \tag{2.21}$$

Ignoring terms second order in ξ we find that $h_{\mu\nu}$ transforms as :

$$h'_{\mu\nu} \to h_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu} \, . \tag{2.22}$$

It is convenient to impose the condition (Hilbert or DeDonder gauge):

$$\partial^{\nu}\bar{h}_{\mu\nu} = 0 \, . \tag{2.23}$$

This gauge is allowed if $\bar{h}_{\mu\nu}$ transforms under coordinate transformations in the following way:

$$\bar{h}_{\mu\nu} \to \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \left(\partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu} - \eta_{\mu\nu}\partial_{\rho}\xi^{\rho}\right) \, . \tag{2.24}$$

Thus we have:

$$\begin{aligned} &\partial^{\nu}\bar{h}_{\mu\nu} \to \left(\partial^{\nu}\bar{h}_{\mu\nu}\right)' \\ &\left(\partial^{\nu}\bar{h}_{\mu\nu}\right)' = \partial^{\nu}\bar{h}_{\mu\nu} - \Box\xi_{\mu} \, . \end{aligned} \tag{2.25}$$

Thus if we can choose:

$$\Box\xi_{\mu} = \partial^{\nu}\bar{h}_{\mu\nu} \, ,$$

then the DeDonder gauge condition holds even after making infinitesimal coordinate changes. The above equation is invertible and always admits solutions, therefore, the EFE in linearized gravity becomes

$$\Box\bar{h}_{\mu\nu} = 0\,. \tag{2.26}$$

In the presence of matter-energy sources, the corresponding equation is

$$\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}\,. \tag{2.27}$$

Hence the form of $\bar{h}_{\mu\nu}$ that satisfies the EFE's in a vacuum ($T_{\mu\nu}$=0) is a plane wave solution, corresponding to the equations of motion for the gravitational waves. We conclude that the variation of the Einstein-Hilbert action, in conjunction with the perturbation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ yields the so-called "trace-reversed" form of the wave equation away from any matter-energy sources.

$$\begin{gathered}\Box\bar{h}_{\mu\nu} = 0 \\ \bar{h}_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\nu}\frac{1}{2}h\,.\end{gathered} \tag{2.28}$$

Thus far, we have been talking of pure gravity only. We now add the sources of energy-momentum to the action, namely, the two neutron stars of mass M_n (n=1,2) that constitute the binary system. These two neutron stars lose energy and angular momentum due to gravitational wave emission and eventually merge. There is now firm evidence from LIGO that such events happen fairly regularly in our Galaxy, as supported by the recent multi-messenger multi-wavelength observation of two merging neutron stars, the famous event called GW170817 [12].

Binary neutron stars may also exchange and emit axions, and our ultimate goal is to determine the impact of axions on the binary inspiral and the gravitational waveform. In this thesis, we will only take the first few steps towards this goal by understanding the formalism presented in [11] and carrying out a leading order calculation within the Post-Newtonian (PN) approximation (described in Chapter 3). We can safely assume the binary stars to be described by point particles of a given rest mass, as the inner structure of the neutron star only enters at the fifth order in the PN expansion. Adding the action term corresponding to the world-line of the two neutron stars to that of Einstein-Hilbert gravity in the harmonic gauge, we have

$$S_{\mathrm{GR}} = -2m_{\mathrm{Pl}}^2 \int d^4x\sqrt{-g}\left[R - \frac{1}{2}\Gamma^\mu\Gamma^\nu g_{\mu\nu}\right] - \sum_{n=1,2} M_n \int d\tau\,. \tag{2.29}$$

To this, we add the axion part of the Lagrangian:

$$S_\phi = \int d^4x\sqrt{-g}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right]. \tag{2.30}$$

Equation (2.30) contains coupling terms between the axion and graviton fields because of the factor $\sqrt{-g}$. The potential $V(\phi)$ for the axion is assumed to have reflection symmetry, which disallows odd powers of ϕ, and we also limit to terms of no higher order than ϕ^2, as this is sufficient to describe the Physics of our problem in the 1PN approximation. The interaction between neutron stars carrying a net axion charge is also terminated at the 1PN approximation, and is given by

$$S_{\mathrm{pp}} = -\sum_{n=1,2}\int d\tau_n\left(M_n + q_n\frac{\phi}{m_{\mathrm{Pl}}} + p_n\left(\frac{\phi}{m_{\mathrm{Pl}}}\right)^2 + \cdots\right). \tag{2.31}$$

The factors of m_{Pl} are required to provide the right dimensionality, and q_n, p_n are couplings in the effective theory that must be determined by matching to an underlying microscopic theory (not attempted in this thesis). The resulting (full) action comes from combining equations (2.29),(2.30),(2.31).

$$S_{full} = S_{GR} + S_{\phi} + S_{pp} \,. \tag{2.32}$$

With our action for free fields (axions/gravitons) and point particles (neutron stars) defined, we are now ready to explore the axion-graviton coupling in an effective theory and develop the power counting scheme of the effective action as it relates to the post-Newtonian approximation.

CHAPTER 3
AN EFFECTIVE FIELD THEORY FOR AXION-GRAVITON INTERACTIONS

3.1 Introduction

In this chapter, we establish the EFT framework, explain how the power counting allows for the systematic computation of the PN corrections to the binary and obtain the propagators and vertex factors for axions and gravitons. The purpose and advantage of using the EFT approach is that it can be extended order-by-order depending on the phenomena under study. Since some of the computations in this Chapter are tedious yet standard in Quantum Field Theory, we have relegated detailed steps to Appendices B and C, choosing to focus more on the Physics of the effective action and the power counting as it pertains to the interacting binary neutron star system.

3.2 Effective Field Theories

The laws of physics and the theories we use to model natural phenomena make use of different length scales. Often, only a few scales are relevant to the phenomenon we are trying to interpret. It is therefore convenient to remove the scales that are not relevant to the problem and study the physics as an effective theory. For example, if we want to calculate the energy spectrum of the Hydrogen atom to some reasonable but not very high precision, using Quantum Field Theory and the possibility of particle-antiparticle creation is difficult but also unnecessary - energy levels in the Hydrogen atoms are of the $E \sim$ eV scale, while the field theoretic approach would be justified at scales $E \sim 2m_ec^2 \sim$ MeV where pair-production can take place. It is far simpler to study the Hydrogen atom using non-relativistic Quantum mechanics, and add systematic relativistic corrections as needed. Such an approach yields, for example, the Darwin term that explains “Zitterbewegung” (jittery motion of the electron in s-orbitals). Of course, if one wants to obtain energy spectra very precisely with no approximations, for example, the $^2S_{1/2} -^2 P_{1/2}$ splitting (Lamb shift), a perturbative calculation in QED should be employed. Similarly, if we were dealing with gravity, calculating the orbital dynamics of a binary system is better addressed in the framework described by the Post-Newtonian expansion approximation of General Relativity, as gravitational fields in such a system are weak, and the speed of the stars is small compared to c until the final moments when the stars merge.

What an effective theory is at its core is the simplest framework that captures the essential physics (symmetries) and can be corrected systematically by perturbative expansion in a ratio of the low to high-energy scale. Suppose for instance there is some known quantum field theory of light scalar modes ϕ and heavy scalar modes ρ near the ultra-violet (UV) scale Λ described by an action functional $S[\phi, \rho]$. Then, the problem of calculating the effects of the short distance physics at low energy scale $\omega \ll \Lambda$ can be resolved in the EFT description by integrating out the heavy modes from the theory [13]:

$$e^{iS_{eff}[\phi]} = \int D\rho e^{iS[\phi,\rho]} \,. \tag{3.1}$$

The effective functional can then be expressed as by a local functional in ϕ

$$S_{eff}[\phi] = \sum_i c_i \int d^4x \mathcal{O}_i(x) \,. \tag{3.2}$$

where c_i, called Wilson coefficients, are associated with the renormalization group and $\mathcal{O}_i(x)$ are operators of dimension l. If c_i is evaluated at the renormalization point μ of order Λ then:

$$c_i(\mu = \Lambda) = \frac{\alpha_i}{\Lambda^{l_i - 4}} \tag{3.3}$$

An observation that can be made is that the effects of the short distance physics appear in two ways in an effective theory, namely,

- The renormalization of operators with mass dimension $l > 4$.
- By creating an infinite perturbative series of terms of increasing irrelevance with dimensional scaling described by c_i.

Therefore, all dependence of UV divergences Λ is encoded in the coefficients of the effective Lagrangian, UV dependence of observables follows from dimensional power counting, and the Lagrangian must respect the symmetries of both the high and low energy action in order to be physical. In the interaction of neutron stars in binary, there are three relevant scales.

- r_s = the radius of the neutron star
- r = the orbital radius of the neutron star binary
- λ = the wavelength of the radiation emitted

All three parameters are controlled by the expansion parameter $v \ll 1$ which is the orbital velocity on neutron binary. Note that v is in units of the speed of light c which is also the speed at which gravitons travel.

$$\frac{r_s}{r} \sim v^2 \quad ; \quad \frac{r}{\lambda} \sim v \tag{3.4}$$

Below, we apply the EFT formalism to the interaction of neutron stars in a binary, since the speed of the stars v until just before the merger is a small fraction of the speed of light c. The interaction is mediated by "potential" modes [14] that are short-distance and carry momentum $k \sim 1/r \gg \Omega$ where Ω is the binary's orbital frequency. On the other hand, radiation modes carry momentum $k \sim \Omega \sim v/r$, so to study the radiation, we can integrate out the potential modes. But first, we need to understand how to effect this separation of scales in an EFT Lagrangian.

3.3 The Lagrangian

We want to start with the full action for the neutron stars in a binary (2.32) and derive an effective action separating low momentum (long-distance) and high momentum (short-distance) modes that can be organized as an infinite perturbative series in v. To create such an effective action, two things must be done. The first is analytically manipulating the terms in the action into a suitable form, and the second is re-scaling the fields so that each term in the series increases homogeneously with v.Recalling that the full action has three sub-components, we begin the derivation by starting with (2.10):

$$S_{\mathrm{pp}} = -\sum_{n=1,2} \int d\tau_n \left(M_n + q_n \frac{\phi}{m_{\mathrm{Pl}}} + p_n \left(\frac{\phi}{m_{\mathrm{Pl}}} \right)^2 + \cdots \right) . \tag{3.5}$$

Notice that each term of S_{pp} has a coupling to the world lines $d\tau$ of the binary star system. For the matter term, the coupling to gravitons is found through an expansion in $\mathbf{v}$:

$$S_{\mathrm{pp}} \supset M \int d\tau \tag{3.6}$$

.

By using the identities:

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + \frac{h_{\mu\nu}}{m_{\mathrm{Pl}}} , \\ v^\mu &= \frac{dx^\mu}{d\tau} = (\gamma, \gamma \mathbf{v}) , \\ \mathbf{v}^i &= -\mathbf{v}_i , \end{aligned} \tag{3.7}$$

and expressing the proper time $d\tau$ in terms of coordinate time dt, Eq.(3.6) becomes

$$S_{\mathrm{pp}} \supset -M \int \sqrt{1 - (\mathbf{v}^2 - \frac{h_{00}}{m_{\mathrm{P}l}} - \frac{2h_{0i}}{m_{\mathrm{P}l}} \mathbf{v}^i - \frac{h_{ij}}{m_{\mathrm{P}l}} \mathbf{v}^i \mathbf{v}^j)}\; dt \tag{3.8}$$

When $|\mathbf{v}| = v \ll 1$, the result of a binomial expansion on (3.8) yields

$$S_{\rm pp} \supset M \int dt \left(\frac{1}{2}\mathbf{v}^2 - \frac{1}{2}\frac{h_{00}}{m_{\rm Pl}} + \frac{h_{0i}}{m_{\rm Pl}}\mathbf{v}_i - \frac{1}{4}\frac{h_{00}}{m_{\rm Pl}}\mathbf{v}^2 - \frac{1}{2}\frac{h_{ij}}{m_{\rm Pl}}\mathbf{v}_i\mathbf{v}_j + \cdots \right) \tag{3.9}$$

Likewise, this expansion can also be done for the remaining terms in Eq.(3.5)

$$S_{\rm pp} \supset \frac{q}{m_{\rm Pl}} \int dt \left(-1 + \frac{1}{2}\mathbf{v}^2 - \frac{1}{2}\frac{h_{00}}{m_{\rm Pl}} + \frac{h_{0i}}{m_{\rm Pl}}\mathbf{v}_i - \frac{1}{4}\frac{h_{00}}{m_{\rm Pl}}\mathbf{v}^2 - \frac{1}{2}\frac{h_{ij}}{m_{\rm Pl}}\mathbf{v}_i\mathbf{v}_j + \cdots \right) \phi \, . \tag{3.10}$$

$$S_{\rm pp} \supset \frac{p}{m_{Pl}^2} \int dt \left(-1 + \frac{1}{2}\mathbf{v}^2 - \frac{1}{2}\frac{h_{00}}{m_{\rm Pl}} + \frac{h_{0i}}{m_{\rm Pl}}\mathbf{v}_i - \frac{1}{4}\frac{h_{00}}{m_{\rm Pl}}\mathbf{v}^2 - \frac{1}{2}\frac{h_{ij}}{m_{\rm Pl}}\mathbf{v}_i\mathbf{v}_j + \cdots \right) \phi^2 \, . \tag{3.11}$$

Thus, the point particle term of the action describing the neutron stars has been organized in a power series in v. Now we turn to the action S_ϕ which describes the axion field to extract its coupling to gravitons and the field propagators.

$$S_\phi = \int d^4x \sqrt{-g} \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_s\phi^2 \right] \, . \tag{3.12}$$

In order to see the coupling between axions and gravitons the Jacobian of transformation $\sqrt{-g}$ must be expanded.

$$\begin{aligned} -\det(g) = -g &= \exp\{\log\det(I+h)\} \\ &= \exp\{\mathrm{Tr}\log(I+h)\} \\ &= \exp\left\{\mathrm{Tr}\left[h + O\left(h^2\right)\right]\right\} \\ &= 1 + \mathrm{Tr}\, h + O\left(h^2\right) \\ &= 1 + h + O\left(h_{\mu\nu}^2\right) \, , \end{aligned} \tag{3.13}$$

where $h = \dfrac{h^\mu_\mu}{m_{\mathrm{Pl}}}$, and after keeping only linear terms

$$\sqrt{-g} = \sqrt{1+h}\,. \tag{3.14}$$

Using the binomial expansion,

$$\sqrt{-g} = 1 + \frac{h_{00}}{2m_{\mathrm{Pl}}} + \frac{h_{ij}\eta^{ij}}{2m_{\mathrm{Pl}}}\,. \tag{3.15}$$

Next consider the term inside the brackets from (3.12)

$$[\cdots] = \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_s\phi^2\right]\,, \tag{3.16}$$

which has the alternate form:

$$\begin{aligned}
[\cdots] &= \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_s\phi^2\right] \\
[\cdots] &= \left[\frac{1}{2}g_{\mu\nu}\partial^\mu\phi\partial^\nu\phi - \frac{1}{2}m_s\phi^2\right] \\
[\cdots] &= \left[\frac{1}{2}\left(\eta_{\mu\nu} + \frac{h_{\mu\nu}}{m_{\mathrm{Pl}}}\right)\partial^\mu\phi\partial^\nu\phi - \frac{1}{2}m_s\phi^2\right]\,.
\end{aligned} \tag{3.17}$$

From Eq.(3.17), we have

$$\sqrt{-g}\,[\cdots] = \frac{1}{2}\left(1 + \frac{h_{00}}{2m_{\mathrm{Pl}}} + \frac{h_{ij}\eta^{ij}}{2m_{\mathrm{Pl}}}\right)\left[\left(\eta_{\mu\nu} + \frac{h_{\mu\nu}}{m_{\mathrm{Pl}}}\right)\partial^\mu\phi\partial^\nu\phi - m_s\phi^2\right] \tag{3.18}$$

.

Plugging Eq.(3.18) into the action S_ϕ from Eq.(3.12), the latter can be split into two terms - the pure axion action which will be denoted as $S_\phi^{(0)}$ for future reference and the coupling term $S_{h\phi}$. At this stage, it is convenient to expand ϕ in Fourier space

$$\phi(x) = \int_{\mathbf{k}} \tilde{\phi}_{\mathbf{k}}(x^0) e^{-i\mathbf{k}\cdot\mathbf{x}}\,, \tag{3.19}$$

where $\int_k$ is short hand for $\int \frac{d^3k}{(2\pi)^3}$. After suppressing the time dependent terms of ϕ (due to scaling arguments discussed below), and omitting terms $\mathcal{O}(h^2)$ the subset of (3.17) corresponding to $S_{h\phi}$ becomes:

$$\begin{aligned} S_{h\phi} = \frac{1}{4m_{\text{Pl}}} \int_{\mathbf{k},\mathbf{q}} d^4x\; e^{-i(\mathbf{k}+\mathbf{q})\cdot\mathbf{x}}\; &\left[\left(h_{00}\eta_{mn} + h_{ij}\eta^{ij}\eta_{mn} + h_{mn}\right) \partial^m \phi_{\mathbf{k}} \partial^n \phi_{\mathbf{q}} \right. \\ &\left. - \left(h_{00} + h_{ij}\eta^{ij}\right) m_s \phi_{\mathbf{k}} \phi_{\mathbf{q}} \right]. \end{aligned} \tag{3.20}$$

Noting that

$$\begin{aligned} \eta_{mn}\partial^m\phi_{\mathbf{k}}\partial^n\phi_{\mathbf{q}} &\to (\mathbf{k}\cdot\mathbf{q})\,\phi_{\mathbf{k}}\phi_{\mathbf{q}}\,, \\ h_{mn}\partial^m\phi_{\mathbf{k}}\partial^n\phi_{\mathbf{q}} &\to -h_{mn}\mathbf{k}^m\mathbf{q}^n\phi_{\mathbf{k}}\phi_{\mathbf{q}}\,, \end{aligned} \tag{3.21}$$

and after relabeling any contracted indices $m \to i$ or $n \to j$, (3.20) becomes

$$\begin{aligned} S_{h\phi} = \frac{1}{4m_{\text{Pl}}} \int_{\mathbf{k},\mathbf{q}} d^4x\; e^{-i(\mathbf{k}+\mathbf{q})\cdot\mathbf{x}}\; &\left(\left[(\mathbf{k}\cdot\mathbf{q}) - m_s^2 \right] h_{00}\phi_{\mathbf{k}}\phi_{\mathbf{q}} \right. \\ &\left. + \left[\left(\mathbf{k}\cdot\mathbf{q} - m_{\text{s}}^2\right)\eta^{ij} + 2\mathbf{k}^i\mathbf{q}^j \right] h_{ij}\phi_{\mathbf{k}}\phi_{\mathbf{q}} \right) \end{aligned} \tag{3.22}$$

If we amputate the external fields lines by multiplying by $\delta(k)$ and $\delta(q)$ then integrate over k and q we arrive at the form of the interaction action integral for gravitons coupled to axions given below.

$$S_{h\phi} = \int d^4x \frac{1}{4m_{\text{Pl}}} \left(\mathbf{k}\cdot\mathbf{q} - m_{\text{s}}^2\right) h_{00}\phi^2 + \frac{1}{4m_{\text{Pl}}} \left[\left(\mathbf{k}\cdot\mathbf{q} - m_{\text{s}}^2\right)\eta^{ij} + 2\mathbf{k}^i\mathbf{q}^j \right] h_{ij}\phi^2\,. \tag{3.23}$$

3.4 Propagators

Using the actions for the free fields $h_{\mu\nu}$ and ϕ in S_{GR} and $S_\phi^{(0)}$ respectively, we want to determine the graviton and axion propagators. However, there is an issue. Unlike neutron stars which can have a velocity anywhere from 0 to c, the axion and graviton fields are entirely relativistic. Meaning there is no distinction between radiation modes and potential modes which mediate instantaneous exchange between point particles. Since the exchange is instantaneous those degrees of freedom can never be on-shell and therefore should not be included in the non-relativistic limit. If they did, the set of Feynman rules would not scale homogeneously with v, appearing to thwart our intention to organize a perturbative expansion in v. The fix to this is decomposing the original fields into radiation and potential terms, but then adjusting the scaling on the potential fields such that all fields scale homogeneously with v. That is, start with

$$\begin{aligned} h_{\mu\nu}(x) &= \bar{h}_{\mu\nu}(x) + H_{\mu\nu}(x)\,, \\ \phi(x) &= \bar{\phi}(x) + \Phi(x)\,, \end{aligned} \tag{3.24}$$

where $\bar{h}_{\mu\nu}$ and $\bar{\phi}(x)$ are the radiation fields (with on-shell modes) and $H_{\mu\nu}(x)$ and $\Phi(x)$ are the potential fields (with off-shell modes). We notice that the separated fields scale in the following manner:

$$\partial_\alpha \bar{h}_{\mu\nu} \sim \frac{v}{r}\bar{h}_{\mu\nu} \tag{3.25}$$

$$\partial_i H_{\mu\nu} \sim \frac{1}{r}H_{\mu\nu} \quad ; \quad \partial_0 H_{\mu\nu} \sim \frac{v}{r}H_{\mu\nu} \tag{3.26}$$

$$\partial_\alpha \bar{\phi} \sim \frac{v}{r}\bar{\phi} \tag{3.27}$$

$$\partial_i \Phi \sim \frac{1}{r}\Phi \quad ; \quad \partial_0 \Phi \sim \frac{v}{r}\Phi \tag{3.28}$$

It is useful to transform the potential fields using the 3 component Fourier transform:

$$\begin{aligned} H_{\mu\nu} &= \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} H_{\mathbf{k}\mu\nu}(x^0)\,, \\ \Phi &= \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \Phi_{\mathbf{k}}(x^0)\,. \end{aligned} \tag{3.29}$$

This transformation gives the advantage of choosing the transformed fields $H_{\mathbf{k}\mu\nu}(x^0)$ and $\Phi_{\mathbf{k}}(x^0)$ in a manner such that the diagrams containing the force governed by the range of the heavy modes, with momenta $\mathbf{k} \sim 1/r$, can be set to have definitive powers of v. The actual scaling of the fields will now be determined by the propagators. Recall that the action for pure gravity is:

$$S_{\mathrm{GR}} = -2m_{\mathrm{Pl}}^2 \int d^4x\sqrt{-g}\left[R - \frac{1}{2}\Gamma^\mu\Gamma^\nu g_{\mu\nu}\right]\,. \tag{3.30}$$

In order to determine the propagators (see Appendices B and C for mathematical details) only the terms quadratic in $\bar{h}_{\mu\nu}$ and $H_{\mu\nu}$ that are on the order of $\mathcal{O}(v^0)$ are needed. Hence the subset of the action needed to find potential graviton propagators is (after suppressing time dependent operators due to the scaling above):

$$S_{H^2} = -\frac{1}{2}\int dx^0 \int_{\mathbf{k}} \left[\mathbf{k}^2 H_{\mathbf{k}\mu\nu} H^{\mu\nu}_{-\mathbf{k}} - \frac{\mathbf{k}^2}{2} H_{\mathbf{k}} H_{-\mathbf{k}}\right]\,. \tag{3.31}$$

Also, the subset of the action needed to find the radiation graviton is:

$$S_{\bar{h}^2} = \int d^4x \left[\frac{1}{2}\partial_\rho \bar{h}_{\mu\nu}\partial^\rho \bar{h}^{\mu\nu} - \frac{1}{2}\partial_\rho \bar{h}\partial^\rho \bar{h}\right]\,. \tag{3.32}$$

The potential graviton propagator (derived in appendix C) comes out to be

$$\left\langle H_{\mathbf{k}\mu\nu}\left(x^0\right) H_{\mathbf{q}\alpha\beta}(0)\right\rangle = -(2\pi)^3\delta^3(\mathbf{k}+\mathbf{q})\frac{i}{\mathbf{k}^2}\delta\left(x^0\right) P_{\mu\nu;\alpha\beta}\,. \tag{3.33}$$

where

$$P_{\mu\nu;\alpha\beta} = I_{\mu\nu\alpha\beta} - \frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}\,. \tag{3.34}$$

which is represented as a curvy line in the diagrams:

$$\overset{H_{\mu\nu}}{\sim\!\sim\!\sim\!\sim\!\sim\!\sim\!\sim} = \left\langle H_{\mathbf{k}\mu\nu}\left(x^0\right) H_{\mathbf{q}\alpha\beta}(x'^0)\right\rangle\,. \tag{3.35}$$

Given the form of the potential propagator, one can choose the scaling of $H_{\mathbf{k}\mu\nu}$ to ensure the terms in the perturbative series scale homogeneously in v. Recalling that the potential graviton's three-momentum scales like $\mathbf{k} \sim 1/r$, and $\int dx^\mu \sim r/v$

$$S_{H^2} \sim \int dx^0 \int d^3\mathbf{k}\;\mathbf{k}^2 (H_\mathbf{k})^2 \sim \left(\frac{r}{v}\right) \times \left(\frac{1}{r}\right)^3 \times \left(\frac{1}{r}\right)^2 \times (H_\mathbf{k})^2 \tag{3.36}$$

In order for the action to remain unit-less the Fourier transformed must scale like $H_\mathbf{k} \sim r^2 v^{1/2}$. Now, given the free field axion action

$$S_\phi^{(0)} = \int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_s\phi^2\right]\,, \tag{3.37}$$

It is possible to calculate the propagator for both the radiation axion field and potential axion field. Decomposition of the field into its potential and radiation components yields:

$$S_{\Phi^2} = \int dx^0 \int_\mathbf{k} \left[\frac{1}{2}\partial_m\Phi_\mathbf{k}\partial^m\Phi_{-\mathbf{k}}\frac{1}{2}m_s\Phi_\mathbf{k}\Phi_{-\mathbf{k}}\right]\,, \tag{3.38}$$

for the potential axion (after suppressing time dependent operators due to scaling), and for the radiation axion one finds

$$S_{\bar{\phi}^2} = \int d^4x \left[\frac{1}{2}\partial_\mu\bar{\phi}\partial^\mu\bar{\phi} - \frac{1}{2}m_s\bar{\phi}^2\right]\,. \tag{3.39}$$

The potential axion propagator (derived in Appendix C) is

$$\langle \Phi_{\mathbf{k}}(x^0)\Phi_{\mathbf{q}}(0)\rangle = \frac{-(2\pi)^3 i\delta^3(\mathbf{k}+\mathbf{q})\delta\left(x^0\right)}{\mathbf{k^2}+m_s^2}\,. \tag{3.40}$$

which is represented as a dashed line in the diagrams:

$$\overset{\Phi}{\text{- - - - - - - - - - - -}} = \langle \Phi_{\mathbf{k}}(x^0)\Phi_{\mathbf{q}}(x'^0)\rangle \tag{3.41}$$

With it, we can set the scaling of $\Phi_{\mathbf{k}}$ to leave the action unit less.

$$S_{\Phi^2} \sim \int dx^0 \int d^3\mathbf{k}\; \mathbf{k}^2\; \Phi_{\mathbf{k}}^2 \sim \left(\frac{r}{v}\right) \times \left(\frac{1}{r}\right)^3 \times \left(\frac{1}{r}\right)^2 \times \left(\Phi_{\mathbf{k}}\right)^2\,. \tag{3.42}$$

This sets the scaling of $\Phi_{\mathbf{k}} \sim r^2 v^{1/2}$. By noting that $L \sim Mvr$ is the typical orbital angular momentum. The following table summarizes the power counting rules that were derived from the propagators.

TABLE 3.1. Non Relativistic Power Counting Rules

x^0	$\mathbf{k}$	$\bar{\phi}$	$\Phi_{\mathbf{k}}$	$\bar{h}_{\mu\nu}$	$H_{\mathbf{k}\mu\nu}$	m_n/m_{Pl}	q_n/m_{Pl}
r/v	$1/r$	v/r	$r^2v^{1/2}$	v/r	$r^2v^{1/2}$	$\sqrt{Lv}$	$\sqrt{Lv}$

3.5 Power Counting in the Effective Action

Now, we can explain how the interactions encoded in the effective action emerge in a perturbative series that can be related to some power $\mathcal{O}(v^n)$. At leading order, the binary star system exchanges axions Φ and gravitons $H_{\mu\nu}$. The leading order interactions come from treating the couplings in point particle action (2.10) as sources.

Expansion of (2.10) leads to an infinite series of terms of the form:

$$S_{\text{pp}} \supset \sum_{n=1,2} M_n \int dt_n \left(-1 + \frac{1}{2}\mathbf{v}_n^2 - \frac{1}{2}\frac{h_{00}}{m_{\text{Pl}}} + \frac{h_{0i}}{m_{\text{Pl}}}\mathbf{v}_{ni} + \cdots \right) , \tag{3.43}$$

$$S_{\text{pp}} \supset \sum_{n=1,2} \frac{q_n}{m_{Pl}} \int dt_n \left(-1 + \frac{1}{2}\mathbf{v}_n^2 - \frac{1}{2}\frac{h_{00}}{m_{\text{Pl}}} + \frac{h_{0i}}{m_{\text{Pl}}}\mathbf{v}_{ni} + \cdots \right) \phi \, , \tag{3.44}$$

$$S_{\text{pp}} \supset \sum_{n=1,2} \frac{p_n}{m_{Pl}^2} \int dt_n \left(-1 + \frac{1}{2}\mathbf{v}_n^2 - \frac{1}{2}\frac{h_{00}}{m_{\text{Pl}}} + \frac{h_{0i}}{m_{\text{Pl}}}\mathbf{v}_{ni} + \cdots \right) \phi^2 \, , \tag{3.45}$$

The lowest order terms in post Newtonian expansion ($\mathcal{O}(v^0)$) have source couplings of the form

$$\begin{aligned} S_h^{J_1} &= \int d^4x J_1^{\mu\nu}(x) h_{\mu\nu}(x) \, , \\ S_\phi^{J_2} &= \int d^4x J_2(x) \phi(x) \, , \end{aligned} \tag{3.46}$$

By comparing (3.46) to (3.43) and (3.44) we see that $J_1^{\mu\nu}(x)$ and $J_2(x)$ are:

$$\begin{aligned} J_1^{\mu\nu}(x) &= -\sum_{n=1,2} \frac{M_n}{2m_{\text{Pl}}} \int dt_n \delta^4(x - x_n) \eta^{\mu 0} \eta^{\nu 0} \, , \\ J_2(x) &= -\sum_{n=1,2} \frac{q_n}{m_{\text{Pl}}} \int dt_n \delta^4(x - x_n) \, , \end{aligned} \tag{3.47}$$

We now introduce the terms

$$\begin{aligned} S_H &= S_{H^2} + S_H^{J_1} \, , \\ S_\Phi &= S_{\Phi^2} + S_\Phi^{J_2} \, , \end{aligned} \tag{3.48}$$

which comes from disentangling the potential and radiation fields and associating then with zeroth-order Post Newtonian sources. Using steps detailed in Appendix B, we see that these actions can be expressed as

$$\begin{aligned} S_H &= \frac{1}{2}\int d^4x d^4x' J_1^{\mu\nu}(x)\left\langle H_{\mathbf{k}\mu\nu}\left(x^0\right) H_{\mathbf{q}\alpha\beta}(x^{0\prime})\right\rangle J_1^{\alpha\beta}\left(x'\right) , \\ S_\Phi &= \frac{1}{2}\int d^4x d^4x' J_2(x)\langle \Phi_{\mathbf{k}}(x^0)\Phi_{\mathbf{q}}(x^{0\prime})\rangle J_2\left(x'\right) , \end{aligned} \tag{3.49}$$

where the propagators are given by the equations (3.33) and (3.40). Plugging equations (3.33), (3.40), and (3.47) into (3.49) yields the terms:

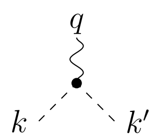
$$\begin{aligned} S_H &= \sum_{n,m} \frac{iM_n M_m}{8m_{\mathrm{Pl}}^2}\int dx^0 \int_{\mathbf{k}} \frac{e^{-ik(x_n - x_m)}}{\mathbf{k}^2} , \\ S_\Phi &= \sum_{n,m} \frac{iq_n q_m}{2m_{\mathrm{Pl}}^2}\int dx^0 \int_{\mathbf{k}} \frac{e^{-ik(x_n - x_m)}}{\mathbf{k}^2 + m_s^2} . \end{aligned} \tag{3.50}$$

When $n \neq m$ we get the leading order diagrams. Furthermore, using the NR power counting rules it can be shown that these diagrams are proportional to Lv^0. Hence, the inclusion of any perturbative corrections from the point particle action will appear as additional powers of v, which yields the result that the potential modes appear in the action as terms proportional to the some power of angular momentum multiplied by some power of the relative binary star system velocity $L^i v^j$. In this way, the interaction terms for the potential modes obtained from the effective action correspond to a series expansion in v.

3.6 Vertex Factors for Axion-Graviton Coupling

Using the perturbative approach that treats interacting terms as functional derivatives (see discussion in Appendix B), the vertex factors for interacting terms in $S_{h\phi}$

can be calculated. The result of this is that $H_{00}\Phi^2$ term shows up in Feynman Diagrams



$$= \frac{1}{4m_{Pl}} \frac{-m_s^2}{\mathbf{q}^2(\mathbf{k}^2 + m_s^2)(\mathbf{k}'^2 + m_s^2)} (2\pi)^3 \delta^3(\mathbf{k} + \mathbf{k}' + \mathbf{q}) \tag{3.51}$$

Now we have the pieces of the effective action, and can compute the Feynman diagrams that contribute at leading order to the axion and gravitational interactions of two neutron stars.

CHAPTER 4

RESULTS

4.1 Leading Order Feynman Diagrams for the Binary Neutron Star System

In Chapter 3, we have obtained the mathematical expressions for the Feynman propagators of the graviton, the axion, and the axion-graviton vertex. We now proceed to write down the expressions for the tree-level Feynman diagrams obtained from the EFT that describes the interaction between the two neutron stars up to 1PN order. This constitutes the main result of this thesis. We will begin with the leading order diagrams. These diagrams simply describe two neutron stars interacting via the exchange of gravitons or axions.

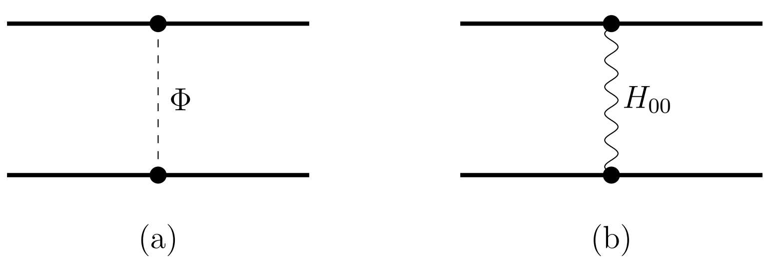


FIGURE 4.1. Leading order diagrams for the interaction of two neutron stars with axion and graviton interactions.

Fig.1a represents the scalar force between two neutron stars mediated by the exchange of the massive axion. The amplitude is (using G=$1/(32\pi m_{\text{Pl}}^2)$):

$$\mathrm{F}ig\,1a \equiv 32\pi G q_1 q_2 \left(\frac{i}{k^2 + m_s^2}\right) \tag{4.1}$$

where q_1, q_2 are the axion charges of the two stars, and k is the exchanged axion's momentum. Fig.1b represents the gravitational force between two neutron stars mediated by the exchange of the potential graviton H_{00}.The amplitude is:

$$Fig\,1b \equiv 4\pi G M_1 M_2 \left(\frac{i}{k^2}\right) \tag{4.2}$$

where M_1, M_2 are the masses of the two stars, and k is the exchanged potential graviton's momentum.

4.2 Order 1PN Diagrams for the Binary Neutron Star System

At the 1PN level, there are 3 sets of diagrams that arise from the power counting. The first set is corrections of $\mathcal{O}(Gv^2)$ purely from the insertion of post-Newtonian effects in the neutron star's propagator on the exchange of either a graviton or an axion. The second is corrections of $\mathcal{O}(G^2)$ arising from the simultaneous exchange of two gravitons or an axion and a graviton. The third set is axion-graviton interactions of the exchanged axion or graviton itself which brings corrections of $\mathcal{O}(Gv^2)$ as well, and which are all 3-vertex diagrams. The first set of diagrams (Fig.4.2), appearing at $\mathcal{O}(Gv^2)$ are:

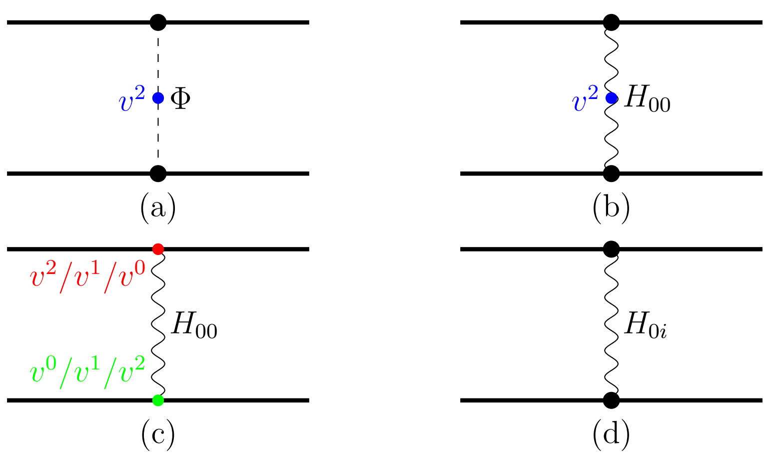


FIGURE 4.2. Corrections of order $\mathcal{O}(Gv^2)$ to the leading order interaction, coming from 1PN insertions in the propagator of the two neutron stars.

$$Fig\,2a \equiv 16\pi G q_1 q_2 \frac{-i}{(k^2 + m_s^2)^2} \tag{4.3}$$

from post-Newtonian effects on the axion exchange, whereas

$$\mathrm{F}ig\,2b \equiv 4\pi G M_1 M_2 \left(\frac{-i}{k^4}\right) \tag{4.4}$$

$$\mathrm{F}ig\,2c \equiv 8\pi G M_1 M_2 \left(\frac{i}{k^2}\right) \left(P_{00,00}\delta_{ij} + P_{00,ij}\right) \tag{4.5}$$

and

$$\mathrm{F}ig\,2d \equiv 32\pi G M_1 M_2 \left(\frac{i}{k^2}\right) P_{0i,0j} \tag{4.6}$$

come from post-Newtonian effects on the graviton exchange. The second set of diagrams (Fig.4.3), appearing at $\mathcal{O}(G^2)$ are:

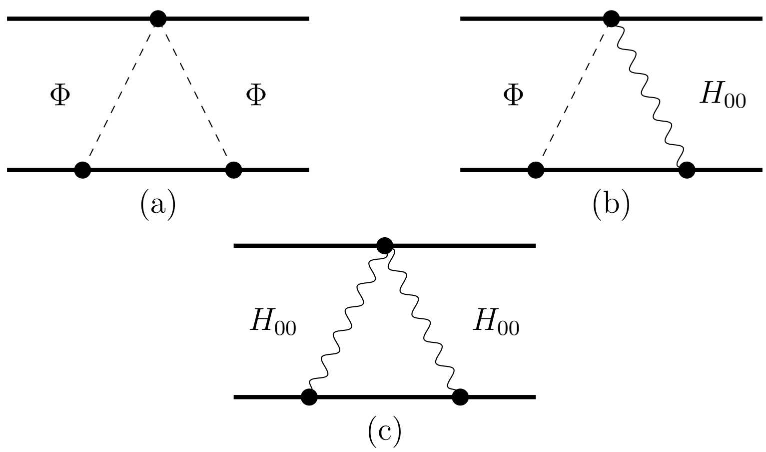


FIGURE 4.3. Corrections of order $\mathcal{O}(G^2)$ to the leading order interaction from "seagull" diagrams.

$$\mathrm{F}ig\,3a \equiv 2(32\pi G)^2 p_1 q_2^2 \frac{-i}{(k^2+m_s^2)} \frac{1}{(l^2+m_s^2)} \tag{4.7}$$

from the simultaneous exchange of two axions, k, l being the axion momenta,

$$\mathrm{F}ig\,3b \equiv \frac{(32\pi G)^2}{4} q_1 M_2 q_2 \frac{-i}{(k^2+m_s^2)} \frac{1}{l^2} P_{00,00} \tag{4.8}$$

from the simultaneous exchange of an axion and a graviton, k, l being the axion and graviton momenta respectively, and

$$\mathrm{F}ig\,3c \equiv 32(\pi G)^2 M_1 M_2^2 \frac{i}{k^2}\frac{1}{l^2} P_{00,00}^2 \tag{4.9}$$

from the simultaneous exchange of two gravitons. We have not shown the 3 additional diagrams in this set corresponding to the interchange of neutron star labels 1 and 2 in Fig.4.3 as their expressions are easily obtained by applying the interchange $(1 \leftrightarrow 2)$ to the three equations above. The third and final set of diagrams (Fig.4.4), appearing at $\mathcal{O}(Gv^2)$ and which are all 3-vertex diagrams are:

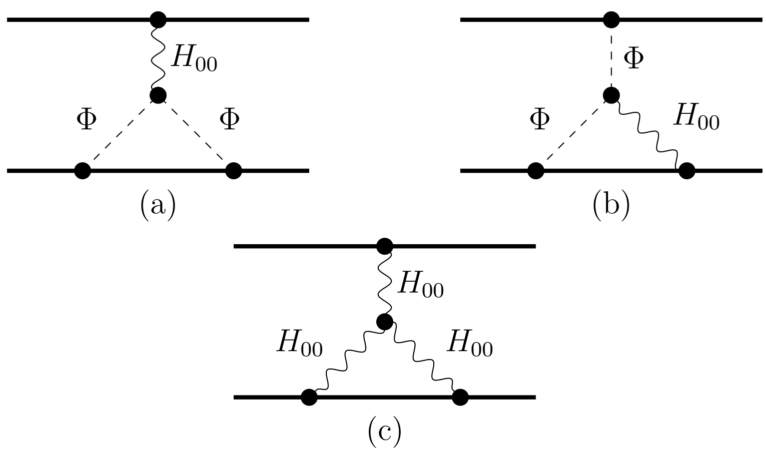


FIGURE 4.4. Corrections at 1PN order to the leading order result, coming solely from 3-vertex diagrams representing axion-graviton and graviton-graviton interaction during the exchange.

$$\mathrm{F}ig\,4a \equiv 8(4\pi G)^2 M_1 q_2^2 \frac{i}{k^2}\frac{-m_s^2}{(l^2+m_s^2)}\frac{1}{(l'^2+m_s^2)}(2\pi)^3\delta(k+l-l') \tag{4.10}$$

with l, l' being the axion momenta and k the graviton momentum,

$$\mathrm{F}ig\,4b \equiv 8(4\pi G)^2 M_2 q_1 q_2 \frac{1}{k^2}\frac{-m_s^2}{(l^2+m_s^2)}\frac{1}{(l'^2+m_s^2)}(2\pi)^3\delta(l+l'-k) \tag{4.11}$$

and

$$Fig\, 4c \equiv (4\pi G)^2 M_1 M_2^2 \frac{1}{k_1^2}\frac{1}{k_2^2}\frac{1}{k_3^2}(k_1^2 + k_2^2 + k_3^2)(2\pi)^3\delta(k_1 - k_2 - k_3) \tag{4.12}$$

We have not shown the 2 additional diagrams in this set corresponding to the interchange of neutron star labels 1 and 2 in the first two diagrams of Fig.4.4.

While all of these diagrams contribute to the interaction between the two neutron stars up to 1PN order, Fig. 1b, 2b, 2c, 2d, 3c, and 4c contribute to the binding energy in the gravitational sector. The corresponding expressions are obtained by evaluating their Fourier transforms and integrating over equal time coordinates of the two neutron stars, a procedure that is not explicitly shown here but is shown in [14]. It leads to the effective 1PN Lagrangian for the gravitational sector

$$L_{gravity} = \sum_{i=1,2} \frac{1}{2} M_i v_i^2 + \frac{G M_1 M_2}{r} + L_{\mathrm{E}IH} \tag{4.13}$$

with

$$\begin{aligned} L_{\mathrm{E}IH} = &\sum_{i=1,2} \frac{1}{8} M_i v_i^4 + \frac{G M_1 M_2}{2r}\left[3(\mathbf{v}_1^2 + \mathbf{v}_2^2) - 7(\mathbf{v}_1.\mathbf{v}_2) - \frac{(\mathbf{v}_1.\mathbf{r})(\mathbf{v}_2.\mathbf{r})}{r^2}\right] \\ &- \frac{G^2 M_1 M_2 (M_1 + M_2)}{2r^2} \end{aligned} \tag{4.14}$$

being the Einstein-Infeld-Hoffman Lagrangian ($L_{\mathrm{E}IH}$).

Similarly, Fig. 1a, 2a, 3a, 3b, 4a and 4b all involve the axion, and give the effective

Lagrangian for the axion sector

$$
\begin{aligned}
L_\phi =& 8Gq_1q_2\frac{e^{-m_s r}}{r}\times \\
&[1-\frac{G\left(M_1+M_2\right)}{r}-\frac{\left(\mathbf{v}_1\cdot\mathbf{r}\right)\left(\mathbf{v}_2\cdot\mathbf{r}\right)}{2r^2}\left(1+m_s r\right) \\
&+\frac{\left(\mathbf{v}_1\cdot\mathbf{v}_2\right)}{2}-16G\left(\frac{q_1^2p_2+q_2^2p_1}{q_1q_2}\right)\frac{e^{-m_s r}}{r}] \\
&-\frac{2G^2\left(M_1q_2^2+M_2q_1^2\right)}{r}m_s\left[e^{-2m_s r}+2m_s r\,\mathrm{Ei}\left(-2m_s r\right)\right] \\
&+\frac{16G^2q_1q_2\left(M_1+M_2\right)}{r}m_s\mathcal{I}\left(m_s r\right)
\end{aligned}
\tag{4.15}
$$

where

$$
\mathrm{Ei}(x)=-\int_{-x}^{\infty}dte^{-t}/t \tag{4.16}
$$

and

$$
\mathcal{I}(x)\equiv\frac{2}{\pi}\int_0^\infty\frac{dk}{k^2+1}\sin(kx)\arctan k \tag{4.17}
$$

From this effective Lagrangian $L_{gravity}+L_\phi$, one can proceed to compute the radiated power in gravitational waves from a binary neutron star system that is also interacting via axion exchange, at least to 1PN order, which is sufficient until the very last moments before the merger. We will briefly outline this calculation in the next chapter but it is not part of the calculations performed for this thesis.

CHAPTER 5

CONCLUSIONS AND FUTURE WORK

5.1 Summary of Work Done

In this thesis, we have worked through the formalism of the effective action describing axion and graviton interactions between two neutron stars in a binary, essentially following the work of Huang et al. [11]. This lays the foundation for the calculation of gravitational radiation power from a binary that is interacting through axion as well as graviton exchange. The formalism is general enough to explore other "fifth" force interactions between neutron stars or other objects bound by gravity.

Since this is an exploratory work without numerical calculations, our main results are simply the successful execution and understanding of the theoretical steps in obtaining the effective Lagrangian and the leading order Feynman diagrams. These are summarized below.

- We analyzed the three main scales used to organize the power counting in the effective Lagrangian for the radiating binary neutron star system, namely $r_s \ll r \ll \lambda_{\mathrm{GW}}$. By treating neutron stars as world lines interacting with gravitons and axions, the power counting in v organizes the effective Lagrangian in correspondence with the post-Newtonian expansion after rescaling the Fourier-transformed fields.

- We derived the axion-graviton coupling, as well as the propagators for the potential axion and graviton fields, using the effective Lagrangian. The potential axion and graviton modes must eventually be integrated out to obtain the radiation power.

- Finally, we used Feynman diagrams to derive the expressions corresponding to the interaction of two neutron stars with gravitons and axions up to 1PN order, including axion-graviton coupling during the exchange. These amplitudes are useful for subsequent calculations of the radiation power.

5.2 Summary of Future Work

The formalism presented in [11] can be used to calculate the radiation power in gravitons from the neutron star binary. In this way, the aim is to estimate the effect of axion interactions on an observable, namely, the gravitational wave power. The diagrams below show the effect of the axion on the radiated power in gravitons at 1PN order.

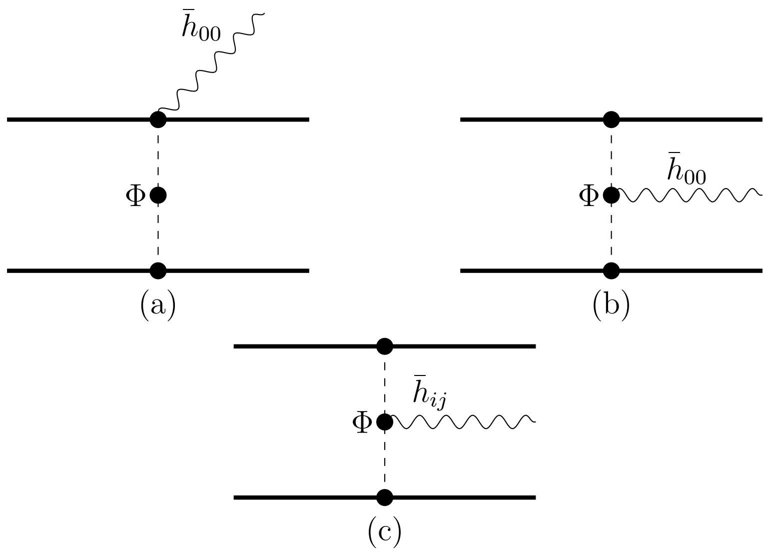


FIGURE 5.1. Corrections to the gravitational wave emission from axion exchange at 1PN order.

When we are interested in calculating the radiated gravitons, the short distance (potential) modes can be integrated out of the theory. The procedure involves integrating over the potential gravitons and axions

$$e^{iS_{\text{eff}}[\bar{h},\bar{\phi},\mathbf{x}]} = \int \mathcal{D}H_{\mu\nu}\mathcal{D}\Phi e^{iS_{\text{full}}[h,\phi,\mathbf{x}]} \tag{5.1}$$

to get an effective source term

$$S_{\text{eff}}^{\text{source}} = -\frac{1}{2m_{\text{Pl}}} \int d^4x T^{\mu\nu} \left(\mathbf{x}, \bar{h}_{\mu\nu}, \bar{\phi}\right) \bar{h}_{\mu\nu} \tag{5.2}$$

where $T^{\mu\nu}$ is the stress-energy tensor for the effective theory. The radiated power can be calculated using the optical theorem, which relates the scattering cross-section to the forward scattering amplitude from the Feynman diagrams. Similar diagrams and procedures apply to the axion radiation from the system, which could be of interest to direct detection of axions in the future.

APPENDICES

APPENDIX A

CONVENTIONS USED

A.1 Introduction

In this appendix, I briefly cover the conventions used for convenience of the reader.

A.2 Conventions

$$c = \hbar = 1\,, \quad G = 1/(32\pi m_{\mathrm{Pl}}^2)$$

$$x^\mu = \left(x^0, x^1, x^2, x^3\right) \quad x_\mu = (x_0, -x_1, -x_2, -x_3) \quad x^\mu = \eta^{\mu\nu} x_\nu \quad x_\mu = \eta_{\mu\nu} x^\nu \tag{A.1}$$

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{A.2}$$

$$\eta_{\mu\rho}\eta^{\rho\nu} = \delta_\mu^\nu \quad \eta_{\mu\nu}\eta^{\mu\nu} = \eta_\mu^\mu = 4 \tag{A.3}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{m_{Pl}} \tag{A.4}$$

$$S = \int d^d x \mathcal{L} \tag{A.5}$$

$$\Phi_{\mathbf{k}}(x^0) = \int d^3x e^{-ikx}\Phi(x) \quad \Phi(x) = \int \frac{d^3k}{(2\pi)^3} e^{ikx}\Phi_{\mathbf{k}}(x^0) \tag{A.6}$$

$$H_{\mathbf{k}\mu\nu}(x^0) = \int d^3x e^{-ikx} H_{\mu\nu}(x) \quad H_{\mu\nu}(x) = \int \frac{d^3k}{(2\pi)^3} e^{ikx} H_{\mathbf{k}\mu\nu}(x^0) \tag{A.7}$$

$$\int_{\mathbf{k}} = \int \frac{d^3k}{(2\pi)^3} \tag{A.8}$$

APPENDIX B

GENERATING FEYNMAN RULES

B.1 Introduction

In the following appendix, I will provide methods for generating the Feynman rules of example field theories. I will begin with the procedure used to findpropagators, then explain how to find vertex factors for an interacting theory, and lastly I will explain how to find the correlation function [15].

B.2 Finding Propagators

Consider the action:

$$S[\varphi] = \int d^4x \left(-\frac{1}{2}\varphi\left(-\partial_\mu\partial^\mu + m^2\right)\varphi + J\varphi\right) .. \tag{B.1}$$

The generating functional (path integral) for this action is:

$$Z_0(J) = \int \mathcal{D}\varphi \exp\left(i\int d^4x \left(-\frac{1}{2}\varphi\left(-\partial_\mu\partial^\mu + m^2\right)\varphi + J\varphi\right)\right) . \tag{B.2}$$

Here the path integral is also normalized such that $Z_0(0) = 1$. Now in order to find the propagator we can write the equation of motion satisfied by some free field as $D_{AB}\varphi_B$ where D_{AB} is some differential operator, then the propagator Δ_{BC} satisfies the equation:

$$D_{AB}\Delta_{BC}\left(x - x'\right) = \delta_{AC}\delta^4\left(x - x'\right) . \tag{B.3}$$

In practice, one is able to determine the form of the differential operator D_{AB} by looking at the Lagrangian corresponding to the free field terms and finding the expression that allows the Lagrangian to match the form:

$$\mathcal{L}_{free} = -\frac{1}{2}\varphi_A D_{AB}\varphi_B . \tag{B.4}$$

Comparison of (B.4) to (B.1) reveals that the the differential operation used to find the propagator would be $D_{AB} \;\rightarrow\; (-\partial_\mu\partial^\mu + m^2)$, therefore the propagator satisfies the equation:

$$\left(-\partial_\mu\partial^\mu + m^2\right)\Delta\left(x - x'\right) = \delta^4\left(x - x'\right)\,. \tag{B.5}$$

It is often most convenient to solve (B.5) by first taking the propagator in momentum space $\Delta\left(x - x'\right) \rightarrow \Delta(k)$, where the form of the propagator for the simplest field theory becomes:

$$\Delta\left(x - x'\right) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-x')}}{k^2 + m^2 - i\varepsilon}\,. \tag{B.6}$$

Equation (B.6) can be integrated in the usual manner by computing the contour integral around the poles.

B.3 Interacting Field Theories

In this section I will use the fact that free fields are Gaussian and henceforth solvable to show the perturbative approach to calculating the interactions of a field theory. First recall that Gaussian integrals have the solution:

$$\int_{-\infty}^{\infty} dx\; e^{-\alpha x^2/+Jx} = \sqrt{\frac{2\pi}{\alpha}}e^{J^2/2\alpha}\,. \tag{B.7}$$

This result can now be used to calculate interactions in a perturbative manner. Consider now an interacting theory where the path integral has an extra term in the exponential $gx^3/3!$ making the integral no longer exactly solvable.

$$Z(J) = \int_{-\infty}^{\infty} dx e^{-\alpha x^2/2 + gx^3/3! + Jx}\,. \tag{B.8}$$

Using a simple trick we can re-express (B.8) as:

$$Z(J) = \exp\left(\frac{g}{3!}\left(\frac{\partial}{\partial J}\right)^3\right)\int_{-\infty}^{\infty} dx e^{-\alpha x^2/2+Jx}\,, \tag{B.9}$$

which when combined with (B.8) becomes:

$$Z(J) = \sqrt{\frac{2\pi}{\alpha}}\exp\left(\frac{g}{3!}\left(\frac{\partial}{\partial J}\right)^3\right)\exp\bigl(J^2/2\alpha\bigr)\,. \tag{B.10}$$

The next step is to expand both exponentials to generate a power series in J and g

$$\left(1+\frac{g}{3!}\left(\frac{\partial}{\partial J}\right)^3+\frac{1}{2}\frac{g^2}{3!^2}\left(\frac{\partial}{\partial J}\right)^6+\dots\right)\left(1+\frac{1}{2\alpha}J^2+\frac{1}{2}\frac{1}{4\alpha^4}J^4+\frac{1}{3!}\frac{1}{8\alpha^3}J^6+\dots\right)\,. \tag{B.11}$$

Here, I have dropped the factor of $\sqrt{\frac{2\pi}{\alpha}}$ as it can be ignored.

$$Z(J) = \left(1+\frac{1}{2}\frac{J^2}{\alpha}+\frac{1}{2^3}\frac{J^4}{\alpha^2}+\dots+\frac{1}{2}g\frac{J}{\alpha^2}+\frac{5}{12}g\frac{J^3}{\alpha^3}+\dots\right)\,. \tag{B.12}$$

Equation (B.12) is an infinite series that can be interpreted as a sum of diagrams, where each diagram is composed of sources (J's), propagators (α's) and vertices joining three lines (g's). We can now generalize this approach to finding the diagrams in any theory, but as an example, let us consider the path integral for the interacting field theory:

$$Z(J) = \int \mathcal{D}\varphi \exp\left(i\int d^4x\left(-\frac{1}{2}\varphi\left(-\partial_\mu\partial^\mu+m^2\right)\varphi+\frac{g}{3!}\varphi^3+J\varphi\right)\right)\,. \tag{B.13}$$

We can apply the same trick that leads to the result (B.8)

$$Z(J) = \exp\left(i\frac{g}{3!}\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3\right) \times \int \mathcal{D}\varphi \exp\left(i\int d^4x \left(-\frac{1}{2}\varphi\left(-\partial_\mu\partial^\mu + m^2\right)\varphi + J\varphi\right)\right). \tag{B.14}$$

The integral on the right is the same as equation (B.2) which has the propagator equation (B.6) and can be expressed in an alternate form

$$Z(J) = \exp\left(i\frac{g}{3!}\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3\right) \exp\left(\frac{i}{2}\int d^4x d^4y J(x)\Delta(x-y)J(y)\right). \tag{B.15}$$

We have already seen from the expansion of equations (B.9) and (B.11) that the generating function now becomes an infinite series of diagrams, composed of sources $i\int d^4xJ(x)$, vertices $ig\int d^4x$, and propagators $i\Delta(x-y)$, multiplied by some numerical multiplicity factor. We now introduce a general diagram "D" which is a product of connected sub-connected diagrams "C_I"

$$D = \prod_I \frac{1}{n_I!}C_I\,. \tag{B.16}$$

The partition function is the sum of all diagrams, and after some manipulation it can be seen that the partition function only depends on the connected diagrams.

$$Z(J) = \sum_I \prod_I \frac{1}{n_I!}C_I = \prod_I \sum_I \frac{1}{n_I!}C_I = \prod_I \exp\left(C_I\right) = \exp\left(\sum_I C_I\right). \tag{B.17}$$

It is now to convenient to define a quantity equal to the sum of all connected diagrams:

$$iW(J) \equiv \sum_I C_I\,, \tag{B.18}$$

because after all, what is actually useful for field theories is the time-ordered correlation functions and not the partition function by itself, which is acquired by performing functional derivatives on the partition function.

$$\langle 0 \left| T\varphi\left(x_1\right) \dots \varphi\left(x_n\right) \right| 0 \rangle = \frac{1}{i} \frac{\delta}{\delta J\left(x_1\right)} \dots \frac{1}{i} \frac{\delta}{\delta J\left(x_n\right)} Z(J) \bigg|_{J=0} , \tag{B.19}$$

which can expressed in terms of the sum of all connected diagrams plus disconnected diagrams.

$$\langle 0 \left| T\varphi\left(x_1\right) \dots \varphi\left(x_n\right) \right| 0 \rangle = \frac{1}{i} \frac{\delta}{\delta J\left(x_1\right)} \dots \frac{1}{i} \frac{\delta}{\delta J\left(x_n\right)} iW(J) \bigg|_{J=0} + \text{disconnected terms} , \tag{B.20}$$

After setting $J(x) = 0$ all the disconnected terms become equal to 0, hence they can be excluded from the time-ordered correlation function.

$$\langle 0 \left| T\varphi\left(x_1\right) \dots \varphi\left(x_n\right) \right| 0 \rangle_C = \frac{1}{i} \frac{\delta}{\delta J\left(x_1\right)} \dots \frac{1}{i} \frac{\delta}{\delta J\left(x_n\right)} iW(J) \bigg|_{J=0} . \tag{B.21}$$

APPENDIX C

DETERMINING THE PROPAGATORS

C.1 Introduction

In this appendix, I provide a detailed derivation of the potential graviton and axion propagators stemming from the the actions S_{GR} and $S_{\phi}^{(0)}$.

C.2 Potential Graviton Propagators

$$S_{\mathrm{GR}} = -2m_{\mathrm{Pl}}^2 \int d^4x\sqrt{-g}\left[R - \frac{1}{2}\Gamma^{\mu}\Gamma^{\nu}g_{\mu\nu}\right] . \tag{C.1}$$

Substituting the expanded metric into the Ricci scalar $R(x)$ leads to an infinite series of terms:

$$-2m_{Pl}^2 \int d^4x\sqrt{-g}R(x) \to \int d^4x \left[(\partial h)^2 + \frac{h(\partial h)^2}{m_{Pl}} + \frac{h^2(\partial h)^2}{m_{Pl}^2} + \cdots\right] . \tag{C.2}$$

For the purposes of this paper the gauges of both $\bar{h}_{\mu\nu}$ and $H_{\mu\nu}$ will be taken from the expansion from the term in [16]:

$$\Gamma^{\mu}\Gamma^{\nu}g_{\mu\nu} = \left(D^{\nu}h_{\mu\nu} - \frac{1}{2}D_{\mu}h^{\lambda}_{\lambda}\right)\left(D_{\sigma}h^{\mu\sigma} - \frac{1}{2}D^{\mu}h^{\sigma}_{\sigma}\right) , \tag{C.3}$$

$$\Gamma^{\mu}\Gamma^{\nu}g_{\mu\nu} = \left(D^{\nu}\left[\bar{h}_{\mu\nu} + H_{\mu\nu}\right] D_{\sigma}\left[\bar{h}^{\mu\sigma} + H^{\mu\sigma}\right] + \cdots\right) . \tag{C.4}$$

The gauge fixing gives terms related to the coupling of $\bar{h}^i H^j$ amongst others. However, in order to determine the propagators (see Appendix B) only the terms quadratic in $H_{\mu\nu}$ that are on the order of $\mathcal{O}(v^0)$ are needed. Hence the lowest order Lagrangian and action needed to determine the potential propagators are:

$$\mathcal{L}_{H^2} = -\frac{1}{2}\int_{\mathbf{k}}\left[\mathbf{k}^2 H_{\mathbf{k}\mu\nu}H^{\mu\nu}_{-\mathbf{k}} - \frac{\mathbf{k}^2}{2}H_{\mathbf{k}}H_{-\mathbf{k}}\right] , \tag{C.5}$$

$$S_{H^2} = -\frac{1}{2}\int dx^0 \int_{\mathbf{k}}\left[\mathbf{k}^2 H_{\mathbf{k}\mu\nu}H^{\mu\nu}_{-\mathbf{k}} - \frac{\mathbf{k}^2}{2}H_{\mathbf{k}}H_{-\mathbf{k}}\right] , \tag{C.6}$$

where time dependent terms have been dropped due to scaling with extra powers of v. Referring to Appendix B, the potential graviton propagator can be found first by transforming (C.6) into the form

$$S_{H^2} = -\frac{1}{2}\int dx^0 \int_{\mathbf{k}} H_{\mathbf{k}\mu\nu} D^{\mu\nu\alpha\beta} H_{-\mathbf{k}\alpha\beta}\,. \tag{C.7}$$

The form of the differential operator can be found rather simply by noticing that :

$$S_{H^2} = -\frac{1}{2}\int dx^0 \int_{\mathbf{k}} \frac{\mathbf{k}^2}{2}\left[H_{\mathbf{k}\mu\nu}\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}\right)H_{-\mathbf{k}\alpha\beta} - H_{\mathbf{k}\mu\nu}\left(\eta^{\mu\nu}\eta^{\alpha\beta}\right)H_{-\mathbf{k}\alpha\beta}\right]\,, \tag{C.8}$$

$$D^{\mu\nu\alpha\beta} = \mathbf{k}^2 T^{\mu\nu;\alpha\beta}\,, \tag{C.9}$$

where

$$\begin{aligned} T^{\mu\nu;\alpha\beta} &= \frac{1}{2}[\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta}]\,, \\ T^{\mu\nu;\alpha\beta} &= [I^{\mu\nu\alpha\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}]\,, \end{aligned} \tag{C.10}$$

In general, writing the equations of motion of some free field $\rho_{\alpha\beta}$ as $D^{\mu\nu\alpha\beta}\rho_{\alpha\beta}$ then the propagator $\langle\rho_{\sigma\gamma}(x)\rho_{\alpha\beta}(x')\rangle$ satisfies the equation

$$D^{\mu\nu\sigma\gamma}\langle\rho_{\sigma\gamma}(x)\rho_{\alpha\beta}(x')\rangle = I^{\mu\nu}_{\alpha\beta}\delta^4\left(x-x'\right)\,. \tag{C.11}$$

Hence the potential graviton propagator in co configuration space satisfies the equation:

$$D^{\mu\nu;\sigma\gamma}\langle H_{\sigma\gamma}(x)H_{\alpha\beta}(x')\rangle = I^{\mu\nu}_{\alpha\beta}\delta^4\left(x-x'\right)\,. \tag{C.12}$$

Thus the momentum space representation of potential graviton propagator in Equation (C.12) satisfies:

$$\mathbf{k^2} T^{\mu\nu;\sigma\gamma} \left\langle H_{\mathbf{k}\sigma\gamma}\left(x^0\right) H_{\mathbf{q}\alpha\beta}(0)\right\rangle = -(2\pi)^3 i\delta^3(\mathbf{k}+\mathbf{q})\delta\left(x_0\right) I^{\mu\nu}_{\alpha\beta}\,, \tag{C.13}$$

where the factor of $-i$ is a convention. This leads to the ansatz:

$$\langle H_{\mathbf{k}\sigma\gamma}(x^0) H_{\mathbf{q}\alpha\beta}(0)\rangle = \frac{-(2\pi)^3 i\delta^3(\mathbf{k}+\mathbf{q})\delta\left(x_0\right)}{\mathbf{k^2}}(aI_{\sigma\gamma\alpha\beta} + b\eta_{\sigma\gamma}\eta_{\alpha\beta})\,. \tag{C.14}$$

Plugging Equation (C.14) into Equation (C.13) results in:

$$\begin{aligned}
I^{\mu\nu}_{\alpha\beta} &= T^{\mu\nu;\sigma\gamma}(aI_{\sigma\gamma\alpha\beta} + b\eta_{\sigma\gamma}\eta_{\alpha\beta}) \\
I^{\mu\nu}_{\alpha\beta} &= (I^{\mu\nu\sigma\gamma} - \frac{1}{2}\eta^{\mu\nu}\eta^{\sigma\gamma})(aI_{\sigma\gamma\alpha\beta} + b\eta_{\sigma\gamma}\eta_{\alpha\beta})\,, \\
I^{\mu\nu}_{\alpha\beta} &= (aI^{\mu\nu\sigma\gamma}I_{\sigma\gamma\alpha\beta} - \frac{a}{2}\eta^{\mu\nu}\eta^{\sigma\gamma}I_{\sigma\gamma\alpha\beta} + I^{\mu\nu\sigma\gamma}\eta_{\sigma\gamma}\eta_{\alpha\beta} - \frac{b}{2}\eta^{\mu\nu}\eta^{\sigma\gamma}\eta_{\sigma\gamma}\eta_{\alpha\beta})\,, \\
I^{\mu\nu}_{\alpha\beta} &= (aI^{\mu\nu}_{\alpha\beta} - \frac{a}{2}\eta^{\mu\nu}\eta_{\alpha\beta} + b\eta^{\mu\nu}\eta_{\alpha\beta} - 2b\eta^{\mu\nu}\eta_{\alpha\beta})\,.
\end{aligned} \tag{C.15}$$

Equation (C.15) is satisfied if $a = 1$ and $b = -\frac{1}{2}$ so now;

$$\left\langle H_{\mathbf{k}\mu\nu}\left(x^0\right) H_{\mathbf{q}\alpha\beta}(0)\right\rangle = -(2\pi)^3\delta^3(\mathbf{k}+\mathbf{q})\frac{i}{\mathbf{k}^2}\delta\left(x_0\right) P_{\mu\nu;\alpha\beta}\,, \tag{C.16}$$

where

$$P_{\mu\nu;\alpha\beta} = I_{\mu\nu\alpha\beta} - \frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}\,. \tag{C.17}$$

C.3 Potential Axion Propagators

Given the free field axion Lagrangian $S_{\phi}^{(0)}$ it is possible to calculate the propagator for both the radiation axion field and potential axion field.

$$S_{\phi}^{(0)} = \int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_s\phi^2\right] . \tag{C.18}$$

Extraction of the free quadratic potential axion fields from (C.18) yields the Lagrangian and action :

$$\mathcal{L}_{\Phi^2} = \left[\frac{1}{2}\partial_m\Phi_{\mathbf{k}}\partial^m\Phi_{-\mathbf{k}} - \frac{1}{2}m_s\Phi_{\mathbf{k}}\Phi_{-\mathbf{k}}\right] , \tag{C.19}$$

$$S_{\Phi^2} = \int dx^0 \int_{\mathbf{k}} \left[\frac{1}{2}\partial_m\Phi_{\mathbf{k}}\partial^m\Phi_{-\mathbf{k}} - \frac{1}{2}m_s\Phi_{\mathbf{k}}\Phi_{-\mathbf{k}}\right] , \tag{C.20}$$

for the potential axion, where time dependent terms have been dropped due to scaling with extra powers of v. Referring to Appendix B, the potential axion propagator can first be found be transforming (C.20) into the form:

$$S_{\Phi^2} = \int dx^0 \int_{\mathbf{k}} -\frac{1}{2}\left[\Phi_{\mathbf{k}}D\Phi_{-\mathbf{k}}\right] . \tag{C.21}$$

The propagator then satisfies the following equation in momentum space:

$$D\langle\Phi_{\mathbf{k}}(x^0)\Phi_{\mathbf{q}}(0)\rangle = -(2\pi)^3 i\delta^3(\mathbf{k}+\mathbf{q})\delta\left(x_0\right) . \tag{C.22}$$

Integration by parts on equation (C.20) yields:

$$S_{\Phi^2} = \int dx^0 \int_{\mathbf{k}} \left[-\frac{1}{2}\Phi_{\mathbf{k}}\left(\partial_m\partial^m + \frac{1}{2}m_s\right)\Phi_{-\mathbf{k}}\right] . \tag{C.23}$$

In momentum space we then determine the form of the differential operator D to be:

$$D = \mathbf{k}^2 + m_s^2 \,. \tag{C.24}$$

Hence the momentum space representation of the potential axion propagator is :

$$\langle \Phi_{\mathbf{k}}(x^0)\Phi_{\mathbf{q}}(0)\rangle = \frac{-(2\pi)^3 i\delta^3(\mathbf{k}+\mathbf{q})\delta\,(x_0)}{\mathbf{k^2} + m_s^2} \,. \tag{C.25}$$

APPENDIX D

GLOSSARY OF SYMBOLS

- x_μ is the four vector associated with space time coordinates
- k_μ is the four vector associated with momentum
- ∂_μ is the partial derivative operator
- D_μ is the covariant derivative operator
- $g_{\mu\nu}$ is the generalized space-time metric
- $\eta_{\mu\nu}$ is the space time metric used in flat space, known as the "Minkowski Metric"
- $\Gamma^\alpha_{\mu\nu}$ is the Christoffel symbol composed of derivatives of the generalized metric
- $R_{\mu\nu\alpha\beta}$ is the Riemannian curvature tensor, composed of Christoffel symbols and derivatives of Christoffel Symbols
- $R_{\mu\nu}$ is the Ricci Tensor, composed by contracting two of Riemannian curvature tensor
- $h_{\mu\nu}$ is the graviton field, composed of two different types of gravitons, radiation and potential gravitons
- $\bar{h}_{\mu\nu}$ is the radiation graviton field
- $H_{\mu\nu}$ is the potential graviton field
- $H_{\mathbf{k}\mu\nu}(x^0)$ is the potential graviton field with space like components Fourier transformed
- $\langle H_{\mathbf{k}\mu\nu}(x^0)H_{\mathbf{q}\alpha\beta}(0)\rangle$ is the potential graviton propagator in momentum Space
- ϕ is the axion field, composed of two different types of axions, radiation and potential axions

- $\bar{\phi}$ is the radiation axion field
- Φ is the potential axion field
- $\Phi_{\mathbf{k}}(x^0)$ is the potential axion field with space like components Fourier transformed
- $\langle \Phi_{\mathbf{k}}(x^0)\Phi_{\mathbf{q}}(0)\rangle$ is the potential axion propagator in momentum space
- m_a is the axion mass term
- m_s is the axion mass term in the effective action
- c is the speed of light
- $\hbar$ is Plank's constant
- G_N is Newton's constant
- M_n is the mass of the n^{th} neutron star in the binary star system
- q_n is the scalar coupling constant for the n^{th} neutron star in the binary star system, has dimension equal to mass
- p_n is the second order scalar coupling constant for the n^{th} neutron star in the binary star system, has dimension equal to mass
- $I^{\mu\nu}_{\alpha\beta}$ is the four index identity tensor
- $P_{\mu\nu;\alpha\beta}$ is the four index tensor associated with the graviton propagator
- $T^{\mu\nu;\alpha\beta}$ is the inverse of $P_{\mu\nu;\alpha\beta}$
- r_s is radius of neutron star in binary star system objects
- r is the orbital radius

- λ is the wavelength of emitted radiation
- $\mathbf{v}$ is the relative three velocity of the binary star system
- v is the magnitude of the relative three velocity
- S is the symbol used for the action, there are several actions reference in this paper
- $Z(J)$ is the partition function
- θ_{QCD} is the QCD vacuum expectation angle
- $\mathcal{L}_{\text{QCD}}$ is Lagrangian density of QCD
- γ^{μ} is the gamma matrix
- $G^{\alpha}_{\mu\nu}$ is the gluonic field strength tensor
- $\tilde{G}^{\alpha}_{\mu\nu}$ is the Hodge Dual of the gluonic field strength tensor
- ψ is the quark field
- $\bar{\psi}$ is the conjugate quark field
- m_{ij} is the quark mass matrix

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