Illustrative Mathematics

Solutions

Lesson 13 Practice Problems

1. The population of a town is growing exponentially and can be modeled by the equation $f(t) = 42 \cdot e^{(0.015t)}$. The population is measured in thousands, and time is measured in years since 1950.

   a. What was the population of the town in 1950?

   42,000

   b. What is the approximate percent increase in the population each year?

   1.5%

   c. According to this model, approximately what was the population in 1960?

   $f(10) = 42 \cdot e^{0.015(10)} = 48.797...$ thousand

2. The revenue of a technology company, in thousands of dollars, can be modeled with an exponential function whose starting value is \$395,000 where time t is measured in years after 2010.

   Which function predicts exactly 1.2% of annual growth: $R(t) = 395 \cdot e^{(0.012t)}$ or $S(t) = 395 \cdot (1.012)^t$? Explain your reasoning.

   S(t). The continuous function, R(t) would grow slightly faster

3. How are the functions f and g given by $f(x) = (1.05)^x$ and $g(x) = e^{0.05x}$ similar? How are they different?

   Both have 5% exponential growth starting at zero. f(x) compounds annually (geometric) whereas g(x) is continuous.

4. a. A bond is worth \$100 and grows in value by 4 percent each year. Explain why the value of the bond after t years is given by $100 \cdot 1.04^t$.

From an initial value of 100, it increases by a factor of 1.04 each year

b. A second bond is worth \$100 and grows in value by 2 percent each half year. Explain why the value of the bond after t years is given by $100 \cdot (1.02)^{2t}$.

It compounds semi-annually, so half of the 4%, each six months, or 2%, but ~~twic~~ twice the compounding periods, 2t

c. A third bond is worth \$100 and grows in value by 4 percent each year, but the interest is applied continuously, at every moment. The value of this bond after t years is given by $100 \cdot e^{(0.04t)}$. Order the bonds from slowest growing to fastest growing. Explain how you know.

annually, semi annually, continuous. Because the more frequent the compounding, the greater the value.

5. The population of a country is growing exponentially, doubling every 50 years. What is the annual growth rate? Explain or show your reasoning.

each year $\sqrt[50]{2}$

It is exponential (geometric) and $2^{\frac{1}{50}} = \sqrt[50]{2}$

(From Unit 4, Lesson 6.)

6. Which expression has a greater value: $\log_3 \frac{1}{3}$ or $\log_b \frac{1}{b}$? Explain how you know.

they are both equal (to -1)

(From Unit 4, Lesson 11.)

7. The expression $5 \cdot \left(\frac{1}{2}\right)^d$ models the amount of a radioactive substance, in nanograms, in a sample over time in decades, d. (1 nanogram is a billionth or 1×10^{-9} gram.)

a. What do the 5 and the $\frac{1}{2}$ tell us in this situation?

5 is initial amount

$\frac{1}{2}$ is factor per decade

b. When will the sample have less than 0.5 nanogram of the radioactive substance? Express your answer to the nearest half decade. Show your reasoning.

d	
3	0.625
3.5	0.4419 ←
4	0.3125

$3\frac{1}{2}$ decades

c. Show that only about 5 picograms of the substance will remain one century after the sample is measured. (A picogram is a trillionth or 1×10^{-12} gram.)

(From Unit 4, Lesson 7.)

$f(10) = 5 \cdot \left(\frac{1}{2}\right)^{10}$

$= 0.0048828...$

$\approx$ 5 pico grams

8. Select **all** true statements about the number e.

A. e is a rational number. +

B. e is approximately 2.718.

C. e is an irrational number.

D. e is between π and $\sqrt{2}$ on the number line.

E. e is exactly 2.718. +

(From Unit 4, Lesson 12.)

Solutions

Lesson 14 Practice Problems

1. Solve each equation without using a calculator. Some solutions will need to be expressed using log notation.

   a. $4 \cdot 10^x = 400{,}000$ $x = 5$

   b. $10^{(n+1)} = 1$ $n = -1$

   c. $10^{3n} = 1{,}000{,}000$ $n = 2$

   d. $10^p = 725$ $p = \log 725$

   e. $6 \cdot 10^t = 360$ $t = \log 60$

2. Solve $\frac{1}{4} \cdot 10^{(d+2)} = 0.25$. Show your reasoning.

   $10^{d+2} = 1$

   $d+2 = 0$ $d = -2$

3. Write two equations—one in logarithmic form and one in exponential form—that represent the statement: "the natural logarithm of 10 is y".

   $y = \ln 10$ $e^y = 10$

4. Explain why $\ln 1 = 0$.

   $e^0 = 1$

5. If $\log_{10}(x) = 6$, what is the value of x? Explain how you know.

$x = 100,000$

(From Unit 4, Lesson 9.)

6. For each logarithmic equation, write an equivalent equation in exponential form.

a. $\log_2 16 = 4$ $2^4 = 16$

b. $\log_3 9 = 2$ $3^2 = 9$

c. $\log_5 5 = 1$ $5^1 = 5$

d. $\log_{10} 20 = y$ $10^y = 20$

e. $\log_2 30 = y$ $2^y = 30$

(From Unit 4, Lesson 10.)

7. The function f is given by $f(x) = e^{0.07x}$.

a. What is the continuous growth rate of f? 0.07

b. By what factor does f grow when the input x increases by 1?

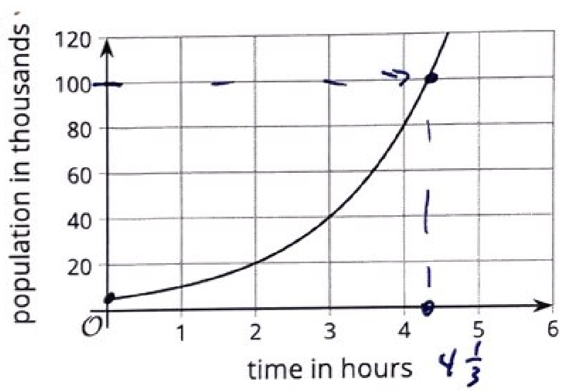
$e^{0.07}$

(From Unit 4, Lesson 13.)

Solutions

Lesson 15 Practice Problems

1. The equation $p(h) = 5{,}000 \cdot 2^h$ represents a bacteria population as a function of time in hours. Here is a graph of the function p.



   a. Use the graph to determine when the population will reach 100,000.

   $4\frac{1}{3}$

   b. Explain why $\log_2 20$ also tells us when the population will reach 100,000.

   $p(h) = 5{,}000 \cdot 2^h = 100{,}000$

   $2^h = 20$

   $h = \log_2 20$

2. *Technology required.* Population growth in the U.S. between 1800 and 1850, in millions, can be represented by the function f, defined by $f(t) = 5 \cdot e^{(0.028t)}$.

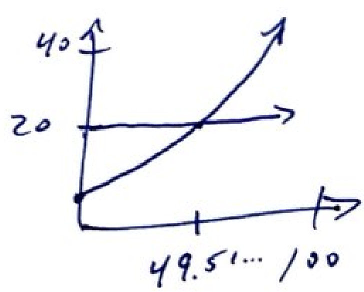
   a. What was the U.S. population in 1800? 5 million

   b. Use graphing technology to graph the equations $y = f(t)$ and $y = 20$. Adjust the graphing window to the following boundaries: $0 < x < 100$ and $0 < y < 40$.

   c. What is the point of intersection of the two graphs, and what does it mean in this situation?

$(49.51..., 20)$

between 1849 and 1850 the population reached 20 million



3. The growth of a bacteria population is modeled by the equation $p(h) = 1{,}000e^{(0.4h)}$. For each question, explain or show how you know.

   a. How long does it take for the population to double?

   $p(h) = 1{,}000e^{0.4h} = 2000$

   $e^{0.4h} = 2$

   $\ln 2 = 0.4h$

   $h = \frac{\ln 2}{0.4} = 1.73286...$

   b. How long does it take for the population to reach 1,000,000?

   $p(h) = 1{,}000e^{0.4h} = 1{,}000{,}000$

   $e^{0.4h} = 1{,}000$

   $0.4h = \ln 1000$

   $h = \frac{\ln 1000}{0.4} = 17.269...$

4. What value of b makes each equation true?

   a. $\log_b 144 = 2$ $b=12$

   b. $\log_b 64 = 2$ $b = 8$

   c. $\log_b 64 = 3$ $b = 4$

   d. $\log_b 64 = 6$ $b = 2$

   e. $\log_b \frac{1}{9} = -2$ $b = 3$

(From Unit 4, Lesson 10.)

5. Put the following expressions in order, from least to greatest.

   (3,4) (1,2) 2 3 (2,3)

   $\log_2 11$ $\log_3 5$ $\log_5 25$ $\log_{10} 1{,}000$ $\log_2 5$

   $\log_3 5$, $\log_5 25$, $\log_2 5$, $\log_{10} 1000$, $\log_2 11$

(From Unit 4, Lesson 11.)

6. Solve $9 \cdot 10^{(0.2t)} = 900$. Show your reasoning.

$10^{0.2t} = 100$

$0.2t = 2$

$t = 10$

(From Unit 4, Lesson 14.)

7. Explain why $\ln 4$ is greater than 1 but is less than 2.

$\cancel{1 = e^0}$

$e^1 \approx 2.718 < e^{\ln 4} = 4 < e^2 \approx 8$

so $1 < \ln 4 < 2$

(From Unit 4, Lesson 14.)