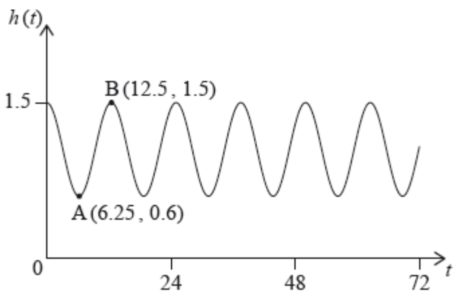
3-4_Periodic-functions-spicy *[198 marks]*

At Grande Anse Beach the height of the water in metres is modelled by the function $h(t) = p\cos(q \times t) + r$, where t is the number of hours after 21:00 hours on 10 December 2017. The following diagram shows the graph of h , for $0 \leqslant t \leqslant 72$.



The point $\mathrm{A}(6.25,\ 0.6)$ represents the first low tide and $\mathrm{B}(12.5,\ 1.5)$ represents the next high tide.

1a. How much time is there between the first low tide and the next high tide? *[2 marks]*

Markscheme

attempt to find the difference of x-values of A and B ***(M1)***

eg $6.25 - 12.5$

6.25 (hours), (6 hours 15 minutes) ***A1*** ***N2***

[2 marks]

1b. Find the difference in height between low tide and high tide. *[2 marks]*

Markscheme

attempt to find the difference of y-values of A and B ***(M1)***

eg $1.5 - 0.6$

$0.9\ (\mathrm{m})$ ***A1*** ***N2***

[2 marks]

1c. Find the value of p; *[2 marks]*

Markscheme

valid approach ***(M1)***

eg $\frac{\text{max}-\text{min}}{2},\ 0.9 \div 2$

$p = 0.45$ ***A1 N2***

[2 marks]

1d. Find the value of q; *[3 marks]*

Markscheme

METHOD 1

period $= 12.5$ (seen anywhere) ***(A1)***

valid approach (seen anywhere) ***(M1)***

eg $\text{period} = \frac{2\pi}{b},\ q = \frac{2\pi}{\text{period}},\ \frac{2\pi}{12.5}$

0.502654

$q = \frac{4\pi}{25},\ 0.503\ \left(\text{or}\ -\frac{4\pi}{25},\ -0.503\right)$ ***A1 N2***

METHOD 2

attempt to use a coordinate to make an equation ***(M1)***

eg $p\cos(6.25q) + r = 0.6,\ p\cos(12.5q) + r = 1.5$

correct substitution ***(A1)***

eg $0.45\cos(6.25q) + 1.05 = 0.6,\ 0.45\cos(12.5q) + 1.05 = 1.5$

0.502654

$q = \frac{4\pi}{25},\ 0.503\ \left(\text{or}\ -\frac{4\pi}{25},\ -0.503\right)$ ***A1 N2***

[3 marks]

1e. Find the value of r. *[2 marks]*

Markscheme

valid method to find r ***(M1)***

eg $\frac{\text{max}+\text{min}}{2},\ 0.6 + 0.45$

$r = 1.05$ ***A1 N2***

[2 marks]

1f. There are two high tides on 12 December 2017. At what time does the second high tide occur? *[3 marks]*

Markscheme

METHOD 1

attempt to find start or end t-values for 12 December ***(M1)***

eg $3 + 24,\ t = 27,\ t = 51$

finds t-value for second max ***(A1)***

$t = 50$

23:00 (or 11 pm) ***A1*** ***N3***

METHOD 2

valid approach to list either the times of high tides after 21:00 or the t-values of high tides after 21:00, showing at least two times ***(M1)***

eg $21{:}00 + 12.5,\ 21{:}00 + 25,\ 12.5 + 12.5,\ 25 + 12.5$

correct time of first high tide on 12 December ***(A1)***

eg 10:30 (or 10:30 am)

time of second high tide = 23:00 ***A1*** ***N3***

METHOD 3

attempt to set **their** h equal to 1.5 ***(M1)***

eg $h(t) = 1.5,\ 0.45\cos\left(\frac{4\pi}{25}t\right) + 1.05 = 1.5$

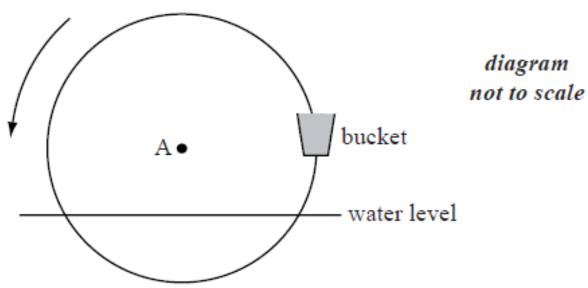
correct working to find second max ***(A1)***

eg $0.503t = 8\pi,\ t = 50$

23:00 (or 11 pm) ***A1*** ***N3***

[3 marks]

The following diagram shows a waterwheel with a bucket. The wheel rotates at aconstant rate in an anticlockwise (counter-clockwise) direction.



The diameter of the wheel is 8 metres. The centre of the wheel, A, is 2 metresabove the water level. After t seconds, the height of the bucket above the water level is given by
$h = a\sin bt + 2$.

2a. Show that $a = 4$. *[2 marks]*

Markscheme

METHOD 1

evidence of recognizing the amplitude is the radius ***(M1)***

e.g. amplitude is half the diameter

$a = \frac{8}{2}$ ***A1***

$a = 4$ ***AG*** ***N0***

METHOD 2

evidence of recognizing the maximum height ***(M1)***

e.g. $h = 6$, $a \sin bt + 2 = 6$

correct reasoning

e.g. $a \sin bt = 4$ and $\sin bt$ has amplitude of 1 ***A1***

$a = 4$ ***AG*** ***N0***

[2 marks]

2b. The wheel turns at a rate of one rotation every 30 seconds. *[2 marks]*

Show that $b = \frac{\pi}{15}$.

Markscheme

METHOD 1

period = 30 ***(A1)***

$b = \frac{2\pi}{30}$ ***A1***

$b = \frac{\pi}{15}$ ***AG*** ***N0***

METHOD 2

correct equation ***(A1)***

e.g. $2 = 4 \sin 30b + 2$, $\sin 30b = 0$

$30b = 2\pi$ ***A1***

$b = \frac{\pi}{15}$ ***AG*** ***N0***

[2 marks]

2c. In the first rotation, there are two values of *t* when the bucket is **descending** at a rate of 0.5 ms^{-1} . *[6 marks]*

Find these values of *t* .

Markscheme

recognizing $h'(t) = -0.5$ (seen anywhere) ***R1***

attempting to solve ***(M1)***

e.g. sketch of h' , finding h'

correct work involving h' ***A2***

e.g. sketch of h' showing intersection, $-0.5 = \frac{4\pi}{15}\cos\left(\frac{\pi}{15}t\right)$

$t = 10.6$, $t = 19.4$ ***A1A1*** ***N3***

[6 marks]

2d. In the first rotation, there are two values of *t* when the bucket is **descending** at a rate of $0.5\ \mathrm{ms}^{-1}$. *[4 marks]*

Determine whether the bucket is underwater at the second value of *t* .

Markscheme

METHOD 1

valid reasoning for **their** conclusion (seen anywhere) ***R1***

e.g. $h(t) < 0$ so underwater; $h(t) > 0$ so not underwater

evidence of substituting into *h* ***(M1)***

e.g. $h(19.4)$, $4\sin\frac{19.4\pi}{15} + 2$

correct calculation ***A1***

e.g. $h(19.4) = -1.19$

correct statement ***A1*** ***N0***

e.g. the bucket is underwater, yes

METHOD 2

valid reasoning for **their** conclusion (seen anywhere) ***R1***

e.g. $h(t) < 0$ so underwater; $h(t) > 0$ so not underwater

evidence of valid approach ***(M1)***

e.g. solving $h(t) = 0$, graph showing region below *x*-axis

correct roots ***A1***

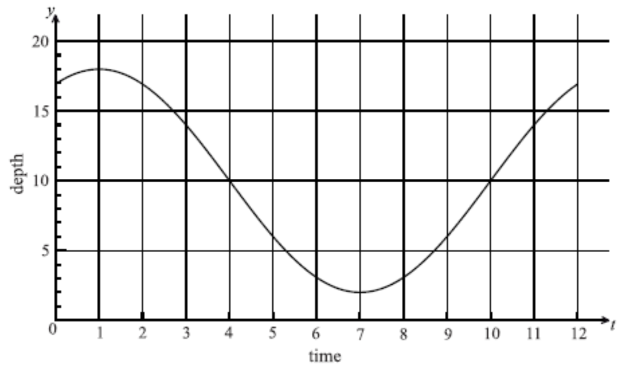
e.g. 17.5, 27.5

correct statement ***A1*** ***N0***

e.g. the bucket is underwater, yes

[4 marks]

The following graph shows the depth of water, y metres , at a point P, during one day. The time t is given in hours, from midnight to noon.



3a. Use the graph to write down an estimate of the value of t when *[3 marks]*

(i) the depth of water is minimum;

(ii) the depth of water is maximum;

(iii) the depth of the water is increasing most rapidly.

Markscheme

(i) 7 ***A1 N1***

(ii) 1 ***A1 N1***

(iii) 10 ***A1 N1***

[3 marks]

3b. The depth of water can be modelled by the function $y = \cos A(B(t-1)) + C$. *[6 marks]*

(i) Show that $A = 8$.

(ii) Write down the value of C.

(iii) Find the value of B.

Markscheme

(i) evidence of appropriate approach ***M1***

e.g. $A = \frac{18-2}{2}$

$A = 8$ ***AG*** ***N0***

(ii) $C = 10$ ***A2*** ***N2***

(iii) **METHOD 1**

$\text{period} = 12$ ***(A1)***

evidence of using $B \times \text{period} = 2\pi$ (accept $360°$) ***(M1)***

e.g. $12 = \frac{2\pi}{B}$

$B = \frac{\pi}{6}$ (accept 0.524 or 30) ***A1*** ***N3***

METHOD 2

evidence of substituting ***(M1)***

e.g. $10 = 8\cos 3B + 10$

simplifying ***(A1)***

e.g. $\cos 3B = 0$ $\left(3B = \frac{\pi}{2}\right)$

$B = \frac{\pi}{6}$ (accept 0.524 or 30) ***A1*** ***N3***

[6 marks]

3c. A sailor knows that he cannot sail past P when the depth of the water is less than 12 m. Calculate the values of *t* between which he cannot sail past P. *[2 marks]*

Markscheme

correct answers ***A1A1***

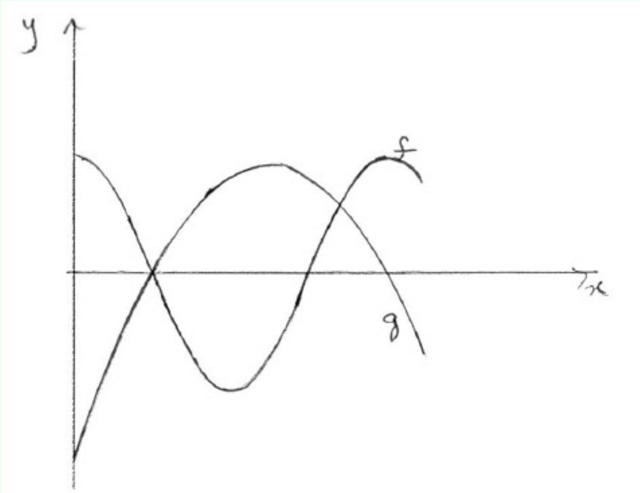
e.g. $t = 3.52$, $t = 10.5$, between 03:31 and 10:29 (accept 10:30) ***N2***

[2 marks]

Let
$f(x) = 5\cos\frac{\pi}{4}x$ and
$g(x) = -0.5x^2 + 5x - 8$ for
$0 \le x \le 9$.

4a. On the same diagram, sketch the graphs of *f* and *g* . *[3 marks]*

Markscheme



A1A1A1 ***N3***

Note: Award ***A1*** for *f* being of sinusoidal shape, with 2 maxima and one minimum, ***A1*** for *g* being a parabola opening down, ***A1*** for **two** intersection points in approximately correct position.

[3 marks]

4b. Consider the graph of f. Write down *[4 marks]*

(i) the *x*-intercept that lies between $x = 0$ and $x = 3$;

(ii) the period;

(iii) the amplitude.

Markscheme

(i) $(2, 0)$ (accept $x = 2$) ***A1*** ***N1***

(ii) $\text{period} = 8$ ***A2*** ***N2***

(iii) $\text{amplitude} = 5$ ***A1*** ***N1***

[4 marks]

4c. Consider the graph of g. Write down *[3 marks]*

(i) the two *x*-intercepts;

(ii) the equation of the axis of symmetry.

Markscheme

(i) $(2, 0)$, $(8, 0)$ (accept $x = 2$, $x = 8$) ***A1A1*** ***N1N1***

(ii) $x = 5$ (must be an equation) ***A1*** ***N1***

[3 marks]

4d. Let R be the region enclosed by the graphs of f and g. Find the area of R. *[5 marks]*

Markscheme

METHOD 1

intersect when $x = 2$ and $x = 6.79$ (may be seen as limits of integration) ***A1A1***

evidence of approach ***(M1)***

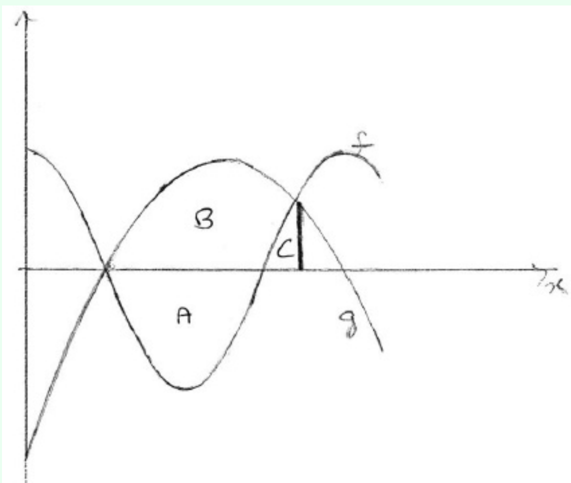
e.g. $\int g - f$, $\int f(x)\mathrm{d}x - \int g(x)\mathrm{d}x$, $\int_2^{6.79}\left((-0.5x^2 + 5x - 8) - \left(5\cos\frac{\pi}{4}x\right)\right)$

$\text{area} = 27.6$ ***A2*** ***N3***

METHOD 2

intersect when $x = 2$ and $x = 6.79$ (seen anywhere) ***A1A1***

evidence of approach using a sketch of g and f, or $g - f$. ***(M1)***



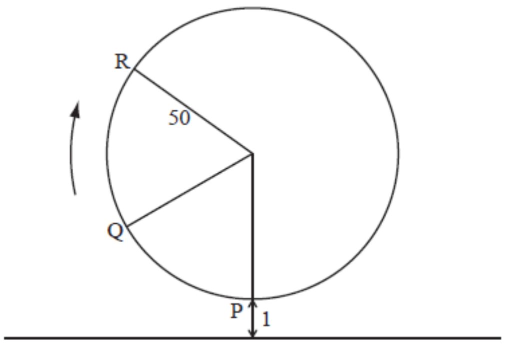
e.g. area = $A + B - C$, $12.7324 + 16.0938 - 1.18129\ldots$

$\text{area} = 27.6$ ***A2*** ***N3***

[5 marks]

The following diagram represents a large Ferris wheel at an amusement park.

The points P, Q and R represent different positions of a seat on the wheel.



The wheel has a radius of 50 metres and rotates clockwise at a rate of one revolutionevery 30 minutes.

A seat starts at the lowest point P, when its height is one metre above the ground.

5a. Find the height of a seat above the ground after 15 minutes. *[2 marks]*

Markscheme

valid approach *(M1)*

e.g. 15 mins is half way, top of the wheel, $d+1$

height $= 101$ (metres) ***A1 N2***

[2 marks]

5b. After six minutes, the seat is at point Q. Find its height above the ground at Q. *[5 marks]*

Markscheme

evidence of identifying rotation angle after 6 minutes ***A1***

e.g. $\frac{2\pi}{5}$, $\frac{1}{5}$ of a rotation, 72°

evidence of appropriate approach *(M1)*

e.g. drawing a right triangle and using cosine ratio

correct working (seen anywhere) ***A1***

e.g. $\cos\frac{2\pi}{5}=\frac{x}{50}$, $15.4(508\dots)$

evidence of appropriate method ***M1***

e.g. height $=\text{radius}+1-15.45\dots$

height $= 35.5$ (metres) (accept 35.6) ***A1 N2***

[5 marks]

5c. The height of the seat above ground after t minutes can be modelled by the function $h(t) = 50\sin(b(t-c)) + 51$. *[6 marks]*

Find the value of b and of c.

Markscheme

METHOD 1

evidence of substituting into $b = \frac{2\pi}{\text{period}}$ ***(M1)***

correct substitution

e.g. period = 30 minutes, $b = \frac{2\pi}{30}$ ***A1***

$b = 0.209 \left(\frac{\pi}{15}\right)$ ***A1*** ***N2***

substituting into $h(t)$ ***(M1)***

e.g. $h(0) = 1$, $h(15) = 101$

correct substitution ***A1***

$1 = 50\sin\left(-\frac{\pi}{15}c\right) + 51$

$c = 7.5$ ***A1*** ***N2***

METHOD 2

evidence of setting up a system of equations ***(M1)***

two correct equations

e.g. $1 = 50\sin b(0-c) + 51$, $101 = 50\sin b(15-c) + 51$ ***A1A1***

attempt to solve simultaneously ***(M1)***

e.g. evidence of combining two equations

$b = 0.209 \left(\frac{\pi}{15}\right)$, $c = 7.5$ ***A1A1*** ***N2N2***

[6 marks]

5d. The height of the seat above ground after t minutes can be modelled by the function $h(t) = 50\sin(b(t-c)) + 51$. *[3 marks]*

Hence find the value of t the first time the seat is 96 m above the ground.

Markscheme

evidence of solving $h(t) = 96$ ***(M1)***

e.g. equation, graph

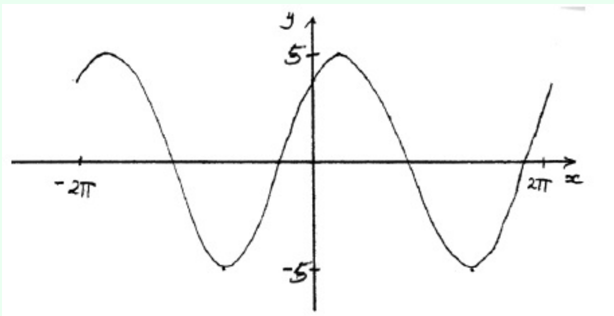
$t = 12.8$ (minutes) ***A2*** ***N3***

[3 marks]

Let
$f(x) = 3\sin x + 4\cos x$, for
$-2\pi \le x \le 2\pi$.

6a. Sketch the graph of f . *[3 marks]*

Markscheme



A1A1A1 N3

Note: Award ***A1*** for approximately sinusoidal shape, ***A1*** for end points approximately correct $(-2\pi, 4)$ $(2\pi, 4)$, ***A1*** for approximately correct position of graph, (*y*-intercept $(0, 4)$, maximum to right of *y*-axis).

[3 marks]

6b. Write down *[3 marks]*

(i) the amplitude;

(ii) the period;

(iii) the *x*-intercept that lies between $-\frac{\pi}{2}$ and 0.

Markscheme

(i) 5 ***A1 N1***

(ii) 2π (6.28) ***A1 N1***

(iii) -0.927 ***A1 N1***

[3 marks]

6c. Hence write $f(x)$ in the form $p\sin(qx + r)$. *[3 marks]*

Markscheme

$f(x) = 5\sin(x + 0.927)$ (accept $p = 5$, $q = 1$, $r = 0.927$) ***A1A1A1*** ***N3***

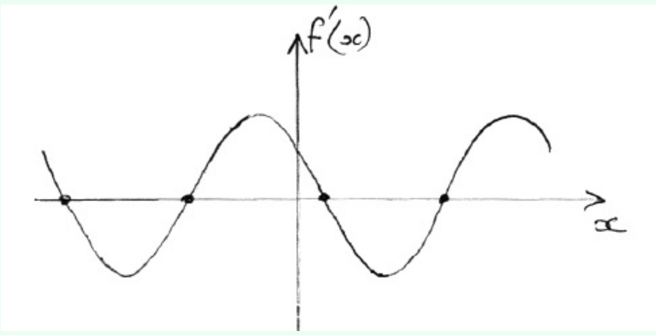
[3 marks]

6d. Write down one value of *x* such that $f'(x) = 0$. *[2 marks]*

Markscheme

evidence of correct approach ***(M1)***

e.g. max/min, sketch of $f'(x)$ indicating roots



one 3 s.f. value which rounds to one of -5.6, -2.5, 0.64, 3.8 ***A1*** ***N2***

[2 marks]

6e. Write down the two values of *k* for which the equation $f(x) = k$ has exactly two solutions. *[2 marks]*

Markscheme

$k = -5$, $k = 5$ ***A1A1*** ***N2***

[2 marks]

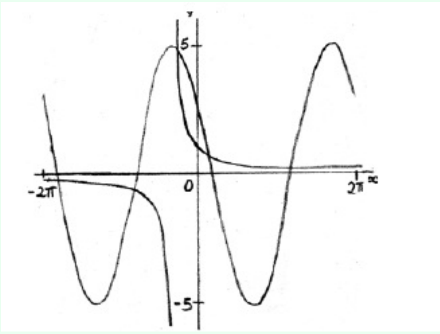
6f. Let $g(x) = \ln(x + 1)$, for $0 \le x \le \pi$. There is a value of *x*, between 0 and 1, for which the gradient of *f* is equal to the gradient of *g*. Find this value of *x*. *[5 marks]*

Markscheme

METHOD 1

graphical approach (but must involve derivative functions) ***M1***

e.g.



each curve ***A1A1***

$x = 0.511$ ***A2*** ***N2***

METHOD 2

$g'(x) = \frac{1}{x+1}$ ***A1***

$f'(x) = 3\cos x - 4\sin x$ $(5\cos(x + 0.927))$ ***A1***

evidence of attempt to solve $g'(x) = f'(x)$ ***M1***

$x = 0.511$ ***A2*** ***N2***

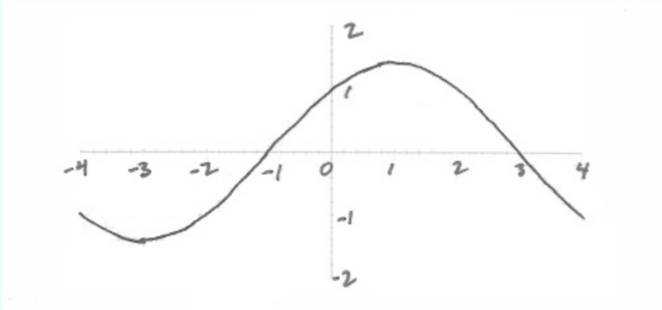
[5 marks]

Let

$f(x) = \cos\left(\frac{\pi}{4}x\right) + \sin\left(\frac{\pi}{4}x\right), \text{ for } -4 \leqslant x \leqslant 4.$

7a. Sketch the graph of f. *[3 marks]*

Markscheme



A1A1A1 ***N3***

Note: Award ***A1*** for approximately correct sinusoidal shape.

Only if this ***A1*** is awarded, award the following:

A1 for correct domain,

A1 for approximately correct range.

[3 marks]

7b. Find the values of x where the function is decreasing. *[5 marks]*

Markscheme

recognizes decreasing to the left of minimum or right of maximum,

eg $f'(x) < 0$ ***(R1)***

x-values of minimum and maximum (may be seen on sketch in part (a)) ***(A1)(A1)***

eg $x = -3,\ (1, 1.4)$

two correct intervals ***A1A1*** ***N5***

eg $-4 < x < -3,\ 1 \leqslant x \leqslant 4;\ x < -3,\ x \geqslant 1$

[5 marks]

7c. The function f can also be written in the form $f(x) = a\sin\left(\frac{\pi}{4}(x+c)\right)$, where $a \in \mathbb{R}$, and $0 \leqslant c \leqslant 2$. Find the value of a; *[3 marks]*

Markscheme

recognizes that a is found from amplitude of wave ***(R1)***

y-value of minimum or maximum ***(A1)***

eg (–3, –1.41) , (1, 1.41)

$a = 1.41421$

$a = \sqrt{2},\ \ \text{(exact)}, 1.41,$ ***A1 N3***

[3 marks]

7d. The function f can also be written in the form $f(x) = a\sin\left(\frac{\pi}{4}(x+c)\right)$, where $a \in \mathbb{R}$, and $0 \leqslant c \leqslant 2$. Find the value of c. *[4 marks]*

Markscheme

METHOD 1

recognize that shift for sine is found at *x*-intercept ***(R1)***

attempt to find *x*-intercept ***(M1)***

eg $\cos\left(\frac{\pi}{4}x\right) + \sin\left(\frac{\pi}{4}x\right) = 0,\ x = 3 + 4k,\ k \in \mathbb{Z}$

$x = -1$ ***(A1)***

$c = 1$ ***A1 N4***

METHOD 2

attempt to use a coordinate to make an equation ***(R1)***

eg $\sqrt{2}\sin\left(\frac{\pi}{4}c\right) = 1,\ \sqrt{2}\sin\left(\frac{\pi}{4}(3-c)\right) = 0$

attempt to solve resulting equation ***(M1)***

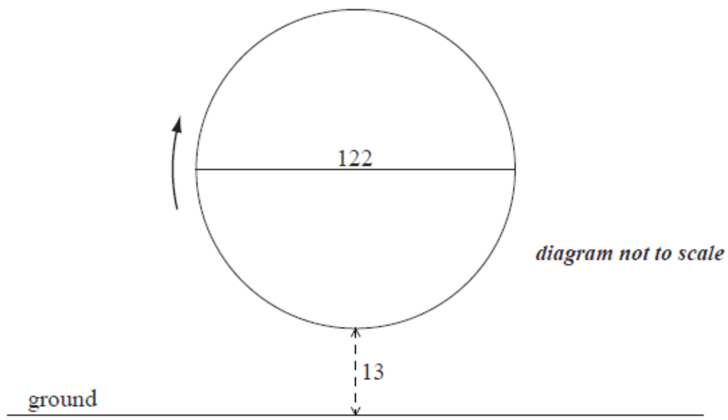
eg sketch, $x = 3 + 4k,\ k \in \mathbb{Z}$

$x = -1$ ***(A1)***

$c = 1$ ***A1 N4***

[4 marks]

A Ferris wheel with diameter
122 metres rotates clockwise at a constant speed.The wheel completes
2.4 rotations every hour. The bottom of the wheel is
13 metres above the ground.



A seat starts at the bottom of the wheel.

8a. Find the maximum height above the ground of the seat. *[2 marks]*

Markscheme

valid approach ***(M1)***

eg $13 + \text{diameter}$, $13 + 122$

maximum height $= 135$ (m) ***A1 N2***

[2 marks]

After ***t*** minutes, the height
h metres above the ground of the seat is given by

$$h = 74 + a\cos bt.$$

8b. (i) Show that the period of h is 25 minutes. *[2 marks]*

(ii) Write down the **exact** value of b .

Markscheme

(i) period $= \frac{60}{2.4}$ ***A1***

period $= 25$ minutes ***AG N0***

(ii) $b = \frac{2\pi}{25}$ $(= 0.08\pi)$ ***A1 N1***

[2 marks]

8c. Find the value of a . *[3 marks]*

Markscheme

METHOD 1

valid approach ***(M1)***

eg $\max - 74$, $|a| = \frac{135-13}{2}$, $74 - 13$

$|a| = 61$ (accept $a = 61$) ***(A1)***

$a = -61$ ***A1*** ***N2***

METHOD 2

attempt to substitute valid point into equation for ***h*** ***(M1)***

eg $135 = 74 + a\cos\left(\frac{2\pi\times12.5}{25}\right)$

correct equation ***(A1)***

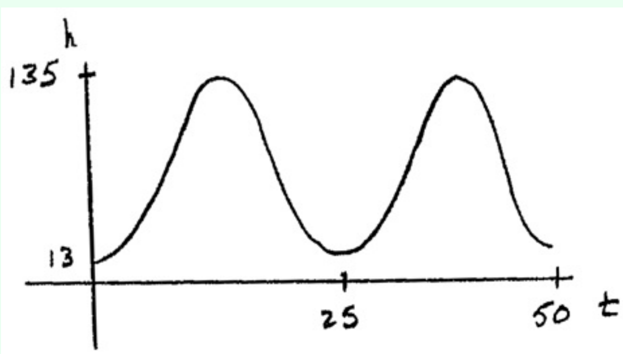
eg $135 = 74 + a\cos(\pi)$, $13 = 74 + a$

$a = -61$ ***A1*** ***N2***

[3 marks]

8d. Sketch the graph of h , for $0 \le t \le 50$. *[4 marks]*

Markscheme



A1A1A1A1 ***N4***

Note: Award ***A1*** for approximately correct domain, ***A1*** for approximately correct range, ***A1*** for approximately correct sinusoidal shape with 2 cycles.

Only if this last ***A1*** awarded, award ***A1*** for max/min in approximately correct positions.

[4 marks]

8e. In one rotation of the wheel, find the probability that a randomly selected seat is at least 105 metres above the ground. *[5 marks]*

Markscheme

setting up inequality (accept equation) ***(M1)***

eg $h > 105$, $105 = 74 + a\cos bt$, sketch of graph with line $y = 105$

any **two** correct values for ***t*** (seen anywhere) ***A1A1***

eg $t = 8.371\ldots$, $t = 16.628\ldots$, $t = 33.371\ldots$, $t = 41.628\ldots$

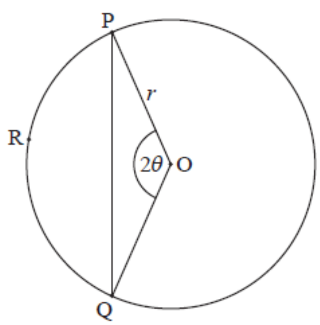
valid approach ***M1***

eg $\frac{16.628-8.371}{25}$, $\frac{t_1-t_2}{25}$, $\frac{2\times 8.257}{50}$, $\frac{2(12.5-8.371)}{25}$

$p = 0.330$ ***A1*** ***N2***

[5 marks]

Consider the following circle with centre O and radius$r$.



The points P, R and Q are on the circumference,
$\mathrm{P\hat{O}Q} = 2\theta$, for
$0 < \theta < \frac{\pi}{2}$.

9a. Use the cosine rule to show that $\mathrm{PQ} = 2r\sin\theta$. *[4 marks]*

Markscheme

correct substitution into cosine rule ***A1***

e.g. $\mathrm{PQ}^2 = r^2 + r^2 - 2(r)(r)\cos(2\theta)$, $\mathrm{PQ}^2 = 2r^2 - 2r^2(\cos(2\theta))$

substituting $1 - 2\sin^2\theta$ for $\cos 2\theta$ (seen anywhere) ***A1***

e.g. $\mathrm{PQ}^2 = 2r^2 - 2r^2(1 - 2\sin^2\theta)$

working towards answer ***(A1)***

e.g. $\mathrm{PQ}^2 = 2r^2 - 2r^2 + 4r^2\sin^2\theta$

recognizing $2r^2 - 2r^2 = 0$ (including crossing out) (seen anywhere)

e.g. $\mathrm{PQ}^2 = 4r^2\sin^2\theta$, $\mathrm{PQ} = \sqrt{4r^2\sin^2\theta}$

$\mathrm{PQ} = 2r\sin\theta$ ***AG*** ***N0***

[4 marks]

9b. Let *l* be the length of the arc PRQ . *[5 marks]*

Given that $1.3\text{PQ} - l = 0$, find the value of θ .

Markscheme

$\text{PRQ} = r \times 2\theta$ (seen anywhere) ***(A1)***

correct set up ***A1***

e.g. $1.3 \times 2r\sin\theta - r \times (2\theta) = 0$

attempt to eliminate *r* ***(M1)***

correct equation in terms of the one variable θ ***(A1)***

e.g. $1.3 \times 2\sin\theta - 2\theta = 0$

1.221496215

$\theta = 1.22$ (accept $70.0°$ (69.9)) ***A1*** ***N3***

[5 marks]

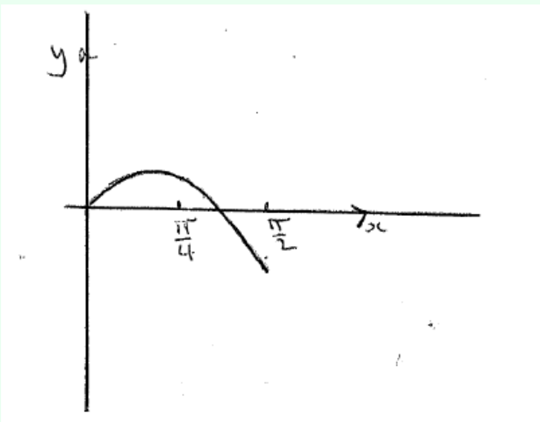
9c. Consider the function $f(\theta) = 2.6\sin\theta - 2\theta$, for $0 < \theta < \frac{\pi}{2}$. *[4 marks]*

(i) Sketch the graph of *f* .

(ii) Write down the root of $f(\theta) = 0$.

Markscheme

(i)



A1A1A1 ***N3***

Note: Award ***A1*** for approximately correct shape, ***A1*** for *x*-intercept in approximately correct position, ***A1*** for domain. Do not penalise if sketch starts at origin.

(ii) 1.221496215

$\theta = 1.22$ ***A1*** ***N1***

[4 marks]

9d. Use the graph of f to find the values of θ for which $l < 1.3\mathrm{PQ}$. *[3 marks]*

Markscheme

evidence of appropriate approach (may be seen earlier) ***M2***

e.g. $2\theta < 2.6\sin\theta$, $0 < f(\theta)$, showing positive part of sketch

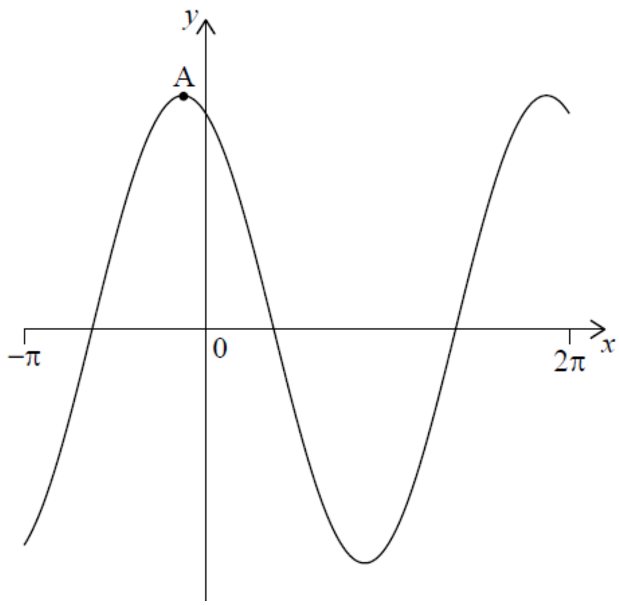
$0 < \theta < 1.221496215$

$0 < \theta = 1.22$ (accept $\theta < 1.22$) ***A1 N1***

[3 marks]

Let $f(x) = 12\cos x - 5\sin x,\ -\pi \leqslant x \leqslant 2\pi$, be a periodic function with $f(x) = f(x + 2\pi)$

The following diagram shows the graph of f.



There is a maximum point at A. The minimum value of f is –13 .

10a. Find the coordinates of A. *[2 marks]*

Markscheme

–0.394791,13

A(–0.395, 13) ***A1A1 N2***

[2 marks]

10b. For the graph of f, write down the amplitude. *[1 mark]*

Markscheme

13 ***A1 N1***

[1 mark]

10c. For the graph of f, write down the period. *[1 mark]*

Markscheme

2π, 6.28 ***A1 N1***

[1 mark]

10d. Hence, write $f(x)$ in the form $p \cos(x + r)$. *[3 marks]*

Markscheme

valid approach ***(M1)***

eg recognizing that amplitude is p or shift is r

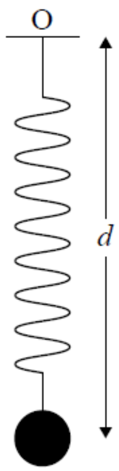
$f(x) = 13 \cos(x + 0.395)$ (accept p = 13, r = 0.395) ***A1A1 N3***

Note: Accept any value of r of the form $0.395 + 2\pi k,\ k \in \mathbb{Z}$

[3 marks]

A ball on a spring is attached to a fixed point O. The ball is then pulled down and released, so that it moves back and forth vertically.

diagram not to scale



The distance, d centimetres, of the centre of the ball from O at time t seconds, is given by $d(t) = f(t) + 17,\ 0 \leqslant t \leqslant 5.$

10e. Find the maximum speed of the ball. *[3 marks]*

Markscheme

recognizing need for $d'(t)$ ***(M1)***

eg $-12\sin(t) - 5\cos(t)$

correct approach (accept any variable for t) ***(A1)***

eg $-13\sin(t + 0.395)$, sketch of d', $(1.18, -13)$, $t = 4.32$

maximum speed = 13 (cms^{-1}) ***A1 N2***

[3 marks]

10f. Find the first time when the ball's speed is changing at a rate of $2\ \text{cm s}^{-2}$. *[5 marks]*

Markscheme

recognizing that acceleration is needed ***(M1)***

eg $a(t)$, $d''(t)$

correct equation (accept any variable for t) ***(A1)***

eg $a(t) = -2,\ \left|\frac{\mathrm{d}}{\mathrm{d}t}(d'(t))\right| = 2,\ -12\cos(t) + 5\sin(t) = -2$

valid attempt to solve **their** equation ***(M1)***

eg sketch, 1.33

1.02154

1.02 ***A2 N3***

[5 marks]

Note: In this question, distance is in millimetres.

Let $f(x) = x + a\sin\left(x - \frac{\pi}{2}\right) + a$, for $x \geqslant 0$.

11a. Show that $f(2\pi) = 2\pi$. *[3 marks]*

Markscheme

substituting $x = 2\pi$ ***M1***

eg $2\pi + a\sin\left(2\pi - \frac{\pi}{2}\right) + a$

$2\pi + a\sin\left(\frac{3\pi}{2}\right) + a$ ***(A1)***

$2\pi - a + a$ ***A1***

$f(2\pi) = 2\pi$ ***AG*** ***N0***

[3 marks]

The graph of f passes through the origin. Let P_k be any point on the graph of f with x-coordinate $2k\pi$, where $k \in \mathbb{N}$. A straight line L passes through all the points P_k.

11b. Find the coordinates of P_0 and of P_1. *[3 marks]*

Markscheme

substituting the value of k ***(M1)***

$\mathrm{P}_0(0,\ 0),\ \mathrm{P}_1(2\pi,\ 2\pi)$ ***A1A1*** ***N3***

[3 marks]

11c. Find the equation of L. *[3 marks]*

Markscheme

attempt to find the gradient ***(M1)***

eg $\frac{2\pi - 0}{2\pi - 0},\ m = 1$

correct working ***(A1)***

eg $\frac{y - 2\pi}{x - 2\pi} = 1,\ b = 0,\ y - 0 = 1(x - 0)$

y = *x* ***A1*** ***N3***

[3 marks]

11d. Show that the distance between the x-coordinates of P_k and P_{k+1} is 2π. *[2 marks]*

Markscheme

subtracting x-coordinates of P_{k+1} and P_k (in any order) ***(M1)***

eg $2(k+1)\pi - 2k\pi,\ 2k\pi - 2k\pi - 2\pi$

correct working (must be in correct order) ***A1***

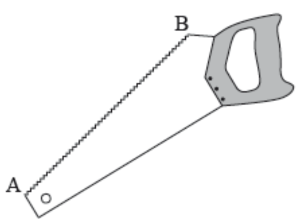
eg $2k\pi + 2\pi - 2k\pi,\ |2k\pi - 2(k+1)\pi|$

distance is 2π ***AG*** ***N0***

[2 marks]

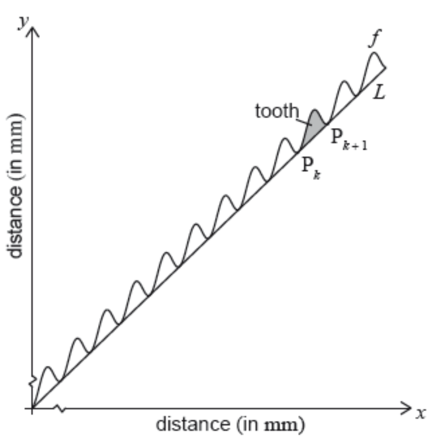
Diagram 1 shows a saw. The length of the toothed edge is the distance AB.

Diagram 1



The toothed edge of the saw can be modelled using the graph of f and the line L. Diagram 2 represents this model.

Diagram 2



The shaded part on the graph is called a tooth. A tooth is represented by the region enclosed by the graph of f and the line L, between P_k and P_{k+1}.

11e. A saw has a toothed edge which is 300 mm long. Find the number of complete teeth on this saw. *[6 marks]*

Markscheme

METHOD 1

recognizing the toothed-edge as the hypotenuse ***(M1)***

eg $300^2 = x^2 + y^2$, sketch

correct working (using their equation of L ***(A1)***

eg $300^2 = x^2 + x^2$

$x = \frac{300}{\sqrt{2}}$ (exact), 212.132 ***(A1)***

dividing their value of x by 2π $\left(\text{do not accept } \frac{300}{2\pi}\right)$ ***(M1)***

eg $\frac{212.132}{2\pi}$

33.7618 ***(A1)***

33 (teeth) ***A1 N2***

METHOD 2

vertical distance of a tooth is 2π (may be seen anywhere) ***(A1)***

attempt to find the hypotenuse for one tooth ***(M1)***

eg $x^2 = (2\pi)^2 + (2\pi)^2$

$x = \sqrt{8\pi^2}$ (exact), 8.88576 ***(A1)***

dividing 300 by their value of x ***(M1)***

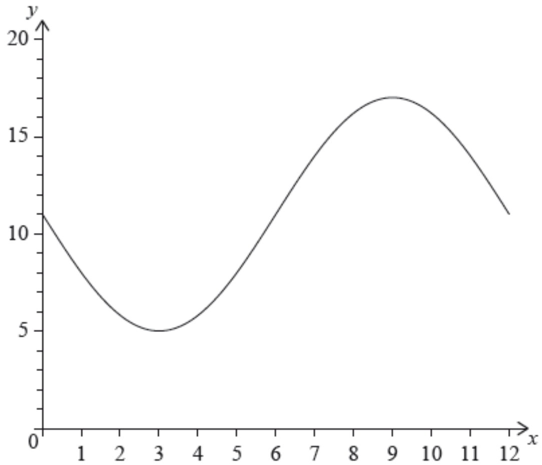
eg

33.7618 ***(A1)***

33 (teeth) ***A1 N2***

[6 marks]

The following diagram shows the graph of $f(x) = a\sin bx + c$, for $0 \leqslant x \leqslant 12$.



The graph of f has a minimum point at $(3,\ 5)$ and a maximum point at $(9,\ 17)$.

12a. (i) Find the value of c. *[6 marks]*

(ii) Show that $b = \frac{\pi}{6}$.

(iii) Find the value of a.

Markscheme

(i) valid approach ***(M1)***

eg $\frac{5+17}{2}$

$c = 11$ ***A1 N2***

(ii) valid approach ***(M1)***

eg period is 12, per $= \frac{2\pi}{b}$, $9 - 3$

$b = \frac{2\pi}{12}$ ***A1***

$b = \frac{\pi}{6}$ ***AG N0***

(iii) **METHOD 1**

valid approach ***(M1)***

eg $5 = a\sin\left(\frac{\pi}{6} \times 3\right) + 11$, substitution of points

$a = -6$ ***A1 N2***

METHOD 2

valid approach ***(M1)***

eg $\frac{17-5}{2}$, amplitude is 6

$a = -6$ ***A1 N2***

[6 marks]

The graph of g is obtained from the graph of f by a translation of $\begin{pmatrix} k \\ 0 \end{pmatrix}$. The maximum point on the graph of g has coordinates $(11.5,\ 17)$.

12b. (i) Write down the value of k. *[3 marks]*

(ii) Find $g(x)$.

Markscheme

(i)

$k = 2.5$ ***A1*** ***N1***

(ii)

$g(x) = -6\sin\left(\frac{\pi}{6}(x - 2.5)\right) + 11$ ***A2*** ***N2***

[3 marks]

The graph of g changes from concave-up to concave-down when $x = w$.

12c. (i) Find w. *[6 marks]*

(ii) Hence or otherwise, find the maximum positive rate of change of g.

Markscheme

(i) **METHOD 1** Using g

recognizing that a point of inflexion is required ***M1***

eg sketch, recognizing change in concavity

evidence of valid approach ***(M1)***

eg $g''(x) = 0$, sketch, coordinates of max/min on g'

$w = 8.5$ (exact) ***A1*** ***N2***

METHOD 2 Using f

recognizing that a point of inflexion is required ***M1***

eg sketch, recognizing change in concavity

evidence of valid approach involving translation ***(M1)***

eg $x = w - k$, sketch, $6 + 2.5$

$w = 8.5$ (exact) ***A1*** ***N2***

(ii) valid approach involving the derivative of g or f (seen anywhere) ***(M1)***

eg $g'(w)$, $-\pi\cos\left(\frac{\pi}{6}x\right)$, max on derivative, sketch of derivative

attempt to find max value on derivative ***M1***

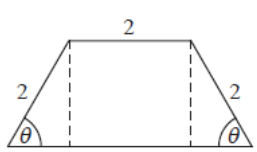
eg $-\pi\cos\left(\frac{\pi}{6}(8.5-2.5)\right)$, $f'(6)$, dot on max of sketch

3.14159

max rate of change $= \pi$ (exact), 3.14 ***A1*** ***N2***

[6 marks]

The diagram below shows a plan for a window in the shape of a trapezium.



Three sides of the window are
2 m long. The angle between the sloping sides of thewindow and the base is
θ , where
$0 < \theta < \frac{\pi}{2}$.

13a. Show that the area of the window is given by $y = 4\sin\theta + 2\sin 2\theta$. *[5 marks]*

Markscheme

evidence of finding height, h ***(A1)***

e.g. $\sin\theta = \frac{h}{2}$, $2\sin\theta$

evidence of finding base of triangle, b ***(A1)***

e.g. $\cos\theta = \frac{b}{2}$, $2\cos\theta$

attempt to substitute valid values into a formula for the area of the window ***(M1)***

e.g. two triangles plus rectangle, trapezium area formula

correct expression (must be in terms of θ) ***A1***

e.g. $2\left(\frac{1}{2}\times 2\cos\theta \times 2\sin\theta\right) + 2\times 2\sin\theta$, $\frac{1}{2}(2\sin\theta)(2+2+4\cos\theta)$

attempt to replace $2\sin\theta\cos\theta$ by $\sin 2\theta$ ***M1***

e.g. $4\sin\theta + 2(2\sin\theta\cos\theta)$

$y = 4\sin\theta + 2\sin 2\theta$ ***AG*** ***N0***

[5 marks]

13b. Zoe wants a window to have an area of $5\ \mathrm{m}^2$. Find the two possible values of θ . *[4 marks]*

Markscheme

correct equation ***A1***

e.g. $y = 5$, $4\sin\theta + 2\sin 2\theta = 5$

evidence of attempt to solve ***(M1)***

e.g. a sketch, $4\sin\theta + 2\sin\theta - 5 = 0$

$\theta = 0.856\ (49.0°)$, $\theta = 1.25\ (71.4°)$ ***A1A1*** ***N3***

[4 marks]

13c. John wants two windows which have the same area A but different values of θ . *[7 marks]*

Find all possible values for A .

Markscheme

recognition that lower area value occurs at $\theta = \frac{\pi}{2}$ ***(M1)***

finding value of area at $\theta = \frac{\pi}{2}$ ***(M1)***

e.g. $4\sin\left(\frac{\pi}{2}\right) + 2\sin\left(2 \times \frac{\pi}{2}\right)$, draw square

$A = 4$ ***(A1)***

recognition that maximum value of y is needed ***(M1)***

$A = 5.19615\ldots$ ***(A1)***

$4 < A < 5.20$ (accept $4 < A < 5.19$) ***A2*** ***N5***

[7 marks]