BECA / Huson / IB Math
22 November 2017
Pre-Exam: Vector algebra and differential calculus

Name:

1a. In the following diagram, $\overrightarrow{OP} = \boldsymbol{p}$, $\overrightarrow{OQ} = \boldsymbol{q}$ and $\overrightarrow{PT} = \frac{1}{2}\overrightarrow{PQ}$.

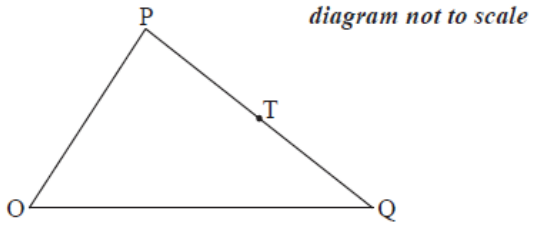


diagram not to scale

Express each of the following vectors in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$,

$\overrightarrow{QP}$; *[2 marks]*

1b. $\overrightarrow{OT}$. *[3 marks]*

2a. Consider the points A$(5, 2, 1)$, B$(6, 5, 3)$, and C$(7, 6, a+1)$, $a \in \mathbb{R}$. Find

(i) $\overrightarrow{AB}$; *[3 marks]*

(ii) $\overrightarrow{AC}$.

2b. Let θ be the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Find the value of a for which $\theta = \frac{\pi}{2}$. *[4 marks]*

2c. Show that $\cos q = \frac{2a+14}{\sqrt{14a^2+280}}$. *[8 marks]*

2d. Hence, find the value of a for which $\theta = 1.2$. *[4 marks]*

3a. The following diagram shows quadrilateral ABCD, with $\overrightarrow{AD} = \overrightarrow{BC}$, $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, and $\overrightarrow{AC} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$.

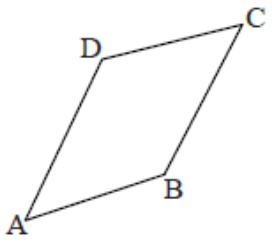


diagram not to scale

Find $\overrightarrow{BC}$. *[2 marks]*

3b. Show that $\overrightarrow{BD} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$. *[2 marks]*

3c. Show that vectors $\overrightarrow{BD}$ and $\overrightarrow{AC}$ are perpendicular. *[3 marks]*

4a. Let $f(x) = ax^3 + bx^2 + c$, where *a*, *b* and *c* are real numbers. The graph of *f* passes through the point (2, 9).

Show that $8a + 4b + c = 9$. *[2 marks]*

4b. The graph of *f* has a local minimum at $(1, 4)$.

Find two other equations in *a*, *b* and *c*, giving your answers in a similar form to part (a). *[7 marks]*

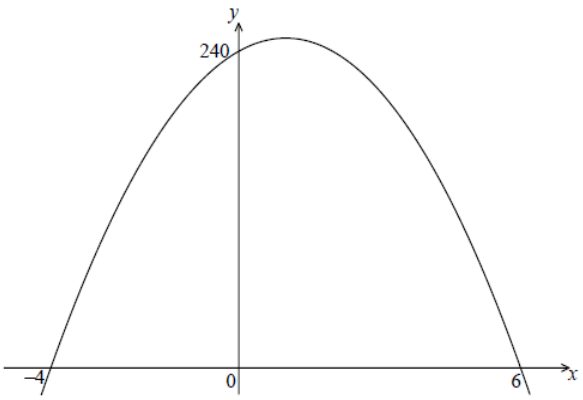
4c. Find the value of *a*, of *b* and of *c*. *[4 marks]*

5a. Let $g(x) = \frac{\ln x}{x^2}$, for $x > 0$.

Use the quotient rule to show that $g'(x) = \frac{1 - 2\ln x}{x^3}$. *[4 marks]*

5b. The graph of *g* has a maximum point at A. Find the *x*-coordinate of A. *[3 marks]*

6a. The following diagram shows part of the graph of a quadratic function f.



The x-intercepts are at $(-4, 0)$ and $(6, 0)$, and the y-intercept is at $(0, 240)$.

Write down $f(x)$ in the form $f(x) = -10(x - p)(x - q)$. *[2 marks]*

6b. Find another expression for $f(x)$ in the form $f(x) = -10(x - h)^2 + k$. *[4 marks]*

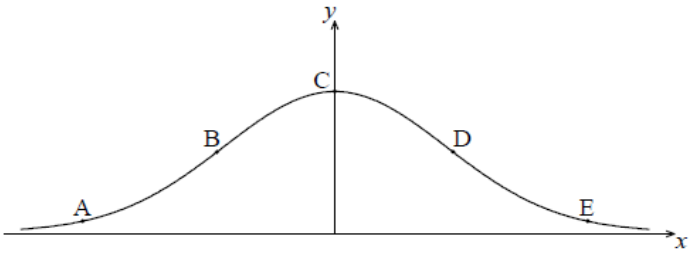
6c. Show that $f(x)$ can also be written in the form $f(x) = 240 + 20x - 10x^2$. *[2 marks]*

6d. A particle moves along a straight line so that its velocity, v ms^{-1}, at time t seconds is given by $v = 240 + 20t - 10t^2$, for $0 \leq t \leq 6$. *[7 marks]*

(i) Find the value of t when the speed of the particle is greatest.

(ii) Find the acceleration of the particle when its speed is zero.

7a. The following diagram shows the graph of $f(x) = \mathrm{e}^{-x^2}$.



The points A, B, C, D and E lie on the graph of f. Two of these are points of inflexion.

Identify the **two** points of inflexion. *[2 marks]*

7b. (i) Find $f'(x)$. *[5 marks]*

(ii) Show that $f''(x) = (4x^2 - 2)\mathrm{e}^{-x^2}$.

7c. Find the *x*-coordinate of each point of inflexion. *[4 marks]*

7d. Use the second derivative to show that one of these points is a point of inflexion. *[4 marks]*

8a. Let $g(x) = 2x \sin x$.

Find $g'(x)$. *[4 marks]*

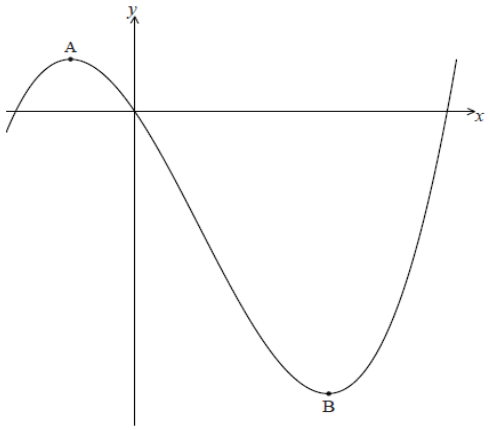
8b. Find the gradient of the graph of *g* at $x = \pi$. *[3 marks]*

9a. Let $f'(x) = -24x^3 + 9x^2 + 3x + 1$. *[3 marks]*

There are two points of inflexion on the graph of *f*. Write down the *x*-coordinates of these points.

9b. Let $g(x) = f''(x)$. Explain why the graph of *g* has no points of inflexion. *[2 marks]*

10a. Let $f(x) = \frac{1}{2}x^3 - x^2 - 3x$. Part of the graph of *f* is shown below.



There is a maximum point at A and a minimum point at B(3, – 9).

Find the coordinates of A. *[8 marks]*

10b. Write down the coordinates of *[6 marks]*

(i) the image of B after reflection in the *y*-axis;

(ii) the image of B after translation by the vector $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$;

(iii) the image of B after reflection in the *x*-axis followed by a horizontal stretch with scale factor $\frac{1}{2}$.

11a. Let $f(x) = \frac{\cos x}{\sin x}$, for $\sin x \neq 0$.

Use the quotient rule to show that $f'(x) = \frac{-1}{\sin^2 x}$. *[5 marks]*

11b. Find $f''(x)$. *[3 marks]*

11c. In the following table, $f'\left(\frac{\pi}{2}\right) = p$ and $f''\left(\frac{\pi}{2}\right) = q$. The table also gives approximate values of $f'(x)$ and $f''(x)$ near $x = \frac{\pi}{2}$.

x	$\frac{\pi}{2} - 0.1$	$\frac{\pi}{2}$	$\frac{\pi}{2} + 0.1$
$f'(x)$	-1.01	p	-1.01
$f''(x)$	0.203	q	-0.203

Find the value of *p* and of *q*. *[3 marks]*

11d. Use information from the table to explain why there is a point of inflexion on the graph of *f* where $x = \frac{\pi}{2}$. *[2 marks]*

12. Let $f(x) = kx^4$. The point $\mathrm{P}(1, k)$ lies on the curve of *f*. At P, the normal to the curve is parallel to $y = -\frac{1}{8}x$. Find the value of *k*. *[6 marks]*