MONOPOLE AND GROUP HIGGS QUANTUM INTEGRABLE SYSTEM, AND Q-OPERS

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1. Group Higgs bundles from Hecke stacks

Let G be a reductive group, X be an algebraic complex curve. We do not assume that G is simple or simply-connected, there could be abelian factors and non-trivial center.

Let $\check{P}$ be the co-weight lattice of G, and let $\check{P}_+ \subset \check{P}$ be the subset of dominant weights. Let $\check{Q} \subset \check{P}$ be the lattice of coroots. Notice that rank of $\check{P}$ differes from the rank of $\check{Q}$ by the dimension of the abelian factor in G. Let $\rho = \sum_{\alpha \in \Delta_+}$, where Δ_+ is the set of positive roots be the Weyl vector.

Following 5.2.1 of Beilinson-Drinfeld [1] let Hecke stack be defined as follows

Definition 1.

$$\mathsf{Hecke}_{G,\underline{x}} = \text{moduli stack of } (P_1, P_2, g) \tag{1.1}$$

- P_1, P_2 are principal G-bundles on X
- $\underline{x} \in X^n$ denotes n marked points $(x_1, \dots, x_n)$ with $x_i \in C$
- $g : P_1 \to P_2$ is an isomorphism over $C \setminus \cup_{i \in I} \{x_i\}$

As explained in 5.2.3 of [1], the stratification teh affine grassmanian Gr_G induces the stratification of the stack $\mathsf{Hecke}_{G,\underline{x}}$ by substacks $\mathsf{Hecke}_{G,\underline{x},\underline{w}}$ where elements $\underline{w}$ of $\check{P}^n_+$ are n-tuples of co-weights $\underline{w} = (w_1, \dots, w_n)$ with $w_i \in \check{P}_{G,+}$.

Framed version is defined similarly. Let x_∞ be a point on $X = \mathbb{P}^1$.

Definition 2.

$$\mathsf{Hecke}^{\mathrm{fr}}_{G,\underline{x}} = \text{moduli stack of } (P_1, P_2, g) \tag{1.2}$$

where

- P_1, P_2 are principal G-bundles on X framed at x_∞.
- $\underline{x} \in X^n$ denotes n marked points $(x_1, \dots, x_n)$ with $x_i \in X \setminus \{x_\infty\}$
- $g : P_1 \to P_2$ is an isomorphism over $X \setminus \cup_{i \in I} \{x_i\}$

Then $\pi^{\mathrm{fr}} : \mathsf{GrHiggs}^{\mathrm{fr}}_{G,\underline{x},\underline{w}} \to G$ is the natural evaluation map at the framing point x_∞

$$\pi^{\mathrm{fr}} : (P_1, P_2, \underline{x}, g) \mapsto g_\infty \tag{1.3}$$ {eq:pifr}

with $g_\infty = g(x_\infty)$.

Let

$$p : \mathsf{Hecke}^{\mathrm{fr}}_{G,\underline{x}} \to \mathsf{Bun}^{\mathrm{fr}}_G \times \mathsf{Bun}^{\mathrm{fr}}_G \tag{1.4}$$

be the natural projection that forgets g. Let

$$\Delta : \mathsf{Bun}^{\mathrm{fr}}_G \to \mathsf{Bun}^{\mathrm{fr}}_G \times \mathsf{Bun}^{\mathrm{fr}}_G \tag{1.5}$$

be the diagonal map

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Definition 3. The moduli stack of multiplicative Higgs bundles is the pullback of p and Δ

$$\mathsf{GrHiggs}^{\mathrm{fr}}_{G,\underline{x},\underline{w}} = \mathsf{Hecke}^{\mathrm{fr}}_{G,\underline{x},\underline{w}} \times_{\mathsf{Bun}^{\mathrm{fr}}_G \times \mathsf{Bun}^{\mathrm{fr}}_G} \times \mathsf{Bun}^{\mathrm{fr}}_G \tag{1.6}$$

For $X = \mathbb{P}^1$, let λ be a section of $K_X = O(-2x_\infty)$ whose only singularity is the second order pole at x_∞.

Let $w = \sum_{i=1}^n w_i$ denote the total coweight. On $X = \mathbb{P}^1$ there is isomorphism

$$\mathsf{Bun}^{\mathrm{fr}}_G = \check{P}/\check{Q} \tag{1.7}$$

Proposition 1. Fix $(X = \mathbb{P}^1, \underline{x}, \underline{w}, x_\infty)$ and λ as above

(1) If $w \notin \check{Q}$ then $\mathsf{GrHiggs}_{G,\underline{x},\underline{w}}$ is empty, otherwise
(2) $\mathsf{GrHiggs}^{\mathrm{fr}}_{G,\underline{x},\underline{w}}$ is affine algebraic variety with canonical Poisson structure induced by λ and Killing form () on $\mathfrak{g}$, and

$$\dim \mathsf{GrHiggs}^{\mathrm{fr}}_{G,\underline{x},\underline{w}} = \dim G + 2(\rho, w) \tag{1.8}$$

Morover, the fibers $\mathsf{GrHiggs}^{\mathrm{fr}}_{G,\underline{x},\underline{w},g_\infty}$ of (1.3) $\pi^{\mathrm{fr}} : \mathsf{GrHiggs}^{\mathrm{fr}}_{G,\underline{x},\underline{w}} \to G$ are symplectic leaves of the dimension $2(\rho, w)$ in the Poisson variety $\mathsf{GrHiggs}^{\mathrm{fr}}_{G,\underline{x},\underline{w}}$
(3) There is birational symplectomorphism

$$\sigma : (\mathbb{C} \times \mathbb{C}^\times)^{(\rho,w)} \to \mathsf{GrHiggs}^{\mathrm{fr}}_{G,\underline{x},\underline{w},g_\infty} \tag{1.9}$$ {eq:coord}

where symplectic form on $\mathbb{C} \times \mathbb{C}^\times$ is $dx \wedge \frac{dy}{y}$
(4) There is natural Poisson embedding $\mathsf{GrHiggs}^{\mathrm{fr}}_{G,\underline{x},\underline{w}} \to G(O_{x_0}) \simeq G[[x]]$ given by the formal series expansion of $g(x)$ near any point x_0 distinct from $\underline{x}$ and x_∞, where x is a coordinate in a formal disc around x_0 such that $x(x_0) = 0$ and $\lambda = dx$, and where Poisson structure on $G[[x]]$ is determined by Manin triple $\mathfrak{g}((x)) = x^{-1}\mathfrak{g}[[x^{-1}]] \oplus \mathfrak{g}[[x]]$ with respect to the pairing $f, g \in \mathfrak{g}((x))$ defined by

$$\langle f, g \rangle = \oint (f, g)\lambda \tag{1.10}$$

where () is Killing form on $\mathfrak{g}$. In particular, the agreement of Poisson structure with $G[[O_{x_0}]]$ does not depend on the choice of point x_0.
(5) the fibers $\mathsf{GrHiggs}^{\mathrm{fr}}_{G,\underline{x},\underline{w},g_\infty}$ are symplectic leaves in Poisson-Lie group $G[[x]]$
(6) To each $\mathsf{GrHiggs}^{\mathrm{fr}}_{G,\underline{x},\underline{w},g_\infty}$ corresponds an ideal in the ϵ-quantized algebra of function $\mathcal{F}_\epsilon(G[[x]])$

$$I_{\mathsf{GrHiggs}^{\mathrm{fr}}_{G,\underline{x},\underline{w},g_\infty}} \subset \mathcal{F}_\epsilon(G[[x]]) \tag{1.11}$$

and there is representation of the quotient algebra

$$\sigma^*_\epsilon : \mathcal{F}_\epsilon(G[[x]])/I_{\mathsf{GrHiggs}^{\mathrm{fr}}_{G,\underline{x},\underline{w},g_\infty}} \to \mathcal{D}_\epsilon(\mathbb{C}^{(\rho,w)}) \tag{1.12}$$

in the algebra of ϵ-difference operators on $\mathbb{C}^{(\rho,\omega)}$ obtained from the coordinate map σ (1.9).

2. First examples of group Higgs bundles

2.1. $GL(2)$-group Higgs bundle. Let $G = GL_2$, let the curve $X = \mathbb{CP}^1$ with the coordinate x, the 1-form $\lambda = dx$ and the framing point $x_\infty = \infty$.

Consider component of $\mathsf{GrHiggs}^{\mathrm{fr}}$ which projects to a trivial G-bundle on X.

Fix framing of trivial G-bundle at infinity. Then trivialization of G-bundle is fixed everywhere on X. Then $\mathsf{GrHiggs}^{\mathrm{fr}}$ is identified with G-valued rational functions $g(x)$ on X with certain conditions that we will identify explicitly.

As a first explicit example we consider the fiber $\mathsf{GrHiggs}^{\mathrm{fr}}_{G,\underline{x},\underline{w},g_\infty}$ with two singularities at points $x_0 \neq x_\infty$ and $x_1 \neq x_\infty$ with $x_0 \neq x_1$. Then $\underline{x} = (x_0, x_1)$ and let co-weights be $\underline{w} = (w_0, w_1)$ where $w_0 = (1, 0)$ and $w_1 = (0, -1)$ in the defining basis of GL_2.

Proposition 2. The fiber $\mathsf{GrHiggs}^{\mathrm{fr}}_{G,\underline{x},\underline{w},g_\infty}$ with $G = GL_2$ and $\underline{x} = (x_0, x_1)$ and $w_0 = (1, 0)$ and $w_{-1} = (0, -1)$ is identified with the space of functions $g(x)$ valued in 2×2 matrices

$$g(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \tag{2.1}$$

where $a(x), b(x), c(x), d(x)$ are rational functions on X such that

(1) $a(x), b(x), c(x), d(x)$ are regular everywhere on $X \setminus x_1$, in particular they are regular at $x_\infty = \infty$ and x_0.

(2) $g(x_\infty) = g_\infty$ where $g_\infty \in GL_2$ is a fixed element

$$g_\infty = \begin{pmatrix} a_\infty & b_\infty \\ c_\infty & d_\infty \end{pmatrix}, \qquad a_\infty d_\infty - c_\infty b_\infty \neq 0 \tag{2.2}$$

where $a_\infty, b_\infty, c_\infty, d_\infty \in \mathbb{C}$

(3)

$$\det g(x) = \frac{x - x_0}{x - x_1} \det g_\infty \tag{2.3}$$

The conditions (1), (2), (3) imply that $a(x), b(x), c(x), d(x)$ have the form

$$a(x) = \frac{a_\infty x - a_0}{x - x_1} \quad b(x) = \frac{b_\infty x - b_0}{x - x_1} \quad c(x) = \frac{c_\infty x - c_0}{x - x_1} \quad d(x) = \frac{d_\infty x - d_0}{x - x_1} \tag{2.4}$$

where $(a_0, b_0, c_0, d_0) \in \mathbb{C}^4$, such that

$$(a_\infty x - a_0)(d_\infty x - d_0) - (b_\infty x - b_0)(c_\infty x - c_0) = (x - x_0)(x - x_1)(a_\infty d_\infty - b_\infty c_\infty) \tag{2.5}$$

The above equation translates into the system of linear equation and quadric equation on $(a_0, b_0, c_0, d_0) \in \mathbb{C}^4$

$$\begin{aligned} \mathsf{GrHiggs}^{\mathrm{fr}}_{G,\underline{x},\underline{w},g_\infty} = \big\{ &(a_0, b_0, c_0, d_0) \in \mathbb{C}^4 | \\ &- a_0 d_\infty - a_\infty d_0 + b_0 c_\infty + b_\infty c_0 = (-x_0 - x_1)(a_\infty d_\infty - b_\infty c_\infty), \\ &a_0 d_0 - b_0 c_0 = x_0 x_1 (a_\infty d_\infty - b_\infty c_\infty) \big\} \end{aligned} \tag{2.6}$$

We conclude, that in this example $\mathsf{GrHiggs}^{\mathrm{fr}}_{G,\underline{x},\underline{w},g_\infty}$ is a complete intersection of a hyperplane and a quadric on $\mathbb{C}^4$, equivalently a quadric on $\mathbb{C}^3$. For example, say $(a_\infty, b_\infty, c_\infty, d_\infty) = (1, 0, 0, 1)$, and $x_0 = -m, x_1 = m$, then the linear equation implies $d_0 = -a_0$, and the quadratic equation gives a canonical form of smooth affine quadric surface

$$a_0^2 + b_0 c_0 = m^2 \tag{2.7}$$

on $\mathbb{C}^3 = (a_0, b_0, c_0)$.

Remark 1. In the limit when the singularities x_0 and x_1 collide, that is $m = 0$, the quadric becomes singular $a_0^2 + b_0 c_0 = 0$. The resolved ingularity on a quadric by blow-up is identified with $T^*\mathbb{CP}^1$. The m-deformed quadric $a_0^2 + b_0 c_0 = m^2$ can be identified with affine line bundle over $\mathbb{CP}^1$. This $\mathbb{CP}^1$ is the orbit of the fundamental miniscule weight in the affine Grassmanian GL_2, so $\mathsf{GrHiggs}^{\mathrm{fr}}_{G,\underline{x},\underline{w},g_\infty}$ with $G = GL_2$ and $\underline{x} = (x_0, x_1)$ and $w_0 = (1, 0)$ and $w_{-1} = (0, -1)$ is identified with the space of functions $g(x)$ valued in 2×2 matrices

$$g(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \tag{2.8}$$

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where $(a_0, b_0, c_0, d_0) \in \mathbb{C}^4$, such that

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The above equation translates into the system of linear equation and quadric equation on $(a_0, b_0, c_0, d_0) \in \mathbb{C}^4$

$$\begin{aligned} \mathsf{GrHiggs}^{\mathrm{fr}}_{G,\underline{x},\underline{w},g_\infty} = \big\{ &(a_0, b_0, c_0, d_0) \in \mathbb{C}^4 | \\ &- a_0 d_\infty - a_\infty d_0 + b_0 c_\infty + b_\infty c_0 = (-x_0 - x_1)(a_\infty d_\infty - b_\infty c_\infty), \\ &a_0 d_0 - b_0 c_0 = x_0 x_1 (a_\infty d_\infty - b_\infty c_\infty) \big\} \end{aligned} \tag{2.13}$$

We conclude, that in this example $\mathsf{GrHiggs}^{\mathrm{fr}}_{G,\underline{x},\underline{w},g_\infty}$ is a complete intersection of a hyperplane and a quadric on $\mathbb{C}^4$, equivalently a quadric on $\mathbb{C}^3$. For example, say $(a_\infty, b_\infty, c_\infty, d_\infty) = (1, 0, 0, 1)$, and $x_0 = -m, x_1 = m$, then the linear equation implies $d_0 = -a_0$, and the quadratic equation gives a canonical form of smooth affine quadric surface

$$a_0^2 + b_0 c_0 = m^2 \tag{2.14}$$

on $\mathbb{C}^3 = (a_0, b_0, c_0)$.

Remark 2. In the limit when the singularities x_0 and x_1 collide, that is $m = 0$, the quadric becomes singular $a_0^2 + b_0 c_0 = 0$. The resolved ingularity on a quadric by blow-up is identified with $T^*\mathbb{CP}^1$. The m-deformed quadric $a_0^2 + b_0 c_0 = m^2$ can be identified with affine line bundle over $\mathbb{CP}^1$. This $\mathbb{CP}^1$ is the orbit of the fundamental miniscule weight in the affine Grassmanian GL_2. We see that $\mathsf{GrHiggs}^{\mathrm{fr}}_{G,\underline{x},\underline{w},g_\infty}$ in the case of two miniscule co-weight singularities for GL_2 is an affine line bundle over the flag variety $\mathbb{CP}^1$, where, locally, the 1-dimensional base comes from the insertion of one singularity, and 1-dimensional fiber comes from the insertion of the other in the iterative definition of Hecke stack.

2.2. $GL(r)$ **group Higgs bundles on** $\mathbb{P}^1$ **framed at infinity with** nr **Dirac singularities of the co-fundamental type** $\omega_1^\vee$**.** For a positive integer n, by $[n]$ we denote the set $[n] = \{1, 2, \dots, n\}$.

Let $G = GL(r, \mathbb{C})$, and T_G be a maximal torus in G. Let $(\omega_i)_{i\in[r]}$ be fundamental weights $\omega_i : T_G \to \mathbb{C}^\times$, and $\omega_i^\vee$ be fundamental co-weights $\omega_i^\vee : \mathbb{C}^\times \to T_G$. We will use multiplicative notations so that $\omega_i^\vee : z \mapsto z^{\omega_i^\vee}$. In the standard defining representation of $GL(r)$ we have $\omega_i^\vee = (\underbrace{1, \dots, 1}_{i}, \underbrace{0, 0, \dots, 0}_{r-i})$

Definition 4. We consider the moduli space $\mathsf{GrHiggs}_{G,C,D_{nr[\omega_1^\vee]},g_\infty}$ labelled by the data (group G, curve $C = \mathbb{P}^1$ with marked point $z_\infty \in C$ called 'infinity', degree nr divisor $D_{nr[\omega_1^\vee]} = (z_{k,i})_{k\in[n],i\in[r]} \in C$ called 'positions of Dirac singularities', fixed value $g_\infty \in G$ called 'value at infinity') of the pairs (P, g) where

- P is a principal G-bundle on C with a fixed framing at z_∞
- $g(z)$ is a rational section of the group adjoint bundle $g \in \Gamma(C, P \times_G \operatorname{Ad} G)$ such that
  - $g(z)$ is holomorphic everywhere away from $D_{nr[\omega_1^\vee]}$ and z_∞
  - near $z_{i,k}$ there exist holomorphic sections $\tilde{g}^L_{i,k}(z), \tilde{g}^R_{i,k}(z)$ such that

$$g(z) = \tilde{g}^L_{i,k} \cdot (z - z_{i,k})^{\omega_1^\vee} \cdot \tilde{g}^R_{i,k}(z) \qquad \text{as} \qquad z \to z_{i,k} \tag{2.15}$$

  - near z_∞ there exists holomorphic section $\tilde{g}_\infty(z)$ such that

$$g(z) = (z - z_\infty)^{-n\omega_r^\vee} \tilde{g}_\infty \qquad \text{as} \qquad z \to z_\infty \tag{2.16}$$

  and $\tilde{g}_\infty(z_\infty) = g_\infty(z)$

Notice that in the definition of the moduli space we fix values $\tilde{g}_\infty(z_\infty)$ but not $\tilde{g}^L_{i,k}(z_{i,k}), \tilde{g}^R_{i,k}(z_{i,k})$. Also notice that $(z - z_\infty)^{-n\omega_r^\vee}$ is central so it would not matter if we had written either $(z - z_\infty)^{-n\omega_r^\vee} \tilde{g}_\infty(z)$ or $\tilde{g}_\infty(z)(z - z_\infty)^{-n\omega_r^\vee}$. Also we shall assume that g_∞ is simple regular element, and that all $z_{i,k} \in D_{nr[\omega_1^\vee]}$ are different.

Proposition 3. The moduli space $\mathsf{GrHiggs}_{G,C,D_{nr[\omega_1^\vee]},g_\infty}$ of the definition 4 is isomorphic to the space of $r \times r$ complex matrix valued polynomials of degree n

$$p(z) = \sum_{k=0}^{n} p_k z^{n-k}, \qquad g_k \in \operatorname{Mat}_{r\times r}(\mathbb{C}) \tag{2.17}$$

such that

- $p_0 = \rho_{\omega_1}(g_\infty)$ where ρ_{ω_1} is the fundamental representation
- the determinant is fixed

$$\det p(z) = \det p_0 \prod_{z_* \in D_{nr[\omega_1^\vee]}} (z - z_*) \tag{2.18}$$ {eq:det}

Proposition 4. The moduli space $\mathsf{GrHiggs}_{G,C,D_{nr[\omega_1^\vee]},g_\infty}$ is affine algebraic variety defined in the affine space $\mathbb{C}^{nr^2}$ of the coefficients $(p_{k,\alpha,\beta})_{k\in[n],\alpha,\beta\in[r]}$ by the set of nr polynomial equations of degree r (2.18).

Proposition 5. The complex dimension of $\mathsf{GrHiggs}_{G,C,D_{nr[\omega_1^\vee]},g_\infty}$ is $n(r^2-r)$.

Proposition 6. The variety $\mathsf{GrHiggs}_{G,C,D_{nr[\omega_1^\vee]},g_\infty}$ is algebraic symplectic leaf in the meromorphic Poisson Lie loop group of G-valued meromorphic functions on $\mathbb{C} = \mathbb{P}^1 \backslash z_\infty$ with symplectic structure induced from the classical rational r-matrix of Manin triple $\mathfrak{g}[z,z^{-1}] = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ with $\mathfrak{g}_+ = \mathfrak{g}[[z]]$ and $\mathfrak{g}_- = z^{-1}\mathfrak{g}[[z^{-1}]]]$.

Lemma 1. The moduli space $\mathsf{GrHiggs}_{G,C,D_{nr[\omega_1^\vee]},g_\infty}$ is algebraic integrable system. The complete system of Poisson commuting Hamiltonians is formed by the coefficients $(t_{i,k})_{i\in[r],k\in[ni]}$ of the polynomial characters

$$T_i(z) = \operatorname{tr}_{\rho_{\omega_i}}(p(z)), \qquad i \in [r-1] \tag{2.19}$$

in the z-expansion so that $T_i(z) = \sum_{k=0}^{in} t_{i,k} z^{in-k}$. The character map $t : \mathsf{GrHiggs}_{G,C,D_{nr[\omega_1^\vee]},g_\infty} \to \mathfrak{B}$ defines the structure of fibration of algebraic integrable system $\mathsf{GrHiggs}_{G,C,D_{nr[\omega_1^\vee]},g_\infty}$ over the affine base space $\mathfrak{B}$ with coordinates $(t_{i,k})_{i\in[r],k\in[ni]}$. The complex dimension of the base $\mathfrak{B}$ is $\sum_{i=1}^{r-1} ni = \frac{1}{2}n(r^2-r)$.

Question 1. Let $G = GL(n,\mathbb{C})$, $C = \mathbb{P}^1$ with marked point $z_\infty \in C$, $C' = C \setminus z_\infty$.

For arbitrary divisor D valued in the co-weight lattice (a set D of pairs $(z,\omega^\vee)$ where $z \in C'$, $\omega^\vee$ is in the co-weight lattice), co-weight $\omega^\vee_{z_\infty}$ at the point z_∞, and element $g_\infty \in G$ what is a precise definition of the framed moduli space $\mathsf{GrHiggs}_{G,C,D,\omega^\vee_{z_\infty},g_\infty}$? What is the dimension of $\mathsf{GrHiggs}$, is it symplectic, is is algebraic integrable system? Can we describe it explicitly as an affine variety? What if G is an arbitrary reductive group?

Question 2. How to modify the above definition of $\mathsf{GrHiggs}$ if C is an elliptic curve, possibly with nodal or cusp singularity, and prove the similar propositions?

2.3. **Moduli space of framed $GL(r)$ group Higgs bundles on $\mathbb{CP}^1$ regular at the framing point.** Let $G = GL(r,\mathbb{C})$, let $C = \mathbb{CP}^1$ with a marked point $z_\infty \in \mathbb{CP}^1$. Let $C' = C \setminus z_\infty$.

A co-weight lattice of $GL(r)$ in a defining basis of the fundamental representation is identified with $\mathbb{Z}^r$ on which the Weyl group S_r acts by the permutations of the components.

A coweight $\omega^\vee$ with components $(w_1,\dots,w_r) \in \mathbb{Z}^r$ maps $\mathbb{C}^\times \to (\mathbb{C}^\times)^r$ as

$$z \mapsto (z^{w_1},\dots,z^{w_r}) \tag{2.20}$$

Let det be the determinant morphism $GL(r,\mathbb{C}) \to \mathbb{C}^\times$, which induces trace morphism from the co-weight lattice $\mathbb{Z}$ of $GL(r,\mathbb{C})$ to the co-weight lattice $\mathbb{Z}$ of $\mathbb{C}^\times$ given by the sum of the components

$$\operatorname{tr} : (w_1,\dots w_r) \mapsto \sum_{k=1}^r w_k \tag{2.21}$$

We call a co-weight $\omega^\vee$ of $GL(r)$ dominant if its components form non-increasing sequence of integers of length r

$$w_1 \ge w_2 \dots \ge w_r, \qquad (w_i)_{i\in[r]} \in \mathbb{Z} \tag{2.22}$$

which is a 'generalized' (what is a proper adjective for this notion?) partition in a sence that the components may be non-positive integers. For any co-weight $\omega^\vee$ there is a unique dominant co-weight representative $\operatorname{dom}[\omega^\vee]$ in the Weyl orbit $[\omega^\vee]$.

Let ρ denote the Weyl vector $\rho := \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ where Δ^+ is the set of positive roots. In the $\mathbb{Z}^r$ basis dual to the co-weight lattice ρ has components

$$\rho = \tfrac{1}{2}(r-1, r-3, \ldots, 1-r) \tag{2.23}$$

Now let D be a divisor on C' valued in the co-weight lattice of G. An element of D is a pair $(z, \omega^\vee)$ with $z \in C'$ and $\omega^\vee$ in the co-weight lattice.

Lemma 2. If and only if D is such that

$$\sum_{(z,\omega^\vee)\in D} \operatorname{tr}\omega^\vee = 0 \tag{2.24}$$

then there is 'classical' (vs derived) moduli space $\mathsf{GrHiggs}_{G,C,D,g_\infty}$ of group Higgs bundles with Dirac singularities at D and framing g_∞ at z_∞ of complex dimension

$$\dim \mathsf{GrHiggs}_{G,C,D,g_\infty} = 2\sum_{(z,\omega^\vee)\in D}(\rho, \operatorname{dom}[\omega^\vee]) \tag{2.25}$$ {eq:dim}

I think the equation (2.25) comes from index theorems and in the derived sense it should hold universally? However, can we classify D such that $\mathsf{GrHiggs}_{G,C,D}$ is a classical algebraic variety?

Educated guess: The (derived) tangent bundle to $\mathsf{GrHiggs}_{G,C,D,g_\infty}$ is identified with

$$T\mathsf{GrHiggs} = \sum_{\alpha\in\Delta^+} H^\bullet(C, F_\alpha) - H^\bullet(z_\infty, i^*_{z_\infty}F_\alpha) \tag{2.26}$$ {eq:THiggs}

where Δ^+ is the set of positive roots, and F_α is the line bundle

$$F_\alpha = L(\alpha\cdot\operatorname{dom}[D]) \tag{2.27}$$

where $(\alpha\cdot\operatorname{dom}[D])$ is $\mathbb{Z}$-valued divisor obtained by evaluation by a root α on the co-weight valued divisor $\operatorname{dom}[D]$.

I don't know how to arrive to 2.26 and at which step of index computation we would get $\alpha > 0$ and $\operatorname{dom}[\omega^\vee]$? In fact I would prefer some Weyl invariant formula (without operation dom)

2.4. The moduli space $\mathsf{GrHiggs}_{C,G,D,\omega^\vee_\infty,g_\infty}$ with singularity at infinity.

Definition 5. Let G be a reductive group and $g_\infty \in G$ be a semi-simple regular element. Let $T \subset G$ be a maximal torus in G defined as the adjoint centralizer of $g_\infty \in G$, then $g_\infty \in T \subset G$. The torus T defines the weight and co-weight lattice.

Let $C = \mathbb{CP}^1$ be a curve with marked point $z_\infty \in C$, $C' = C \setminus \{z_\infty\}$, D be a divisor on C' valued in the co-weight lattice of G, and $D_\infty = (z_\infty, \omega^\vee_\infty)$ be a co-weight valued divisor supported at z_∞. Then $\mathsf{GrHiggs}_{C,D,\omega^\vee_\infty,g_\infty}$ is the moduli space of group Higgs bundles (P, g) where P is a G-principal bundle on C framed at z_∞ and g is a rational section of group adjoint bundle AdP such that

(1) restriction of $g(z)$ to $C' \setminus D$ is holomorphic
(2) For each $(z_i, \omega^\vee_i) \in D$ there exists sufficiently small punctured disc $\mathbb{D}_i$ around z_i and holomorphic sections $\tilde{g}_{i,L}, \tilde{g}_{i,R} \in \Gamma(\mathbb{D}_i, \mathrm{Ad}P)$ such that in $\mathbb{D}^\times_i = \mathbb{D}_i \setminus z_i$ it holds

$$g(z) = \tilde{g}_{i,L}(z)(z-z_i)^{\omega^\vee_i}\tilde{g}_{i,R}(z) \tag{2.28}$$

(3) There exists sufficiently small disc $\mathbb{D}_\infty$ around z_∞ and a holomorphic section $\tilde{g}_\infty(z) \in \Gamma(\mathbb{D}_\infty, \mathrm{Ad}P)$ such that on $\mathbb{D}_\infty^\times$ it holds that

$$g(z) = \tilde{g}_\infty(z)(z - z_\infty)^{\omega_\infty^\vee} \tag{2.29}$$

and $\tilde{g}_\infty(z_\infty) = g_\infty$.

For a co-weight $\omega^\vee$ let $\bar{\omega}^\vee$ denote a (unique) dominant weight in the Weyl orbit $[\omega^\vee]$. An element of co-weight lattice

$$\alpha = \sum_{(z,\omega^\vee)\in D+D_\infty} \bar{\omega}^\vee \tag{2.30}$$

is called total charge.

Proposition 7. If α belongs to a co-root lattice then the moduli space $\mathsf{GrHiggs}_{C,D,\omega^\infty,g_\infty}$ is a classical symplectic algebraic variety of dimension

$$\dim_{\mathbb{C}} \mathsf{GrHiggs}_{C,D,\omega^\infty,g_\infty} = 2(\rho, \alpha) \tag{2.31}$$

3. Group Higgs bundles from derived geometry

Let G be an arbitrary reductive group, and let (C, D_∞) be a curve with a divisor such that there exists an effective divisor E making $2D_\infty + E$ anticanonical. Concretely there are three possibilities: either C is an elliptic curve and $D_\infty = 0$, $C = \mathbb{CP}^1$ and $D_\infty = 0 - \infty$ (the nodal case), or $C = \mathbb{CP}^1$ and $D_\infty = (\infty)$ (the cuspidal case). According to Spaide [**?**], there is an AKSZ theorem for maps out of C framed along D_∞. I claim something stronger will be true.

(In general this condition on the existence of anticanonical $2D_\infty + E$ is necessary for Spaide's proof: he shows that $\mathrm{Map}_{\mathrm{fr}}(D_\infty + E, D_\infty, X) \to \mathrm{Map}(D_\infty + E, X)$ is Lagrangian precisely when this condition is satisfied.)

Claim 1. If X is k-shifted symplectic and $D \subseteq C$ is a reduced effective divisor of degree d disjoint from D_∞ then the canonical restriction map from the mapping space framed along D_∞

$$\mathrm{Map}_{\mathrm{fr}}(C\backslash D, D_\infty, X) \to \mathrm{Map}((\mathbb{D}^\times)^D, X)$$

is k-shifted Lagrangian. Here $(\mathbb{D}^\times)^D$ is a formal punctured neighbourhood of D.

In out case, take X to be the adjoint quotient G/G, a 1-shifted symplectic stack. Denote the framing at D_∞ by g_∞. We choose a point in $\mathrm{Map}((\mathbb{D}^\times)^D, G/G)$, namely a set of d germs of G-valued meromorphic functions, or d conjugacy classes in LG. Take this point to be given by a set of conjugacy classes of coweights: $([\omega_{z_1}^\vee], \dots, [\omega_{z_d}^\vee])$. The following is true in general.

Claim 2. The point

$$([\omega_{z_1}^\vee], \dots, [\omega_{z_d}^\vee]) \to \mathrm{Map}((\mathbb{D}^\times)^D, G/G)$$

is 1-shifted Lagrangian.

Combining the two claims we have the following.

Definition 6. The space of multiplicative G-Higgs bundles on C, framed by g_∞ along D_∞, with Dirac singularities at the divisor D with prescribed residues conjugate to $\omega^\vee_{z_1}, \ldots, \omega^\vee_{z_d}$, is the derived fiber product

$$\mathrm{GHiggs}_{G,\{\omega_{z_i}\}}(C, g_\infty) = \mathrm{Map}((\mathbb{D}^\times)^D, G/G) \times_{\mathrm{Map}((\mathbb{D}^\times)^D, G/G)} ([\omega^\vee_{z_1}], \ldots, [\omega^\vee_{z_d}]).$$

This fiber product is 0-shifted symplectic according to the two claims above. If $C = \mathbb{CP}^1$ then the moduli space admits connected components indexed by G-bundles on $\mathbb{CP}^1$, or equivalently by dominant coweights.

Each connected component should be an irreducible affine variety provided that the derived fiber product above is actually classical (i.e. provided the higher cohomology of the tangent complex vanishes). This will happen as long as d is sufficiently large.

We can define a natural map

$$\mathrm{GHiggs}_{G,\{\omega_{z_i}\}}(C, g_\infty) \to \mathfrak{B} = \mathrm{Map}((\mathbb{D}^\times)^D, T/W) \times_{\mathrm{Map}((\mathbb{D}^\times)^D, T/W)} ([\omega^\vee_{z_1}], \ldots, [\omega^\vee_{z_d}]).$$

Claim 3. This map makes $\mathrm{GHiggs}_{G,\{\omega_{z_i}\}}(C, g_\infty)$ into an algebraic integrable system.

This claim is clear away from the discriminant locus, where $G_{\mathrm{rss}}/G \cong T_{\mathrm{reg}}/W \times BT/W$. It's also clear in the case where $d = 0$, but in general in the case where $d > 0$ it needs further argument.

4. Monopoles

Let $C' = \mathbb{CP}^1 \backslash z_\infty$ be complex affine line which we identify with $\mathbb{C} = \mathbb{R}^2$ Let $X = C' \times S^1$ be the real three-dimensional flat Riemannian manifold with flat Riemannian metric $dz d\bar{z} + dy^2$ where z is a coordinate on C' and $y \in \mathbb{R}/2\pi\mathbb{Z}$ is a coordinate on S^1. Let $x = (z, y)$ with $z \in \mathbb{C}, y \in \mathbb{R}$ denote a coordinate on X. Let $G_c = U(r)$ be the maximal compact subgroup of $G = GL(r, \mathbb{C})$. Let P_c be a principal G_c bundle on X with a connection $d_A = d + A$ and fixed framing at the boundary at infinity $\partial C' \times S^1$. Let ϕ be $\mathfrak{g}_c$ valued Higgs field $\phi \in \Gamma(X, P_c \times_{G_c} \mathrm{ad}\, \mathfrak{g}_c)$.

Definition 7. A monopole on X is a solution to Bogomolny equation

$$\star F_A = d_A \phi \tag{4.1}$$

Definition 8. A monopole has fixed Dirac singularity at point $x_* \in X$ of co-weight $\omega^\vee$ if near x_* there exists a gauge such that $\mathfrak{g}_c$-valued Higgs field $\phi(x)$ has singularity of the form

$$\phi(x) = \frac{\omega^\vee(\sqrt{-1})}{2|x - x_*|} + \text{finite analytic} \tag{4.2}$$

Here $\omega^\vee(\sqrt{-1})$ denotes the image of $\sqrt{-1} \in \mathrm{Lie}(U(1))$, where $U(1)$ is identified with the group of unitary complex numbers, and co-weight $\omega^\vee$ is a Lie algebra homomorphism $\omega^\vee : \mathfrak{u}(1) \to \mathfrak{g}_c$.

Definition 9. A monopole has charge $\omega^\vee_{x_\infty}$ and asymptotics $g_\infty \in G$ if

$$A_y + \sqrt{-1}\phi = \frac{1}{2\pi}\left(\omega^\vee_{x_\infty}(\log z) + \log g_\infty + \mathcal{O}(|z|^{-1})\right), \qquad z \to \infty \tag{4.3}$$

Let $D_{nr[\omega_1^\vee]}$ be a set of pairwise different nr points in X_3 colored by the fundamental co-weight $\omega_1^\vee$.

Definition 10. The moduli space $\mathsf{Mon}(G_c, X, D_{nr[\omega_1^\vee]}, \omega_{x_\infty}^\vee, g_\infty)$ is the moduli space of framed at infinity monopole solutions to Bogomolny equations which are smooth away from $D_{nr[\omega_1^\vee]}$, have Dirac singularities of fundamental co-weight $\omega_1^\vee$ at $D_{nr[\omega_1^\vee]}$ and have charge $\omega_{x_\infty}^\vee = n\omega_r^\vee$ and asymptotics g_∞ at infinity.

Proposition 8. The moduli space $\mathsf{Mon}(G_c, X, D_{nr[\omega_1^\vee]}, \omega_{x_\infty}^\vee, g_\infty)$ is HyperKähler variety. The twistor sphere $\mathbb{P}^1_\zeta$ is identified with a unit sphere $S^2 \simeq \mathbb{P}^1$ in the tangent space T_X.

Definition 11. In complex structure $\zeta = \partial_y$ the holomorphic variables of the hyperKähler reduction are $A_y + \sqrt{-1}\phi$ and $A_{\bar z}$. By Mon_ζ we denote holomorphic symplectic variety constructed from the HyperKähler structure on Mon restricted to the complex structure ζ.

Lemma 3. The moduli space $\mathsf{Mon}_{\partial_y}(G, C' \times S^1, D_{nr[\omega_1^\vee]}, \omega_{x_\infty}^\vee, g_\infty)$ is complex symplectomorphic to $\mathsf{GrHiggs}(G, C, D'_{nr[\omega_1^\vee]}, \omega_{x_\infty}^\vee, g_\infty)$ where D' denotes the image of D under the S^1-forgetting projection map $C' \times S^1 \to C'$, so that $(z, y) \mapsto z$. The complex dimension of $\mathsf{Mon}(G, C' \times S^1, D_{nr[\omega_1^\vee]}, \omega_{x_\infty}^\vee, g_\infty)$ is $n(r^2 - r)$.

For $X = C' \times S^1$ and a point $y \in S^1$ let $C'_y = C' \times \{y\}$ be a plane at point y. The holomorphic structure of the principal holomorphic vector bundle P in $\mathsf{GrHiggs}$ is induced by the connection $A_{\bar z}$ in the monopole solution restricted to C'_y. The group Higgs field $g(z)$ is identified with the holonomy of the monopole connection $\partial_y + A_y + \sqrt{-1}\phi$ around S^1 fiber in X at point z

$$g(z) = \mathrm{hol}_{S^1_z}(A_y + \sqrt{-1}\phi) \tag{4.4}$$

References

[1] A. Beilinson and V. Drinfeld, "Quantization of hitchins integrable system and hecke eigensheaves,". 1