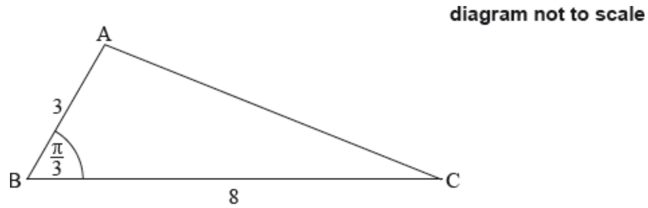
0417Trig functions *[102 marks]*

The following diagram shows triangle ABC, with $\mathrm{AB} = 3\ \mathrm{cm}$, $\mathrm{BC} = 8\ \mathrm{cm}$, and $\mathrm{A\hat{B}C} = \frac{\pi}{3}$.



1a. Show that $\mathrm{AC} = 7\ \mathrm{cm}$. *[4 marks]*

Markscheme

evidence of choosing the cosine rule **(M1)**

eg $c^2 = a^2 + b^2 - ab\cos C$

correct substitution into RHS of cosine rule **(A1)**

eg $3^2 + 8^2 - 2 \times 3 \times 8 \times \cos\frac{\pi}{3}$

evidence of correct value for $\cos\frac{\pi}{3}$ (may be seen anywhere, including in cosine rule) **A1**

eg $\cos\frac{\pi}{3} = \frac{1}{2}$, $\mathrm{AC}^2 = 9 + 64 - \left(48 \times \frac{1}{2}\right)$, $9 + 64 - 24$

correct working clearly leading to answer **A1**

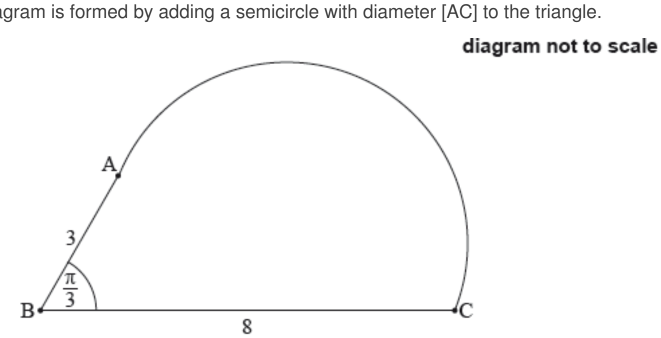
eg $\mathrm{AC}^2 = 49$, $b = \sqrt{49}$

$\mathrm{AC} = 7\ (\mathrm{cm})$ **AG** **N0**

Note: Award no marks if the only working seen is $\mathrm{AC}^2 = 49$ or $\mathrm{AC} = \sqrt{49}$ (or similar).

[4 marks]

1b. The shape in the following diagram is formed by adding a semicircle with diameter [AC] to the triangle. *[3 marks]*



Find the exact perimeter of this shape.

Markscheme

correct substitution for semicircle ***(A1)***

eg $\text{semicircle} = \frac{1}{2}(2\pi \times 3.5),\ \frac{1}{2} \times \pi \times 7,\ 3.5\pi$

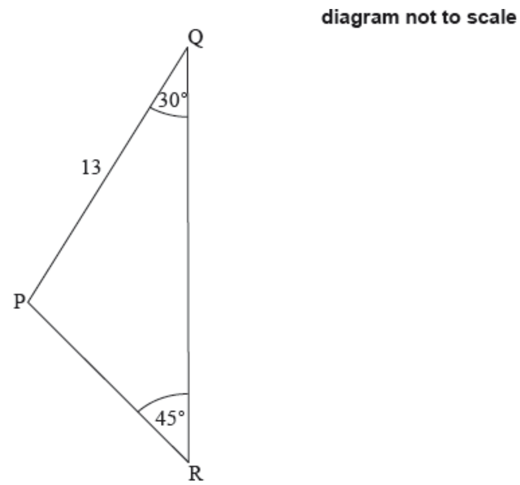
valid approach (seen anywhere) ***(M1)***

eg $\text{perimeter} = \text{AB} + \text{BC} + \text{semicircle},\ 3 + 8 + \left(\frac{1}{2} \times 2 \times \pi \times \frac{7}{2}\right),\ 8 + 3 + 3.5\pi$

$11 + \frac{7}{2}\pi\ (= 3.5\pi + 11)\ (\text{cm})$ ***A1*** ***N2***

[3 marks]

2. The following diagram shows triangle PQR. *[6 marks]*



$\hat{\text{PQR}} = 30°,\ \hat{\text{QRP}} = 45°$ and $\text{PQ} = 13\,\text{cm}$.

Find PR.

Markscheme

METHOD 1

evidence of choosing the sine rule ***(M1)***

eg $\frac{a}{\sin A} = \frac{b}{\sin B}$

correct substitution ***A1***

eg $\frac{x}{\sin 30} = \frac{13}{\sin 45}$, $\frac{13 \sin 30}{\sin 45}$

$\sin 30 = \frac{1}{2}$, $\sin 45 = \frac{1}{\sqrt{2}}$ ***(A1)(A1)***

correct working ***A1***

eg $\frac{1}{2} \times \frac{13}{\frac{1}{\sqrt{2}}}$, $\frac{1}{2} \times 13 \times \frac{2}{\sqrt{2}}$, $13 \times \frac{1}{2} \times \sqrt{2}$

correct answer ***A1 N3***

eg $\text{PR} = \frac{13\sqrt{2}}{2}$, $\frac{13}{\sqrt{2}}$ (cm)

METHOD 2 (using height of ΔPQR)

valid approach to find height of ΔPQR ***(M1)***

eg $\sin 30 = \frac{x}{13}$, $\cos 60 = \frac{x}{13}$

$\sin 30 = \frac{1}{2}$ or $\cos 60 = \frac{1}{2}$ ***(A1)***

$\text{height} = 6.5$ ***A1***

correct working ***A1***

eg $\sin 45 = \frac{6.5}{\text{PR}}$, $\sqrt{6.5^2 + 6.5^2}$

correct working ***(A1)***

eg $\sin 45 = \frac{1}{\sqrt{2}}$, $\cos 45 = \frac{1}{\sqrt{2}}$, $\sqrt{\frac{169 \times 2}{4}}$

correct answer ***A1 N3***

eg $\text{PR} = \frac{13\sqrt{2}}{2}$, $\frac{13}{\sqrt{2}}$ (cm)

[6 marks]

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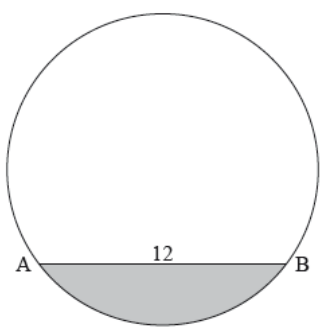
3. The following diagram shows the chord [AB] in a circle of radius 8 cm, where $\text{AB} = 12\text{ cm}$. *[7 marks]*

diagram not to scale



Find the area of the shaded segment.

Markscheme

attempt to find the central angle or half central angle *(M1)*

eg , cosine rule, right triangle

correct working *(A1)*

eg $\cos\theta = \frac{8^2+8^2-12^2}{2\bullet 8\bullet 8}$, $\sin^{-1}\left(\frac{6}{8}\right)$, $0.722734,\ 41.4096°,\ \frac{\pi}{2}-\sin^{-1}\left(\frac{6}{8}\right)$

correct angle $\mathrm{A\hat{O}B}$ (seen anywhere)

eg $1.69612,\ 97.1807°,\ 2\times\sin^{-1}\left(\frac{6}{8}\right)$ *(A1)*

correct sector area

eg $\frac{1}{2}(8)(8)(1.70),\ \frac{97.1807}{360}(64\pi),\ 54.2759$ *(A1)*

area of triangle (seen anywhere) *(A1)*

eg $\frac{1}{2}(8)(8)\sin 1.70,\ \frac{1}{2}(8)(12)\sin 0.722,\ \frac{1}{2}\times\sqrt{64-36}\times 12,\ 31.7490$

appropriate approach (seen anywhere) *(M1)*

eg $A_{\text{triangle}} - A_{\text{sector}}$, their sector-their triangle

22.5269

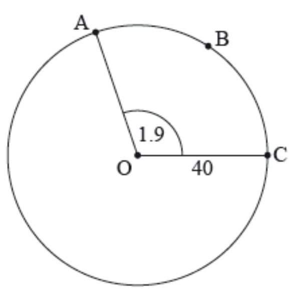
area of shaded region $= 22.5\ (\mathrm{cm}^2)$ ***A1*** ***N4***

Note: Award ***M0A0A0A0A1*** then ***M1A0*** (if appropriate) for correct triangle area without any attempt to find an angle in triangle OAB.

[7 marks]

The following diagram shows a circle with centre O and radius 40 cm.

diagram not to scale



The points A, B and C are on the circumference of the circle and $\mathrm{A\hat{O}C} = 1.9$ radians.

4a. Find the length of arc ABC. *[2 marks]*

Markscheme

correct substitution into arc length formula *(A1)*

eg $(40)(1.9)$

$\text{arc length} = 76\ (\text{cm})$ ***A1*** ***N2***

[2 marks]

4b. Find the perimeter of sector OABC. *[2 marks]*

Markscheme

valid approach ***(M1)***

eg $\text{arc} + 2r,\ 76 + 40 + 40$

$\text{perimeter} = 156\ (\text{cm})$ ***A1 N2***

[2 marks]

4c. Find the area of sector OABC. *[2 marks]*

Markscheme

correct substitution into area formula ***(A1)***

eg $\frac{1}{2}(1.9)(40)^2$

$\text{area} = 1520\ (\text{cm}^2)$ ***A1 N2***

[2 marks]

The depth of water in a port is modelled by the function $d(t) = p\cos qt + 7.5$, for $0 \leqslant t \leqslant 12$, where t is the number of hours after high tide.

At high tide, the depth is 9.7 metres.

At low tide, which is 7 hours later, the depth is 5.3 metres.

5a. Find the value of p. *[2 marks]*

Markscheme

valid approach ***(M1)***

eg $\frac{\text{max}-\text{min}}{2}$, sketch of graph, $9.7 = p\cos(0) + 7.5$

$p = 2.2$ ***A1 N2***

[2 marks]

5b. Find the value of q. *[2 marks]*

Markscheme

valid approach ***(M1)***

eg $B = \frac{2\pi}{\text{period}}$, period is 14, $\frac{360}{14}$, $5.3 = 2.2\cos 7q + 7.5$

0.448798

$q = \frac{2\pi}{14}\ \left(\frac{\pi}{7}\right)$, (do not accept degrees) ***A1 N2***

[2 marks]

5c. Use the model to find the depth of the water 10 hours after high tide. *[2 marks]*

Markscheme

valid approach ***(M1)***

eg $d(10),\ 2.2\cos\left(\frac{20\pi}{14}\right)+7.5$

7.01045

7.01 (m) ***A1*** ***N2***

[2 marks]

Let $\sin\theta=\frac{\sqrt{5}}{3}$, where θ is acute.

6a. Find $\cos\theta$. *[3 marks]*

Markscheme

evidence of valid approach ***(M1)***

eg
right triangle, $\cos^2\theta=1-\sin^2\theta$

correct working ***(A1)***

eg
missing side is 2,
$\sqrt{1-\left(\frac{\sqrt{5}}{3}\right)^2}$

$\cos\theta=\frac{2}{3}$ ***A1*** ***N2***

[3 marks]

6b. Find $\cos 2\theta$. *[2 marks]*

Markscheme

correct substitution into formula for $\cos 2\theta$ ***(A1)***

eg $2\times\left(\frac{2}{3}\right)^2-1,\ 1-2\left(\frac{\sqrt{5}}{3}\right)^2,\ \left(\frac{2}{3}\right)^2-\left(\frac{\sqrt{5}}{3}\right)^2$

$\cos 2\theta=-\frac{1}{9}$ ***A1*** ***N2***

[2 marks]

Let $f(x)=3\sin\left(\frac{\pi}{2}x\right)$, for $0\leqslant x\leqslant 4$.

7a. (i) Write down the amplitude of f. *[3 marks]*

(ii) Find the period of f.

Markscheme

(i) 3 ***A1*** ***N1***

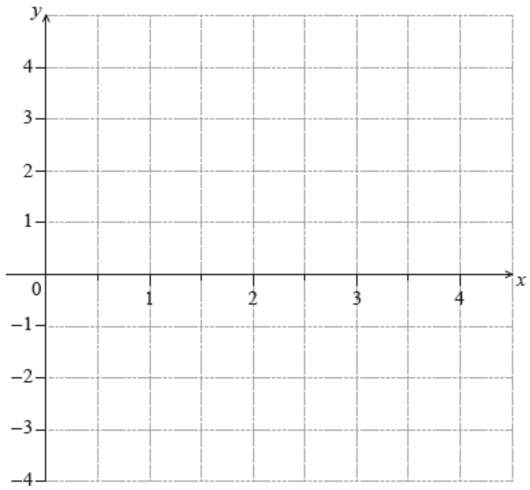
(ii) valid attempt to find the period ***(M1)***

eg $\frac{2\pi}{b},\ \frac{2\pi}{\frac{\pi}{2}}$

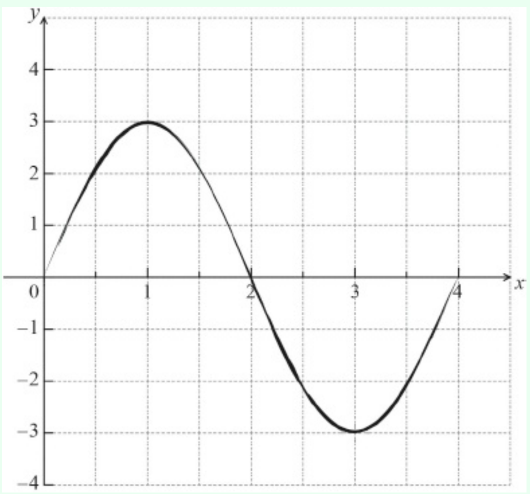
period $=4$ ***A1*** ***N2***

[3 marks]

7b. On the following grid sketch the graph of f. *[4 marks]*



Markscheme



A1A1A1A1 N4

[4 marks]

Let
$f(x) = 6x\sqrt{1-x^2}$, for $-1 \leqslant x \leqslant 1$, and
$g(x) = \cos(x)$, for $0 \leqslant x \leqslant \pi$.

Let $h(x) = (f \circ g)(x)$.

8a. Write $h(x)$ in the form $a\sin(bx)$, where $a,\ b \in \mathbb{Z}$. *[5 marks]*

Markscheme

attempt to form composite in any order ***(M1)***

eg $f(g(x)),\ \cos\left(6x\sqrt{1-x^2}\right)$

correct working ***(A1)***

eg $6\cos x\sqrt{1-\cos^2 x}$

correct application of Pythagorean identity (do not accept $\sin^2 x+\cos^2 x=1$) ***(A1)***

eg $\sin^2 x=1-\cos^2 x,\ 6\cos x\sin x,\ 6\cos x\left|\sin x\right|$

valid approach (do not accept $2\sin x\cos x=\sin 2x$) ***(M1)***

eg $3(2\cos x\sin x)$

$h(x)=3\sin 2x$ ***A1*** ***N3***

[5 marks]

8b. Hence find the range of h. *[2 marks]*

Markscheme

valid approach ***(M1)***

eg amplitude $=3$, sketch with max and min y-values labelled, $-3<y<3$

correct range ***A1*** ***N2***

eg

$-3\leqslant y\leqslant 3,\ [-3,\ 3]$ from -3 to 3

Note: Do not award ***A1*** for $-3<y<3$ or for "between -3 and 3".

[2 marks]

The height, h metros, of a seat on a Ferris wheel after t minutes is given by

$$h(t)=-15\cos 1.2t+17,\ \text{for } t\geqslant 0.$$

9a. Find the height of the seat when $t=0$. *[2 marks]*

Markscheme

valid approach ***(M1)***

eg $h(0),\ -15\cos(1.2\times 0)+17,\ -15(1)+17$

$h(0)=2\ (\text{m})$ ***A1*** ***N2***

[2 marks]

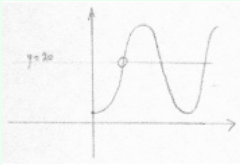
9b. The seat first reaches a height of 20 m after k minutes. Find k. *[3 marks]*

Markscheme

correct substitution into equation ***(A1)***

eg $20 = -15\cos 1.2t + 17,\ -15\cos 1.2k = 3$

valid attempt to solve for k ***(M1)***

eg 

, $\cos 1.2k = -\frac{3}{15}$

1.47679

$k = 1.48$ ***A1 N2***

[3 marks]

9c. Calculate the time needed for the seat to complete a full rotation, giving your answer correct to one decimal place. *[3 marks]*

Markscheme

recognize the need to find the period (seen anywhere) ***(M1)***

eg next t value when $h = 20$

correct value for period ***(A1)***

eg $\text{period} = \frac{2\pi}{1.2},\ 5.23598,\ 6.7 - -1.48$

5.2 (min) (must be 1 dp) ***A1 N2***

[3 marks]

Let $f(x) = 3\sin(\pi x)$.

10a. Write down the amplitude of f. *[1 mark]*

Markscheme

amplitude is 3 ***A1 N1***

10b. Find the period of f. *[2 marks]*

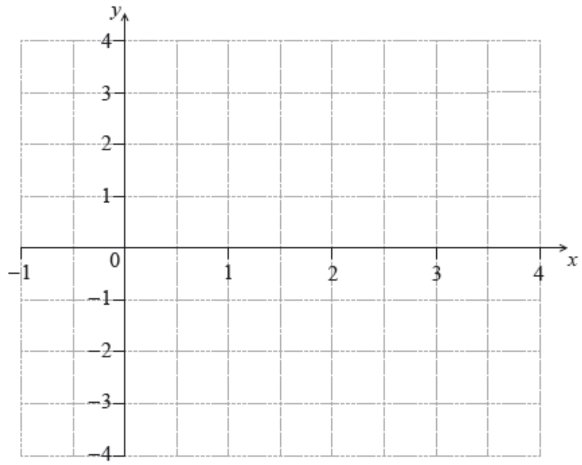
Markscheme

valid approach ***(M1)***

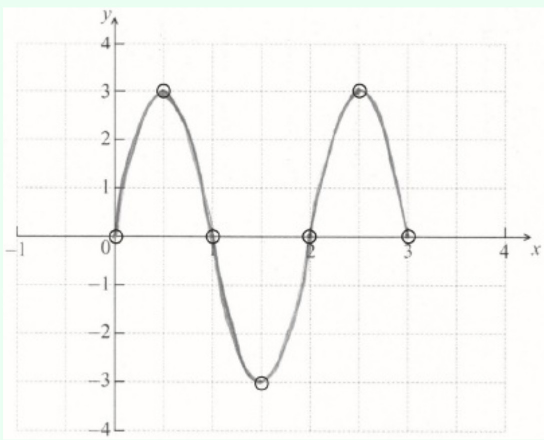
eg $\text{period} = \frac{2\pi}{\pi},\ \frac{360}{\pi}$

period is 2 ***A1 N2***

10c. On the following grid, sketch the graph of $y = f(x)$, for $0 \leq x \leq 3$. *[4 marks]*



Markscheme



A1

A1A1A1 ***N4***

Note: Award ***A1*** for sine curve starting at $(0, 0)$ and correct period.

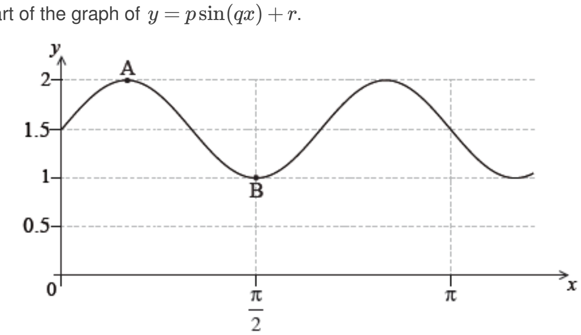
Only if this ***A1*** is awarded, award the following for points in circles:

A1 for correct *x*-intercepts;

A1 for correct max and min points;

A1 for correct domain.

The following diagram shows part of the graph of $y = p \sin(qx) + r$.



The point $\mathrm{A}\left(\frac{\pi}{6}, 2\right)$ is a maximum point and the point $\mathrm{B}\left(\frac{\pi}{6}, 1\right)$ is a minimum point.

Find the value of

11a. p; *[2 marks]*

Markscheme

valid approach *(M1)*

eg $\frac{2-1}{2},\ 2-1.5$

$p=0.5$ ***A1*** ***N2***

[2 marks]

11b. r; *[2 marks]*

Markscheme

valid approach *(M1)*

eg $\frac{1+2}{2}$

$r=1.5$ ***A1*** ***N2***

[2 marks]

11c. q. *[2 marks]*

Markscheme

METHOD 1

valid approach (seen anywhere) ***M1***

eg $q=\frac{2\pi}{\text{period}},\ \frac{2\pi}{\left(\frac{2\pi}{3}\right)}$

period $=\frac{2\pi}{3}$ (seen anywhere) ***(A1)***

$q=3$ ***A1*** ***N2***

METHOD 2

attempt to substitute one point and **their** values for p and r into y ***M1***

eg $2=0.5\sin\left(q\frac{\pi}{6}\right)+1.5,\ \frac{\pi}{2}=0.5\sin(q1)+1.5$

correct equation in q ***(A1)***

eg $q\frac{\pi}{6}=\frac{\pi}{2},\ q\frac{\pi}{2}=\frac{3\pi}{2}$

$q=3$ ***A1*** ***N2***

METHOD 3

valid reasoning comparing the graph with that of $\sin x$ ***R1***

eg position of max/min, graph goes faster

correct working ***(A1)***

eg max at $\frac{\pi}{6}$ not at $\frac{\pi}{2}$, graph goes 3 times as fast

$q=3$ ***A1*** ***N2***

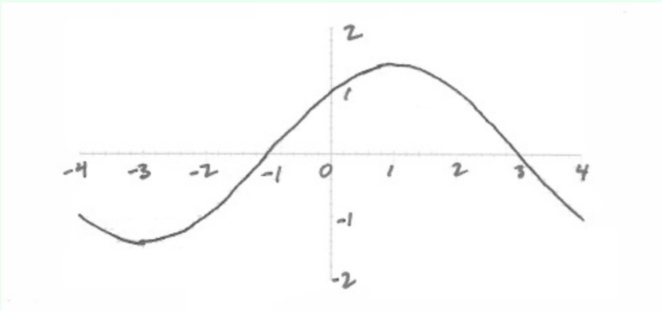
[3 marks]

Total [7 marks]

Let
$f(x) = \cos\left(\frac{\pi}{4}x\right) + \sin\left(\frac{\pi}{4}x\right), \text{ for } -4 \leqslant x \leqslant 4.$

12a. Sketch the graph of f. *[3 marks]*

Markscheme



A1A1A1 N3

Note: Award ***A1*** for approximately correct sinusoidal shape.

Only if this ***A1*** is awarded, award the following:

A1 for correct domain,

A1 for approximately correct range.

[3 marks]

12b. Find the values of x where the function is decreasing. *[5 marks]*

Markscheme

recognizes decreasing to the left of minimum or right of maximum,

eg

$f'(x) < 0$ ***(R1)***

x-values of minimum and maximum (may be seen on sketch in part (a)) ***(A1)(A1)***

eg

$x = -3,\ (1,\ 1.4)$

two correct intervals ***A1A1 N5***

eg

$-4 < x < -3,\ 1 \leqslant x \leqslant 4;\ x < -3,\ x \geqslant 1$

[5 marks]

12c. The function f can also be written in the form $f(x) = a\sin\left(\frac{\pi}{4}(x+c)\right)$, where $a \in \mathbb{R}$, and $0 \leqslant c \leqslant 2$. Find the value of a; *[3 marks]*

Markscheme

recognizes that

a is found from amplitude of wave ***(R1)***

y-value of minimum or maximum ***(A1)***

eg (–3, –1.41) , (1, 1.41)

$a = 1.41421$

$a = \sqrt{2}, \quad (\text{exact}), 1.41,$ ***A1 N3***

[3 marks]

12d. The function f can also be written in the form $f(x) = a\sin\left(\frac{\pi}{4}(x+c)\right)$, where $a \in \mathbb{R}$, and $0 \leqslant c \leqslant 2$. Find the value of c.

[4 marks]

Markscheme

METHOD 1

recognize that shift for sine is found at *x*-intercept ***(R1)***

attempt to find *x*-intercept ***(M1)***

eg

$\cos\left(\frac{\pi}{4}x\right) + \sin\left(\frac{\pi}{4}x\right) = 0,\ x = 3 + 4k,\ k \in \mathbb{Z}$

$x = -1$ ***(A1)***

$c = 1$ ***A1 N4***

METHOD 2

attempt to use a coordinate to make an equation ***(R1)***

eg

$\sqrt{2}\sin\left(\frac{\pi}{4}c\right) = 1,\ \sqrt{2}\sin\left(\frac{\pi}{4}(3-c)\right) = 0$

attempt to solve resulting equation ***(M1)***

eg sketch,

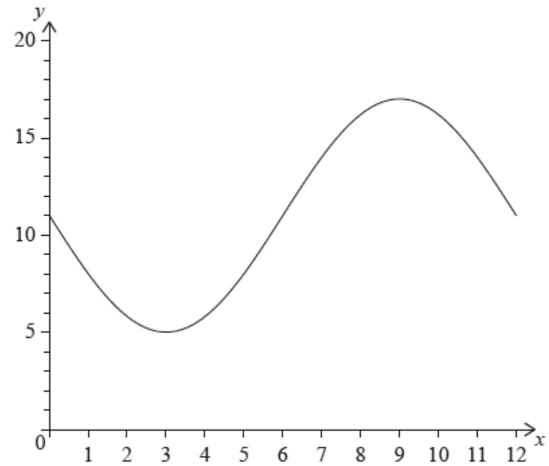
$x = 3 + 4k,\ k \in \mathbb{Z}$

$x = -1$ ***(A1)***

$c = 1$ ***A1 N4***

[4 marks]

The following diagram shows the graph of $f(x) = a\sin bx + c$, for $0 \leqslant x \leqslant 12$.



The graph of f has a minimum point at $(3,\ 5)$ and a maximum point at $(9,\ 17)$.

13a. (i) Find the value of c. *[6 marks]*

(ii) Show that $b = \frac{\pi}{6}$.

(iii) Find the value of a.

Markscheme

(i) valid approach ***(M1)***

eg $\frac{5+17}{2}$

$c = 11$ ***A1 N2***

(ii) valid approach ***(M1)***

eg period is 12, per $= \frac{2\pi}{b}$, $9 - 3$

$b = \frac{2\pi}{12}$ ***A1***

$b = \frac{\pi}{6}$ ***AG N0***

(iii) **METHOD 1**

valid approach ***(M1)***

eg

$5 = a\sin\left(\frac{\pi}{6} \times 3\right) + 11$, substitution of points

$a = -6$ ***A1 N2***

METHOD 2

valid approach ***(M1)***

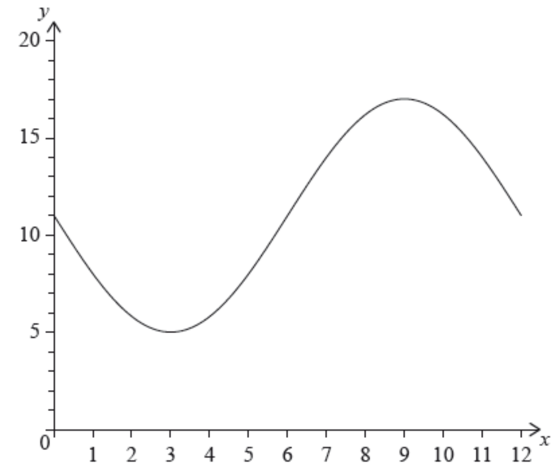
eg

$\frac{17-5}{2}$, amplitude is 6

$a = -6$ ***A1 N2***

[6 marks]

The following diagram shows the graph of $f(x) = a\sin bx + c$, for $0 \leqslant x \leqslant 12$.



The graph of f has a minimum point at $(3,\ 5)$ and a maximum point at $(9,\ 17)$.

The graph of g is obtained from the graph of f by a translation of $\begin{pmatrix} k \\ 0 \end{pmatrix}$. The maximum point on the graph of g has coordinates $(11.5,\ 17)$.

13b. (i) Write down the value of k. *[3 marks]*

(ii) Find $g(x)$.

Markscheme

(i)

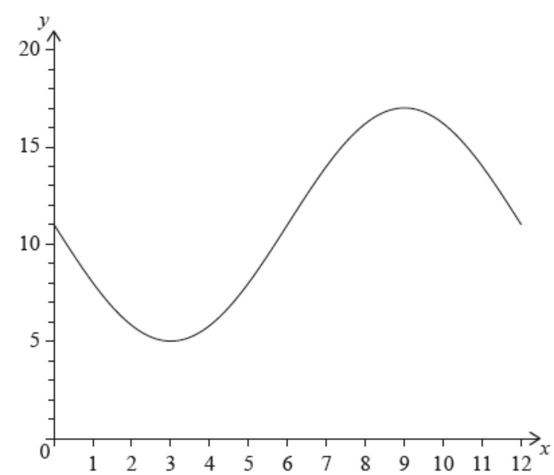
$k = 2.5$ ***A1*** ***N1***

(ii)

$g(x) = -6\sin\left(\frac{\pi}{6}(x - 2.5)\right) + 11$ ***A2*** ***N2***

[3 marks]

The following diagram shows the graph of $f(x) = a\sin bx + c$, for $0 \leqslant x \leqslant 12$.



The graph of f has a minimum point at $(3,\ 5)$ and a maximum point at $(9,\ 17)$.

The graph of g changes from concave-up to concave-down when $x = w$.

13c. (i) Find w. *[6 marks]*

(ii) Hence or otherwise, find the maximum positive rate of change of g.

Markscheme

(i) **METHOD 1** Using g

recognizing that a point of inflexion is required ***M1***

eg
sketch, recognizing change in concavity

evidence of valid approach ***(M1)***

eg

$g''(x) = 0$, sketch, coordinates of max/min on g'

$w = 8.5$ (exact) ***A1*** ***N2***

METHOD 2 Using f

recognizing that a point of inflexion is required ***M1***

eg sketch, recognizing change in concavity

evidence of valid approach involving translation ***(M1)***

eg

$x = w - k$, sketch, $6 + 2.5$

$w = 8.5$ (exact) ***A1*** ***N2***

(ii) valid approach involving the derivative of g or f (seen anywhere) ***(M1)***

eg

$g'(w),\ -\pi\cos\left(\frac{\pi}{6}x\right)$, max on derivative, sketch of derivative

attempt to find max value on derivative ***M1***

eg

$-\pi\cos\left(\frac{\pi}{6}(8.5 - 2.5)\right),\ f'(6)$, dot on max of sketch

3.14159

max rate of change $= \pi$ (exact), 3.14 ***A1*** ***N2***

[6 marks]