BECA / Huson / 12.1 IB Math SL Name:
10 December 2018
**Pretest**: Vector and calculus, plus review

**1a.** Line $L_1$ passes through points $\mathrm{A}(1,\ -1,\ 4)$ and $\mathrm{B}(2,\ -2,\ 5)$.

Find $\overrightarrow{\mathrm{AB}}$. *[2 marks]*

**1b.** Find an equation for $L_1$ in the form $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$. *[2 marks]*

**1c.** Line $L_2$ has equation $\boldsymbol{r} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$.

Find the angle between $L_1$ and $L_2$. *[7 marks]*

**1d.** The lines $L_1$ and $L_2$ intersect at point C. Find the coordinates of C. *[6 marks]*

**2a.** The diagram shows quadrilateral ABCD with vertices A(1, 0), B(1, 5), C(5, 2) and D(4, −1).

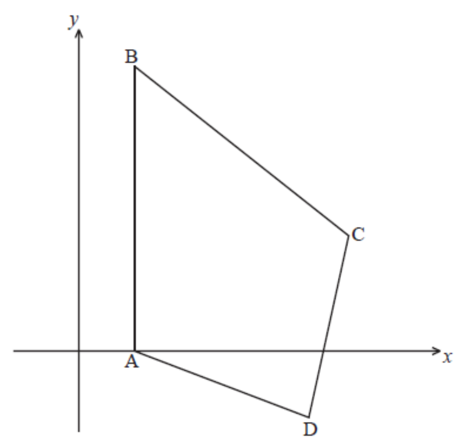



*diagram not to scale*

(i) Show that $\overrightarrow{\mathrm{AC}} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

(ii) Find $\overrightarrow{\mathrm{BD}}$.

(iii) Show that $\overrightarrow{\mathrm{AC}}$ is perpendicular to $\overrightarrow{\mathrm{BD}}$.

**2b.** The line (AC) has equation $\boldsymbol{r} = \boldsymbol{u} + s\boldsymbol{v}$.

(i) Write down vector ***u*** and vector ***v***.

(ii) Find a vector equation for the line (BD). *[4 marks]*

**2c.** The lines (AC) and (BD) intersect at the point $\mathrm{P}(3,\ k)$.

Show that $k = 1$. *[3 marks]*

**2d.** The lines (AC) and (BD) intersect at the point $\mathrm{P}(3,\ k)$.

**Hence** find the area of triangle ACD. *[5 marks]*

**3.** Let $\boldsymbol{v} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} k \\ -2 \\ 4 \end{pmatrix}$, for $k > 0$. The angle between ***v*** and ***w*** is $\frac{\pi}{3}$.

Find the value of $k$. *[7 marks]*

**4a.**

The line $L_1$ is represented by the vector equation $\boldsymbol{r} = \begin{pmatrix} -3 \\ -1 \\ -25 \end{pmatrix} + p\begin{pmatrix} 2 \\ 1 \\ -8 \end{pmatrix}$.

A second line $L_2$ is parallel to $L_1$ and passes through the point B$(-8, -5, 25)$.

Write down a vector equation for $L_2$ in the form $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$. *[2 marks]*

**4b.** A third line $L_3$ is perpendicular to $L_1$ and is represented by $\boldsymbol{r} = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix} + q\begin{pmatrix} -7 \\ -2 \\ k \end{pmatrix}$.

Show that $k = -2$. *[5 marks]*

**4c.** The lines $L_1$ and $L_3$ intersect at the point A.

Find the coordinates of A. *[6 marks]*

**4d.** The lines $L_2$ and $L_3$ intersect at point C where $\overrightarrow{\mathrm{BC}} = \begin{pmatrix} 6 \\ 3 \\ -24 \end{pmatrix}$.

(i) Find $\overrightarrow{\mathrm{AB}}$.

(ii) Hence, find $|\overrightarrow{\mathrm{AC}}|$. *[5 marks]*

**5a.** Let $\overrightarrow{AB} = \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix}$.

Find $\overrightarrow{BC}$. *[2 marks]*

**5b.** *[3 marks]*

Find a unit vector in the direction of $\overrightarrow{AB}$.

**5c.** *[3 marks]*

Show that $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{AC}$.

**6a.** *[4 marks]*

*In this question, distance is in metres.*

Toy airplanes fly in a straight line at a constant speed. Airplane 1 passes through a point A.

Its position, *p* seconds after it has passed through A, is given by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} + p \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$.

(i) Write down the coordinates of A.

(ii) Find the speed of the airplane in $\mathrm{ms}^{-1}$.

**6b.** *[5 marks]*

After seven seconds the airplane passes through a point B.

(i) Find the coordinates of B.

(ii) Find the distance the airplane has travelled during the seven seconds.

**6c.** Airplane 2 passes through a point C. Its position *q* seconds after it passes through C is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 8 \end{pmatrix} + q \begin{pmatrix} -1 \\ 2 \\ a \end{pmatrix}, a \in \mathbb{R}.$$

The angle between the flight paths of Airplane 1 and Airplane 2 is $40^\circ$. Find the two values of *a*.

*[7 marks]*

**7a.** Let $f(x) = \frac{6x}{x+1}$, for $x > 0$.

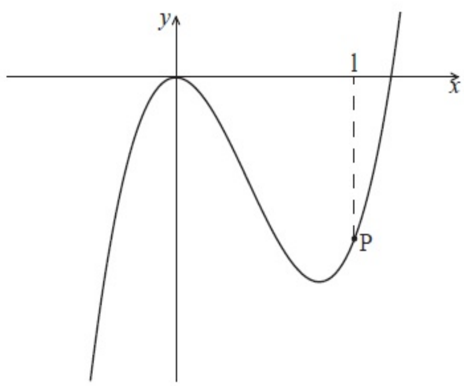
Find $f'(x)$. *[5 marks]*

**7b.** Let $g(x) = \ln\left(\frac{6x}{x+1}\right)$, for $x > 0$.

Show that $g'(x) = \frac{1}{x(x+1)}$. *[4 marks]*

**7c.** Let $h(x) = \frac{1}{x(x+1)}$. The area enclosed by the graph of *h*, the *x*-axis and the lines $x = \frac{1}{5}$ and $x = k$ is $\ln 4$. Given that $k > \frac{1}{5}$, find the value of *k*. *[7 marks]*

**8a.** Part of the graph of $f(x) = ax^3 - 6x^2$ is shown below.




The point P lies on the graph of $f$. At P, *x* = 1.

Find $f'(x)$. *[2 marks]*

**8b.** The graph of $f$ has a gradient of $3$ at the point P. Find the value of $a$. *[4 marks]*

**9a.** In this question, you are given that $\cos\frac{\pi}{3} = \frac{1}{2}$, and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

The displacement of an object from a fixed point, O is given by $s(t) = t - \sin 2t$ for $0 \leq t \leq \pi$.

Find $s'(t)$. *[3 marks]*

**9b.** In this interval, there are only two values of *t* for which the object is not moving. One value is $t = \frac{\pi}{6}$.

Find the other value. *[4 marks]*

**9c.** Show that $s'(t) > 0$ between these two values of *t*. *[3 marks]*

**9d.** Find the distance travelled between these two values of *t*. *[5 marks]*

**10a.** Let $f(x) = \mathrm{e}^{6x}$.

Write down $f'(x)$. *[1 mark]*

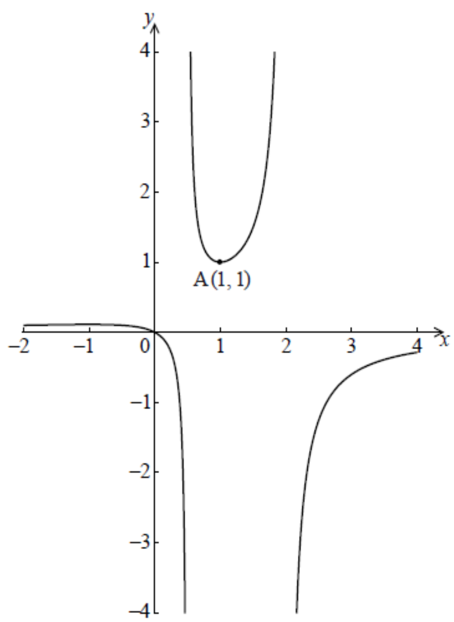
**10b.** The tangent to the graph of *f* at the point $\mathrm{P}(0, b)$ has gradient *m*.

(i) Show that $m = 6$.

(ii) Find *b*. *[4 marks]*

**10c.** Hence, write down the equation of this tangent. *[1 mark]*

**11a.** Let $f(x) = \frac{x}{-2x^2+5x-2}$ for $-2 \le x \le 4$, $x \ne \frac{1}{2}$, $x \ne 2$. The graph of $f$ is given below.



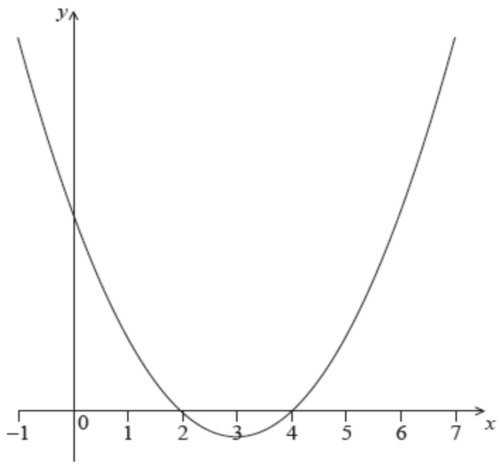

The graph of $f$ has a local minimum at A(1, 1) and a local maximum at B.

Use the quotient rule to show that $f'(x) = \frac{2x^2-2}{(-2x^2+5x-2)^2}$. *[6 marks]*

**11b.** Hence find the coordinates of B. *[7 marks]*

**11c.** Given that the line $y = k$ does not meet the graph of *f*, find the possible values of *k*. *[3 marks]*

**12a.** The following diagram shows part of the graph of a quadratic function $f$.




The vertex is at $(3,\ -1)$ and the $x$-intercepts at 2 and 4.

The function $f$ can be written in the form $f(x) = (x-h)^2 + k$.

Write down the value of $h$ and of $k$. *[2 marks]*

**12b.** The function can also be written in the form $f(x) = (x-a)(x-b)$.

Write down the value of $a$ and of $b$. *[2 marks]*

**12c.** Find the $y$-intercept. *[2 marks]*

**13.** Three consecutive terms of a geometric sequence are $x-3$, 6 and $x+2$.

Find the possible values of $x$. *[6 marks]*

**14a.** Let $f(x) = x^2$ and $g(x) = 3\ln(x+1)$, for $x > -1$.

Solve $f(x) = g(x)$. *[3 marks]*

**14b.** Find the area of the region enclosed by the graphs of $f$ and $g$. *[3 marks]*

**15a.** A population of rare birds, $P_t$, can be modelled by the equation $P_t = P_0 \mathrm{e}^{kt}$, where $P_0$ is the initial population, and $t$ is measured in decades. After one decade, it is estimated that $\frac{P_1}{P_0} = 0.9$.

(i) Find the value of $k$.

(ii) Interpret the meaning of the value of $k$. *[3 marks]*

**15b.** Find the least number of **whole** years for which $\frac{P_t}{P_0} < 0.75$. *[5 marks]*

**16a.** The price of a used car depends partly on the distance it has travelled. The following table shows the distance and the price for seven cars on 1 January 2010.

| **Distance, $x$ km** | 11 500 | 7500 | 13 600 | 10 800 | 9500 | 12 200 | 10 400 |
|---|---|---|---|---|---|---|---|
| **Price, $y$ dollars** | 15 000 | 21 500 | 12 000 | 16 000 | 19 000 | 14 500 | 17 000 |

The relationship between $x$ and $y$ can be modelled by the regression equation $y = ax + b$.

(i) Find the correlation coefficient.

(ii) Write down the value of $a$ and of $b$. *[4 marks]*

**16b.** On 1 January 2010, Lina buys a car which has travelled $11\,000$ km.

Use the regression equation to estimate the price of Lina's car, giving your answer to the nearest 100 dollars. *[3 marks]*

**16c.** The price of a car decreases by 5% each year.

Calculate the price of Lina's car after 6 years. *[4 marks]*

**16d.** Lina will sell her car when its price reaches $10\,000$ dollars.

Find the year when Lina sells her car. *[4 marks]*

**17a.** Let $f(x) = \frac{1}{x-1} + 2$, for $x > 1$.

Write down the equation of the horizontal asymptote of the graph of $f$. *[2 marks]*

**17b.** Find $f'(x)$. *[2 marks]*

**17c.** Let $g(x) = ae^{-x} + b$, for $x \geqslant 1$. The graphs of $f$ and $g$ have the same horizontal asymptote.

Write down the value of $b$. *[2 marks]*

**17d.** Given that $g'(1) = -e$, find the value of $a$. *[4 marks]*

**17e.** There is a value of $x$, for $1 < x < 4$, for which the graphs of $f$ and $g$ have the same gradient. Find this gradient. *[4 marks]*

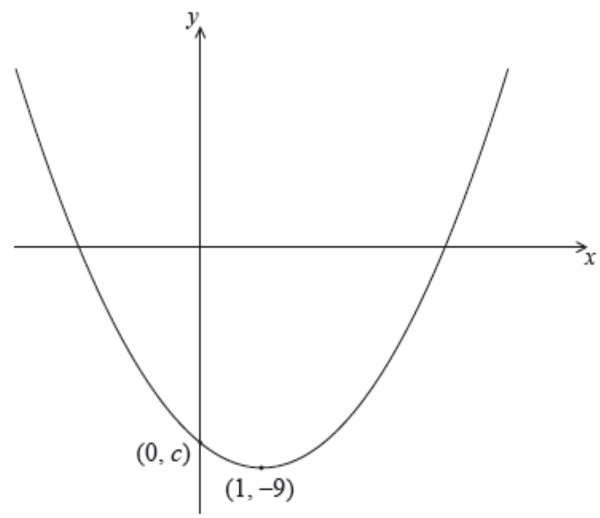
**18a.** Let $f(x) = (x-5)^3$, for $x \in \mathbb{R}$.

Find $f^{-1}(x)$. *[3 marks]*

**18b.** Let $g$ be a function so that $(f \circ g)(x) = 8x^6$. Find $g(x)$. *[3 marks]*

**19a.** The following diagram shows part of the graph of a quadratic function $f$.

The vertex is at $(1,\ -9)$, and the graph crosses the $y$-axis at the point $(0,\ c)$.



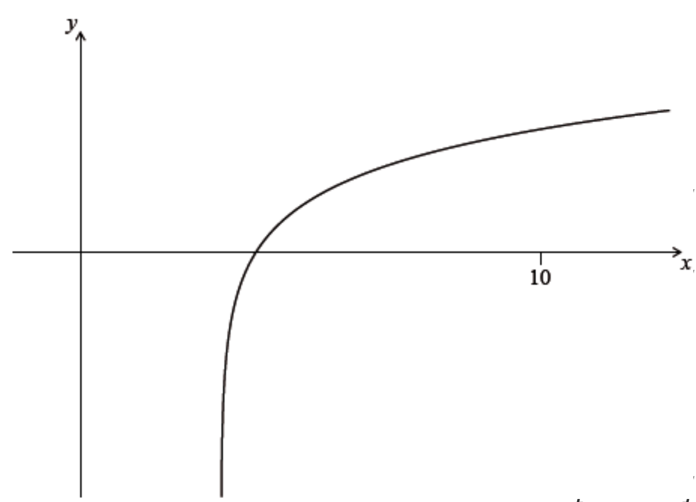

The function can be written in the form $f(x) = (x-h)^2 + k$.

Write down the value of $h$ and of $k$.

**19b.** Let $g(x) = -(x-3)^2 + 1$. The graph of $g$ is obtained by a reflection of the graph of $f$ in the $x$-axis, followed by a translation of $\begin{pmatrix} p \\ q \end{pmatrix}$.

Find the value of $p$ and of $q$. *[5 marks]*

**20a.** Let $f(x) = 2\ln(x-3)$, for $x > 3$. The diagram shows part of the graph of $f$. Find the equation of the vertical asymptote to the graph of $f$.




**20b.** Find the $x$-intercept of the graph of $f$.

**21a.** The first three terms of a geometric sequence are $u_1 = 0.64,\ u_2 = 1.6$, and $u_3 = 4$.

Find the value of $r$.

**21b.** Find the value of $S_6$. *[2 marks]*

**21c.** Find the least value of $n$ such that $S_n > 75\,000$. *[3 marks]*