Notes on Higgs Moduli

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November 10, 2017

1 Properties of the Multiplicative Higgs Stack

Let me try to refine and correct the description I gave before of multiplicative Higgs bundles, which wasn't quite correct.

Definition 1.1. The derived moduli stack $\mathrm{GpHiggs}_G(C;D)$ of *multiplicative Higgs bundles* on C with singularities at the effective divisor $D = \{z_1, \ldots, z_k\}$ (where the z_i are distinct) is the derived fiber product

$$\mathrm{GpHiggs}_G(C;D) = \underline{\mathrm{Map}}(C, BG) \times_{\underline{\mathrm{Map}}(C \backslash D, BG)} \underline{\mathrm{Map}}(C \backslash D, G/G),$$

modelling a G-bundle on C with a section of the multiplicative adjoint bundle with singularities permitted at the divisor D.

Remark 1.2. One can replace all the mapping spaces with relative mapping spaces for a divisor D_∞ in C disjoint from D to define a moduli stack of multiplicative Higgs bundles with a *framing* at D_∞. We'll be most interested in the example $C = \mathbb{CP}^1$ with a framing at ∞.

Definition 1.3. The derived moduli stack of multiplicative Higgs bundles with prescribed singularities $(\omega^\vee_{z_1}, \ldots, \omega^\vee_{z_k})$ at D is the derived fiber product

$$\mathrm{GpHiggs}_G(C; D, \omega^\vee) = \mathrm{GpHiggs}_G(C;D) \times_{\mathrm{Bun}_G(\mathbb{B})^k} (B\Lambda_1 \times \cdots \times B\Lambda_k)$$

where $\mathrm{GpHiggs}_G(C;D)$ first maps down to the adjoint quotient $G(K)/_{\mathrm{ad}}G(\mathcal{O})$ by restricting to a punctured neighbourhood of each singularity, and then in turn to the double quotient stack $\mathrm{Bun}_G(\mathbb{B}) = G(\mathcal{O}) \setminus G(K)/G(\mathcal{O})$. We take the fiber product with the k-tuple of left $G(\mathcal{O})$-orbits in $\mathrm{Gr}_G = G(K)/G(\mathcal{O})$ corresponding to the k-tuple of coweights $\omega^\vee = (\omega^\vee_{z_1}, \ldots, \omega^\vee_{z_k})$, whose $G(\mathcal{O})$-stabilizers are $(\Lambda_1, \ldots, \Lambda_k)$ (infinite-dimensional groups whose quotients by the unipotent subgroup $G(z\mathcal{O})$ are Levi subgroups of G).

There's a natural map *into* the moduli stack of multiplicative Higgs bundles from the stack of k^{th} order Hecke modifications with equal source and target, defined as follows.

Definition 1.4. The stack of *Hecke modifications* on a curve C at a divisor $D = \{z_1, \ldots, z_k\}$ is the fiber product

$$\mathrm{Hecke}_G(C;D) = \mathrm{Bun}_G(C) \times_{BG(\mathcal{O}_{z_1})} \mathrm{Bun}_G(C) \times_{BG(\mathcal{O}_{z_2})} \cdots \times_{BG(\mathcal{O}_{z_k})} \mathrm{Bun}_G(C)$$

modelling sequences of $(k+1)$ G-bundles $P_1, \ldots, P_{k+1}$ on C with isomorphisms $P_i|_{C \setminus z_i} \cong P_{i+1}|_{C \setminus z_i}$. We can view this stack as a k-iterated Gr_G-bundle over $\mathrm{Bun}_G(C)$ whose projection to a $G(\mathcal{O}) \setminus \mathrm{Gr}_G$-bundle is trivial – at each step one can restrict to a $G(\mathcal{O})$-orbit in the fiber direction – say at the i^{th} step the orbit corresponding to the coweight $\omega^\vee_{z_i}$ – to obtain a stack $\mathrm{Hecke}_G(C; D, \omega^\vee)$ of Hecke modifications on D of type $\omega^\vee$.

If we want we can modify this definition to include a *framing* at a divisor D_∞ on C disjoint from D by replacing G-bundles by framed G-bundles (or D_∞-pointed maps into BG) at the initial and final steps. We no longer just have an iterated Gr_G-bundle over $\mathrm{Bun}^{\mathrm{fr}}_G(C)$, but additionally a G^{D_∞}-bundle at the last step corresponding to the choice of framing.

Proposition 1.5. There is a natural map of derived stacks

$$c\colon \mathrm{Hecke}_G(C; D, \omega^\vee) \times_{\mathrm{Bun}_G(C)^2} \mathrm{Bun}_G(C) \to \mathrm{GpHiggs}_G(C; D, \omega^\vee),$$

where on the left we map the Hecke stack into $\mathrm{Bun}_G(C^2)$ by projecting onto the first and last G-bundles in the chain, the form the derived intersection with the diagonal (i.e. set the first and last G-bundles to be equal). The map c is given by composing the k bundle isomorphisms to obtain an automorphism of the restriction $P_1|_{C\setminus D}$.

Proof sketch. There's an obvious projection map $\mathrm{Hecke}_G(C; D, \omega^\vee) \times_{\mathrm{Bun}_G(C)^2} \mathrm{Bun}_G(C) \to \mathrm{Bun}_G(C)$. Before fixing the coweights we need to give a map $\mathrm{Hecke}_G(C; D) \to \underline{\mathrm{Map}}(C\setminus D, G/G)$. We identify this latter stack with the derived loop space $\mathcal{L}\,\mathrm{Bun}_G(C\setminus D)$ of $\mathrm{Bun}_G(C\setminus D)$. Convolution defines a map $\mathrm{Hecke}_G(C; D) \to (\mathrm{Bun}_G(C) \times_{BG(\mathcal{O}_D)} \mathrm{Bun}_G(C)) \times_{\mathrm{Bun}_G(C)^2} \mathrm{Bun}_G(C)$, which maps to $\mathrm{Bun}_G(C\setminus D) \times_{\mathrm{Bun}_G(C\setminus D)^2} \mathrm{Bun}_G(C\setminus D) = \mathcal{L}\,\mathrm{Bun}_G(C\setminus D)$ as required.

Now, turning on the i^{th} coweight on the Hecke side means fixing a component after projecting the Gr_G-bundle to a $G(\mathcal{O}) \setminus \mathrm{Gr}_G$-bundle. Viewing this i^{th} trivial $G(\mathcal{O}) \setminus \mathrm{Gr}_G$-bundle structure as a projection map to $G(\mathcal{O}) \setminus \mathrm{Gr}_G$ we identify the choice of a $G(\mathcal{O})$-orbit with the choice of a coweight in definition 1.3.

We finally need to check that the two maps down to $\underline{\mathrm{Map}}(C\setminus D, BG)$ coincide, but this is clear: both maps send a Hecke modification from P_1 to P_1 to the G-bundle $P_1|_{C\setminus D}$. □

Question 1.6. Is this map clearly an equivalence? Checking this means checking that the construction via the Hecke stack satisfies the universal property characterising the pullback. This isn't obvious to me. Equivalently one could exhibit an explicit quasi-inverse.

Example 1.7. Let's restrict to the case of $C = \mathbb{CP}^1$ with a framing at ∞. The stack $\mathrm{Bun}_G^{\mathrm{fr}}(\mathbb{CP}^1)$ is classical and has virtual dimension $\dim(G)$, so the Hecke stack $\mathrm{Hecke}_G^{\mathrm{fr}}(\mathbb{CP}^1; D, \omega^\vee) \times_{\mathrm{Bun}_G^{\mathrm{fr}}(\mathbb{CP}^1)^2} \mathrm{Bun}_G^{\mathrm{fr}}(\mathbb{CP}^1)$ also has virtual dimension $\dim(G) + 2\sum_{i=1}^k \langle \rho, \mathrm{dom}(\omega_{z_i}^\vee)\rangle$ as expected as long as the classical intersection is non-empty, which occurs whenever the composition of the Hecke modifications corresponding to $\omega_{z_i}^\vee$ is trivial, or in turn whenever $\sum_{i=1}^k \omega_{z_i}^\vee$ is a coroot.

This stack is also naturally 0-shifted Poisson. Because $(\mathrm{Bun}_G^{\mathrm{fr}}(\mathbb{CP}^1))^2$ is 1-shifted symplectic and the diagonal map is always derived coisotropic by Melani-Safronov, to we only need to check that the Hecke stack is also 1-shifted coisotropic. Actually something stronger is true which we can see in general.

Claim. If $L_1 \to X_{12}$, $L_2 \to X_{12} \times X_{23}$ and $L_3 \to X_{23}$ are n-shifted coisotropic maps, then there is a composed coisotropic correspondence from pt_n to pt_n $L_1 \times_{X_{12}} L_2 \times_{X_{23}} L_3$, which is then canonically $(n-1)$-Poisson. We can forget this structure down to a $(n-1)$-shifted coisotropic correspondence

$$\begin{tikzcd} & L_1 \times_{X_{12}} L_2 \times_{X_{23}} L_3 \arrow[dl] \arrow[dr] & \\ L_1 \times_{X_{12}} L_2 & & L_2 \times_{X_{23}} L_3. \end{tikzcd}$$

To recover our example for $|D| = 1$ we just set $n = 2$, $X_{12} = X_{23} = BG(K)$, $L_1 = L_3 = BG(z^{-1}\mathbb{C}[z^{-1}])$ and $L_2 = BG(\mathcal{O})$, so $L_1 \times_{X_{12}} L_2$ and $L_2 \times_{X_{23}} L_3$ both model the 1-shifted symplectic stack $(\mathrm{Bun}_G^{\mathrm{fr}}(\mathbb{CP}^1))^2$. One can iterate this in the obvious way for more punctures. So actually the Hecke stack itself is 1-shifted Poisson, but only a 0-shifted Poisson structure survives on the intersection with the diagonal.

This was the case corresponding to a cuspidal curve, but a similar argument should work for a nodal or smooth elliptic curve.

Question 1.8. Is this Poisson structure witnessed by the 1-shifted coisotropic forgetful map to $\mathrm{Bun}_G^{\mathrm{fr}}(\mathbb{CP}^1)$ – i.e. is this map the inclusion into the Poisson center? Essentially this is just asking: is this map not just coisotropic but Lagrangian? If so then the symplectic leaves of the Poisson stack correspond to a choice of a framed bundle.

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