Problem 8.16 extended

EM simluation

```
import numpy as np
import matplotlib.pyplot as plt

np.random.seed(10)

# actual values
_p = 0.5
_mu0 = -1
_mu1 = 2
_sigma = 1

# randomly generate data
n = 200
r = np.random.binomial(200,0.5)
X = np.random.binomial(1,_p,size=n)
Y = np.zeros(n)
Y[np.where(X==0)] = np.random.normal(loc=_mu0,scale=_sigma,size=Y[np.where(X==0)].size)
Y[np.where(X==1)] = np.random.normal(loc=_mu1,scale=_sigma,size=Y[np.where(X==1)].size)
```

x_i is randomly generated and each y_i is generaterd according to x_i. Simplily removing the last n-r values of X yields a MCAR parttern.

```
# EM's control parameters
cache_size = 1000 # record the last $cache_size$ iteration's results
delta = np.inf    # difference between (p,mu0,mu1,sigma)[t] and (p,mu0,mu1,sigma)[t+1]
c_limit = 1e-6    # stop iteration if delta < $c_limit$
```

Define the control parameters to be used in EM algorithm.

```
# f(y|x=1;p,mu0,mu1,sigma)
def f(y,loc,scale):
    return np.exp(-(y-loc)**2/(2*scale**2))/(np.sqrt(2*np.pi)*scale)
```

Define a function to calculate the density of Gaussian distribution

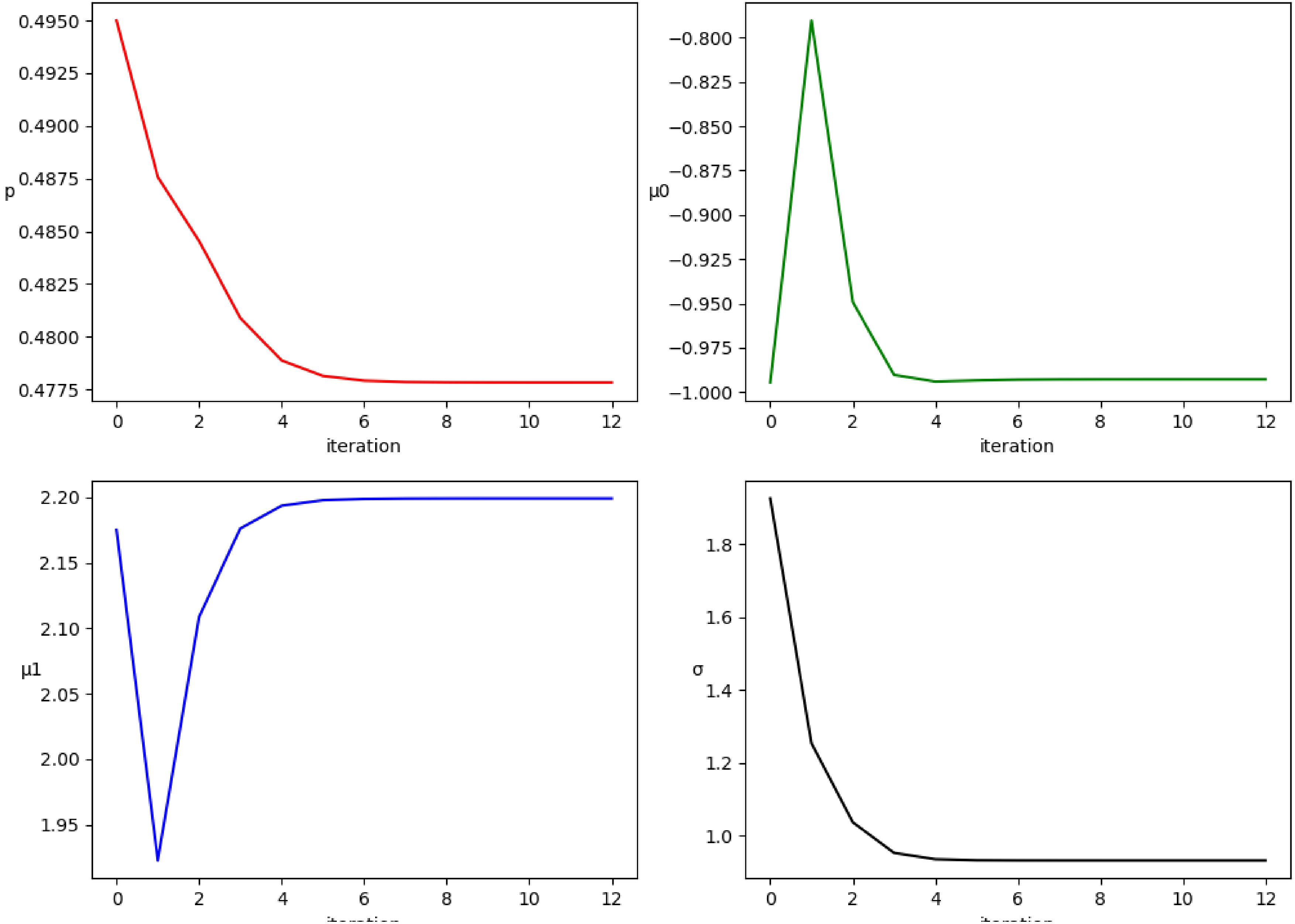
```
w = np.zeros(n)
w[0:r] = X[0:r]
p = np.zeros(cache_size)
mu0 = np.zeros(cache_size)
mu1 = np.zeros(cache_size)
sigma = np.zeros(cache_size)
t = 0
p[0] = np.mean(X)
mu0[0] = np.mean(Y[np.intersect1d(np.where(X==0),range(r))])
mu1[0] = np.mean(Y[np.intersect1d(np.where(X==1),range(r))])
sigma[0] = np.sqrt(np.mean(Y[0:r]*Y[0:r]))
```

Initialize the variables. Use the available cases to get initial value of $(p, \mu_0, \mu_1, \sigma)$

```
while (delta > c_limit and t<cache_size):
    # E-step
    w[r:n] = (f(Y[r:n],mu1[t],sigma[t])*p[t]) / (f(Y[r:n],mu1[t],sigma[t])*p[t] + f(Y[r:n],mu0[t],sigma[t])*(1-p[t]))
    # M-step
    p[t+1] = np.mean(w)
    mu0[t+1] = sum((1-w)*Y) / sum(1-w)
    mu1[t+1] = sum(w*Y) / sum(w)
    sigma[t+1] = np.sqrt((sum((1-w)*(Y-mu0[t+1])**2) + sum(w*(Y-mu1[t+1])**2))/n)
    delta = max([p[t+1]-p[t],mu0[t+1]-mu0[t],mu1[t+1]-mu1[t],sigma[t+1]-sigma[t]])
    t += 1
```

Results

iteration	p	μ_0	μ_1	σ
0	0.49500000	-0.99457691	2.17524187	1.92607354
1	0.48758075	-0.79025558	1.92259111	1.25311736
2	0.48454403	-0.94919517	2.10867222	1.03616975
3	0.48089607	-0.99030215	2.17624148	0.95182042
4	0.47887747	-0.99403613	2.19365271	0.93447625
5	0.47814558	-0.99338580	2.19782225	0.93164587
6	0.47792149	-0.99295901	2.19885236	0.93118387
7	0.47785747	-0.99280857	2.19911555	0.93110293
8	0.47783976	-0.99276315	2.19918427	0.93108730
9	0.47783493	-0.99275025	2.19920243	0.93108396
10	0.47783362	-0.99274669	2.19920727	0.93108318
11	0.47783327	-0.99274572	2.19920856	0.93108299
12	0.47783318	-0.99274546	2.19920890	0.93108294



p
0.4950
0.4925
0.4900
0.4875
0.4850
0.4825
0.4800
0.4775
iteration
μ0
−0.800
−0.825
−0.850
−0.875
−0.900
−0.925
−0.950
−0.975
−1.000
iteration
μ1
2.20
2.15
2.10
2.05
2.00
1.95
σ
1.8
1.6
1.4
1.2
1.0
0
2
4
6
8
10
12

iteration iteration

Previously

X is Bernoulli with $Pr(X=1)=1-Pr(X=0)=p$. Y given $X=j$ is normal with mean μ_j and variance σ^2.

Consider now the monotone missing-data pattern with Y completely observed but n-r values of X missing and an ignorable mechanism.

Describe the E and M steps of EM algorithm for this problem.

preparation

Let $\theta=(p,\mu_0,\mu_1,\sigma)$, we have

$$f(y_i|x_i=j;\theta)=\frac{1}{\sqrt{2\pi}\sigma}exp\left[-\frac{(y_i-\mu_j)^2}{2\sigma_2}\right]$$

$$f(x_i=1|\theta)=1-f(x_i=0|\theta)=p$$

E-step

$$w_i=E[X_i|x,y,\theta^{(t)}]=\begin{cases} x_i & ,i\leq r \\ \frac{f(y_i|x_i=1;\theta^{(t)})f(x_i=1;\theta^{(t)})}{f(y_i|x_i=0;\theta^{(t)})f(x_i=0;\theta^{(t)})+f(y_i|x_i=1;\theta^{(t)})f(x_i=1;\theta^{(t)})} & ,i>r \end{cases}$$

M-step

$$
\begin{aligned}
p^{(t+1)} &= \sum_{i=1}^{n} w_i/n \\
\mu_0^{(t+1)} &= \sum_{i=1}^{n}(1-w_i)y_i/\sum_{i=1}^{n}(1-w_i) \\
\mu_1^{(t+1)} &= \sum_{i=1}^{n} w_i y_i/\sum_{i=1}^{n} w_i \\
\sigma^{(t+1)} &= \sqrt{\sum_{i=0}^{n}[(1-w_i)(y_i-\mu_0^{(t+1)})^2 + w_i(y_i-\mu_1^{(t+1)})^2]/n}
\end{aligned}
$$