The instantaneous absorbed fraction $A(x, y, t)$ for a given column density $n(x, y, t)$ can be found by solving:

$$\sigma_0 n(x, y, t) = -\alpha^\star \log(1 - A(x, y, t)) + \frac{I}{I_{\text{sat}}} A(x, y, t). \tag{1}$$

This will serve as our expression for $A(x, y, t)$ and will need to be solved numerically. The integrated absorbed fraction after a time τ is just the average of this:

$$A_{\text{av}}(x, y) = \frac{1}{\tau} \int_0^\tau A(x, y, t)\, \mathrm{d}t. \tag{2}$$

But because of diffraction we don't have access to this quantity, only its integral along the y direction:

$$A_{\text{meas}}(x) = \int A_{\text{av}}(x, y) dy \tag{3}$$

$$\Rightarrow A_{\text{meas}}(x) = \frac{1}{\tau} \int \mathrm{d}y \int_0^\tau A(x, y, t)\, \mathrm{d}t. \tag{4}$$

Given a model for $n(x, y, t)$ with a single parameter for each x position and a measurement $A_{\text{meas}}(x)$ for each x position, we can invert (4) and (1) to find the parameter at each x position.

The model for $n(x, y, t)$ is that it's a Gaussian with a time-dependent width in the y direction, parametrised by its linear density $n_{\text{1D}}(x)$ at each x position:

$$n(x, y, t) = \frac{n_{\text{1D}}(x)}{\sqrt{2\pi\sigma_y^2(t)}} \exp\left[-\frac{y^2}{2\sigma_y^2(t)}\right] \tag{5}$$

Where $\sigma_y^2(t)$ is the mean squared y position of the scatterers, which, assuming pure momentum diffusion starting from a size $\sqrt{\sigma_0/\pi}$ equal to the radius of a circle with area equal to the scattering cross section $\sigma_0 = 3\lambda^2/2\pi$ is:

$$\sigma_y^2(t) = \frac{1}{3}\sigma_{v_y}^2(t)\, t^2 + \frac{\sigma_0}{\pi}. \tag{6}$$

The mean squared y velocity $\sigma_{v_y}^2(t)$ is given by the scattering rate $R_{\text{scat}}$and recoil velocity v_{rec}:

$$\sigma_{v_y}^2(t) = \frac{1}{3}(2\pi)^{-1} R_{\text{scat}} v_{\text{rec}}^2 t \tag{7}$$

which is assuming isotropic scattering such that the per scattering event expected squared change in y velocity is $v_{\text{rec}}^2/3$.

The scattering rate, assuming resonance (ignoring Doppler shifting out of resonance) is:

$$R_{\text{scat}} = \frac{\Gamma}{2} \frac{I/I_{\text{sat}}}{1 + I/I_{\text{sat}}} \tag{8}$$

Putting it all together for the mean squared y position:

$$\sigma_y^2(t) = \frac{\sigma_0}{\pi} + \frac{1}{18\pi} R_{\text{scat}} v_{\text{rec}}^2 t^3 \tag{9}$$