0226HW_Function-graphs *[70 marks]*

A function f has its derivative given by $f'(x) = 3x^2 - 2kx - 9$, where k is a constant.

1a. Find $f''(x)$. *[2 marks]*

Markscheme

$f''(x) = 6x - 2k$ ***A1A1*** ***N2***

[2 marks]

1b. The graph of f has a point of inflexion when $x = 1$. *[3 marks]*

Show that $k = 3$.

Markscheme

substituting $x = 1$ into f'' ***(M1)***

eg $f''(1),\ 6(1) - 2k$

recognizing $f''(x) = 0$ (seen anywhere) ***M1***

correct equation ***A1***

eg $6 - 2k = 0$

$k = 3$ ***AG*** ***N0***

[3 marks]

1c. Find $f'(-2)$. *[2 marks]*

Markscheme

correct substitution into $f'(x)$ ***(A1)***

eg $3(-2)^2 - 6(-2) - 9$

$f'(-2) = 15$ ***A1*** ***N2***

[2 marks]

1d. Find the equation of the tangent to the curve of f at $(-2,\ 1)$, giving your answer in the form $y = ax + b$. *[4 marks]*

Markscheme

recognizing gradient value (may be seen in equation) ***M1***

eg $a = 15,\ y = 15x + b$

attempt to substitute $(-2,\ 1)$ into equation of a straight line ***M1***

eg $1 = 15(-2) + b,\ (y-1) = m(x+2),\ (y+2) = 15(x-1)$

correct working ***(A1)***

eg $31 = b,\ y = 15x + 30 + 1$

$y = 15x + 31$ ***A1*** ***N2***

[4 marks]

1e. Given that $f'(-1) = 0$, explain why the graph of f has a local maximum when $x = -1$. *[3 marks]*

Markscheme

METHOD 1 (2^{nd} derivative)

recognizing $f'' < 0$ (seen anywhere) ***R1***

substituting $x = -1$ into f'' ***(M1)***

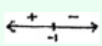
eg $f''(-1),\ 6(-1) - 6$

$f''(-1) = -12$ ***A1***

therefore the graph of f has a local maximum when $x = -1$ ***AG*** ***N0***

METHOD 2 (1^{st} derivative)

recognizing change of sign of $f'(x)$ (seen anywhere) ***R1***

eg sign chart 

correct value of f' for $-1 < x < 3$ ***A1***

eg $f'(0) = -9$

correct value of f' for x value to the left of -1 ***A1***

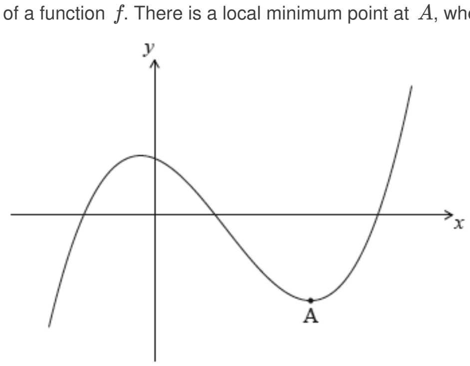
eg $f'(-2) = 15$

therefore the graph of f has a local maximum when $x = -1$ ***AG*** ***N0***

[3 marks]

Total [14 marks]

The following diagram shows the graph of a function f. There is a local minimum point at A, where $x > 0$.



The derivative of f is given by $f'(x) = 3x^2 - 8x - 3$.

2a. Find the x-coordinate of A. *[5 marks]*

Markscheme

recognizing that the local minimum occurs when $f'(x) = 0$ ***(M1)***

valid attempt to solve $3x^2 - 8x - 3 = 0$ ***(M1)***

eg factorization, formula

correct working ***A1***

$(3x + 1)(x - 3),\ x = \frac{8 \pm \sqrt{64+36}}{6}$

$x = 3$ ***A2*** ***N3***

Note: Award ***A1*** if both values $x = \frac{-1}{3},\ x = 3$ are given.

[5 marks]

2b. The y-intercept of the graph is at $(0, 6)$. Find an expression for $f(x)$. *[6 marks]*

The graph of a function g is obtained by reflecting the graph of f in the y-axis, followed by a translation of $\begin{pmatrix} m \\ n \end{pmatrix}$.

Markscheme

valid approach ***(M1)***

$f(x) = \int f'(x)\mathrm{d}x$

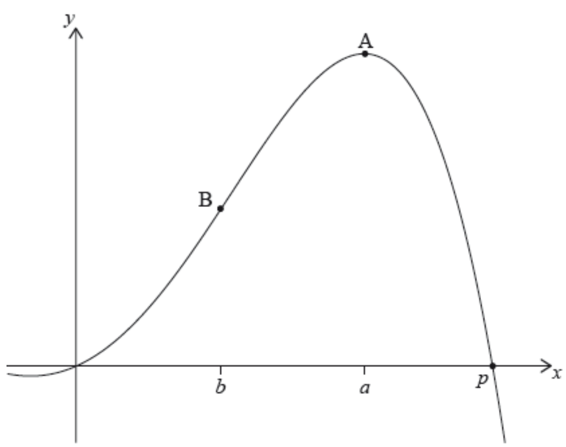
$f(x) = x^3 - 4x^2 - 3x + c$ (do not penalize for missing "$+c$") ***A1A1A1***

$c = 6$ ***(A1)***

$f(x) = x^3 - 4x^2 - 3x + 6$ ***A1*** ***N6***

[6 marks]

Let $f(x) = -0.5x^4 + 3x^2 + 2x$. The following diagram shows part of the graph of f.



There are x-intercepts at $x = 0$ and at $x = p$. There is a maximum at A where $x = a$, and a point of inflexion at B where $x = b$.

3a. Find the value of p. *[2 marks]*

Markscheme

evidence of valid approach ***(M1)***

eg $f(x) = 0,\ y = 0$

2.73205

$p = 2.73$ ***A1*** ***N2***

[2 marks]

3b. Write down the coordinates of A. *[2 marks]*

Markscheme

1.87938, 8.11721

$(1.88,\ 8.12)$ ***A2*** ***N2***

[2 marks]

3c. Write down the rate of change of f at A. *[1 mark]*

Markscheme

rate of change is 0 (do not accept decimals) ***A1*** ***N1***

[1 marks]

3d. Find the coordinates of B. *[4 marks]*

Markscheme

METHOD 1 (using GDC)

valid approach ***M1***

eg $f'' = 0$, max/min on f', $x = -1$

sketch of either f' or f'', with max/min or root (respectively) ***(A1)***

$x = 1$ ***A1*** ***N1***

Substituting **their** x value into f ***(M1)***

eg $f(1)$

$y = 4.5$ ***A1*** ***N1***

METHOD 2 (analytical)

$f'' = -6x^2 + 6$ ***A1***

setting $f'' = 0$ ***(M1)***

$x = 1$ ***A1*** ***N1***

substituting **their** x value into f ***(M1)***

eg $f(1)$

$y = 4.5$ ***A1*** ***N1***

[4 marks]

3e. Find the the rate of change of f at B. *[3 marks]*

Markscheme

recognizing rate of change is f' ***(M1)***

eg $y',\ f'(1)$

rate of change is 6 ***A1*** ***N2***

[3 marks]

3f. Let R be the region enclosed by the graph of f, the x-axis, the line $x = b$ and the line $x = a$. The region R is rotated 360° about the x-axis. Find the volume of the solid formed. *[3 marks]*

Markscheme

attempt to substitute either limits or the function into formula ***(M1)***

involving f^2 (accept absence of π and/or $\mathrm{d}x$)

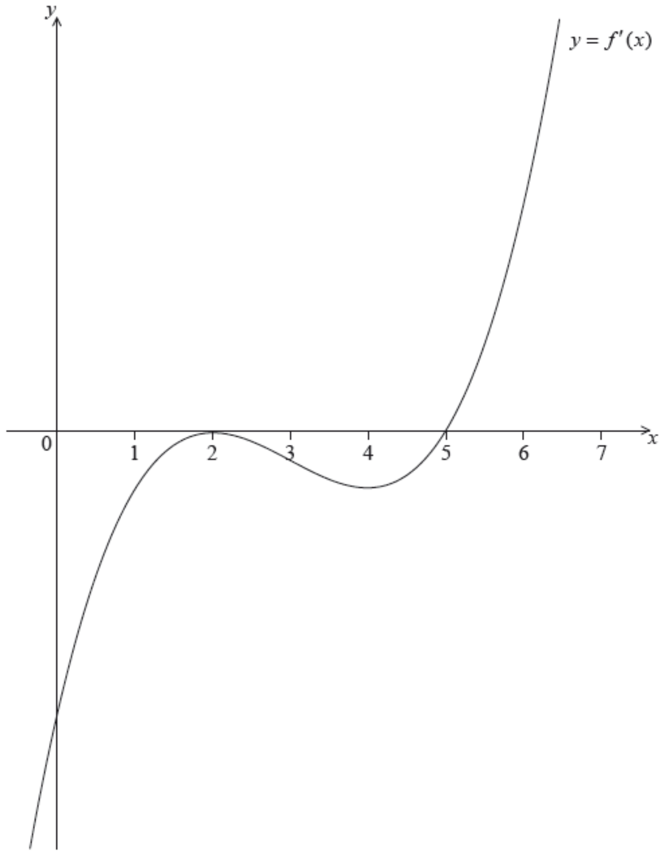
eg $\pi \int (-0.5x^4 + 3x^2 + 2x)^2 \mathrm{d}x,\ \int_1^{1.88} f^2$

128.890

$\text{volume} = 129$ ***A2*** ***N3***

[3 marks]

Let $y = f(x)$, for
$-0.5 \leq$ x
$\leq$
6.5. The following diagram shows the graph of f', the derivative of f.



The graph of f' has a local maximum when $x = 2$, a local minimum when $x = 4$, and it crosses the x-axis at the point
$(5,\ 0)$.

4a. Explain why the graph of f has a local minimum when $x = 5$. *[2 marks]*

Markscheme

METHOD 1

$f'(5) = 0$ ***(A1)***

valid reasoning including reference to the graph of f' ***R1***

eg f' changes sign from negative to positive at $x = 5$, labelled sign chart for f'

so f has a local minimum at $x = 5$ ***AG*** ***N0***

Note: It must be clear that any description is referring to the graph of f', simply giving the conditions for a minimum without relating them to f' does not gain the ***R1***.

METHOD 2

$f'(5) = 0$ ***A1***

valid reasoning referring to second derivative ***R1***

eg $f''(5) > 0$

so f has a local minimum at $x = 5$ ***AG*** ***N0***

[2 marks]

4b. Find the set of values of x for which the graph of f is concave down. *[2 marks]*

Markscheme

attempt to find relevant interval ***(M1)***

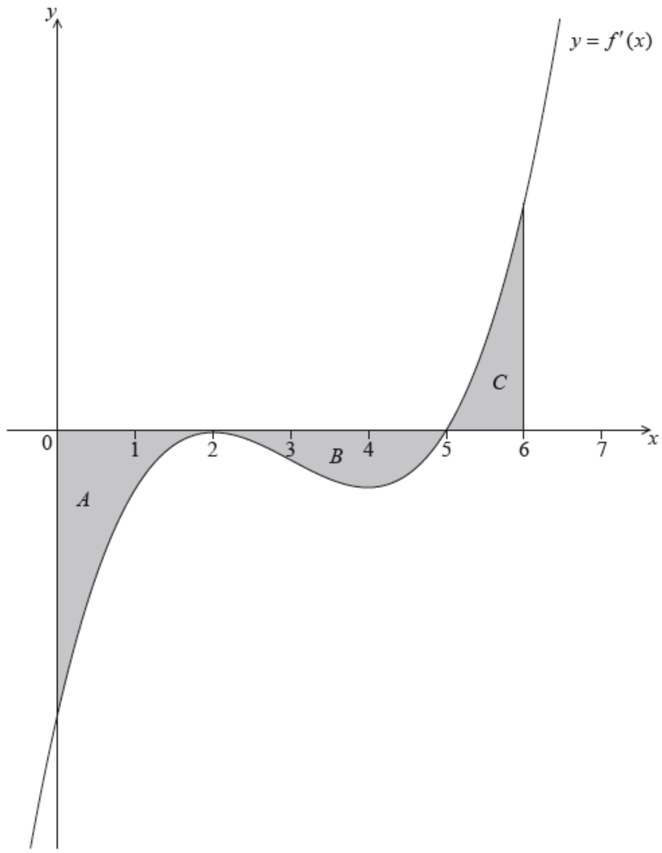
eg f' is decreasing, gradient of f' is negative, $f'' < 0$

$2 < x < 4$ (accept "between 2 and 4") ***A1*** ***N2***

Notes: If no other working shown, award ***M1A0*** for incorrect inequalities such as $2 \le x \le 4$, or "from 2 to 4"

[2 marks]

4c. The following diagram shows the shaded regions A, B and C. *[5 marks]*



The regions are enclosed by the graph of f', the x-axis, the y-axis, and the line $x = 6$.

The area of region A is 12, the area of region B is 6.75 and the area of region C is 6.75.

Given that $f(0) = 14$, find $f(6)$.

Markscheme

METHOD 1 (one integral)

correct application of Fundamental Theorem of Calculus ***(A1)***

eg $\int_0^6 f'(x)\mathrm{d}x = f(6) - f(0),\ f(6) = 14 + \int_0^6 f'(x)\mathrm{d}x$

attempt to link definite integral with areas ***(M1)***

eg $\int_0^6 f'(x)\mathrm{d}x = -12 - 6.75 + 6.75,\ \int_0^6 f'(x)\mathrm{d}x = \text{Area } A + \text{Area } B + \text{ Area } C$

correct value for $\int_0^6 f'(x)\mathrm{d}x$ ***(A1)***

eg $\int_0^6 f'(x)\mathrm{d}x = -12$

correct working ***A1***

eg $f(6) - 14 = -12,\ f(6) = -12 + f(0)$

$f(6) = 2$ ***A1*** ***N3***

METHOD 2 (more than one integral)

correct application of Fundamental Theorem of Calculus ***(A1)***

eg $\int_0^2 f'(x)\mathrm{d}x = f(2) - f(0),\ f(2) = 14 + \int_0^2 f'(x)$

attempt to link definite integrals with areas ***(M1)***

eg $\int_0^2 f'(x)\mathrm{d}x = 12,\ \int_2^5 f'(x)\mathrm{d}x = -6.75,\ \int_0^6 f'(x) = 0$

correct values for integrals ***(A1)***

eg $\int_0^2 f'(x)\mathrm{d}x = -12,\ \int_5^2 f'(x)\mathrm{d}x = 6.75,\ f(6) - f(2) = 0$

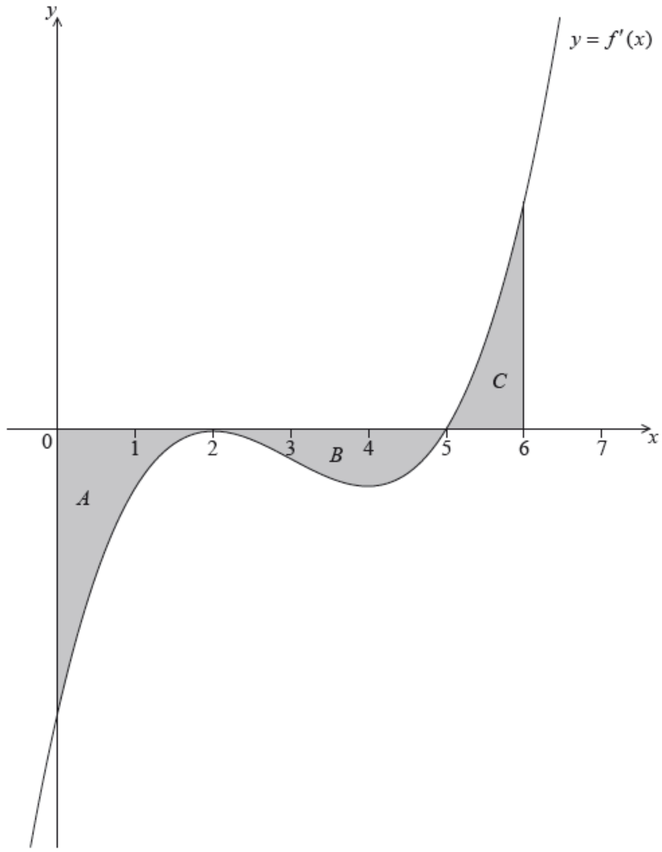
one correct intermediate value ***A1***

eg $f(2) = 2,\ f(5) = -4.75$

$f(6) = 2$ ***A1*** ***N3***

[5 marks]

4d. The following diagram shows the shaded regions A, B and C. *[6 marks]*



The regions are enclosed by the graph of f', the *x*-axis, the *y*-axis, and the line $x = 6$.

The area of region A is 12, the area of region B is 6.75 and the area of region C is 6.75.

Let $g(x) = (f(x))^2$. Given that $f'(6) = 16$, find the equation of the tangent to the graph of g at the point where $x = 6$.

Markscheme

correct calculation of $g(6)$ (seen anywhere) ***A1***

eg $2^2,\ g(6) = 4$

choosing chain rule or product rule ***(M1)***

eg $g'(f(x))\,f'(x),\ \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx},\ f(x)f'(x) + f'(x)f(x)$

correct derivative ***(A1)***

eg $g'(x) = 2f(x)f'(x),\ f(x)f'(x) + f'(x)f(x)$

correct calculation of $g'(6)$ (seen anywhere) ***A1***

eg $2(2)(16),\ g'(6) = 64$

attempt to substitute **their** values of $g'(6)$ and $g(6)$ (in any order) into equation of a line ***(M1)***

eg $2^2 = (2 \times 2 \times 16)6 + b,\ y - 6 = 64(x - 4)$

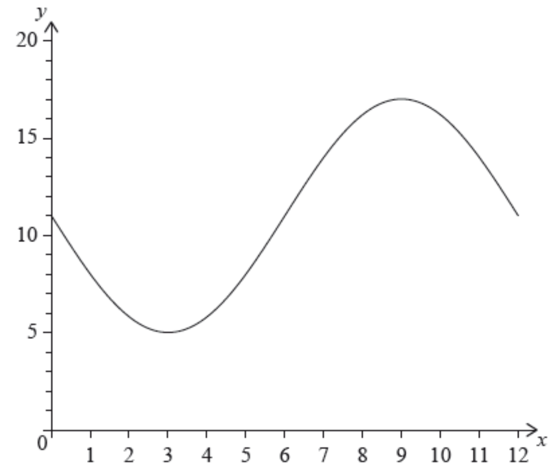
correct equation in any form ***A1*** ***N2***

eg $y - 4 = 64(x - 6),\ y = 64x - 380$

[6 marks]

[Total 15 marks]

The following diagram shows the graph of $f(x) = a\sin bx + c$, for $0 \leqslant x \leqslant 12$.



The graph of f has a minimum point at $(3,\ 5)$ and a maximum point at $(9,\ 17)$.

5a. (i) Find the value of c. *[6 marks]*

(ii) Show that $b = \frac{\pi}{6}$.

(iii) Find the value of a.

Markscheme

(i) valid approach ***(M1)***

eg $\frac{5+17}{2}$

$c = 11$ ***A1 N2***

(ii) valid approach ***(M1)***

eg period is 12, per $= \frac{2\pi}{b}$, $9 - 3$

$b = \frac{2\pi}{12}$ ***A1***

$b = \frac{\pi}{6}$ ***AG N0***

(iii) **METHOD 1**

valid approach ***(M1)***

eg

$5 = a\sin\left(\frac{\pi}{6} \times 3\right) + 11$, substitution of points

$a = -6$ ***A1 N2***

METHOD 2

valid approach ***(M1)***

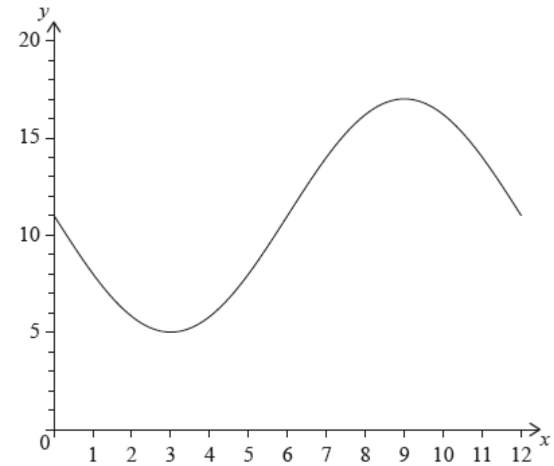
eg

$\frac{17-5}{2}$, amplitude is 6

$a = -6$ ***A1 N2***

[6 marks]

The following diagram shows the graph of $f(x) = a\sin bx + c$, for $0 \leqslant x \leqslant 12$.



The graph of f has a minimum point at $(3,\ 5)$ and a maximum point at $(9,\ 17)$.

The graph of g is obtained from the graph of f by a translation of $\begin{pmatrix} k \\ 0 \end{pmatrix}$. The maximum point on the graph of g has coordinates $(11.5,\ 17)$.

5b. (i) Write down the value of k. *[3 marks]*

(ii) Find $g(x)$.

Markscheme

(i)

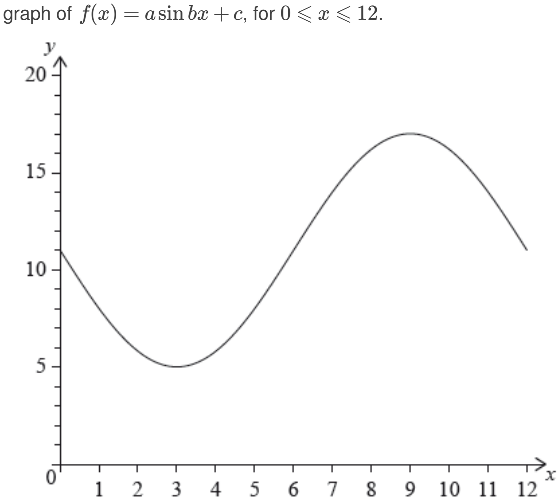
$k = 2.5$ ***A1 N1***

(ii)

$g(x) = -6\sin\left(\frac{\pi}{6}(x - 2.5)\right) + 11$ ***A2 N2***

[3 marks]

The following diagram shows the graph of $f(x) = a\sin bx + c$, for $0 \leqslant x \leqslant 12$.



The graph of f has a minimum point at $(3,\ 5)$ and a maximum point at $(9,\ 17)$.

The graph of g changes from concave-up to concave-down when $x = w$.

5c. (i) Find w. *[6 marks]*

(ii) Hence or otherwise, find the maximum positive rate of change of g.

Markscheme

(i) **METHOD 1** Using g

recognizing that a point of inflexion is required ***M1***

eg
sketch, recognizing change in concavity

evidence of valid approach ***(M1)***

eg

$g''(x) = 0$, sketch, coordinates of max/min on g'

$w = 8.5$ (exact) ***A1*** ***N2***

METHOD 2 Using f

recognizing that a point of inflexion is required ***M1***

eg sketch, recognizing change in concavity

evidence of valid approach involving translation ***(M1)***

eg

$x = w - k$, sketch, $6 + 2.5$

$w = 8.5$ (exact) ***A1*** ***N2***

(ii) valid approach involving the derivative of g or f (seen anywhere) ***(M1)***

eg

$g'(w),\ -\pi\cos\left(\frac{\pi}{6}x\right)$, max on derivative, sketch of derivative

attempt to find max value on derivative ***M1***

eg

$-\pi\cos\left(\frac{\pi}{6}(8.5 - 2.5)\right),\ f'(6)$, dot on max of sketch

3.14159

max rate of change $= \pi$ (exact), 3.14 ***A1*** ***N2***

[6 marks]