Vector problems (simple) *[61 marks]*

Let $u = 6i + 3j + 6k$ and $v = 2i + 2j + k$.

1a. Find *[5 marks]*

(i)
$u \bullet v$;

(ii) $|u|$;

(iii) $|v|$.

Markscheme

(i) correct substitution ***(A1)***

eg $6 \times 2 + 3 \times 2 + 6 \times 1$

$u \bullet v = 24$ ***A1 N2***

(ii) correct substitution into magnitude formula for u or v ***(A1)***

eg $\sqrt{6^2 + 3^2 + 6^2}$, $\sqrt{2^2 + 2^2 + 1^2}$, correct value for $|v|$

$|u| = 9$ ***A1 N2***

(iii) $|v| = 3$ ***A1 N1***

[5 marks]

1b. Find the angle between u and v. *[2 marks]*

Markscheme

correct substitution into angle formula ***(A1)***

eg $\frac{24}{9 \times 3}$, $0.\bar{8}$

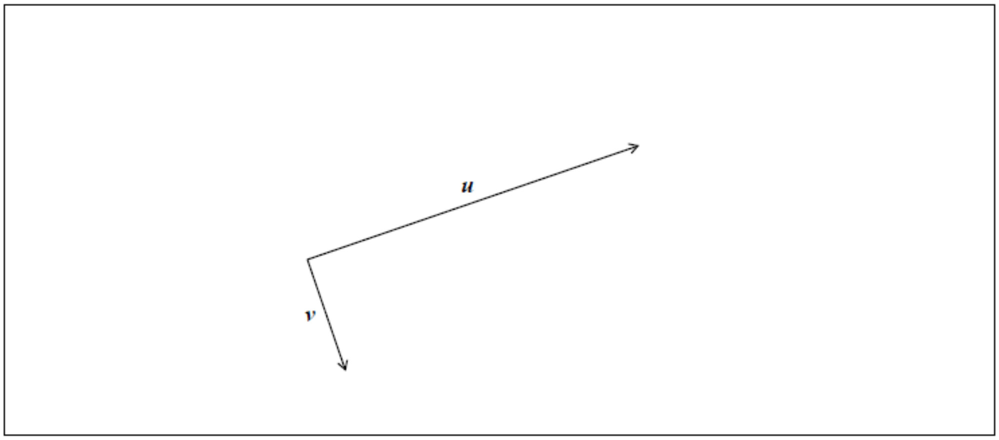
$0.475882,\ 27.26604^\circ$ ***A1 N2***

$0.476,\ 27.3^\circ$

[2 marks]

Total [7 marks]

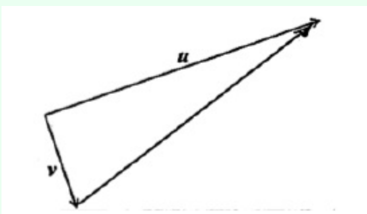
The following diagram shows two perpendicular vectors ***u*** and ***v***.



2a. Let $w = u - v$. Represent w on the diagram above. *[2 marks]*

Markscheme

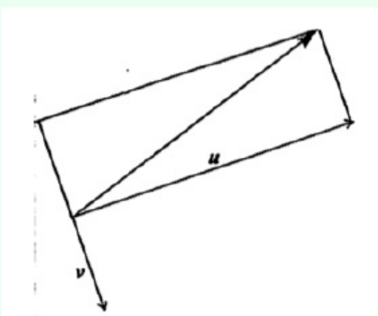
METHOD 1



A1A1 ***N2***

Note: Award ***A1*** for segment connecting endpoints and ***A1*** for direction (must see arrow).

METHOD 2



A1A1 ***N2***

Notes: Award ***A1*** for segment connecting endpoints and ***A1*** for direction (must see arrow).

Additional lines not required.

[2 marks]

2b. Given that [4 marks]

$u = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ and

$v = \begin{pmatrix} 5 \\ n \\ 3 \end{pmatrix}$, where

$n \in \mathbb{Z}$, find \(n\).

Markscheme

evidence of setting scalar product equal to zero (seen anywhere) ***R1***

eg ***u***

· ***v***

$= 0,\ 15 + 2n + 3 = 0$

correct expression for scalar product ***(A1)***

eg

$3 \times 5 + 2 \times n + 1 \times 3,\ 2n + 18 = 0$

attempt to solve equation ***(M1)***

eg

$2n = -18$

$n = -9$ ***A1*** ***N3***

[4 marks]

The vectors ***a*** = $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and ***b*** = $\begin{pmatrix} k+3 \\ k \end{pmatrix}$ are perpendicular to each other.

3a. Find the value of k. *[4 marks]*

Markscheme

evidence of scalar product ***M1***

eg ***a*** • ***b***, $4(k+3) + 2k$

recognizing scalar product must be zero ***(M1)***

eg ***a*** • ***b*** $= 0,\ 4k + 12 + 2k = 0$

correct working (must involve combining terms) ***(A1)***

eg $6k + 12,\ 6k = -12$

$k = -2$ ***A1*** ***N2***

[4 marks]

3b. Given that ***c*** = ***a*** + 2***b***, find ***c***. *[3 marks]*

Markscheme

attempt to substitute **their** value of k (seen anywhere) ***(M1)***

eg $\boldsymbol{b} = \begin{pmatrix} -2+3 \\ -2 \end{pmatrix}$, $2\boldsymbol{b} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

correct working ***(A1)***

eg $\begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix}$, $\begin{pmatrix} 4+2k+6 \\ 2+2k \end{pmatrix}$

$\boldsymbol{c} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$ ***A1 N2***

[3 marks]

4. Let $\boldsymbol{u} = -3\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{v} = m\boldsymbol{j} + n\boldsymbol{k}$, where $m,\ n \in \mathbb{R}$. Given that $\boldsymbol{v}$ is a unit vector perpendicular to $\boldsymbol{u}$, find the possible values of m and of n. *[7 marks]*

Markscheme

correct scalar product ***(A1)***

eg $m+n$

setting up their scalar product equal to 0 (seen anywhere) ***(M1)***

eg $\boldsymbol{u} \bullet \boldsymbol{v} = 0,\ -3(0)+1(m)+1(n) = 0,\ m = -n$

correct interpretation of unit vector ***(A1)***

eg $\sqrt{0^2+m^2+n^2} = 1,\ m^2+n^2 = 1$

valid attempt to solve their equations (must be in one variable) ***M1***

eg $(-n)^2+n^2 = 1,\ \sqrt{1-n^2}+n = 0,\ m^2+(-m)^2 = 1,\ m-\sqrt{1-m^2} = 0$

correct working ***A1***

eg $2n^2 = 1,\ 2m^2 = 1,\ \sqrt{2} = \frac{1}{n},\ m = \pm\frac{1}{\sqrt{2}}$

both correct pairs ***A2 N3***

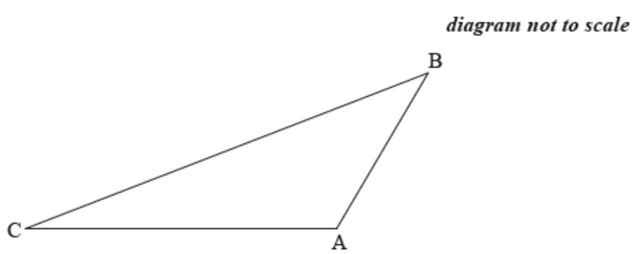
eg $m = \frac{1}{\sqrt{2}}$ and $n = -\frac{1}{\sqrt{2}}$, $m = -\frac{1}{\sqrt{2}}$ and $n = \frac{1}{\sqrt{2}}$,

$m = (0.5)^{\frac{1}{2}}$ and $n = -(0.5)^{\frac{1}{2}}$, $m = -\sqrt{\frac{1}{2}}$ and $n = \sqrt{\frac{1}{2}}$

Note: Award ***A0*** for $m = \pm\frac{1}{\sqrt{2}},\ n = \pm\frac{1}{\sqrt{2}}$, or any other answer that does not clearly indicate the correct pairs.

[7 marks]

5. The following diagram shows triangle ABC. *[6 marks]*



Let $\overrightarrow{AB} \bullet \overrightarrow{AC} = -5\sqrt{3}$ and $\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right| = 10$. Find the area of triangle ABC.

Markscheme

attempt to find $\cos \mathrm{C\hat{A}B}$ (seen anywhere) ***(M1)***

eg $\cos\theta = \frac{\overrightarrow{\mathrm{AB}} \bullet \overrightarrow{\mathrm{AC}}}{\left|\overrightarrow{\mathrm{AB}}\right|\left|\overrightarrow{\mathrm{AC}}\right|}$

$\cos \mathrm{C\hat{A}B} = \frac{-5\sqrt{3}}{10}$ $\left(= -\frac{\sqrt{3}}{2}\right)$ ***A1***

valid attempt to find $\sin \mathrm{C\hat{A}B}$ ***(M1)***

eg triangle, Pythagorean identity, $\mathrm{C\hat{A}B} = \frac{5\pi}{6},\ 150^\circ$

$\sin \mathrm{C\hat{A}B} = \frac{1}{2}$ ***(A1)***

correct substitution into formula for area ***(A1)***

eg $\frac{1}{2} \times 10 \times \frac{1}{2},\ \frac{1}{2} \times 10 \times \sin\frac{\pi}{6}$

$\mathrm{area} = \frac{10}{4}$ $\left(= \frac{5}{2}\right)$ ***A1 N3***

[6 marks]

Consider the vectors
$a = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and
$b = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

6a. (a) Find *[6 marks]*

(i) $2a + b$;

(ii) $|2a + b|$.

Let $2a + b + c = 0$, where 0 is the zero vector.

(b) Find c .

Markscheme

(a) (i)

$2a = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$ ***(A1)***

correct expression for
$2a + b$ ***A1*** ***N2***

eg
$\begin{pmatrix} 5 \\ -2 \end{pmatrix}$,
$(5, -2)$,
$5i - 2j$

(ii) correct substitution into length formula ***(A1)***

eg
$\sqrt{5^2 + 2^2}$,
$\sqrt{5^2 + -2^2}$

$|2a + b| = \sqrt{29}$ ***A1*** ***N2***

[4 marks]

(b) valid approach ***(M1)***

eg
$c = -(2a + b)$,
$5 + x = 0$,
$-2 + y = 0$

$c = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$ ***A1 N2***

[2 marks]

6b. Find *[4 marks]*

(i)
$2a + b$;

(ii)
$|2a + b|$.

Markscheme

(i)

$2a = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$ ***(A1)***

correct expression for
$2a + b$ ***A1*** ***N2***

eg
$\begin{pmatrix} 5 \\ -2 \end{pmatrix}$,
$(5, -2)$,
$5i - 2j$

(ii) correct substitution into length formula ***(A1)***

eg
$\sqrt{5^2 + 2^2}$,
$\sqrt{5^2 + -2^2}$

$|2a + b| = \sqrt{29}$ ***A1*** ***N2***

[4 marks]

Let
$2a + b + c = 0$, where
0 is the zero vector.

6c. Find
c . *[2 marks]*

Markscheme

valid approach ***(M1)***

eg
$c = -(2a + b)$,
$5 + x = 0$,
$-2 + y = 0$

$c = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$ ***A1 N2***

[2 marks]

Note: In this question, distance is in metres and time is in seconds.

Two particles P_1 and P_2 start moving from a point A at the same time, along different straight lines.

After t seconds, the position of P_1 is given by ***r*** = $\begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

7a. Find the coordinates of A. *[2 marks]*

Markscheme

recognizing $t = 0$ at A ***(M1)***

A is $(4,\ -1,\ 3)$ ***A1 N2***

[2 marks]

Two seconds after leaving A, P_1 is at point B.

7b. Find $\overrightarrow{\mathrm{AB}}$; *[3 marks]*

Markscheme

METHOD 1

valid approach ***(M1)***

eg $\begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$, $(6,\ 3,\ -1)$

correct approach to find $\overrightarrow{\mathrm{AB}}$ ***(A1)***

eg $\mathrm{AO} + \mathrm{OB},\ \mathrm{B} - \mathrm{A},\ \begin{pmatrix} 6 \\ 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$

$\overrightarrow{\mathrm{AB}} = \begin{pmatrix} 2 \\ 4 \\ -4 \end{pmatrix}$ ***A1 N2***

METHOD 2

recognizing $\overrightarrow{\mathrm{AB}}$ is two times the direction vector ***(M1)***

correct working ***(A1)***

eg $\overrightarrow{\mathrm{AB}} = 2\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

$\overrightarrow{\mathrm{AB}} = \begin{pmatrix} 2 \\ 4 \\ -4 \end{pmatrix}$ ***A1 N2***

[3 marks]

7c. Find $\left|\overrightarrow{\mathrm{AB}}\right|$. *[2 marks]*

Markscheme

correct substitution ***(A1)***

eg $\left|\overrightarrow{\mathrm{AB}}\right| = \sqrt{2^2 + 4^2 + 4^2},\ \sqrt{4 + 16 + 16},\ \sqrt{36}$

$\left|\overrightarrow{\mathrm{AB}}\right| = 6$ ***A1 N2***

[2 marks]

Two seconds after leaving A, P_2 is at point C, where $\overrightarrow{AC} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$.

7d. Find $\cos \mathrm{B\hat{A}C}$. *[5 marks]*

Markscheme

METHOD 1 (vector approach)

valid approach involving $\overrightarrow{AB}$ and $\overrightarrow{AC}$ ***(M1)***

eg $\overrightarrow{AB} \bullet \overrightarrow{AC}$, $\frac{\overrightarrow{BA} \bullet \overrightarrow{AC}}{AB \times AC}$

finding scalar product and $\left|\overrightarrow{AC}\right|$ ***(A1)(A1)***

scalar product $2(3) + 4(0) - 4(4)\ (= -10)$

$\left|\overrightarrow{AC}\right| = \sqrt{3^2 + 0^2 + 4^2}\ (= 5)$

substitution of **their** scalar product and magnitudes into cosine formula ***(M1)***

eg $\cos \mathrm{B\hat{A}C} = \frac{6+0-16}{6\sqrt{3^2+4^2}}$

$\cos B\hat{A}C = -\frac{10}{30} \left(= -\frac{1}{3}\right)$ ***A1 N2***

METHOD 2 (triangle approach)

valid approach involving cosine rule ***(M1)***

eg $\cos \mathrm{B\hat{A}C} = \frac{AB^2 + AC^2 - BC^2}{2 \times AB \times AC}$

finding lengths AC and BC ***(A1)(A1)***

$\mathrm{AC} = 5,\ \mathrm{BC} = 9$

substitution of **their** lengths into cosine formula ***(M1)***

eg $\cos \mathrm{B\hat{A}C} = \frac{5^2+6^2-9^2}{2 \times 5 \times 6}$

$\cos \mathrm{B\hat{A}C} = -\frac{20}{60} \left(= -\frac{1}{3}\right)$ ***A1 N2***

[5 marks]

7e. Hence or otherwise, find the distance between P_1 and P_2 two seconds after they leave A. *[4 marks]*

Markscheme

Note: Award relevant marks for working seen to find BC in part (c) (if cosine rule used in part (c)).

METHOD 1 (using cosine rule)

recognizing need to find BC ***(M1)***

choosing cosine rule ***(M1)***

eg $c^2 = a^2 + b^2 - 2ab\cos\mathrm{C}$

correct substitution into RHS ***A1***

eg $\mathrm{BC}^2 = (6)^2 + (5)^2 - 2(6)(5)\left(-\frac{1}{3}\right)$, $36 + 25 + 20$

distance is 9 ***A1*** ***N2***

METHOD 2 (finding magnitude of $\overrightarrow{BC}$)

recognizing need to find BC ***(M1)***

valid approach ***(M1)***

eg attempt to find $\overrightarrow{\mathrm{OB}}$ or $\overrightarrow{\mathrm{OC}}$, $\overrightarrow{\mathrm{OB}} = \begin{pmatrix} 6 \\ 3 \\ -1 \end{pmatrix}$ or $\overrightarrow{\mathrm{OC}} = \begin{pmatrix} 7 \\ -1 \\ 7 \end{pmatrix}$, $\overrightarrow{\mathrm{BA}} + \overrightarrow{\mathrm{AC}}$

correct working ***A1***

eg $\overrightarrow{\mathrm{BC}} = \begin{pmatrix} 1 \\ -4 \\ 8 \end{pmatrix}$, $\overrightarrow{\mathrm{CB}} = \begin{pmatrix} -1 \\ 4 \\ -8 \end{pmatrix}$, $\sqrt{1^2 + 4^2 + 8^2} = \sqrt{81}$

distance is 9 ***A1*** ***N2***

METHOD 3 (finding coordinates and using distance formula)

recognizing need to find BC ***(M1)***

valid approach ***(M1)***

eg attempt to find coordinates of B or C, $\mathrm{B}(6,\ 3,\ -1)$ or $\mathrm{C}(7,\ -1,\ 7)$

correct substitution into distance formula ***A1***

eg $\mathrm{BC} = \sqrt{(6-7)^2 + (3-(-1))^2 + (-1-7)^2}$, $\sqrt{1^2 + 4^2 + 8^2} = \sqrt{81}$

distance is 9 ***A1*** ***N2***

[4 marks]