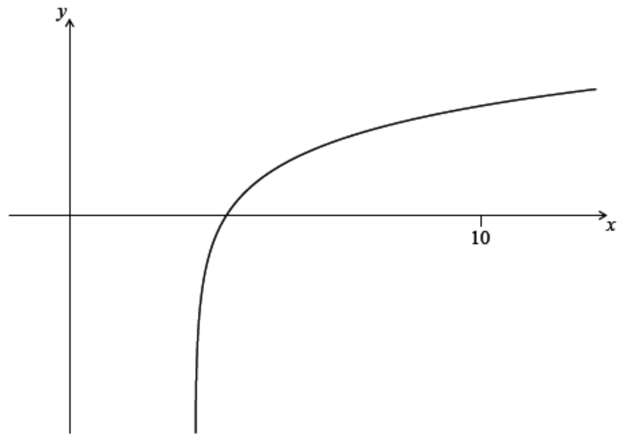
0416HW-integration-solids-rotation *[103 marks]*

Let $f(x) = 2\ln(x-3)$, for $x > 3$. The following diagram shows part of the graph of f.



1a. Find the equation of the vertical asymptote to the graph of f. *[2 marks]*

Markscheme

valid approach ***(M1)***

eg horizontal translation 3 units to the right

$x = 3$ (must be an equation) ***A1 N2***

[2 marks]

1b. Find the x-intercept of the graph of f. *[2 marks]*

Markscheme

valid approach ***(M1)***

eg $f(x) = 0,\ e^0 = x - 3$

$4,\ x = 4,\ (4,\ 0)$ ***A1 N2***

[2 marks]

1c. The region enclosed by the graph of f, the x-axis and the line $x = 10$ is rotated $360°$ about the x-axis. Find the volume of the solid formed. *[3 marks]*

Markscheme

attempt to substitute either **their correct** limits or the function into formula involving f^2 ***(M1)***

eg $\int_4^{10} f^2,\ \pi\int(2\ln(x-3))^2\mathrm{d}x$

141.537

volume = 142 ***A2 N3***

[3 marks]

Total [7 marks]

Let $f(x) = -x^4 + 2x^3 - 1$, for $0 \leq x \leq 2$.

2a. Sketch the graph of f on the following grid. *[3 marks]*

Markscheme

A1A1A1 ***N3***

Note: Award ***A1*** for both endpoints in circles,

A1 for approximately correct shape (concave up to concave down).

Only if this ***A1*** for shape is awarded, award ***A1*** for maximum point in circle.

2b. Solve $f(x) = 0$. *[2 marks]*

Markscheme

$x = 1 \quad x = 1.83928$

$x = 1 \text{ (exact)} \quad x = 1.84 \; [1.83,\ 1.84]$ ***A1A1*** ***N2***

[2 marks]

2c. The region enclosed by the graph of f and the x-axis is rotated 360° about the x-axis. *[3 marks]*

Find the volume of the solid formed.

Markscheme

attempt to substitute either (***FT***) limits or function into formula with f^2 ***(M1)***

eg $V = \pi \int_1^{1.84} f^2,\ \int \left(-x^4 + 2x^3 - 1\right)^2 \mathrm{d}x$

0.636581

$V = 0.637 \; [0.636,\ 0.637]$ ***A2*** ***N3***

[3 marks]

Total [8 marks]

Let
$f(x) = x^2$.

3a. Find
$\int_1^2 \left(f(x)\right)^2 \mathrm{d}x$. [no calculator on this problem] *[4 marks]*

Markscheme

substituting for
$(f(x))^2$ (may be seen in integral) ***A1***

eg
$(x^2)^2, x^4$

correct integration,
$\int x^4 \mathrm{d}x = \frac{1}{5}x^5$ ***(A1)***

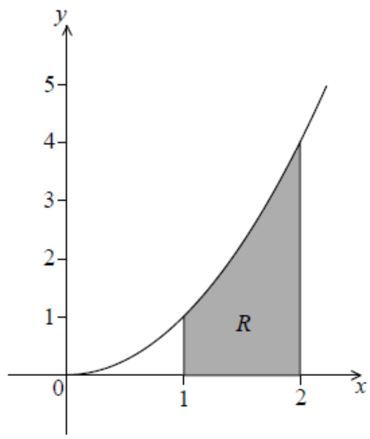
substituting limits into **their integrated** function and subtracting (in any order) ***(M1)***

eg
$\frac{2^5}{5} - \frac{1}{5}, \frac{1}{5}(1-4)$

$\int_1^2 (f(x))^2 \mathrm{d}x = \frac{31}{5} (= 6.2)$ ***A1 N2***

[4 marks]

3b. The following diagram shows part of the graph of f. *[2 marks]*



The shaded region R is enclosed by the graph of f, the x-axis and the lines $x = 1$ and $x = 2$.

Find the volume of the solid formed when R is revolved $360°$ about the x-axis.

Markscheme

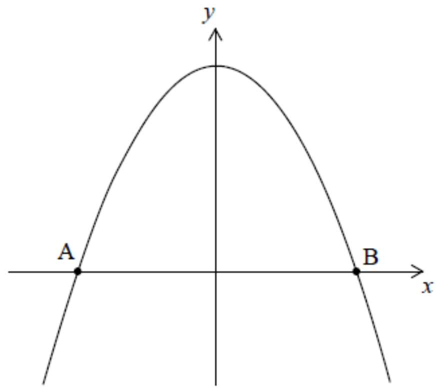
attempt to substitute limits or function into formula involving
f^2 ***(M1)***

eg
$\int_1^2 (f(x))^2 \mathrm{d}x, \pi \int x^4 \mathrm{d}x$

$\frac{31}{5}\pi (= 6.2\pi)$ ***A1 N2***

[2 marks]

Let
$f(x) = 5 - x^2$. Part of the graph of
$f$is shown in the following diagram.



The graph crosses the
x-axis at the points
A and
$\mathrm{B}.$

4a. Find the
x-coordinate of
A and of
$\mathrm{B}.$ *[3 marks]*

Markscheme

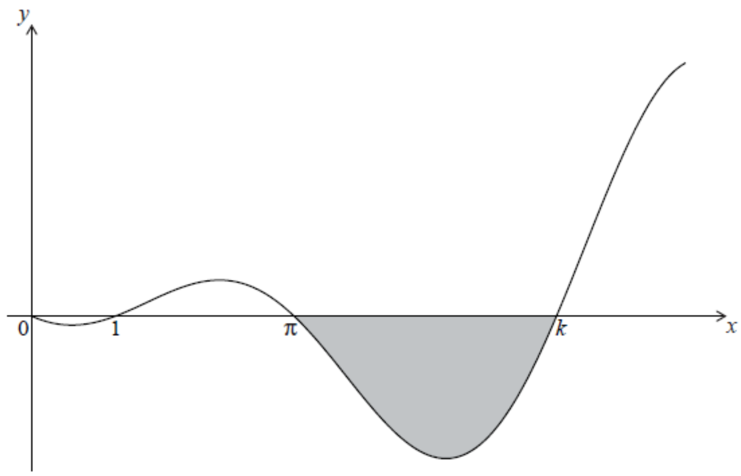
recognizing
$f(x) = 0$ ***(M1)***
eg
$f = 0,\ x^2 = 5$
$x = \pm 2.23606$
$x = \pm\sqrt{5}\,(\mathrm{exact}), x = \pm 2.24$ ***A1A1*** ***N3***
[3 marks]

4b. The region enclosed by the graph of
f and the
x-axis is revolved
360° about the
x-axis.
Find the volume of the solid formed. *[3 marks]*

Markscheme

attempt to substitute either limits or the function into formula involving
f^2 ***(M1)***
eg
$\pi\int\left(5 - x^2\right)^2 \mathrm{d}x,\ \pi\int_{-2.24}^{2.24}\left(x^4 - 10x^2 + 25\right),\ 2\pi\int_0^{\sqrt{5}} f^2$
187.328
volume
$= 187$ ***A2*** ***N3***
[3 marks]

The graph of $y = (x - 1)\sin x$, for $0 \le x \le \frac{5\pi}{2}$, is shown below.



The graph has x-intercepts at 0, 1, π and k.

5a. Find *k* . *[2 marks]*

Markscheme

evidence of valid approach ***(M1)***

e.g. $y = 0$, $\sin x = 0$

$2\pi = 6.283185\ldots$

$k = 6.28$ ***A1*** ***N2***

[2 marks]

5b. The shaded region is rotated 360° about the *x*-axis. Let *V* be the volume of the solid formed. *[3 marks]*

Write down an expression for *V* .

Markscheme

attempt to substitute either limits or the function into formula ***(M1)***

(accept absence of
$\mathrm{d}x$)

e.g.
$V = \pi \int_{\pi}^{k} (f(x))^2 \mathrm{d}x$,
$\pi \int ((x-1)\sin x)^2$,
$\pi \int_{\pi}^{6.28\ldots} y^2 \mathrm{d}x$

correct expression ***A2 N3***

e.g.
$\pi \int_{\pi}^{6.28} (x-1)^2 \sin^2 x \mathrm{d}x$,
$\pi \int_{\pi}^{2\pi} ((x-1)\sin x)^2 \mathrm{d}x$

[3 marks]

5c. The shaded region is rotated
360° about the *x*-axis. Let *V* be the volume of the solid formed. *[2 marks]*

Find *V* .

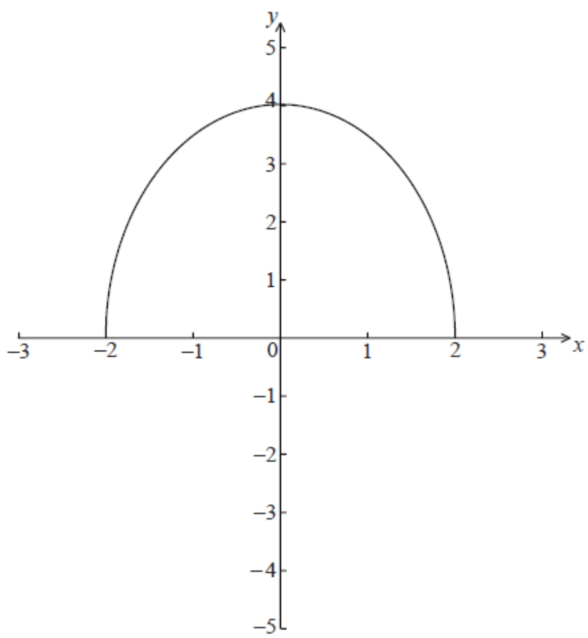
Markscheme

$V = 69.60192562\ldots$

$V = 69.6$ ***A2 N2***

[2 marks]

The graph of
$f(x) = \sqrt{16 - 4x^2}$, for
$-2 \le x \le 2$, is shown below.



6. The region enclosed by the curve of *f* and the *x*-axis is rotated
360° about the *x*-axis. [no calculator on this problem] *[6 marks]*

Find the volume of the solid formed.

Markscheme

attempt to set up integral expression ***M1***

e.g.
$\pi\int\sqrt{16-4x^2}^2\mathrm{d}x$,
$2\pi\int_0^2(16-4x^2)$,
$\int\sqrt{16-4x^2}^2\mathrm{d}x$

$\int 16\mathrm{d}x = 16x$,
$\int 4x^2\mathrm{d}x = \frac{4x^3}{3}$ (seen anywhere) ***A1A1***

evidence of substituting limits into the integrand ***(M1)***

e.g.
$\left(32-\frac{32}{3}\right)-\left(-32+\frac{32}{3}\right)$,
$64-\frac{64}{3}$

volume
$=\frac{128\pi}{3}$ ***A2*** ***N3***

[6 marks]

Let
$f(x)=\sqrt{x}$. Line *L* is the normal to the graph of *f* at the point (4, 2) .

7a. Show that the equation of *L* is
$y=-4x+18$. [no calculator on this problem] *[4 marks]*

Markscheme

finding derivative ***(A1)***

e.g.
$f'(x)=\frac{1}{2}x^{\frac{1}{2}}, \frac{1}{2\sqrt{x}}$

correct value of derivative or its negative reciprocal (seen anywhere) ***A1***

e.g.
$\frac{1}{2\sqrt{4}}$,
$\frac{1}{4}$

gradient of normal =
$\frac{1}{\text{gradient of tangent}}$ (seen anywhere) ***A1***

e.g.
$-\frac{1}{f'(4)}=-4$,
$-2\sqrt{x}$

substituting into equation of line (for normal) ***M1***

e.g.
$y-2=-4(x-4)$

$y=-4x+18$ ***AG*** ***N0***

[4 marks]

7b. Point A is the *x*-intercept of *L* . Find the *x*-coordinate of A. *[2 marks]*

Markscheme

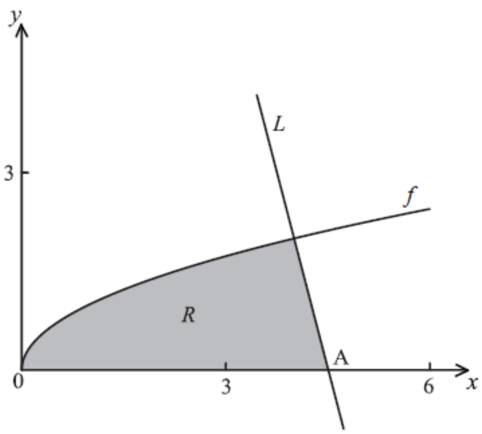
recognition that
$y = 0$ at A ***(M1)***

e.g.
$-4x + 18 = 0$

$x = \frac{18}{4}$
$\left(= \frac{9}{2}\right)$ ***A1*** ***N2***

[2 marks]

In the diagram below, the shaded region *R* is bounded by the *x*-axis, the graph of *f* and the line *L* .



7c. Find an expression for the area of *R* . *[3 marks]*

Markscheme

splitting into two appropriate parts (areas and/or integrals) ***(M1)***

correct expression for area of *R* ***A2*** ***N3***

e.g. area of *R* =
$\int_0^4 \sqrt{x}\mathrm{d}x + \int_4^{4.5} (-4x + 18)\mathrm{d}x$,
$\int_0^4 \sqrt{x}\mathrm{d}x + \frac{1}{2} \times 0.5 \times 2$ (triangle)

Note: Award ***A1*** if d*x* is missing.

[3 marks]

7d. The region *R* is rotated
360° about the *x*-axis. Find the volume of the solid formed,giving your answer in terms of
π . *[8 marks]*

Markscheme

correct expression for the volume from
$x = 0$ to
$x = 4$ ***(A1)***

e.g.
$V = \int_0^4 \pi \left[f(x)^2\right] \mathrm{d}x$,
$\int_0^4 \pi \sqrt{x}^2 \mathrm{d}x$,
$\int_0^4 \pi x \mathrm{d}x$

$V = \left[\frac{1}{2}\pi x^2\right]_0^4$ ***A1***

$V = \pi\left(\frac{1}{2} \times 16 - \frac{1}{2} \times 0\right)$ ***(A1)***

$V = 8\pi$ ***A1***

finding the volume from
$x = 4$ to
$x = 4.5$

EITHER

recognizing a cone ***(M1)***

e.g.
$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi(2)^2 \times \frac{1}{2}$ ***(A1)***

$= \frac{2\pi}{3}$ ***A1***

total volume is
$8\pi + \frac{2}{3}\pi$
$\left(= \frac{26}{3}\pi\right)$ ***A1*** ***N4***

OR

$V = \pi \int_4^{4.5} (-4x + 18)^2 \mathrm{d}x$ ***(M1)***

$= \int_4^{4.5} \pi(16x^2 - 144x + 324)\mathrm{d}x$

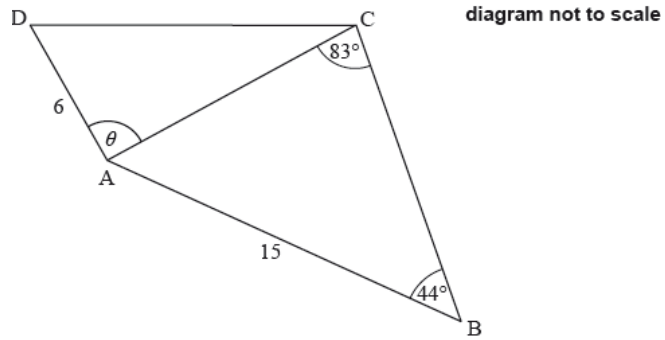
$= \pi\left[\frac{16}{3}x^3 - 72x^2 + 324x\right]_4^{4.5}$ ***A1***

$= \frac{2\pi}{3}$ ***A1***

total volume is
$8\pi + \frac{2}{3}\pi$
$\left(= \frac{26}{3}\pi\right)$ ***A1*** ***N4***

[8 marks]

The following diagram shows the quadrilateral $ABCD$.



$$\mathrm{AD} = 6\ \mathrm{cm},\ \mathrm{AB} = 15\ \mathrm{cm}, \mathrm{A\hat{B}C} = 44^\circ, \mathrm{A\hat{C}B} = 83^\circ \text{and} \mathrm{D\hat{A}C} = \theta$$

8a. Find AC. *[3 marks]*

Markscheme

evidence of choosing sine rule ***(M1)***

eg $\frac{\mathrm{AC}}{\sin \mathrm{C\hat{B}A}} = \frac{\mathrm{AB}}{\sin \mathrm{A\hat{C}B}}$

correct substitution ***(A1)***

eg $\frac{\mathrm{AC}}{\sin 44^\circ} = \frac{15}{\sin 83^\circ}$

10.4981

$\mathrm{AC} = 10.5\ (\mathrm{cm})$ ***A1 N2***

[3 marks]

8b. Find the area of triangle ABC. *[3 marks]*

Markscheme

finding $\mathrm{C\hat{A}B}$ (seen anywhere) ***(A1)***

eg $180^\circ - 44^\circ - 83^\circ, \mathrm{C\hat{A}B} = 53^\circ$

correct substitution for area of triangle ABC ***A1***

eg $\frac{1}{2} \times 15 \times 10.4981 \times \sin 53^\circ$

62.8813

$\mathrm{area} = 62.9\ (\mathrm{cm}^2)$ ***A1 N2***

[3 marks]

8c. The area of triangle ACD is half the area of triangle ABC. *[5 marks]*

Find the possible values of θ.

Markscheme

correct substitution for area of triangle DAC ***(A1)***

eg $\frac{1}{2} \times 6 \times 10.4981 \times \sin\theta$

attempt to equate area of triangle ACD to half the area of triangle ABC ***(M1)***

eg $\text{area ACD} = \frac{1}{2} \times \text{ area ABC; 2ACD} = \text{ABC}$

correct equation ***A1***

eg $\frac{1}{2} \times 6 \times 10.4981 \times \sin\theta = \frac{1}{2}(62.9),\ 62.9887\sin\theta = 62.8813,\ \sin\theta = 0.998294$

$86.6531,\ 93.3468$

$\theta = 86.7^\circ$, $\theta = 93.3^\circ$ ***A1A1*** ***N2***

[5 marks]

8d. Given that θ is obtuse, find CD. *[3 marks]*

Markscheme

Note: Note: If candidates use an acute angle from part (c) in the cosine rule, award ***M1A0A0*** in part (d).

evidence of choosing cosine rule ***(M1)***

eg $\text{CD}^2 = \text{AD}^2 + \text{AC}^2 - 2 \times \text{AD} \times \text{AC} \times \cos\theta$

correct substitution into rhs ***(A1)***

eg $\text{CD}^2 = 6^2 + 10.498^2 - 2(6)(10.498)\cos 93.336^\circ$

12.3921

$12.4\ \text{(cm)}$ ***A1*** ***N2***

[3 marks]

Total [14 marks]

Let L_x be a family of lines with equation given by $r = \begin{pmatrix} x \\ \frac{2}{x} \end{pmatrix} + t\begin{pmatrix} x^2 \\ -2 \end{pmatrix}$, where $x > 0$.

9a. Write down the equation of L_1. *[2 marks]*

Markscheme

attempt to substitute $x = 1$ ***(M1)***

eg $\boldsymbol{r} = \begin{pmatrix} 1 \\ \frac{2}{1} \end{pmatrix} + t\begin{pmatrix} 1^2 \\ -2 \end{pmatrix},\ L_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

correct equation (vector or Cartesian, but do not accept "L_1")

eg $\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \end{pmatrix},\ y = -2x + 4$ (must be an equation) ***A1*** ***N2***

[2 marks]

9b. A line L_a crosses the y-axis at a point P. *[6 marks]*

Show that P has coordinates $\left(0,\ \frac{4}{a}\right)$.

Markscheme

appropriate approach *(M1)*

eg $\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} a \\ \frac{2}{a} \end{pmatrix} + t\begin{pmatrix} a^2 \\ -2 \end{pmatrix}$

correct equation for x-coordinate ***A1***

eg $0 = a + ta^2$

$t = \frac{-1}{a}$ ***A1***

substituting **their** parameter to find y *(M1)*

eg $y = \frac{2}{a} - 2\left(\frac{-1}{a}\right)$, $\begin{pmatrix} a \\ \frac{2}{a} \end{pmatrix} - \frac{1}{a}\begin{pmatrix} a^2 \\ -2 \end{pmatrix}$

correct working ***A1***

eg $y = \frac{2}{a} + \frac{2}{a}$, $\begin{pmatrix} a \\ \frac{2}{a} \end{pmatrix} - \begin{pmatrix} a \\ -\frac{2}{a} \end{pmatrix}$

finding correct expression for y ***A1***

eg $y = \frac{4}{a}$, $\begin{pmatrix} 0 \\ \frac{4}{a} \end{pmatrix}$ $\mathrm{P}\left(0,\ \frac{4}{a}\right)$ ***AG*** ***N0***

[6 marks]

9c. The line L_a crosses the x-axis at $\mathrm{Q}(2a,\ 0)$. Let $d = \mathrm{PQ}^2$. *[2 marks]*

Show that $d = 4a^2 + \frac{16}{a^2}$.

Markscheme

valid approach ***M1***

eg distance formula, Pythagorean Theorem, $\overrightarrow{\mathrm{PQ}} = \begin{pmatrix} 2a \\ -\frac{4}{a} \end{pmatrix}$

correct simplification ***A1***

eg $(2a)^2 + \left(\frac{4}{a}\right)^2$

$d = 4a^2 + \frac{16}{a^2}$ ***AG*** ***N0***

[2 marks]

9d. There is a minimum value for d. Find the value of a that gives this minimum value. *[7 marks]*

Markscheme

recognizing need to find derivative ***(M1)***

eg $d',\ d'(a)$

correct derivative ***A2***

eg $8a - \frac{32}{a^3},\ 8x - \frac{32}{x^3}$

setting **their** derivative equal to 0 ***(M1)***

eg $8a - \frac{32}{a^3} = 0$

correct working ***(A1)***

eg $8a = \frac{32}{a^3},\ 8a^4 - 32 = 0$

working towards solution ***(A1)***

eg $a^4 = 4,\ a^2 = 2,\ a = \pm\sqrt{2}$

$a = \sqrt[4]{4} \quad (a = \sqrt{2}) \quad (\text{do not accept } \pm\sqrt{2})$ ***A1 N3***

[7 marks]

Total [17 marks]

The first two terms of a geometric sequence u_n are $u_1 = 4$ and $u_2 = 4.2$.

10a. (i) Find the common ratio. *[5 marks]*

(ii) Hence or otherwise, find u_5.

Markscheme

(i) valid approach ***(M1)***

eg $r = \frac{u_2}{u_1},\ \frac{4}{4.2}$

$r = 1.05 \quad (\text{exact})$ ***A1 N2***

(ii) attempt to substitute into formula, with **their** r ***(M1)***

eg $4 \times 1.05^n,\ 4 \times 1.05 \times 1.05 \ldots$

correct substitution ***(A1)***

eg $4 \times 1.05^4,\ 4 \times 1.05 \times 1.05 \times 1.05 \times 1.05$

$u_5 = 4.862025\ (\text{exact}),\ 4.86\ [4.86,\ 4.87]$ ***A1 N2***

[5 marks]

10b. Another sequence v_n is defined by $v_n = an^k$, where $a,\ k \in \mathbb{R}$, and $n \in \mathbb{Z}^+$, such that $v_1 = 0.05$ and $v_2 = 0.25$. *[5 marks]*

(i) Find the value of a.

(ii) Find the value of k.

Markscheme

(i) attempt to substitute $n = 1$ ***(M1)***

eg $0.05 = a \times 1^k$

$a = 0.05$ ***A1 N2***

(ii) correct substitution of $n = 2$ into v_2 ***A1***

eg $0.25 = a \times 2^k$

correct work ***(A1)***

eg finding intersection point, $k = \log_2\left(\frac{0.25}{0.05}\right), \frac{\log 5}{\log 2}$

2.32192

$k = \log_2 5 \quad (\text{exact}), 2.32\ [2.32,\ 2.33]$ ***A1 N2***

[5 marks]

10c. Find the smallest value of n for which $v_n > u_n$. *[5 marks]*

Markscheme

correct expression for u_n ***(A1)***

eg $4 \times 1.05^{n-1}$

EITHER

correct substitution into inequality (accept equation) ***(A1)***

eg $0.05 \times n^k > 4 \times 1.05^{n-1}$

valid approach to solve inequality (accept equation) ***(M1)***

eg finding point of intersection, $n = 7.57994\ (7.59508 \text{ from } 2.32)$

$n = 8$ (must be an integer) ***A1 N2***

OR

table of values

when $n = 7,\ u_7 = 5.3604,\ v_7 = 4.5836$ ***A1***

when $n = 8,\ u_8 = 5.6284,\ v_8 = 6.2496$ ***A1***

$n = 8$ (must be an integer) ***A1 N2***

[4 marks]

Total [14 marks]