0413HW_Integration *[74 marks]*

Let $f(x) = 6 - \ln(x^2 + 2)$, for $x \in \mathbb{R}$. The graph of f passes through the point $(p,\ 4)$, where $p > 0$.

1a. Find the value of p. *[2 marks]*

Markscheme

valid approach ***(M1)***

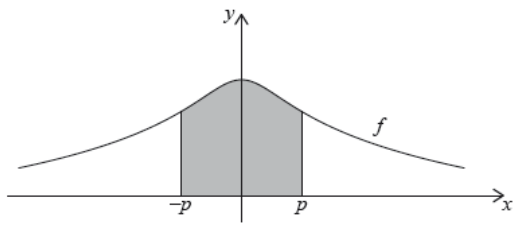
eg $f(p) = 4$, intersection with $y = 4,\ \pm 2.32$

2.32143

$p = \sqrt{e^2 - 2}$ (exact), 2.32 ***A1 N2***

[2 marks]

1b. The following diagram shows part of the graph of f. *[3 marks]*



The region enclosed by the graph of f, the x-axis and the lines $x = -p$ and $x = p$ is rotated 360° about the x-axis. Find the volume of the solid formed.

Markscheme

attempt to substitute **either their** limits **or** the function into volume formula (must involve f^2, accept reversed limits and absence of π and/or $\mathrm{d}x$, but do not accept any other errors) ***(M1)***

eg $\int_{-2.32}^{2.32} f^2,\ \pi \int \left(6 - \ln(x^2 + 2)\right)^2 \mathrm{d}x,\ 105.675$

331.989

$\text{volume} = 332$ ***A2 N3***

[3 marks]

2a. Find $\int x e^{x^2 - 1} \mathrm{d}x$. *[4 marks]*

Markscheme

valid approach to set up integration by substitution/inspection ***(M1)***

eg $u = x^2 - 1,\ \mathrm{d}u = 2x,\ \int 2x\mathrm{e}^{x^2-1}\mathrm{d}x$

correct expression ***(A1)***

eg $\frac{1}{2}\int 2x\mathrm{e}^{x^2-1}\mathrm{d}x,\ \frac{1}{2}\int \mathrm{e}^{u}\mathrm{d}u$

$\frac{1}{2}\mathrm{e}^{x^2-1} + c$ ***A2 N4***

Notes: Award ***A1*** if missing "$+c$".

[4 marks]

2b. Find $f(x)$, given that $f'(x) = x\mathrm{e}^{x^2-1}$ and $f(-1) = 3$. *[3 marks]*

Markscheme

substituting $x = -1$ into **their** answer from (a) ***(M1)***

eg $\frac{1}{2}\mathrm{e}^0,\ \frac{1}{2}\mathrm{e}^{1-1} = 3$

correct working ***(A1)***

eg $\frac{1}{2} + c = 3,\ c = 2.5$

$f(x) = \frac{1}{2}\mathrm{e}^{x^2-1} + 2.5$ ***A1 N2***

[3 marks]

3. Let $f'(x) = \frac{3x^2}{(x^3+1)^5}$. Given that $f(0) = 1$, find $f(x)$. *[6 marks]*

Markscheme

valid approach ***(M1)***

eg $\int f'\mathrm{d}x,\ \int \frac{3x^2}{(x^3+1)^5}\mathrm{d}x$

correct integration by substitution/inspection ***A2***

eg $f(x) = -\frac{1}{4}(x^3+1)^{-4} + c,\ \frac{-1}{4(x^3+1)^4}$

correct substitution into **their** integrated function (must include c) ***M1***

eg $1 = \frac{-1}{4(0^3+1)^4} + c,\ -\frac{1}{4} + c = 1$

Note: Award ***M0*** if candidates substitute into f' or f''.

$c = \frac{5}{4}$ ***(A1)***

$f(x) = -\frac{1}{4}(x^3+1)^{-4} + \frac{5}{4}\ \left(= \frac{-1}{4(x^3+1)^4} + \frac{5}{4},\ \frac{5(x^3+1)^4 - 1}{4(x^3+1)^4}\right)$ ***A1 N4***

[6 marks]

4. Let $f'(x) = \sin^3(2x)\cos(2x)$. Find $f(x)$, given that $f\left(\frac{\pi}{4}\right) = 1$. *[7 marks]*

Markscheme

evidence of integration ***(M1)***

eg $\int f'(x)\mathrm{d}x$

correct integration (accept missing C) ***(A2)***

eg $\frac{1}{2} \times \frac{\sin^4(2x)}{4},\ \frac{1}{8}\sin^4(2x) + C$

substituting initial condition into their integrated expression (must have $+C$) ***M1***

eg $1 = \frac{1}{8}\sin^4\left(\frac{\pi}{2}\right) + C$

Note: Award ***M0*** if they substitute into the original or differentiated function.

recognizing $\sin\left(\frac{\pi}{2}\right) = 1$ ***(A1)***

eg $1 = \frac{1}{8}(1)^4 + C$

$C = \frac{7}{8}$ ***(A1)***

$f(x) = \frac{1}{8}\sin^4(2x) + \frac{7}{8}$ ***A1 N5***

[7 marks]

Let $f(x) = x\mathrm{e}^{-x}$ and $g(x) = -3f(x) + 1$.

The graphs of f and g intersect at $x = p$ and $x = q$, where $p < q$.

5a. Find the value of p and of q. *[3 marks]*

Markscheme

valid attempt to find the intersection ***(M1)***

eg

$f = g$, sketch, one correct answer

$p = 0.357402,\ q = 2.15329$

$p = 0.357,\ q = 2.15$ ***A1A1 N3***

[3 marks]

5b. Hence, find the area of the region enclosed by the graphs of f and g. *[3 marks]*

Markscheme

attempt to set up an integral involving subtraction (in any order) ***(M1)***

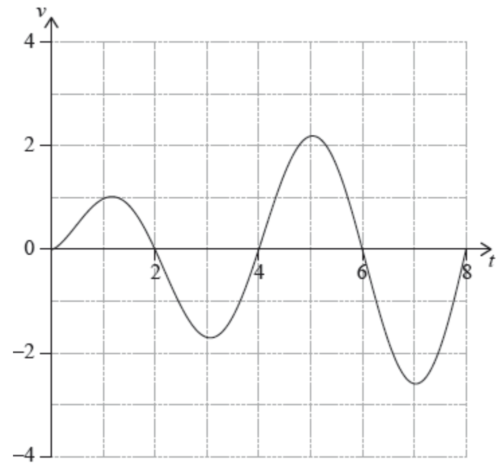
eg $\int_p^q [f(x) - g(x)]\,\mathrm{d}x,\ \int_p^q f(x)\mathrm{d}x - \int_p^q g(x)\mathrm{d}x$

0.537667

$\text{area} = 0.538$ ***A2 N3***

[3 marks]

A particle P moves along a straight line. Its velocity $v_{\mathrm{P}}\ \mathrm{m\,s^{-1}}$ after t seconds is given by $v_{\mathrm{P}}=\sqrt{t}\sin\left(\frac{\pi}{2}t\right)$, for $0\leqslant t\leqslant 8$. The following diagram shows the graph of v_{P}.



6a. Write down the first value of t at which P changes direction. *[1 mark]*

Markscheme

$t=2$ ***A1 N1***

[1 mark]

6b. Find the **total** distance travelled by P, for $0\leqslant t\leqslant 8$. *[2 marks]*

Markscheme

substitution of limits or function into formula or correct sum ***(A1)***

eg $\int_0^8|v|\,\mathrm{d}t,\ \int|v_Q|\,\mathrm{d}t,\ \int_0^2 v\mathrm{d}t-\int_2^4 v\mathrm{d}t+\int_4^6 v\mathrm{d}t-\int_6^8 v\mathrm{d}t$

9.64782

distance $=9.65\ (\text{metres})$ ***A1 N2***

[2 marks]

6c. A second particle Q also moves along a straight line. Its velocity, $v_{\mathrm{Q}}\ \mathrm{m\,s^{-1}}$ after t seconds is given by $v_{\mathrm{Q}}=\sqrt{t}$ for $0\leqslant t\leqslant 8$. After k seconds Q has travelled the same total distance as P. *[4 marks]*

Find k.

Markscheme

correct approach ***(A1)***

eg $s=\int\sqrt{t},\ \int_0^k\sqrt{t}\mathrm{d}t,\ \int_0^k|v_{\mathrm{Q}}|\,\mathrm{d}t$

correct integration ***(A1)***

eg $\int\sqrt{t}=\frac{2}{3}t^{\frac{3}{2}}+c,\ \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^k,\ \frac{2}{3}k^{\frac{3}{2}}$

equating their expression to the distance travelled by their P ***(M1)***

eg $\frac{2}{3}k^{\frac{3}{2}}=9.65,\ \int_0^k\sqrt{t}\mathrm{d}t=9.65$

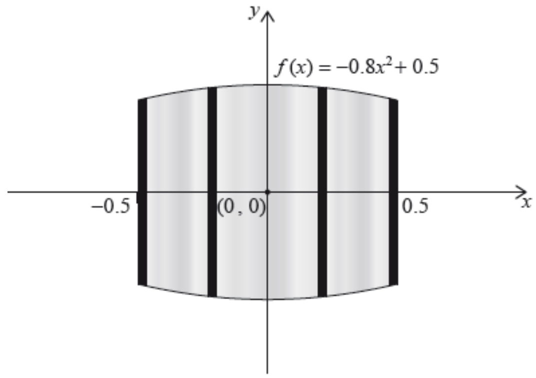
5.93855

5.94 (seconds) ***A1 N3***

[4 marks]

All lengths in this question are in metres.

Let $f(x) = -0.8x^2 + 0.5$, for $-0.5 \leqslant x \leqslant 0.5$. Mark uses $f(x)$ as a model to create a barrel. The region enclosed by the graph of f, the x-axis, the line $x = -0.5$ and the line $x = 0.5$ is rotated 360° about the x-axis. This is shown in the following diagram.



7a. Use the model to find the volume of the barrel. *[3 marks]*

Markscheme

attempt to substitute correct limits or the function into the formula involving

y^2

eg $\pi\int_{-0.5}^{0.5} y^2\mathrm{d}x,\ \pi\int(-0.8x^2+0.5)^2\mathrm{d}x$

0.601091

volume $= 0.601\ (\mathrm{m}^3)$ ***A2*** ***N3***

[3 marks]

7b. The empty barrel is being filled with water. The volume $V\ \mathrm{m}^3$ of water in the barrel after t minutes is given by $V = 0.8(1 - \mathrm{e}^{-0.1t})$. How long will it take for the barrel to be half-full? *[3 marks]*

Markscheme

attempt to equate half **their** volume to V ***(M1)***

eg

$0.30055 = 0.8(1 - \mathrm{e}^{-0.1t})$, graph

4.71104

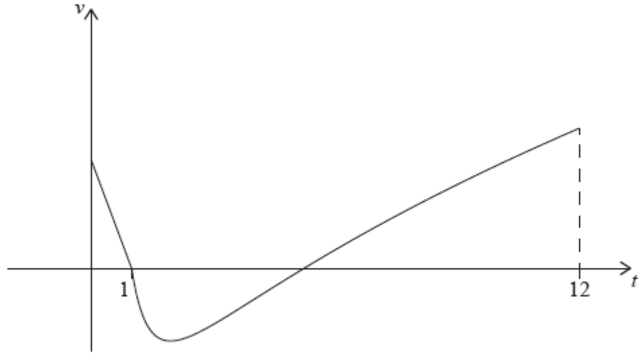
4.71 (minutes) ***A2*** ***N3***

[3 marks]

A particle P starts from a point A and moves along a horizontal straight line. Its velocity $v\text{ cm}\,\text{s}^{-1}$ after t seconds is given by

$$v(t) = \begin{cases} -2t + 2, & \text{for } 0 \leqslant t \leqslant 1 \\ 3\sqrt{t} + \frac{4}{t^2} - 7, & \text{for } 1 \leqslant t \leqslant 12 \end{cases}$$

The following diagram shows the graph of v.



8a. Find the initial velocity of P. *[2 marks]*

Markscheme

valid attempt to substitute $t = 0$ into the correct function ***(M1)***

eg $-2(0) + 2$

2 ***A1*** ***N2***

[2 marks]

P is at rest when $t = 1$ and $t = p$.

8b. Find the value of p. *[2 marks]*

Markscheme

recognizing $v = 0$ when P is at rest ***(M1)***

5.21834

$p = 5.22\text{ (seconds)}$ ***A1*** ***N2***

[2 marks]

When $t = q$, the acceleration of P is zero.

8c. (i) Find the value of q. *[4 marks]*

(ii) Hence, find the **speed** of P when $t = q$.

Markscheme

(i) recognizing that $a = v'$ ***(M1)***

eg

$v' = 0$, minimum on graph

1.95343

$q = 1.95$ ***A1 N2***

(ii) valid approach to find **their** minimum ***(M1)***

eg

$v(q),\ -1.75879$, reference to min on graph

1.75879

speed $= 1.76\ (\text{cm s}^{-1})$ ***A1 N2***

[4 marks]

8d. (i) Find the total distance travelled by P between $t = 1$ and $t = p$. *[6 marks]*

(ii) Hence or otherwise, find the displacement of P from A when $t = p$.

Markscheme

(i) substitution of **correct** $v(t)$ into distance formula, ***(A1)***

eg $\int_1^p \left|3\sqrt{t} + \frac{4}{t^2} - 7\right| \mathrm{d}t,\ \left|\int 3\sqrt{t} + \frac{4}{t^2} - 7 \mathrm{d}t\right|$

4.45368

distance $= 4.45\ (\text{cm})$ ***A1 N2***

(ii) displacement from $t = 1$ to $t = p$ (seen anywhere) ***(A1)***

eg $-4.45368,\ \int_1^p \left(3\sqrt{t} + \frac{4}{t^2} - 7\right) \mathrm{d}t$

displacement from $t = 0$ to $t = 1$ ***(A1)***

eg $\int_0^1 (-2t + 2) \mathrm{d}t,\ 0.5 \times 1 \times 2,\ 1$

valid approach to find displacement for $0 \leqslant t \leqslant p$ ***M1***

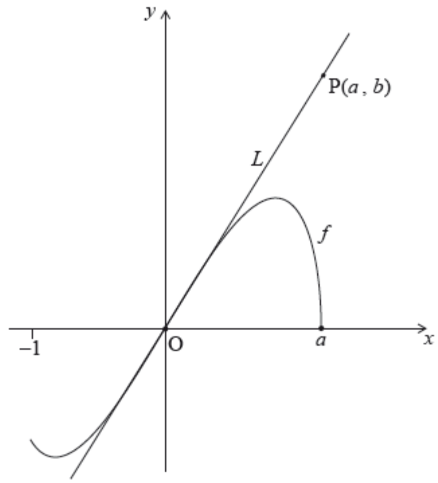
eg $\int_0^1 (-2t + 2) \mathrm{d}t + \int_1^p \left(3\sqrt{t} + \frac{4}{t^2} - 7\right) \mathrm{d}t,\ \int_0^1 (-2t + 2) \mathrm{d}t - 4.45$

-3.45368

displacement $= -3.45\ (\text{cm})$ ***A1 N2***

[6 marks]

The following diagram shows the graph of $f(x) = 2x\sqrt{a^2 - x^2}$, for $-1 \leqslant x \leqslant a$, where $a > 1$.



The line L is the tangent to the graph of f at the origin, O. The point $\mathrm{P}(a,\, b)$ lies on L.

9a. (i) Given that $f'(x) = \frac{2a^2 - 4x^2}{\sqrt{a^2 - x^2}}$, for $-1 \leqslant x < a$, find the equation of L. *[6 marks]*

(ii) Hence or otherwise, find an expression for b in terms of a.

Markscheme

(i) recognizing the need to find the gradient when $x = 0$ (seen anywhere) ***R1***

eg $f'(0)$

correct substitution ***(A1)***

$f'(0) = \frac{2a^2 - 4(0)}{\sqrt{a^2 - 0}}$

$f'(0) = 2a$ ***(A1)***

correct equation with gradient 2
a (do not accept equations of the form $L = 2ax$) ***A1 N3***

eg $y = 2ax,\ y - b = 2a(x - a),\ y = 2ax - 2a^2 + b$

(ii) **METHOD 1**

attempt to substitute $x = a$ into **their** equation of L ***(M1)***

eg $y = 2a \times a$

$b = 2a^2$ ***A1 N2***

METHOD 2

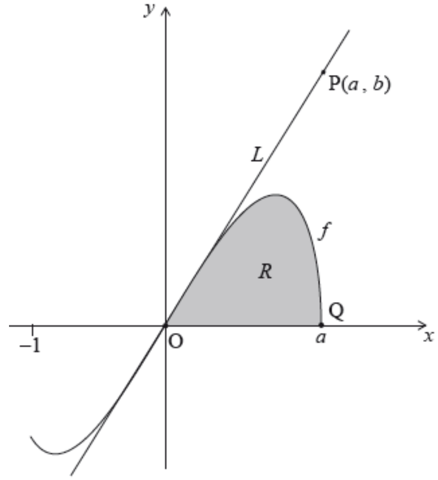
equating gradients ***(M1)***

eg $\frac{b}{a} = 2a$

$b = 2a^2$ ***A1 N2***

[6 marks]

The point $\mathrm{Q}(a,\ 0)$ lies on the graph of f. Let R be the region enclosed by the graph of f and the x-axis. This information is shown in the following diagram.



Let A_R be the area of the region R.

9b. Show that $A_R = \frac{2}{3}a^3$. *[6 marks]*

Markscheme

METHOD 1

recognizing that area $= \int_0^a f(x)\mathrm{d}x$ (seen anywhere) ***R1***

valid approach using substitution or inspection ***(M1)***

eg $\int 2x\sqrt{u}\mathrm{d}x,\ u = a^2 - x^2,\ \mathrm{d}u = -2x\mathrm{d}x,\ \frac{2}{3}(a^2 - x^2)^{\frac{3}{2}}$

correct working ***(A1)***

eg $\int 2x\sqrt{a^2 - x^2}\mathrm{d}x = \int -\sqrt{u}\mathrm{d}u$

$\int -\sqrt{u}\mathrm{d}u = -\frac{u^{\frac{3}{2}}}{\frac{3}{2}}$ ***(A1)***

$\int f(x)\mathrm{d}x = -\frac{2}{3}(a^2 - x^2)^{\frac{3}{2}} + c$ ***(A1)***

substituting limits and subtracting ***A1***

eg $A_R = -\frac{2}{3}(a^2 - a^2)^{\frac{3}{2}} + \frac{2}{3}(a^2 - 0)^{\frac{3}{2}},\ \frac{2}{3}(a^2)^{\frac{3}{2}}$

$A_R = \frac{2}{3}a^3$ ***AG N0***

METHOD 2

recognizing that area $= \int_0^a f(x)\mathrm{d}x$ (seen anywhere) ***R1***

valid approach using substitution or inspection ***(M1)***

eg $\int 2x\sqrt{u}\mathrm{d}x,\ u = a^2 - x^2,\ \mathrm{d}u = -2x\mathrm{d}x,\ \frac{2}{3}(a^2 - x^2)^{\frac{3}{2}}$

correct working ***(A1)***

eg $\int 2x\sqrt{a^2 - x^2}\mathrm{d}x = \int -\sqrt{u}\mathrm{d}u$

$\int -\sqrt{u}\mathrm{d}u = -\frac{u^{\frac{3}{2}}}{\frac{3}{2}}$ ***(A1)***

new limits for *u* (even if integration is incorrect) ***(A1)***

eg $u = 0 \text{ and } u = a^2,\ \int_0^{a^2} u^{\frac{1}{2}}\mathrm{d}u,\ \left[-\frac{2}{3}u^{\frac{3}{2}}\right]_{a^2}^{0}$

substituting limits and subtracting ***A1***

eg $A_R = -\left(0 - \frac{2}{3}a^3\right),\ \frac{2}{3}(a^2)^{\frac{3}{2}}$

$A_R = \frac{2}{3}a^3$ ***AG N0***

[6 marks]

9c. Let A_T be the area of the triangle OPQ. Given that $A_T = kA_R$, find the value of k. *[4 marks]*

Markscheme

METHOD 1

valid approach to find area of triangle ***(M1)***

eg $\frac{1}{2}(\mathrm{OQ})(\mathrm{PQ}),\ \frac{1}{2}ab$

correct substitution into formula for A_T (seen anywhere) ***(A1)***

eg $A_T = \frac{1}{2} \times a \times 2a^2,\ a^3$

valid attempt to find k (must be in terms of a) ***(M1)***

eg $a^3 = k\frac{2}{3}a^3,\ k = \frac{a^3}{\frac{2}{3}a^3}$

$k = \frac{3}{2}$ ***A1 N2***

METHOD 2

valid approach to find area of triangle ***(M1)***

eg $\int_0^a (2ax)\mathrm{d}x$

correct working ***(A1)***

eg $[ax^2]_0^a,\ a^3$

valid attempt to find k (must be in terms of a) ***(M1)***

eg $a^3 = k\frac{2}{3}a^3,\ k = \frac{a^3}{\frac{2}{3}a^3}$

$k = \frac{3}{2}$ ***A1 N2***

[4 marks]