3-4_Periodic-functions-mild *[86 marks]*

Let $f(x) = 3\sin\left(\frac{\pi}{2}x\right)$, for $0 \leqslant x \leqslant 4$.

1a. (i) Write down the amplitude of f. *[3 marks]*

(ii) Find the period of f.

Markscheme

(i) 3 ***A1*** ***N1***

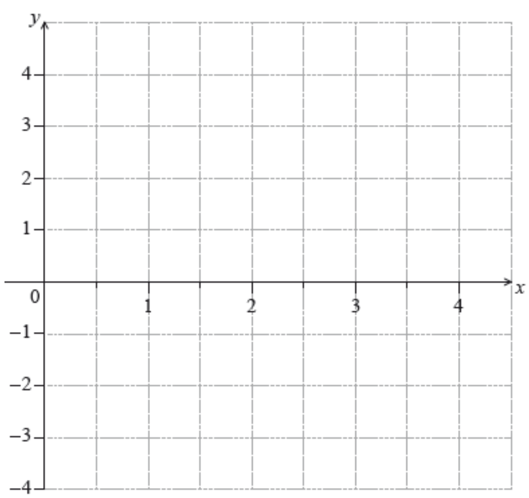
(ii) valid attempt to find the period ***(M1)***

eg $\frac{2\pi}{b}, \frac{2\pi}{\frac{\pi}{2}}$

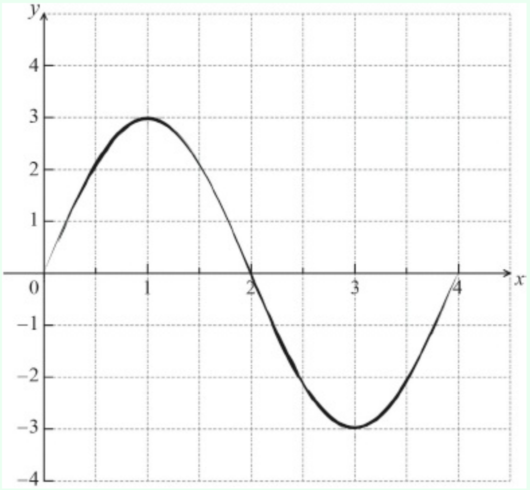
period $= 4$ ***A1*** ***N2***

[3 marks]

1b. On the following grid sketch the graph of f. *[4 marks]*



Markscheme



A1A1A1A1 N4

[4 marks]

Let
$f(x) = \frac{3x}{2} + 1$,
$g(x) = 4\cos\left(\frac{x}{3}\right) - 1$. Let
$h(x) = (g \circ f)(x)$.

2a. Find an expression for $h(x)$. *[3 marks]*

Markscheme

attempt to form any composition (even if order is reversed) ***(M1)***

correct composition $h(x) = g\left(\frac{3x}{2} + 1\right)$ ***(A1)***

$h(x) = 4\cos\left(\frac{\frac{3x}{2}+1}{3}\right) - 1 \quad \left(4\cos\left(\frac{1}{2}x + \frac{1}{3}\right) - 1, 4\cos\left(\frac{3x+2}{6}\right) - 1\right)$ ***A1 N3***

[3 marks]

2b. Write down the period of h . *[1 mark]*

Markscheme

period is $4\pi (12.6)$ ***A1 N1***

[1 mark]

2c. Write down the range of h . *[2 marks]*

Markscheme

range is $-5 \leq h(x) \leq 3$ $([-5,3])$ ***A1A1*** ***N2***

[2 marks]

Let $f(x) = 3\sin(\pi x)$.

3a. Write down the amplitude of f. *[1 mark]*

Markscheme

amplitude is 3 ***A1*** ***N1***

3b. Find the period of f. *[2 marks]*

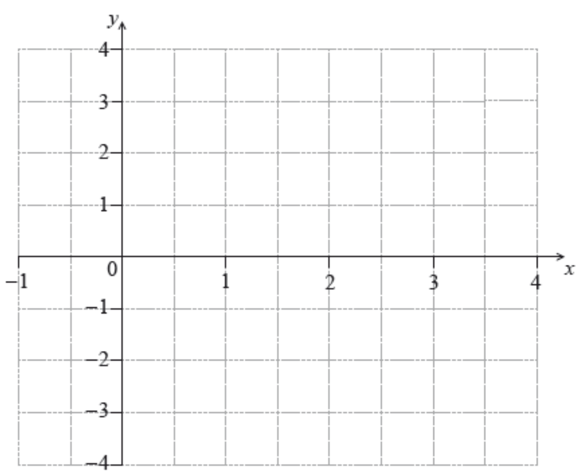
Markscheme

valid approach ***(M1)***

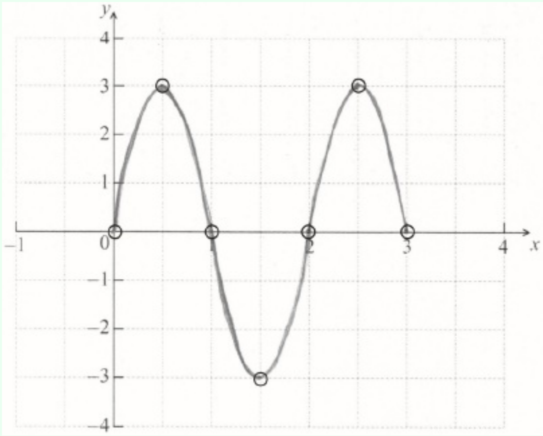
eg $\text{period} = \frac{2\pi}{\pi}, \frac{360}{\pi}$

period is 2 ***A1*** ***N2***

3c. On the following grid, sketch the graph of $y = f(x)$, for $0 \leq x \leq 3$. *[4 marks]*



Markscheme



A1

A1A1A1 ***N4***

Note: Award ***A1*** for sine curve starting at (0, 0) and correct period.

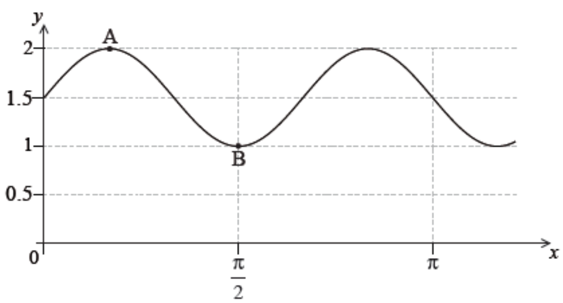
Only if this ***A1*** is awarded, award the following for points in circles:

A1 for correct *x*-intercepts;

A1 for correct max and min points;

A1 for correct domain.

The following diagram shows part of the graph of $y = p\sin(qx) + r$.



The point $\mathrm{A}\left(\frac{\pi}{6},\, 2\right)$ is a maximum point and the point $\mathrm{B}\left(\frac{\pi}{6},\, 1\right)$ is a minimum point.

Find the value of

4a. p; *[2 marks]*

Markscheme

valid approach ***(M1)***

eg $\frac{2-1}{2},\ 2-1.5$

$p = 0.5$ ***A1*** ***N2***

[2 marks]

4b. r; *[2 marks]*

Markscheme

valid approach ***(M1)***

eg $\frac{1+2}{2}$

$r = 1.5$ ***A1*** ***N2***

[2 marks]

4c. q. *[2 marks]*

Markscheme

METHOD 1

valid approach (seen anywhere) ***M1***

eg $q = \frac{2\pi}{\text{period}}, \frac{2\pi}{\left(\frac{2\pi}{3}\right)}$

period $= \frac{2\pi}{3}$ (seen anywhere) ***(A1)***

$q = 3$ ***A1*** ***N2***

METHOD 2

attempt to substitute one point and **their** values for p and r into y ***M1***

eg $2 = 0.5\sin\left(q\frac{\pi}{6}\right) + 1.5,\ \frac{\pi}{2} = 0.5\sin(q1) + 1.5$

correct equation in q ***(A1)***

eg $q\frac{\pi}{6} = \frac{\pi}{2},\ q\frac{\pi}{2} = \frac{3\pi}{2}$

$q = 3$ ***A1*** ***N2***

METHOD 3

valid reasoning comparing the graph with that of $\sin x$ ***R1***

eg position of max/min, graph goes faster

correct working ***(A1)***

eg max at $\frac{\pi}{6}$ not at $\frac{\pi}{2}$, graph goes 3 times as fast

$q = 3$ ***A1*** ***N2***

[3 marks]

Total [7 marks]

The depth of water in a port is modelled by the function $d(t) = p\cos qt + 7.5$, for $0 \leqslant t \leqslant 12$, where t is the number of hours after high tide.

At high tide, the depth is 9.7 metres.

At low tide, which is 7 hours later, the depth is 5.3 metres.

5a. Find the value of p. *[2 marks]*

Markscheme

valid approach ***(M1)***

eg $\frac{\text{max}-\text{min}}{2}$, sketch of graph, $9.7 = p\cos(0) + 7.5$

$p = 2.2$ ***A1 N2***

[2 marks]

5b. Find the value of q. *[2 marks]*

Markscheme

valid approach ***(M1)***

eg $B = \frac{2\pi}{\text{period}}$, period is 14, $\frac{360}{14}$, $5.3 = 2.2\cos 7q + 7.5$

0.448798

$q = \frac{2\pi}{14}\ \left(\frac{\pi}{7}\right)$, (do not accept degrees) ***A1 N2***

[2 marks]

5c. Use the model to find the depth of the water 10 hours after high tide. *[2 marks]*

Markscheme

valid approach ***(M1)***

eg $d(10),\ 2.2\cos\left(\frac{20\pi}{14}\right) + 7.5$

7.01045

7.01 (m) ***A1 N2***

[2 marks]

The height, h metros, of a seat on a Ferris wheel after t minutes is given by

$$h(t) = -15\cos 1.2t + 17,\ \text{for } t \geqslant 0.$$

6a. Find the height of the seat when $t = 0$. *[2 marks]*

Markscheme

valid approach ***(M1)***

eg $h(0),\ -15\cos(1.2\times 0)+17,\ -15(1)+17$

$h(0)=2\ (\mathrm{m})$ ***A1*** ***N2***

[2 marks]

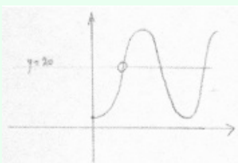
6b. The seat first reaches a height of 20 m after k minutes. Find k. *[3 marks]*

Markscheme

correct substitution into equation ***(A1)***

eg $20=-15\cos 1.2t+17,\ -15\cos 1.2k=3$

valid attempt to solve for k ***(M1)***

eg  , $\cos 1.2k=-\frac{3}{15}$

1.47679

$k=1.48$ ***A1*** ***N2***

[3 marks]

6c. Calculate the time needed for the seat to complete a full rotation, giving your answer correct to one decimal place. *[3 marks]*

Markscheme

recognize the need to find the period (seen anywhere) ***(M1)***

eg next t value when $h=20$

correct value for period ***(A1)***

eg $\text{period}=\frac{2\pi}{1.2},\ 5.23598,\ 6.7--1.48$

5.2 (min) (must be 1 dp) ***A1*** ***N2***

[3 marks]

The population of deer in an enclosed game reserve is modelled by the function
$P(t) = 210\sin(0.5t - 2.6) + 990$, where
t is in months, and
$t = 1$ corresponds to 1 January 2014.

7a. Find the number of deer in the reserve on 1 May 2014. *[3 marks]*

Markscheme

$t = 5$ ***(A1)***

correct substitution into formula ***(A1)***

eg $210\sin(0.5 \times 5 - 2.6) + 990,\ P(5)$

$969.034982\ldots$

969 (deer) (must be an integer) ***A1*** ***N3***

[3 marks]

7b. Find the rate of change of the deer population on 1 May 2014. *[2 marks]*

Markscheme

evidence of considering derivative ***(M1)***

eg P'

104.475

104 (deer per month) ***A1*** ***N2***

[2 marks]

7c. Interpret the answer to part (i) with reference to the deer population size on 1 May 2014. *[1 mark]*

Markscheme

(the deer population size is) **increasing** ***A1*** ***N1***

[1 mark]

Let
$f(x) = \sin\left(x + \frac{\pi}{4}\right) + k$. The graph of *f* passes through the point
$\left(\frac{\pi}{4},\ 6\right)$.

8a. Find the value of k. *[3 marks]*

Markscheme

METHOD 1

attempt to substitute both coordinates (in any order) into f ***(M1)***

eg $f\left(\frac{\pi}{4}\right) = 6,\ \frac{\pi}{4} = \sin\left(6 + \frac{\pi}{4}\right) + k$

correct working ***(A1)***

eg $\sin\frac{\pi}{2} = 1,\ 1 + k = 6$

$k = 5$ ***A1 N2***

[3 marks]

METHOD 2

recognizing shift of $\frac{\pi}{4}$ left means maximum at 6 ***R1)***

recognizing k is difference of maximum and amplitude ***(A1)***

eg $6 - 1$

$k = 5$ ***A1 N2***

[3 marks]

8b. Find the minimum value of $f(x)$. *[2 marks]*

Markscheme

evidence of appropriate approach ***(M1)***

eg minimum value of $\sin x$ is $-1,\ -1 + k,\ f'(x) = 0,\ \left(\frac{5\pi}{4},\ 4\right)$

minimum value is 4 ***A1 N2***

[2 marks]

8c. Let $g(x) = \sin x$. The graph of g is translated to the graph of f by the vector $\begin{pmatrix} p \\ q \end{pmatrix}$. *[2 marks]*

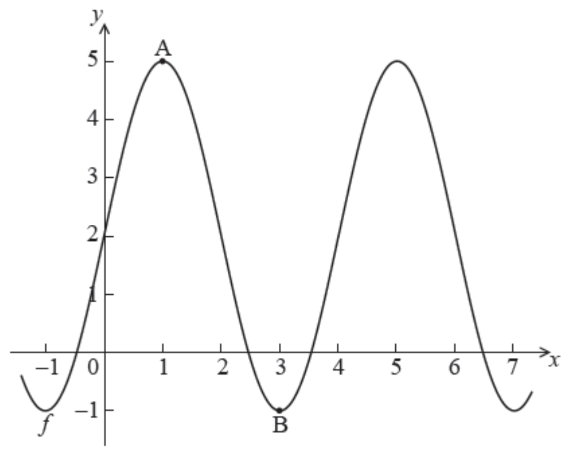
Write down the value of p and of q.

Markscheme

$p = -\frac{\pi}{4},\ q = 5\ \left(\text{accept}\ \begin{pmatrix} -\frac{\pi}{4} \\ 5 \end{pmatrix}\right)$ ***A1A1 N2***

[2 marks]

The diagram below shows part of the graph of a function
f .



The graph has a maximum at A(
$1,$
$5)$ and a minimum at B(
$3,$
$-1)$.

The function
f can be written in the form
$f(x) = p\sin(qx) + r$. Find the value of

9a. (a) p *[6 marks]*

(b) q

(c) r .

Markscheme

(a) valid approach to find p ***(M1)***

eg amplitude $= \frac{\max - \min}{2}$, $p = 6$

$p = 3$ ***A1 N2***

[2 marks]

(b) valid approach to find q ***(M1)***

eg period = 4 , $q = \frac{2\pi}{\text{period}}$

$q = \frac{\pi}{2}$ ***A1 N2***

[2 marks]

(c) valid approach to find r ***(M1)***

eg axis = $\frac{\max + \min}{2}$, sketch of horizontal axis, $f(0)$

$r = 2$ ***A1 N2***

[2 marks]

Total [6 marks]

9b. p *[2 marks]*

Markscheme

valid approach to find p ***(M1)***

eg amplitude $= \frac{\max - \min}{2}$, $p = 6$

$p = 3$ ***A1 N2***

[2 marks]

9c. q *[2 marks]*

Markscheme

valid approach to find q ***(M1)***

eg period = 4 , $q = \frac{2\pi}{\text{period}}$

$q = \frac{\pi}{2}$ ***A1*** ***N2***

[2 marks]

9d. r . *[2 marks]*

Markscheme

valid approach to find r ***(M1)***

eg axis = $\frac{\text{max}+\text{min}}{2}$, sketch of horizontal axis, $f(0)$

$r = 2$ ***A1*** ***N2***

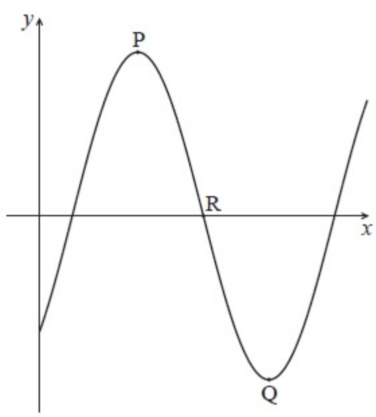
[2 marks]

Total [6 marks]

Let

$f(x) = a\cos(b(x-c))$. The diagram below shows part of the graph of f , for

$0 \le x \le 10$.



The graph has a local maximum at P(3, 5) , a local minimum at Q(7, − 5) , and crosses the x-axis at R.

10a. Write down the value of *[2 marks]*

(i) a ;

(ii) c .

Markscheme

(i) $a = 5$ (accept -5) ***A1*** ***N1***

(ii) $c = 3$ (accept $c = 7$, if $a = -5$) ***A1*** ***N1***

Note: Accept other correct values of *c*, such as 11, -5, etc.

[2 marks]

10b. Find the value of *b* . *[2 marks]*

Markscheme

attempt to find period ***(M1)***

e.g. 8 , $b = \frac{2\pi}{\text{period}}$

$0.785398\ldots$

$b = \frac{2\pi}{8}$ (exact), $\frac{\pi}{4}$, 0.785 $[0.785, 0.786]$ (do not accept 45) ***A1*** ***N2***

[2 marks]

10c. Find the *x*-coordinate of R. *[2 marks]*

Markscheme

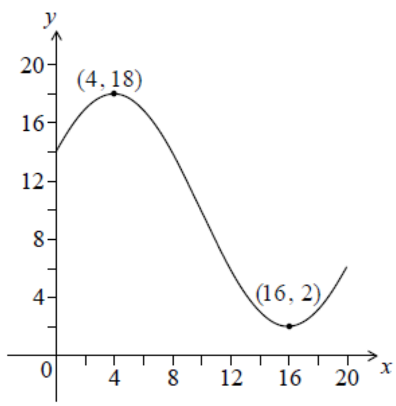
valid approach ***(M1)***

e.g. $f(x) = 0$, symmetry of curve

$x = 5$ (accept $(5\ ,0)$) ***A1*** ***N2***

[2 marks]

Let
$f(x) = p\cos(q(x + r)) + 10$, for
$0 \leqslant x \leqslant 20$. The following diagram shows the graph of
f.



The graph has a maximum at
$(4, 18)$ and a minimum at
$(16, 2)$.

11a. Write down the value of r. *[2 marks]*

Markscheme

$r = -4$ ***A2*** ***N2***

Note: Award ***A1*** for $r = 4$.

[2 marks]

11b. Find p. *[2 marks]*

Markscheme

evidence of valid approach ***(M1)***

eg $\frac{\max y \text{ value} -- y \text{ value}}{2}$, distance from $y = 10$

$p = 8$ ***A1*** ***N2***

[2 marks]

11c. Find q. *[2 marks]*

Markscheme

valid approach *(M1)*

eg period is $24,\ \frac{360}{24}$, substitute a point into **their** $f(x)$

$q = \frac{2\pi}{24}\left(\frac{\pi}{12},\ \text{exact}\right),\ 0.262$ (do not accept degrees) ***A1 N2***

[2 marks]

11d. Solve $f(x) = 7$. *[2 marks]*

Markscheme

valid approach *(M1)*

eg line on graph at $y = 7,\ 8\cos\left(\frac{2\pi}{24}(x - 4)\right) + 10 = 7$

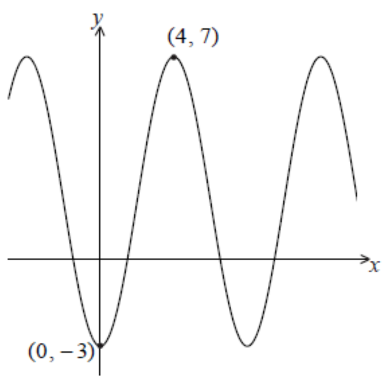
$x = 11.46828$

$x = 11.5$ (accept $(11.5, 7)$) ***A1 N2***

[2 marks]

Note: Do not award the final ***A1*** if additional values are given. If an incorrect value of q leads to multiple solutions, award the final ***A1*** only if **all** solutions within the domain are given.

The graph of $y = p\cos qx + r$, for $-5 \leq x \leq 14$, is shown below.



There is a minimum point at (0, −3) and a maximum point at (4, 7) .

12a. Find the value of *[6 marks]*

(i) *p* ;

(ii) *q* ;

(iii) *r*.

Markscheme

(i) evidence of finding the amplitude ***(M1)***

e.g. $\frac{7+3}{2}$, amplitude $= 5$

$p = -5$ ***A1*** ***N2***

(ii) period $= 8$ ***(A1)***

$q = 0.785 \left(= \frac{2\pi}{8} = \frac{\pi}{4}\right)$ ***A1*** ***N2***

(iii) $r = \frac{7-3}{2}$ ***(A1)***

$r = 2$ ***A1*** ***N2***

[6 marks]

12b. The equation $y = k$ has exactly **two** solutions. Write down the value of k. *[1 mark]*

Markscheme

$k = -3$ (accept $y = -3$) ***A1*** ***N1***

[1 mark]