Exercises: Notes on Calculus-Based Probability Theory

1. Suppose X is a random variable with pdf $f(x) = \begin{cases} cx & \text{if } 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$

a. Find c re.

$$\int_0^4 cx = 1$$

$$\left.\frac{cx^2}{2}\right|_0^4 = 1$$

$$8c = 1 \qquad c = \frac{1}{8}$$

$$f(x) = \frac{x}{8}$$

$$c = \frac{1}{8}$$

b. Find $P(-1 \le X \le 1)$

$$\text{pdf} = \frac{x}{8} \qquad \text{cdf} = \frac{x^2}{16}$$

$$P(-1 \le x \le 1) = P(x \le 1) = \frac{1}{16}$$

c. Find $P(X > 2)$

$$\left.\frac{x^2}{16}\right|_2^4 = f(x>2)$$

$$= 1 - \frac{1}{4} = \frac{3}{4}$$

d. Find $P(X < 3 \mid X > 1)$

$$P(x>1) = \left.\frac{x^2}{16}\right|_1^4 = 1 - \frac{1}{16} = \frac{15}{16}$$

$$P(x>3) = \frac{x^2}{16} = \frac{9}{16}$$

$$\frac{9/16}{15/16} = \frac{3}{5}$$

e. Find $E[X]$

$$E[X] = \int_0^4 \frac{x^2}{8}\,dx = \frac{x^3}{24}\Big|_0^4 = \frac{4^3}{24} = \frac{64}{24} = \frac{8}{3}$$

f. Find $Var(X)$ and σ_x

$$Var(X) = E[x^2] - E[X]^2$$

$$E[x^2] = \int_0^4 \frac{x^3}{8}\,dx = \frac{x^4}{32}\Big|_0^4 = \frac{16}{2} = 8$$

$$Var(X) = 8 - \left(\frac{8}{3}\right)^2 = 8 - \frac{64}{9} = \frac{8}{9}$$

g. Find the CDF of X, $F(x)$

$$cdf = \int_0^4 \frac{x}{8} = \frac{x^2}{16}$$

2. Suppose X is a random variable with pdf $f(x) = \begin{cases} \frac{3}{2}x^2 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

a. Find $P(-2 < X < 0)$

$$cdf = \int_{-1}^{1} \frac{3}{2}x^2\,dx = \frac{3x^3}{6}\Big|_{-1}^{1} = \frac{3}{6} + \frac{3}{6} = 1$$

$$cdf = \frac{x^3}{2}$$

$$\frac{x^3}{2}\Big|_{-1}^{0} = 0 - \frac{-1}{2} = \frac{1}{2}$$

b. Find $P(X > -.5)$

$$\frac{x^3}{2}\Big|_{-.5}^{1} = \frac{1}{2} + \frac{1}{16} = \frac{9}{16}$$

c. Find $(P\,X \geq .5 \mid X > -.5)$

$$P(X > -.5) = \frac{9}{16} \qquad P(X \geq .5) = \frac{x^3}{2}\Big|_{.5}^{1} = \frac{1}{2} - \frac{1}{16} = \frac{7}{16} \qquad \frac{7/16}{9/16} = \frac{7}{9}$$

d. Find $E|X|$

$$= \int x\, f(x) = \int_{-1}^{1} \frac{3}{2} x^3\, dx$$

$$\frac{3x^4}{8} \Big|_{-1}^{1} = \frac{3}{8} - \frac{3}{8} = 0$$

e. Find $Var(X)$ and σ_X

$$Var(x) = E(x^2) - E|X|^2$$

$$E(x^2) = \int x^2 f(x)\, dx$$

$$\int_{-1}^{1} \frac{3x^4}{2} = \frac{3x^5}{10} \Big|_{-1}^{1} = \frac{3}{10} + \frac{3}{10} = \frac{6}{10} = \frac{3}{5}$$

$$Var(x) = \frac{3}{5} - 0^2 = \frac{3}{5}$$

$$\sigma = \frac{\sqrt{3}}{\sqrt{5}}$$

f. Find the CDF $F(x)$

$$\int_{-1}^{1} \frac{3}{2} x^2\, dx = \frac{x^3}{2}$$

3. Suppose X is a random variable with pdf $f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 2 - x & \text{if } 1 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$

a. Find $P(1 < X < 3)$

$$cdf = \begin{cases} \frac{x^2}{2} & 0 < x < 1 \\ 2x - \frac{x^2}{2} & 1 \le x < 2 \end{cases}$$

$$2x - \frac{x^2}{2} \Big|_{1}^{2} = (4 - \cancel{2}^{\,2}) - (2 - \tfrac{1}{2})^{\frac{3}{2}} = \frac{1}{2}$$

b. Find $P(x > .5)$

$$\frac{x^2}{2} \Big|_{.5}^{1} = \frac{1}{2} - \frac{1}{8} = \frac{6}{16} = \frac{3}{8}$$

$$2x - \frac{x^2}{2} \Big|_{1}^{2} = (4 - 2)^{2} - (2 - \tfrac{1}{2}) = \frac{1}{2}$$

$$P(X > .5) = \frac{3}{8} + \frac{1}{2} = \frac{7}{8}$$

c. Find $P(x > 1 \mid X > .5)$

$P(x>1) = 2x - \frac{x^2}{2} \Big|_1^2 = \frac{1}{2}$

$P(x > .5) = \frac{7}{8}$

$\frac{1/2}{7/8}$

d. Find $E[X]$

$E[X] = \int_0^1 x^2 + \int_1^2 2x - x^2$

$= \frac{1}{3} + \left(x^2 - \frac{x^3}{3}\right)\Big|_1^2$

$= \frac{1}{3} + \cancel{\left(4 - \frac{8}{3}\right)}^{4/3} - \cancel{\left(1 - \frac{1}{3}\right)}^{2/3} = \frac{3}{3} = 1$

e. Find $Var(X)$ and σ_X

$Var(x) = E[x^2] - E[x]^2$

$E(x^2) = \int_0^1 x^3 + \int_1^2 2x^2 - x^3$

$= \frac{1}{4} + \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)\Big|_1^2$

$= \frac{1}{4} + \left(\frac{16}{3} - \cancel{\frac{32}{4}}\,4\right) - \cancel{\left(\frac{2}{3} - \frac{1}{4}\right)}^{5/12} =$

$= \frac{3}{12} + \left(\frac{16}{12}\right) - \frac{5}{12} = \frac{14}{12}$

$Var(x) = \frac{14}{12} - 1 = \frac{1}{6}$

$\sigma_x = \frac{\sqrt{6}}{6} = \frac{1}{\sqrt{6}}$

f. Find the CDF $F(x)$

$$\begin{cases} \frac{x^2}{2} & 0 < x < 1 \\ 2x - \frac{x^2}{2} & 1 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$$

4. Suppose X is an exponential random variable with mean 4

a. Find the pdf of X

$f(x) = \lambda e^{-\lambda x}$

$E(x) = \int \lambda x e^{-\lambda x}$ $\quad \sigma = 4$

$u = x \quad du = 1 \quad v = -e^{-\lambda x} \quad dv = \lambda e^{-\lambda x} + \frac{e^{-\lambda x}}{-\lambda}$

$4 = -e^{-\lambda x}\left(x - \frac{1}{\lambda}\right)\Big|_0^{\infty}$

$\lim_{x \to \infty} \frac{x + \frac{1}{\lambda}}{e^{\lambda x}} = 0$

$4 = e^{0}\left(0 + \frac{1}{\lambda}\right)$

$\lambda = .25$

$f(x) = .25e^{-.25x}$

b. Find $P(1 < x < 3)$

$cdf = -e^{-.25x}\Big|_1^3 = \frac{-1}{e^{.75}} + \frac{1}{e^{.25}} = \frac{-1}{e^{.5}} = \sqrt{e}^{-1}$

c. Find $P(x > 1 \mid x > .5)$

$-e^{-.25x}\Big|_3^{\infty} = \frac{-1}{e^{.25\infty}} + \frac{1}{e^{.75}} = \frac{1}{e^{.75}}$

d. Find $P(x > 6 \mid x > 3)$

$P(x > 6) = \frac{1}{e^{1.5}}$

$P(x > 3) = \frac{1}{e^{.75}}$

$\frac{1/e^{1.5}}{1/e^{.75}} = \frac{e^{.75}}{e^{1.5}} = \frac{1}{e^{.75}}$

e. Find $Var(x)$ and σ_x

$Var(x) = E(x^2) - E(x)^2 =$

$E(x)^2 = 16$

$E(x^2) = \int \lambda x^2 e^{-\lambda x}$

$Var(x) = \frac{1}{\lambda^2} = \frac{1}{1/16} = 16$

$\sigma = 4$ (given)

f. Find CDF $F(x) = \int_a^b \lambda e^{-\lambda x}$

$= \frac{1}{e^{.25x}}\Big|_a^b \quad x > 0$