Reproduce 11.2

Preparation

$(Y_1, ..., Y_K)$ have a K-variate normal distribution with mean $\mu = (\mu_1, ..., \mu_K)$ and covariance matrix $\Sigma = (\sigma_{jk})$.
$Y = (Y_{obs}, Y_{mis})$ where Y represents a random sample of size n on $(Y_1, ..., Y_K)$, Y_{obs} is the observed values, and Y_{mis} the missing data.

Let $Y_{obs} = (y_{obs,1}, ..., y_{obs,n})$, where y_{obs,i} represents the set of variables observed for case *i, i=1,...,n*.

E-step

$$y_{ij}^{(t)} = \begin{cases} y_{ij} & , if\ y_{ij}\ is\ observed \\ E[y_{ij}, y_{obs,i}, \theta^{(t)}] = \mu_{mis} + \Sigma_{mis-obs,i}\Sigma_{obs-obs,i}^{-1}(y_{obs,i} - \mu_{obs}) & , if\ y_{ij}\ is\ missing \end{cases}$$

$$c_{jki}^{(t)} = \begin{cases} 0 & , if\ y_{ij}\ is\ observed \\ Cov(E_{ij}, y_{obs,i}, \theta^{(t)}) = \Sigma_{mis-mis,i} - \Sigma_{mis-obs,i}\Sigma_{obs-obs,i}^{-1}\Sigma_{obs-mis,i} & , if\ y_{ij}\ is\ missing \end{cases}$$

where
$\Sigma_{mis-mis,i} = (\sigma_{jk})\ ,\ j, k \in \{h|y_{ih}\ is\ missing\}$
$\Sigma_{mis-obs,i} = (\sigma_{jk})\ ,\ j \in \{h|y_{ih}\ is\ missing\}\ ,\ k \in \{h|y_{ih}\ is\ observed\}$
$\Sigma_{obs-obs,i} = (\sigma_{jk})\ ,\ j, k \in \{h|y_{ih}\ is\ observed\}$

M-step

$$\mu_j^{(t+1)} = n^{-1}\sum_{i=1}^{n} y_{ij}^{(t)} \quad , \quad j = 1, ...K$$

$$\sigma_{jk}^{(t+1)} = n^{-1}\sum_{i=1}^{n}(y_{ij}^{(t)} - \mu_j^{(t+1)})(y_{ik}^{(t)} - \mu_k^{(t+1)}) + c_{jki}^{(t)} \quad , \quad j, k = 1, ..., K$$

Implement

Since there is no certain missing pattern, it would be easier to deal with the dataset case by case. In the former sessions, I usually rearrange the dataset to get separate covariance matrices, but it seems too inconvenient in this implement. So I decided to directly retrieve the covariance matrices from the whole covariance matrix Σ.

Here's my implement of EM algorithm:

```python
# y : the input dataset
# mu & sigma : the parameters
def EM(y,mu,sigma):
    r,c = y.shape
    if (mu.shape != (c,) or sigma.shape!=(c,c)):
        raise( NameError('the data and parameter not fit'))
    x = y.copy()
    Covmis = np.zeros([r,c,c])
    # E-step
    for i in range(r):
        # if ther is no missing in case i, then turn to case i+1
        if (np.all(np.isnan(x[i])==False)):continue;

        # store the index of missing values and observed values
        mis = np.where(np.isnan(x[i]))[0]
        obs = np.where(np.isnan(x[i])==False)[0]

        # retrive the covariance matrices (obs-obs , mis-mis , mis-obs)
        Smis = np.matrix(np.zeros([len(mis),len(mis)]))
        for j in range(len(mis)):
            for k in range(len(mis)):
                Smis[j,k] = sigma[mis[j],mis[k]]
        Sobs = np.matrix(np.zeros([len(obs),len(obs)]))
        for j in range(len(obs)):
            for k in range(len(obs)):
                Sobs[j,k] = sigma[obs[j],obs[k]]
        Smvo = np.matrix(np.zeros([len(mis),len(obs)]))
        for j in range(len(mis)):
            for k in range(len(obs)):
                Smvo[j,k] = sigma[mis[j],obs[k]]

        # calculate the Expectations
        x[i][mis] = mu[mis] +
                    np.array(Smvo*np.linalg.inv(Sobs)*
                             np.matrix(x[i][obs]-mu[obs]).transpose()
                             ).reshape(1,len(mis))
        tmp = Smis - Smvo*np.linalg.inv(Sobs)* Smvo.transpose()
        for j in range(len(mis)):
            for k in range(len(mis)):
                Covmis[i,mis[j],mis[k]] = tmp[j,k]

    # M-step
    re_mu = np.mean(x,axis=0)
    re_sigma = np.zeros([c,c])
    for j in range(c):
        for k in range(c):
            tmp = 0
            for i in range(r):
                tmp += (x[i,j]-re_mu[j])*(x[i,k]-re_mu[k]) + Covmis[i][j][k]
            re_sigma[j][k] = tmp/r
    return re_mu,re_sigma
```

Simulation

I randomly generated 3 *C*-variate normal distribution parameters (μ, Σ) and 3 random samples of size *R* that has the distrubution $N(\mu, \Sigma)$, the generated dataset stores in *ORIGIN_Y*

```
np.random.seed(0)   # set it to 0 , 1 , 2 to get 3 different MU and SIGMA
R = 200 # sample size
C = 4   # number of random variables

# generate random mean MU and covariance matrix SIGMA
MU = np.random.randint(-4,4,size=C)

# generate a positive semidefinite matrix as covariance matrix
A = np.matrix(np.random.normal(0,1,size=[C,C]))
SIGMA = A*A.transpose()

# generate multivariate normal sample
ORIGIN_Y = np.random.multivariate_normal(MU,SIGMA,size=R)
```

Copy *ORIGIN_Y* into *Y* and randomly remove some values in Y at a certain missing rate.

```
# randomly remove some obsevations
missing_rate = 0.4
MISS = np.random.binomial(1,missing_rate,size=[R,C])
# minor fix for those entirely deleted obsevations so that evary case has observations
for i in range(100):
    if (np.all(MISS[i] == 1)):
        MISS[i,np.random.randint(0,C)] = 0
Y = ORIGIN_Y.copy()
Y[np.where(MISS==1)] = np.nan
```

Then, use complete cases to acquire starting point and run EM algorithm till convergence.

Results

Random seed	0	1	2
Real μ	(0.00,3.00,1.00,-4.00)	(1.00,-1.00,0.00,-4.00)	(-4.00,3.00,1.00,-4.00)
MLE μ (full Y)	(0.21,3.06,1.23,-3.97)	(1.32,-0.95,-0.01,-3.92)	(-3.98,3.20,0.79,-4.03)
EM MLE μ (Y)	(-0.03,3.07,1.20,-3.92)	(1.20,-1.07,-0.01,-3.85)	(-3.80,3.19,0.68,-4.03)
CC MLE μ (Y)	(0.33,3.24,1.19,-3.81)	(2.48,-1.24,-0.32,-3.89)	(-2.87,2.98,0.60,-4.24)

MLEs of μ and Σ that acquired by EM algorithm is much better than Complete-Case analysis, moreover, it is close to the MLEs calculated using the full generaterd data. The EM algorithm perform well in this circumstance.

$$
\begin{array}{cccc}
 & seed = 0 & seed = 1 & seed = 2 \\
Cov_{real} & \begin{bmatrix} 10.42 & -0.00 & 4.70 & 4.17 \\ -0.00 & 1.10 & -0.11 & 0.13 \\ 4.70 & -0.11 & 2.73 & 1.66 \\ 4.17 & 0.13 & 1.66 & 2.58 \end{bmatrix} & \begin{bmatrix} 7.48 & 0.75 & 2.04 & 2.45 \\ 0.75 & 3.79 & 4.11 & 2.98 \\ 2.04 & 4.11 & 6.63 & 4.32 \\ 2.45 & 2.98 & 4.32 & 3.30 \end{bmatrix} & \begin{bmatrix} 11.18 & -0.45 & 3.45 & -3.08 \\ -0.45 & 3.75 & -1.60 & -0.03 \\ 3.45 & -1.60 & 6.81 & -2.38 \\ -3.08 & -0.03 & -2.38 & 2.03 \end{bmatrix} \\
Cov_{full} & \begin{bmatrix} 9.55 & -0.12 & 4.22 & 3.74 \\ -0.12 & 0.92 & -0.18 & 0.13 \\ 4.22 & -0.18 & 2.53 & 1.33 \\ 3.74 & 0.13 & 1.33 & 2.53 \end{bmatrix} & \begin{bmatrix} 6.57 & 0.49 & 1.67 & 2.06 \\ 0.49 & 3.53 & 4.04 & 2.79 \\ 1.67 & 4.04 & 6.97 & 4.47 \\ 2.06 & 2.79 & 4.47 & 3.26 \end{bmatrix} & \begin{bmatrix} 9.95 & -0.78 & 3.14 & -2.63 \\ -0.78 & 4.42 & -1.82 & 0.08 \\ 3.14 & -1.82 & 6.97 & -2.48 \\ -2.63 & 0.08 & -2.48 & 1.97 \end{bmatrix} \\
Cov_{EM} & \begin{bmatrix} 9.90 & -0.31 & 4.05 & 3.50 \\ -0.31 & 0.94 & -0.35 & 0.22 \\ 4.05 & -0.35 & 2.46 & 1.13 \\ 3.50 & 0.22 & 1.13 & 2.43 \end{bmatrix} & \begin{bmatrix} 6.47 & 0.39 & 1.69 & 2.13 \\ 0.39 & 3.14 & 3.53 & 2.57 \\ 1.69 & 3.53 & 6.80 & 4.61 \\ 2.13 & 2.57 & 4.61 & 3.53 \end{bmatrix} & \begin{bmatrix} 10.34 & -0.77 & 2.72 & -2.64 \\ -0.77 & 4.18 & -1.86 & -0.15 \\ 2.72 & -1.86 & 7.35 & -2.64 \\ -2.64 & -0.15 & -2.64 & 2.10 \end{bmatrix}
\end{array}
$$

$$Cov_{CC} \begin{bmatrix} 8.28 & -0.42 & 3.69 & 3.10 \\ -0.42 & 1.00 & -0.43 & 0.13 \\ 3.69 & -0.43 & 2.50 & 1.05 \\ 3.10 & 0.13 & 1.05 & 2.31 \end{bmatrix} \begin{bmatrix} 4.39 & -0.83 & 0.14 & 1.03 \\ -0.83 & 4.15 & 4.71 & 2.97 \\ 0.14 & 4.71 & 7.07 & 4.49 \\ 1.03 & 2.97 & 4.49 & 3.16 \end{bmatrix} \begin{bmatrix} 11.36 & -0.93 & 2.61 & -2.61 \\ -0.93 & 4.07 & -3.15 & 0.54 \\ 2.61 & -3.15 & 10.30 & -3.74 \\ -2.61 & 0.54 & -3.74 & 2.46 \end{bmatrix}$$