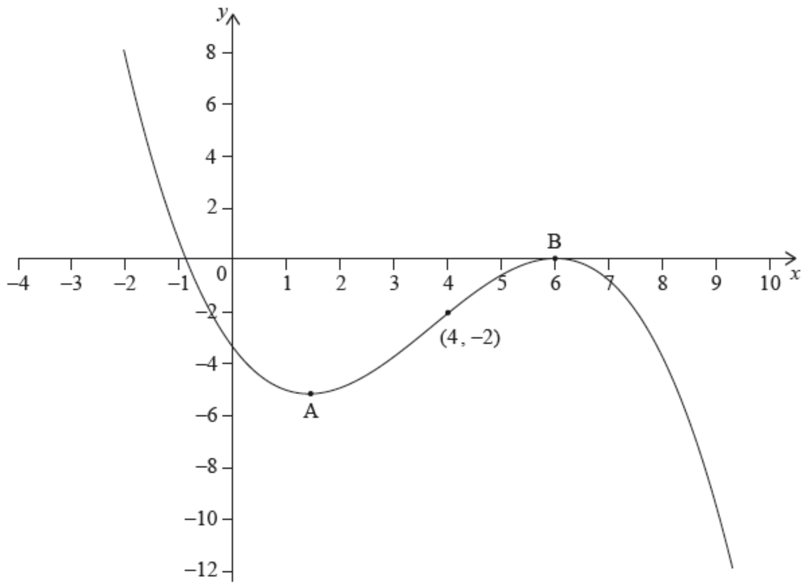
0228HW_Function-graphs *[56 marks]*

The following diagram shows the graph of f', the derivative of f.



The graph of f' has a local minimum at A, a local maximum at B and passes through $(4,\ -2)$.

The point $\mathrm{P}(4,\ 3)$ lies on the graph of the function, f.

1a. Write down the gradient of the curve of f at P. *[1 mark]*

Markscheme

-2 ***A1*** ***N1***

[1 mark]

1b. Find the equation of the normal to the curve of f at P. *[3 marks]*

Markscheme

gradient of normal $= \frac{1}{2}$ ***(A1)***

attempt to substitute their normal gradient and coordinates of P (in any order) ***(M1)***

eg $y - 4 = \frac{1}{2}(x - 3),\ 3 = \frac{1}{2}(4) + b,\ b = 1$

$y - 3 = \frac{1}{2}(x - 4),\ y = \frac{1}{2}x + 1,\ x - 2y + 2 = 0$ ***A1*** ***N3***

[3 marks]

1c. Determine the concavity of the graph of f when $4 < x < 5$ **and** justify your answer. *[2 marks]*

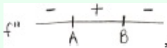
Markscheme

correct answer **and** valid reasoning ***A2*** ***N2***

answer: *eg* graph of f is concave up, concavity is positive (between $4 < x < 5$)

reason: *eg* slope of f' is positive, f' is increasing, $f'' > 0$,

sign chart (must clearly be for f'' and show A and B)



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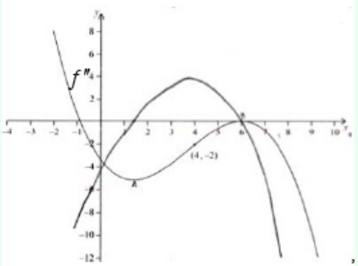

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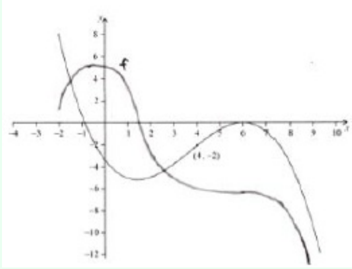
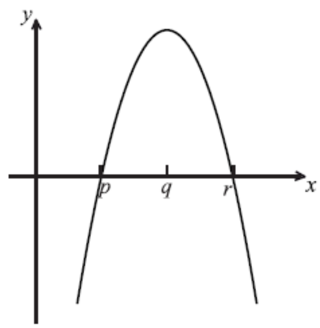


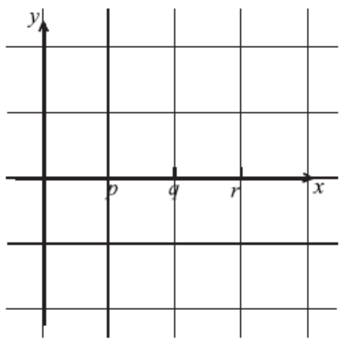
Note: The reason given must refer to a specific function/graph. Referring to “the graph” or “it” is not sufficient.

[2 marks]

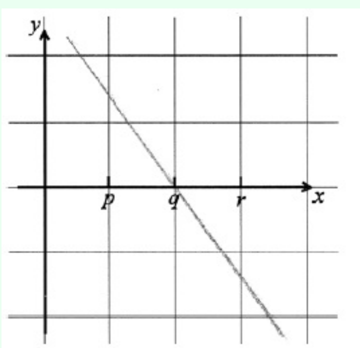
The diagram below shows part of the graph of the gradient function,
$y = f'(x)$.



2a. On the grid below, sketch a graph of $y = f''(x)$, clearly indicating the x-intercept. *[2 marks]*



Markscheme



A1A1 ***N2***

Note: Award ***A1*** for negative gradient throughout, ***A1*** for x-intercept of q. It need not be linear.

[2 marks]

2b. Complete the table, for the graph of $y = f(x)$. *[2 marks]*

	x-coordinate
(i) Maximum point on f	
(ii) Inflexion point on f	

Markscheme

		x-coordinate
(i)	Maximum point on f	r
(ii)	Inflexion point on f	q

A1A1 N1N1

2c. Justify your answer to part (b) (ii). *[2 marks]*

Markscheme

METHOD 1

Second derivative is zero, second derivative changes sign. ***R1R1 N2***

METHOD 2

There is a maximum on the graph of the first derivative. ***R2 N2***

Let
$g(x) = \frac{\ln x}{x^2}$, for
$x > 0$.

3a. Use the quotient rule to show that
$g'(x) = \frac{1-2\ln x}{x^3}$. *[4 marks]*

Markscheme

$\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}$,
$\frac{\mathrm{d}}{\mathrm{d}x}x^2 = 2x$ (seen anywhere) ***A1A1***

attempt to substitute into the quotient rule (do**not** accept product rule) ***M1***

e.g.
$\frac{x^2\left(\frac{1}{x}\right)-2x\ln x}{x^4}$

correct manipulation that clearly leads to result ***A1***

e.g.
$\frac{x-2x\ln x}{x^4}$,
$\frac{x(1-2\ln x)}{x^4}$,
$\frac{x}{x^4}$,
$\frac{2x\ln x}{x^4}$

$g'(x) = \frac{1-2\ln x}{x^3}$ ***AG N0***

[4 marks]

3b. The graph of g has a maximum point at A. Find the$x$-coordinate of A. *[3 marks]*

Markscheme

evidence of setting the derivative equal to zero ***(M1)***

e.g.
$g'(x) = 0$,
$1 - 2\ln x = 0$

$\ln x = \frac{1}{2}$ ***A1***

$x = \mathrm{e}^{\frac{1}{2}}$ ***A1*** ***N2***

[3 marks]

Let
$f'(x) = -24x^3 + 9x^2 + 3x + 1$.

4a. There are two points of inflexion on the graph of *f* . Write down the *x*-coordinates of these points. *[3 marks]*

Markscheme

valid approach ***R1***

e.g.
$f''(x) = 0$, the max and min of
f' gives the points of inflexion on *f*

$-0.114, 0.364$ (accept (
$-0.114, 0.811$) and (
$0.364, 2.13$)) ***A1A1*** ***N1N1***

[3 marks]

4b. Let
$g(x) = f''(x)$. Explain why the graph of *g* has no points of inflexion. *[2 marks]*

Markscheme

METHOD 1

graph of *g* is a quadratic function ***R1*** ***N1***

a quadratic function does not have any points of inflexion ***R1*** ***N1***

METHOD 2

graph of *g* is concave down over entire domain ***R1*** ***N1***

therefore no change in concavity ***R1*** ***N1***

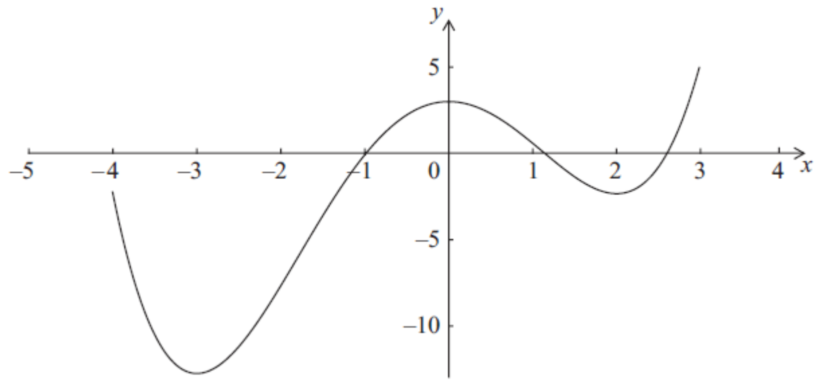
METHOD 3

$g''(x) = -144$ ***R1 N1***

therefore no points of inflexion as
$g''(x) \neq 0$ ***R1*** ***N1***

[2 marks]

A function *f* is defined for $-4 \le x \le 3$. The graph of *f* is given below.



The graph has a local maximum when $x = 0$, and local minima when $x = -3$, $x = 2$.

5a. Write down the *x*-intercepts of the graph of the **derivative** function, f'. *[2 marks]*

Markscheme

x-intercepts at $-3, 0, 2$ ***A2 N2***

[2 marks]

5b. Write down all values of *x* for which $f'(x)$ is positive. *[2 marks]*

Markscheme

$-3 < x < 0$, $2 < x < 3$ ***A1A1 N2***

[2 marks]

5c. At point D on the graph of *f*, the *x*-coordinate is -0.5. Explain why $f''(x) < 0$ at D. *[2 marks]*

Markscheme

correct reasoning ***R2***

e.g. the graph of *f* is **concave-down** (accept convex), the first derivative is decreasing

therefore the second derivative is negative ***AG***

[2 marks]

Let $f'(x) = \frac{6-2x}{6x-x^2}$, for $0 < x < 6$.

The graph of
f has a maximum point at P.

6a. Find the x-coordinate of P. *[3 marks]*

Markscheme

recognizing $f'(x) = 0$ ***(M1)***

correct working ***(A1)***

eg $6 - 2x = 0$

$x = 3$ ***A1 N2***

[3 marks]

Let $f'(x) = \frac{6-2x}{6x-x^2}$, for $0 < x < 6$.

The graph of
f has a maximum point at P.

The
y-coordinate of P is $\ln 27$.

6b. Find $f(x)$, expressing your answer as a single logarithm. *[8 marks]*

Markscheme

evidence of integration ***(M1)***

eg $\int f',\ \int \frac{6-2x}{6x-x^2}\mathrm{d}x$

using substitution ***(A1)***

eg $\int \frac{1}{u}\mathrm{d}u$ where $u = 6x - x^2$

correct integral ***A1***

eg $\ln(u) + c,\ \ln(6x - x^2)$

substituting $(3,\ \ln 27)$ into **their** integrated expression (must have c) ***(M1)***

eg $\ln(6 \times 3 - 3^2) + c = \ln 27,\ \ln(18 - 9) + \ln k = \ln 27$

correct working ***(A1)***

eg $c = \ln 27 - \ln 9$

EITHER

$c = \ln 3$ ***(A1)***

attempt to substitute **their** value of c into $f(x)$ ***(M1)***

eg $f(x) = \ln(6x - x^2) + \ln 3$ ***A1 N4***

OR

attempt to substitute **their** value of c into $f(x)$ ***(M1)***

eg $f(x) = \ln(6x - x^2) + \ln 27 - \ln 9$

correct use of a log law ***(A1)***

eg $f(x) = \ln(6x - x^2) + \ln\left(\frac{27}{9}\right),\ f(x) = \ln\big(27(6x - x^2)\big) - \ln 9$

$f(x) = \ln\big(3(6x - x^2)\big)$ ***A1 N4***

[8 marks]

Let $f'(x) = \frac{6-2x}{6x-x^2}$, for $0 < x < 6$.

The graph of
f has a maximum point at P.

The
y-coordinate of P is $\ln 27$.

6c. The graph of
f is transformed by a vertical stretch with scale factor $\frac{1}{\ln 3}$. The image of P under this transformation has coordinates $(a,\ b)$.

Find the value of a and of b, where $a,\ b \in \mathbb{N}$.

Markscheme

$a = 3$ ***A1*** ***N1***

correct working ***A1***

eg $\frac{\ln 27}{\ln 3}$

correct use of log law ***(A1)***

eg $\frac{3\ln 3}{\ln 3}$, $\log_3 27$

$b = 3$ ***A1*** ***N2***

[4 marks]

Let
$f(x) = \frac{(\ln x)^2}{2}$, for
$x > 0$.

7a. Show that
$f'(x) = \frac{\ln x}{x}$. *[2 marks]*

Markscheme

METHOD 1

correct use of chain rule ***A1A1***

eg
$\frac{2\ln x}{2} \times \frac{1}{x}$, $\frac{2\ln x}{2x}$

Note: Award ***A1*** for
$\frac{2\ln x}{2x}$, ***A1*** for
$\times \frac{1}{x}$.

$f'(x) = \frac{\ln x}{x}$ ***AG*** ***N0***

[2 marks]

METHOD 2

correct substitution into quotient rule, with derivatives seen ***A1***

eg
$\frac{2 \times 2\ln x \times \frac{1}{x} - 0 \times (\ln x)^2}{4}$

correct working ***A1***

eg
$\frac{4\ln x \times \frac{1}{x}}{4}$

$f'(x) = \frac{\ln x}{x}$ ***AG*** ***N0***

[2 marks]

7b. There is a minimum on the graph of f. Find the x-coordinate of this minimum. *[3 marks]*

Markscheme

setting derivative $= 0$ ***(M1)***

eg $f'(x) = 0,\ \frac{\ln x}{x} = 0$

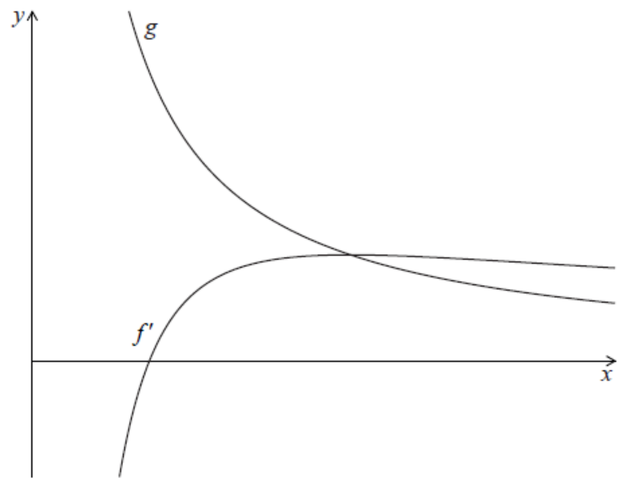
correct working ***(A1)***

eg $\ln x = 0,\ x = \mathrm{e}^0$

$x = 1$ ***A1 N2***

[3 marks]

Let $g(x) = \frac{1}{x}$. The following diagram shows parts of the graphs of f' and g.



The graph of f' has an *x*-intercept at $x = p$.

7c. Write down the value of p. *[2 marks]*

Markscheme

intercept when $f'(x) = 0$ ***(M1)***

$p = 1$ ***A1 N2***

[2 marks]

7d. The graph of g intersects the graph of f' when $x = q$. Find the value of q. *[3 marks]*

Markscheme

equating functions ***(M1)***

eg

$f' = g,\ \frac{\ln x}{x} = \frac{1}{x}$

correct working ***(A1)***

eg

$\ln x = 1$

$q = \mathrm{e}\quad (\text{accept}\, x = \mathrm{e})$ ***A1 N2***

[3 marks]

7e. The graph of g intersects the graph of f' when $x = q$.

[5 marks]

Let R be the region enclosed by the graph of f', the graph of g and the line $x = p$.

Show that the area of R is $\frac{1}{2}$.

Markscheme

evidence of integrating and subtracting functions (in any order, seen anywhere) ***(M1)***

eg

$\int_q^e \left(\frac{1}{x} - \frac{\ln x}{x}\right) \mathrm{d}x,\ \int f' - g$

correct integration

$\ln x - \frac{(\ln x)^2}{2}$ ***A2***

substituting limits into **their** integrated function and subtracting (in any order) ***(M1)***

eg

$(\ln \mathrm{e} - \ln 1) - \left(\frac{(\ln \mathrm{e})^2}{2} - \frac{(\ln 1)^2}{2}\right)$

Note: Do not award ***M1*** if the integrated function has only one term.

correct working ***A1***

eg

$(1-0) - \left(\frac{1}{2} - 0\right),\ 1 - \frac{1}{2}$

$\text{area} = \frac{1}{2}$ ***AG N0***

Notes: Candidates may work with two separate integrals, and only combine them at the end. Award marks in line with the markscheme.

[5 marks]

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