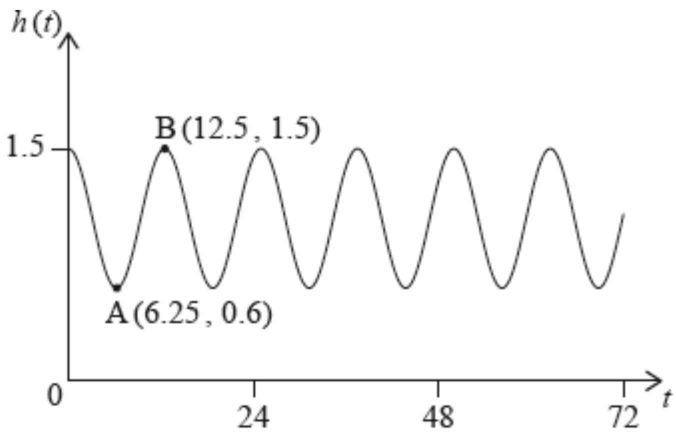
BECA / Huson / 12.1 IB Math SL Name:
3.4 Spiral Review Periodic Function Trig (Calculator, Paper 2)

3.4 Periodic-functions, trigonometry SPICY (Paper 2, with calculator)

1a. *[2 marks]*

At Grande Anse Beach the height of the water in metres is modelled by the function $h(t) = p\cos(q \times t) + r$, where t is the number of hours after 21:00 hours on 10 December 2017. The following diagram shows the graph of h, for $0 \leqslant t \leqslant 72$.



The point $\mathrm{A}(6.25,\ 0.6)$ represents the first low tide and $\mathrm{B}(12.5,\ 1.5)$ represents the next high tide.

How much time is there between the first low tide and the next high tide?

1b. *[2 marks]*

Find the difference in height between low tide and high tide.

1c. *[2 marks]*

Find the value of p;

1d. *[3 marks]*

Find the value of q;

1e. *[2 marks]*

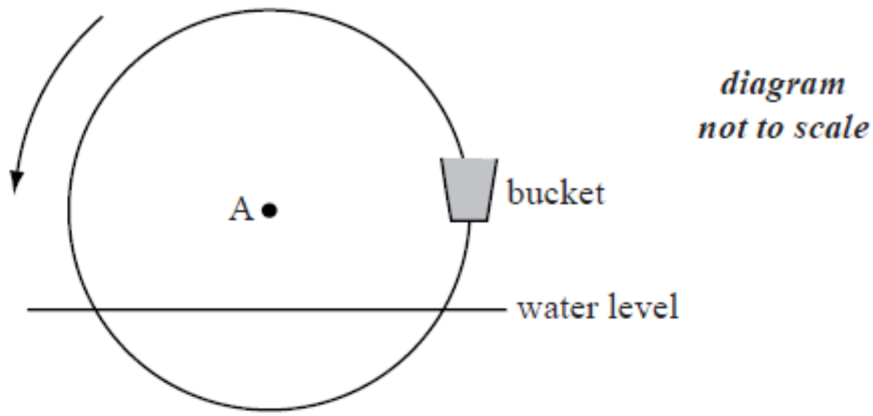
Find the value of r.

1f. *[3 marks]*

There are two high tides on 12 December 2017. At what time does the second high tide occur?

2a. *[2 marks]*

The following diagram shows a waterwheel with a bucket. The wheel rotates at a constant rate in an anticlockwise (counter-clockwise) direction.



The diameter of the wheel is 8 metres. The centre of the wheel, A, is 2 metres above the water level. After t seconds, the height of the bucket above the water level is given by $h = a \sin bt + 2$.

Show that $a = 4$.

2b. *[2 marks]*

The wheel turns at a rate of one rotation every 30 seconds.

Show that $b = \frac{\pi}{15}$.

2c. *[6 marks]*

In the first rotation, there are two values of t when the bucket is **descending** at a rate of 0.5 ms^{-1} .

Find these values of t .

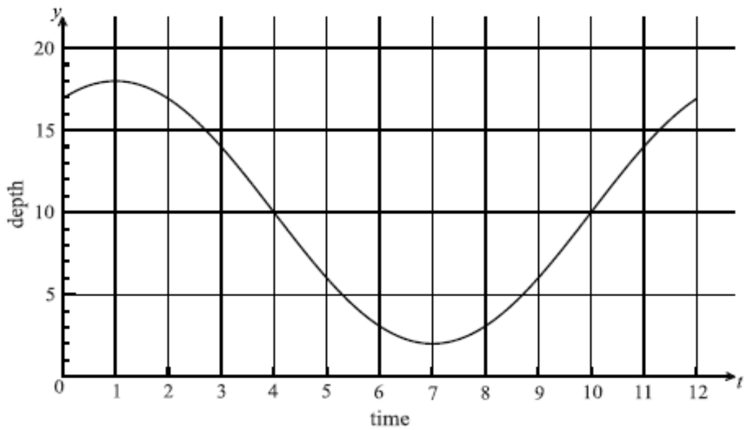
2d. *[4 marks]*

In the first rotation, there are two values of t when the bucket is **descending** at a rate of 0.5 ms^{-1} .

Determine whether the bucket is underwater at the second value of t .

3a. *[3 marks]*

The following graph shows the depth of water, *y* metres , at a point P, during one day. The time *t* is given in hours, from midnight to noon.



Use the graph to write down an estimate of the value of *t* when

(i) the depth of water is minimum;

(ii) the depth of water is maximum;

(iii) the depth of the water is increasing most rapidly.

3b. *[6 marks]*

The depth of water can be modelled by the function $y = \cos A(B(t-1)) + C$.

(i) Show that $A = 8$.

(ii) Write down the value of *C*.

(iii) Find the value of *B*.

3c. *[2 marks]*

A sailor knows that he cannot sail past P when the depth of the water is less than 12 m . Calculate the values of *t* between which he cannot sail past P.

4a. *[3 marks]*

Let $f(x) = 5\cos\frac{\pi}{4}x$ and $g(x) = -0.5x^2 + 5x - 8$ for $0 \leq x \leq 9$.

On the same diagram, sketch the graphs of *f* and *g* .

4b. *[4 marks]*

Consider the graph of f . Write down

(i) the *x*-intercept that lies between $x = 0$ and $x = 3$;

(ii) the period;

(iii) the amplitude.

4c. *[3 marks]*

Consider the graph of *g* . Write down

(i) the two *x*-intercepts;

(ii) the equation of the axis of symmetry.

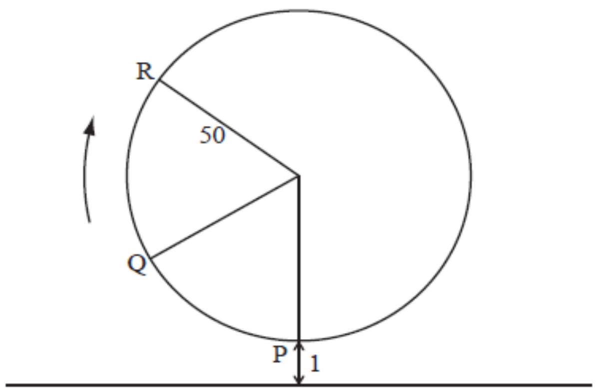
4d. *[5 marks]*

Let *R* be the region enclosed by the graphs of *f* and *g* . Find the area of *R*.

5a. *[2 marks]*

The following diagram represents a large Ferris wheel at an amusement park.

The points P, Q and R represent different positions of a seat on the wheel.



The wheel has a radius of 50 metres and rotates clockwise at a rate of one revolution every 30 minutes.

A seat starts at the lowest point P, when its height is one metre above the ground.

Find the height of a seat above the ground after 15 minutes.

5b. *[5 marks]*

After six minutes, the seat is at point Q. Find its height above the ground at Q.

5c. *[6 marks]*

The height of the seat above ground after t minutes can be modelled by the function $h(t) = 50\sin(b(t - c)) + 51$.

Find the value of b and of c .

5d. *[3 marks]*

The height of the seat above ground after t minutes can be modelled by the function $h(t) = 50\sin(b(t - c)) + 51$.

Hence find the value of t the first time the seat is 96 m above the ground.

6a. *[3 marks]*

Let $f(x) = 3\sin x + 4\cos x$, for $-2\pi \leq x \leq 2\pi$.

Sketch the graph of *f*.

6b. *[3 marks]*

Write down

(i) the amplitude;

(ii) the period;

(iii) the *x*-intercept that lies between $-\frac{\pi}{2}$ and 0.

6c. *[3 marks]*

Hence write $f(x)$ in the form $p\sin(qx + r)$.

6d. *[2 marks]*

Write down one value of *x* such that $f'(x) = 0$.

6e. *[2 marks]*

Write down the two values of *k* for which the equation $f(x) = k$ has exactly two solutions.

6f. *[5 marks]*

Let $g(x) = \ln(x + 1)$, for $0 \leq x \leq \pi$. There is a value of *x*, between 0 and 1, for which the gradient of *f* is equal to the gradient of *g*. Find this value of *x*.

7a. *[3 marks]*

Let $f(x) = \cos\left(\frac{\pi}{4}x\right) + \sin\left(\frac{\pi}{4}x\right),\ \text{for } -4 \leqslant x \leqslant 4.$

Sketch the graph of f.

7b. *[5 marks]*

Find the values of x where the function is decreasing.

7c. *[3 marks]*

The function f can also be written in the form $f(x) = a\sin\left(\frac{\pi}{4}(x+c)\right)$, where $a \in \mathbb{R}$, and $0 \leqslant c \leqslant 2$

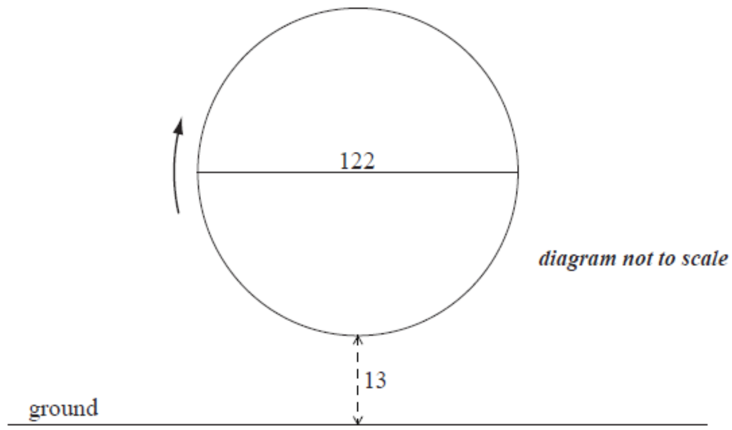
. Find the value of a;

7d. *[4 marks]*

The function f can also be written in the form $f(x) = a\sin\left(\frac{\pi}{4}(x+c)\right)$, where $a \in \mathbb{R}$, and $0 \leqslant c \leqslant 2$

. Find the value of c.

8a. A Ferris wheel with diameter 122 metres rotates clockwise at a constant speed. The wheel completes 2.4 rotations every hour. The bottom of the wheel is 13 metres above the ground.



A seat starts at the bottom of the wheel.

Find the maximum height above the ground of the seat. *[2 marks]*

8b. *[2 marks]*

After t minutes, the height h metres above the ground of the seat is given by

$$h = 74 + a\cos bt.$$

(i) Show that the period of h is 25 minutes.

(ii) Write down the **exact** value of b.

8c. *[3 marks]*

Find the value of a.

8d. *[4 marks]*

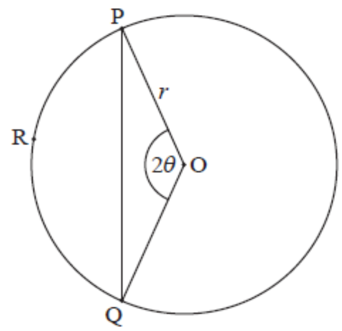
Sketch the graph of h, for $0 \le t \le 50$.

8e. *[5 marks]*

In one rotation of the wheel, find the probability that a randomly selected seat is at least 105 metres above the ground.

9a. *[4 marks]*

Consider the following circle with centre O and radius *r* .



The points P, R and Q are on the circumference, $\mathrm{P\hat{O}Q} = 2\theta$, for $0 < \theta < \frac{\pi}{2}$.

Use the cosine rule to show that $\mathrm{PQ} = 2r\sin\theta$.

9b. *[5 marks]*

Let *l* be the length of the arc PRQ .

Given that $1.3\mathrm{PQ} - l = 0$, find the value of θ .

9c. *[4 marks]*

Consider the function $f(\theta) = 2.6\sin\theta - 2\theta$, for $0 < \theta < \frac{\pi}{2}$.

(i) Sketch the graph of *f* .

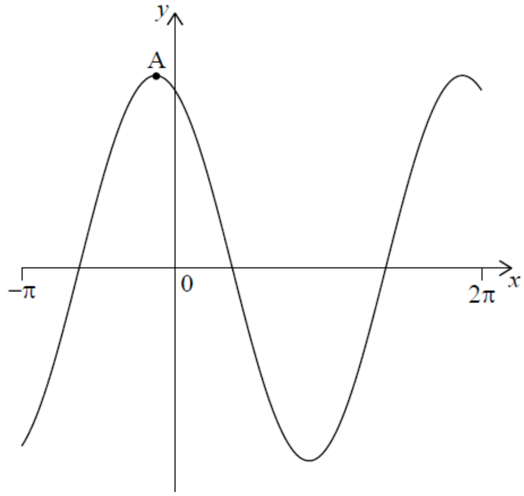
(ii) Write down the root of $f(\theta) = 0$.

9d. *[3 marks]*

Use the graph of *f* to find the values of θ for which $l < 1.3\mathrm{PQ}$.

10a. Let $f(x) = 12\cos x - 5\sin x,\ -\pi \leqslant x \leqslant 2\pi$, be a periodic function with $f(x) = f(x + 2\pi)$

The following diagram shows the graph of f.



There is a maximum point at A. The minimum value of f is −13 .

Find the coordinates of A. *[2 marks]*

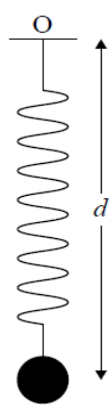
10b. For the graph of f, write down the amplitude. *[1 mark]*

10c. For the graph of f, write down the period. *[1 mark]*

10d. Hence, write $f(x)$ in the form $p\cos(x + r)$. *[3 marks]*

10e. A ball on a spring is attached to a fixed point O. The ball is then pulled down and released, so that it moves back and forth vertically.

diagram not to scale



The distance, *d* centimetres, of the centre of the ball from O at time *t* seconds, is given by

$$d(t) = f(t) + 17,\ 0 \leqslant t \leqslant 5.$$

Find the maximum speed of the ball. *[3 marks]*

10f. Find the first time when the ball's speed is changing at a rate of 2 cm s^{-2}. *[5 marks]*

11a. Note: In this question, distance is in millimetres.

Let $f(x) = x + a\sin\left(x - \frac{\pi}{2}\right) + a$, for $x \geqslant 0$.

Show that $f(2\pi) = 2\pi$. *[3 marks]*

11b. The graph of f passes through the origin. Let P_k be any point on the graph of f with x-coordinate $2k\pi$, where $k \in \mathbb{N}$. A straight line L passes through all the points P_k.

Find the coordinates of P_0 and of P_1. *[3 marks]*

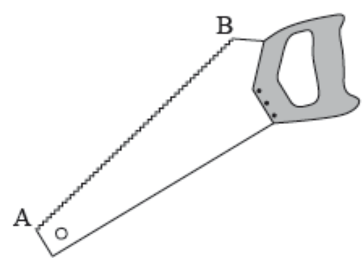
11c. Find the equation of L. *[3 marks]*

11d. Show that the distance between the x-coordinates of P_k and P_{k+1} is 2π. *[2 marks]*

11e. Diagram 1 shows a saw. The length of the toothed edge is the distance AB.

Diagram 1

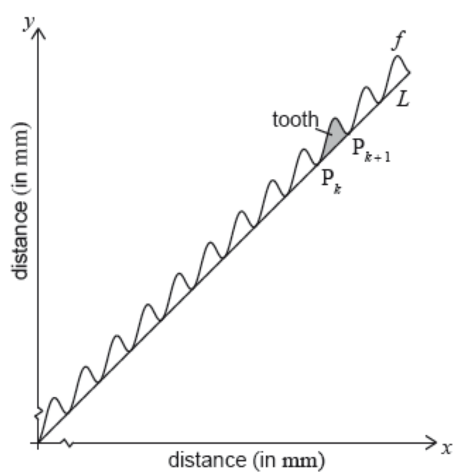
diagram not to scale



The toothed edge of the saw can be modelled using the graph of f and the line L. Diagram 2 represents this model.

Diagram 2

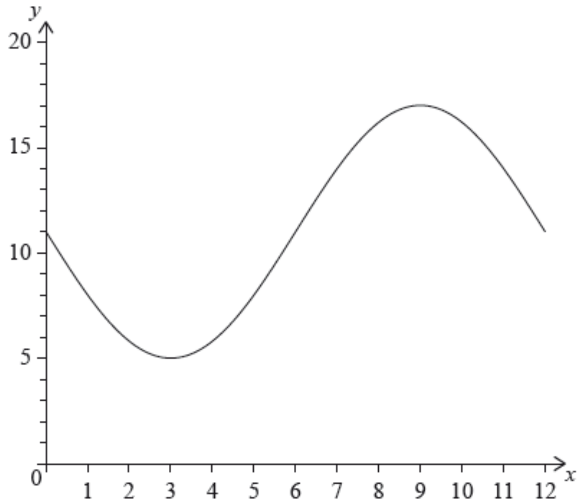
diagram not to scale



The shaded part on the graph is called a tooth. A tooth is represented by the region enclosed by the graph of f and the line L, between P_k and P_{k+1}. A saw has a toothed edge which is 300 mm long. Find the number of complete teeth on this saw. *[6 marks]*

12a. *[6 marks]*

The following diagram shows the graph of $f(x) = a \sin bx + c$, for $0 \leqslant x \leqslant 12$.



The graph of f has a minimum point at $(3,\ 5)$ and a maximum point at $(9,\ 17)$.

(i) Find the value of c.

(ii) Show that $b = \frac{\pi}{6}$.

(iii) Find the value of a.

12b. *[3 marks]*

The graph of g is obtained from the graph of f by a translation of $\begin{pmatrix} k \\ 0 \end{pmatrix}$. The maximum point on the graph of g has coordinates $(11.5,\ 17)$.

(i) Write down the value of k.

(ii) Find $g(x)$.

12c. *[6 marks]*

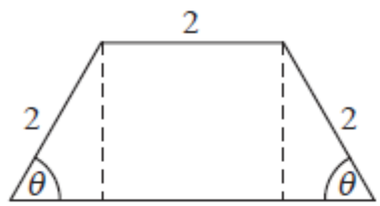
The graph of g changes from concave-up to concave-down when $x = w$.

(i) Find w.

(ii) Hence or otherwise, find the maximum positive rate of change of g.

13a. *[5 marks]*

The diagram below shows a plan for a window in the shape of a trapezium.



Three sides of the window are 2 m long. The angle between the sloping sides of the window and the base is θ , where $0 < \theta < \frac{\pi}{2}$.

Show that the area of the window is given by $y = 4\sin\theta + 2\sin 2\theta$.

13b. *[4 marks]*

Zoe wants a window to have an area of 5 m^2. Find the two possible values of θ .

13c. *[7 marks]*

John wants two windows which have the same area A but different values of θ .

Find all possible values for A .