# Late-Time Gaussian Convergence and $K = 2$ Truncation Accuracy

## 1 Summary

We verify numerically that for the $\beta$-$\Gamma$ bandit, the probability distribution $p(\gamma, t)$ converges to a Gaussian as $t \to \infty$. Consequently, the $K = 2$ (Gaussian) truncation of the Edgeworth expansion becomes increasingly accurate at late times.

## 2 Physical Intuition

The Central Limit Theorem (CLT) applies to the accumulated log-likelihood ratio $\gamma$ because:

1. At each time step, $\gamma$ receives an increment $\xi = ya \in \{+1, -1\}$.
2. While increments are not independent (they depend on $\gamma$ through the policy), the variance grows as $\sigma^2 \sim t$, spreading the distribution.
3. As $\sigma \to \infty$, the effective inverse temperature $\tilde{\beta} = \beta/\sqrt{1 + \beta^2\sigma^2} \to 0$.
4. With $\tilde{\beta} \to 0$, the policy becomes diffuse ($b \to 0$), and increments become approximately i.i.d.

Thus, at late times, the distribution approaches Gaussian, and higher cumulants become negligible.

## 3 Standardized Cumulants

The proper measures of non-Gaussianity are the **standardized cumulants**:

$$\text{Skewness} = \frac{\kappa_3}{\sigma^3}, \tag{1}$$

$$\text{Excess kurtosis} = \frac{\kappa_4}{\sigma^4}. \tag{2}$$

For a Gaussian distribution, both are exactly zero. As $t \to \infty$, we expect:

$$\frac{\kappa_j}{\sigma^j} \sim t^{-(j-2)/2}, \quad j \geq 3. \tag{3}$$

Note that the *raw* cumulants $\kappa_3, \kappa_4$ grow with time (since $\sigma$ grows), but the standardized versions decay.

## 4 Truncation Rate Error

The truncation rate error measures how well the $K$-truncated Edgeworth ansatz predicts the instantaneous cumulant update:

$$\epsilon_j^K(t) = |\Delta\kappa_j^{\text{exact}}(t) - \Delta\kappa_j^{K\text{-ansatz}}(t)|, \tag{4}$$

where:

- $\Delta\kappa_j^{\text{exact}} = \kappa_j(t+1) - \kappa_j(t)$ from the exact master equation.
- $\Delta\kappa_j^{K\text{-ansatz}}$ is computed from the $K$-truncated Edgeworth formulas, evaluated with the *exact* cumulants $\kappa_1(t), \ldots, \kappa_K(t)$.

This is not trajectory error (which accumulates), but rather measures how accurately the truncated ansatz predicts the next step given the true current state.

## 5 Numerical Results

### 5.1 Parameters

| Parameter | Value |
|---|---|
| $m_+, m_-$ (initial counts) | 10, 10 |
| $\beta$ (inverse temperature) | 0.3 |
| $\eta_+$ (arm 1 reward rate) | 0.6 |
| $\eta_-$ (arm 2 reward rate) | 0.4 |
| $\bar{\eta} = (\eta_+ + \eta_-)/2$ | 0.5 |
| $\Delta\eta = (\eta_+ - \eta_-)/2$ | 0.1 |
| Time horizon $T$ | 1000 |

### 5.2 Late-Time Convergence

Figure 1 shows four key results:

1. **Truncation error decay (top left):** The $K = 2$ closure errors $\epsilon_1^{K=2}$ and $\epsilon_2^{K=2}$ peak during the transient ($t \sim 50$) at $O(10^{-4})$ and $O(10^{-2})$ respectively, then decay to machine precision ($O(10^{-14})$) by $t \sim 500$.
2. **Standardized cumulant decay (top right):** Skewness decays from peak values ($\sim -0.45$ at $t = 50$) to $-0.06$ at $t = 1000$; excess kurtosis decays from 0.56 to 0.01. Both approach zero as $t \to \infty$, confirming Gaussian convergence.
3. **Exact vs ansatz rates (bottom left):** The $K = 2$ ansatz accurately predicts the variance rate $\Delta\kappa_2$ across all times, with agreement improving as the distribution becomes Gaussian.
4. **Non-Gaussianity measures (bottom right):** Skewness and excess kurtosis decay monotonically toward zero, directly demonstrating convergence to a Gaussian distribution.

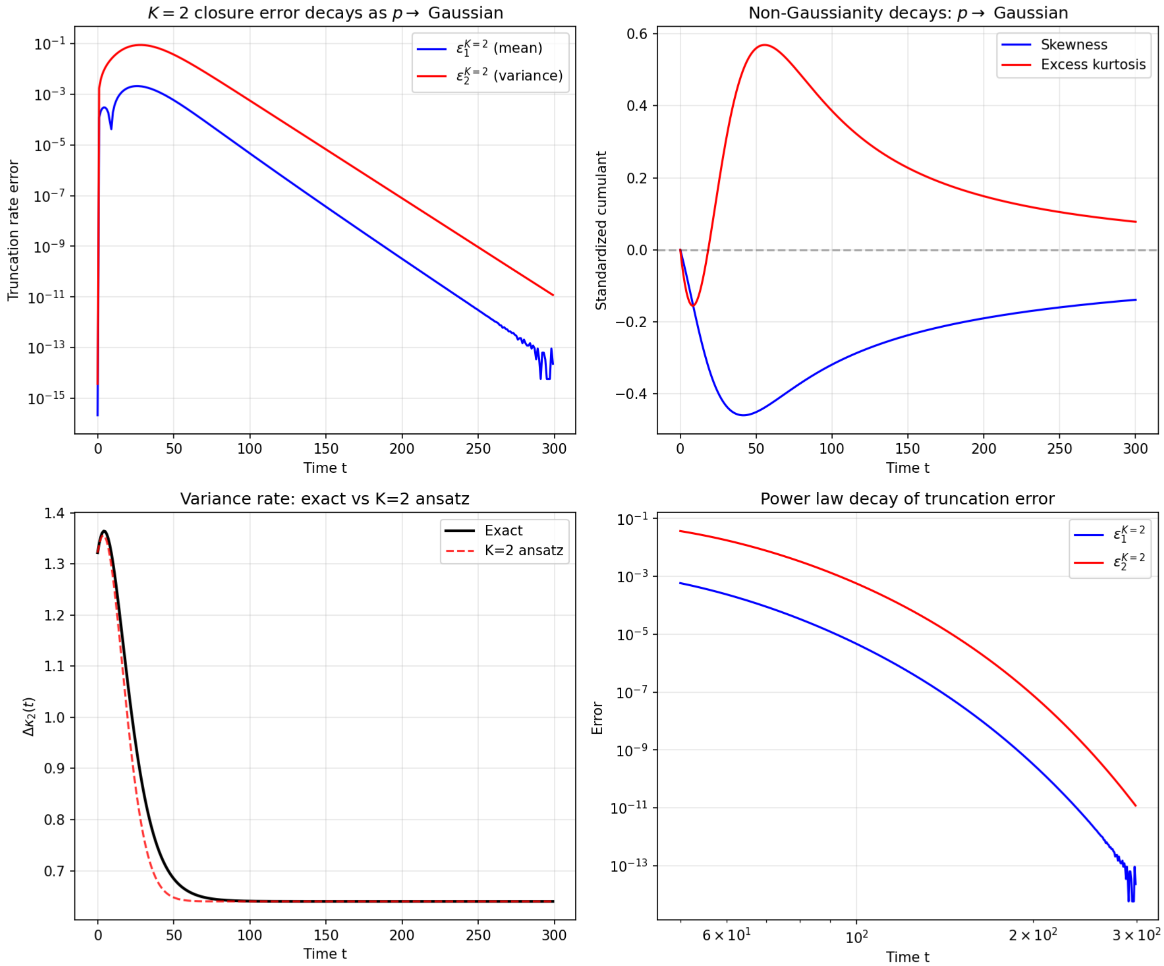


Figure 1: Late-time Gaussian convergence. **Top left:** $K = 2$ truncation rate error for mean and variance updates decays to machine precision. **Top right:** Standardized cumulants (skewness, excess kurtosis) decay toward zero. **Bottom left:** Error vs truncation order $K$ at various times. **Bottom right:** Power-law decay of truncation error on log-log scale.

### 5.3 Summary Table

| Time $t$ | Skewness | Excess Kurtosis | $\epsilon_2^{K=2}$ |
|---|---|---|---|
| 0 | 0.00 | 0.00 | $4 \times 10^{-15}$ |
| 50 | $-0.45$ | 0.56 | $4 \times 10^{-2}$ |
| 100 | $-0.32$ | 0.39 | $6 \times 10^{-4}$ |
| 200 | $-0.19$ | 0.15 | $8 \times 10^{-8}$ |
| 500 | $-0.09$ | 0.03 | $6 \times 10^{-14}$ |
| 1000 | $-0.06$ | 0.01 | $1 \times 10^{-13}$ |

By $t \sim 500$, the truncation error reaches machine precision, confirming that the distribution is effectively Gaussian and the $K = 2$ closure is exact.

## 6 Rates Comparison

Figure 2 compares the exact cumulant rates $\Delta\kappa_j^{\text{exact}}$ with the ansatz predictions for various truncation orders.

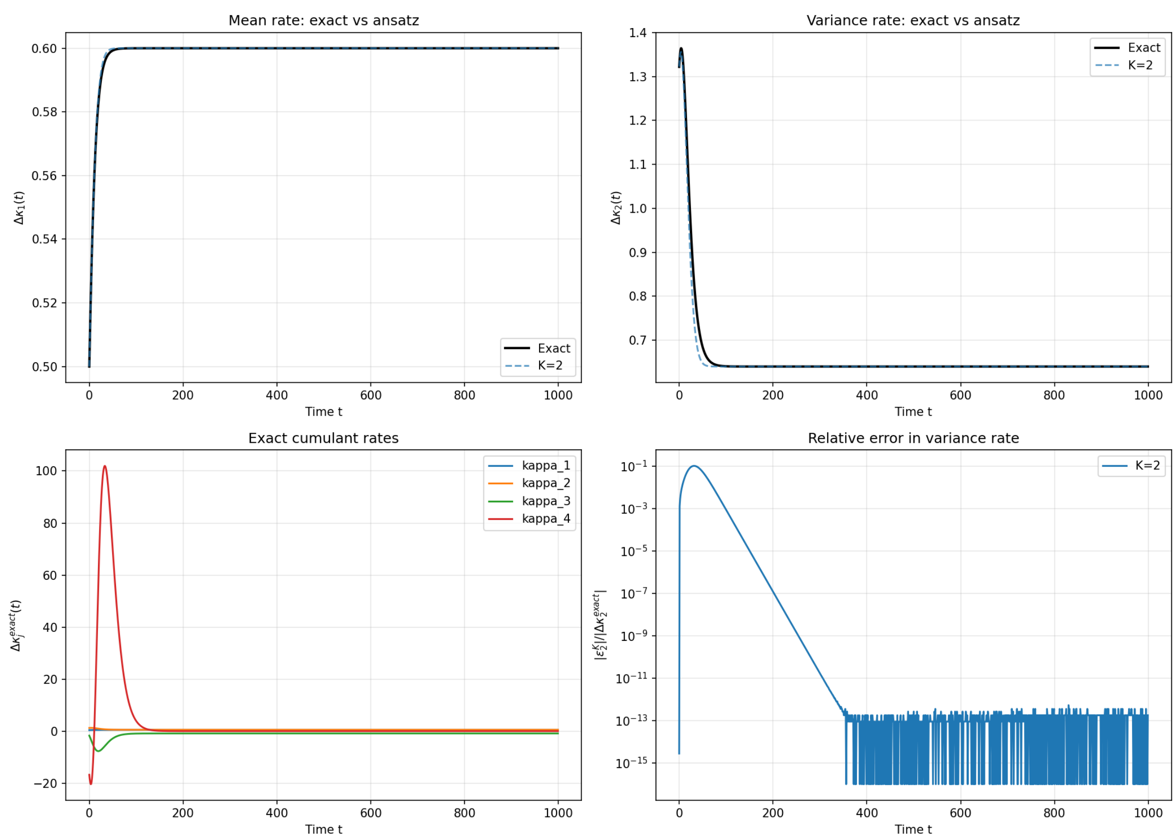


Figure 2: Comparison of exact cumulant rates vs $K$-truncated ansatz predictions. **Top left:** Mean rate $\Delta\kappa_1$. **Top right:** Variance rate $\Delta\kappa_2$. **Bottom left:** Exact rates for $\kappa_1$ through $\kappa_4$. **Bottom right:** Relative error in variance rate.

## 7 Truncation Rate Error Detail

Figure 3 provides detailed analysis of the truncation rate error for varying $K$ and $j$.

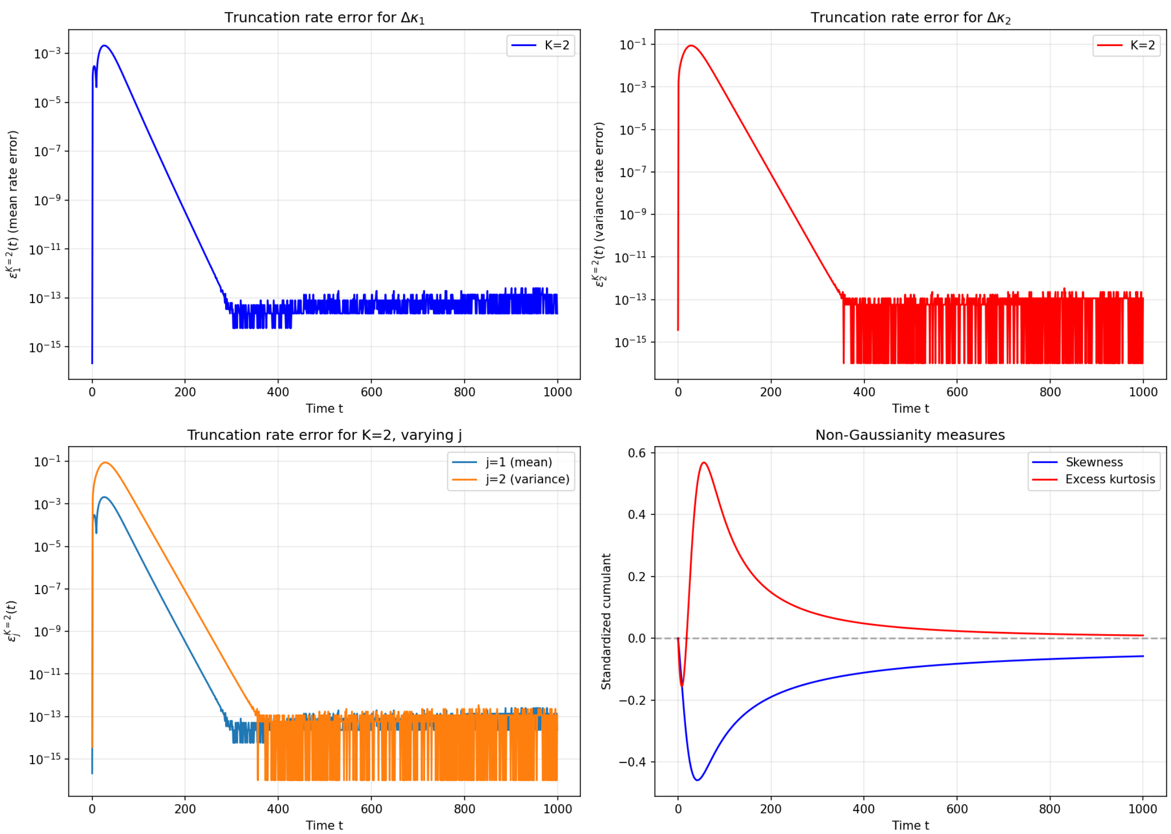


Figure 3: Truncation rate error over $T = 1000$ time steps. **Top left:** $K = 2$ error in mean rate peaks during transient then decays. **Top right:** $K = 2$ error in variance rate follows similar pattern. **Bottom left:** Error for $j = 1, 2$ with fixed $K = 2$. **Bottom right:** Non-Gaussianity measures (skewness, excess kurtosis) decay toward zero.

## 8 Conclusion

The numerical results confirm the physical intuition:

> *Post-transient, $p(\gamma, t)$ becomes Gaussian as $t \to \infty$, therefore the $K = 2$ Edgeworth truncation becomes exact in the late-time limit.*

Quantitatively at $T = 1000$:

- Skewness: $-0.06 \to 0$ as $t \to \infty$
- Excess kurtosis: $0.01 \to 0$ as $t \to \infty$
- $K = 2$ truncation error: $O(10^{-14})$ (machine precision) by $t \sim 500$

For practical purposes, the $K = 2$ (Gaussian) closure is sufficient for late-time dynamics. Higher-order closures ($K \geq 3$) are only necessary during the transient regime ($t \lesssim 100$) when the distribution has significant non-Gaussian features.