Multiplicative Hitchin Systems and Supersymmetric Gauge Theory

Chris Elliott Vasily Pestun

September 19, 2018

Abstract

1 Introduction

2 Multiplicative Higgs Bundles and q-Connections

We'll begin with an abstract definition of the moduli spaces we'll investigate in this paper using the language of derived algebraic geometry.

Let G be a reductive complex algebraic group, let C be a smooth complex algebraic curve and fix a finite set $D = \{z_i, \dots, z_k\}$ of closed points in C. We write $\mathrm{Bun}_G(C)$ for the moduli stack of G-bundles on C, which we view as a mapping stack $\underline{\mathrm{Map}}(C, BG)$ into the classifying stack of G.

Definition 2.1. The moduli stack of *multiplicative G-Higgs fields* on C with singularities at D is the fiber product

$$\mathrm{mHiggs}_G(C, D) = \mathrm{Bun}_G(C) \times_{\mathrm{Bun}_G(C \setminus D)} \underline{\mathrm{Map}}(C \setminus D, G/G)$$

where G/G is the adjoint quotient stack.

Remark 2.2. A closed point of $\mathrm{mHiggs}_G(C, D)$ consists of a principal G-bundle P on C along with an automorphism of the restriction $P|_{C \setminus D}$, i.e. a section of $\mathrm{ad}(P)$ away from D.

The adjoint quotient stack can also be described as the derived loop space $\underline{\mathrm{Map}}(S^1_B, BG)$ of the classifying stack, where S^1_B is the "Betti stack" of S^1, i.e. the constant derived stack at the simplicial set S^1. We can therefore view $\underline{\mathrm{Map}}(C \setminus D, G/G)$ instead as the mapping stack $\underline{\mathrm{Map}}((C \setminus D) \times S^1_B, BG)$, and the moduli stack of multiplicative Higgs bundles instead as

$$\mathrm{mHiggs}_G(C, D) = \underline{\mathrm{Map}}((C \times S^1_B) \setminus (D \times \{0\}), BG).$$

The source of this mapping stack can be q-deformed. Indeed, let q denote an automorphism of the curve C. Write $C \times_q S^1_B$ for the *mapping torus* of q, i.e the derived fiber product

$$C \times_q S^1_B = C \times_{C \times C} C$$

where the two maps $C \to C \times C$ are given by the diagonal and the q-twisted diagonal $(1, q)$ respectively.

Definition 2.3. The moduli stack of q *difference connections* for the group G on C with singularities at D is the mapping space

$$q\text{-}\mathrm{Conn}_G(C, D) = \underline{\mathrm{Map}}((C \times_q S^1_B) \setminus (D \times \{0\}), BG).$$

In particular when $q = 1$ this recovers the moduli stack of multiplicative Higgs bundles.

Remark 2.4. A closed point of $q\text{-Conn}_G(C,D)$ consists of a principal G-bundle P on C along with a q *difference connection*: an isomorphism of G-bundles $P|_{C \setminus D} \to q^*P|_{C \setminus D}$ away from the divisor D. For an introduction to the classical theory of q-difference connections we refer the reader to [STSS98]. (Chris: I'm not sure what to say about q-difference equations. You suggested citing Sauloy but I'm not sure exactly what. He has articles on q-difference modules which don't seem so relevant here?)

2.1 Residue Conditions

These moduli stacks are typically of infinite type. In order to obtain finite type stacks, and later in order to define symplectic rather than only Poisson structure, we can fix the behaviour of our multiplicative Higgs fields and q difference connections near the punctures $D \subseteq C$.

We'll write $\mathbb{D}$ to denote the *formal disk* $\operatorname{Spec} \mathbb{C}[\![z]\!]$. Likewise we'll write $\mathbb{D}^\times$ for the *formal punctured disk* $\operatorname{Spec} \mathbb{C}(\!(z)\!)$. We'll then write $\mathbb{B}$ for the derived pushout $\mathbb{D} \sqcup_{\mathbb{D}^\times} \mathbb{D}$. Let $LG = \underline{\mathrm{Map}}(\mathbb{D}^\times, G)$ and let $L^+G = \underline{\mathrm{Map}}(\mathbb{D}, G)$.

There is a canonical inclusion $\mathbb{B}^{\sqcup k} \to (C \times_q S^1_B) \setminus (D \times \{0\})$, the inclusion of the formal punctured neighbourhood of $D \times \{0\}$. This induces a restriction map on mapping spaces

$$\mathrm{res}_D \colon q\text{-Conn}_G(C,D) \to \mathrm{Bun}_G(\mathbb{B})^k.$$

The following is well-known (see e.g. the expository article [Zhu17]).

Lemma 2.5. The set of closed points of $\mathrm{Bun}_G(\mathbb{B})$ is in canonical bijection with the set of dominant coweights of G.

Definition 2.6. Choose a map from D to the set of dominant coweights and denote it by $\omega^\vee \colon z_i \mapsto \omega^\vee_{z_i}$. Write Λ_i for the isotropy group of the point $\omega^\vee_{z_i}$ in $\mathrm{Bun}_G(\mathbb{B})$. The moduli stack of q difference connections on C with singularities at D and fixed residues given by $\omega^\vee$ is defined to be the fiber product

$$q\text{-Conn}_G(C,D,\omega^\vee) = q\text{-Conn}_G(C,D) \times_{\mathrm{Bun}_G(\mathbb{B})^k} (B\Lambda_1 \times \cdots \times B\Lambda_k).$$

Examples 2.7. The most important examples for our purposes are given by the following rational/trigonometric/elliptic trichotomy.

- **Rational:** We can enhance the definition of our moduli space by including a framing at a point $c \in C$ not contained in D. We always assume that such framed points are fixed by the automorphism q.

  Definition 2.8. The moduli space of q-difference connections on C with a framing at c is defined to be the relative mapping space
  $$q\text{-Conn}^{\mathrm{fr}}_G(C) = \underline{\mathrm{Map}}(C \times_q S^1_B, BG; f)$$
  where $f \colon \{c\} \times S^1_B \to BG$ (or equivalently $f \colon \{c\} \to G/G$) is a choice of adjoint orbit. We can define the framed mapping space with singularities and fixed residues in exactly the same way as above.

  In this paper we'll be most interested in the following example. Let $C = \mathbb{CP}^1$ with framing point $c = \infty$, and consider automorphisms of the form $z \mapsto z + \varepsilon$ for $\varepsilon \in \mathbb{C}$. Choose a finite subset $D \subseteq \mathbb{A}^1$ and label the points $z_i \in D$ by dominant coweights $\omega^\vee_{z_i}$. We can then study the moduli space $\varepsilon\text{-Conn}^{\mathrm{fr}}_G(\mathbb{CP}^1, D, \omega^\vee)$. The main object of study in this paper will be the holomorphic symplectic structure on this moduli space.

  Note that the motivation for this definition comes in part from Spaide's formalism [Spa16] of AKSZ symplectic structures on relative mapping spaces – in this formalism $\mathbb{CP}^1$ with a single framing point is relatively 1-oriented, so mapping spaces out of it with 1-shifted symplectic targets have AKSZ 0-shifted symplectic structures.

- **Trigonometric:** Alternatively, we can enhance our definition by including a reduction of structure group at a point $c \in C$ not contained in D, again fixed by the automorphism q.

Definition 2.9. The moduli space of q-difference connections on C with an H-reduction at c for a subgroup $H \subseteq G$ is defined to be the fiber product

$$q\text{-Conn}_G^{H,c}(C) = \underline{\mathrm{Map}}(C \times_q S^1_B, BG) \times_{G/G} H/H$$

associated to the evaluation at c map $\underline{\mathrm{Map}}(C \times_q S^1_B, BG) \to G/G$. We can define the moduli space with H-reduction with singularities and fixed residues in the same way as above.

Again let $C = \mathbb{CP}^1$. Fix a pair of opposite Borel subgroups B_+ and $B_- \subseteq G$ with unipotent radicals $N_\pm$ and consider the moduli space of q-connections with B_+-reduction at 0 and N_--reduction at ∞. We'll now take q to be an automorphism of the form $z \mapsto qz$ for $q \in \mathbb{C}^\times$. We'll defer in depth analysis of this example to future work.

- **Elliptic:** Finally, let $C = E$ be a smooth curve of genus one. In this case we won't fix any additional boundary data, but just consider the moduli space $q\text{-Conn}_G(E, D, \omega^\vee)$. In the case $q = 1$ this space – or rather its polystable locus – was studied by Hurtubise and Markman [HM02], who proved that it can be given the structure of an algebraic integrable system with symplectic structure related to the elliptic R-matrix of Etingof and Varchenko [EV98].

Remark 2.10. In the elliptic case it's natural to ask to what extent Hurtubise and Markman's integrable system structure can be extended from the variety of polystable multiplicative Higgs bundles to the full moduli stack. If D is empty then it's easy to see that we have a symplectic structure given by the AKSZ construction of Pantev-Toën-Vaquié-Vezzosi [PTVV13]. Indeed, E is compact 1-oriented and the quotient stack G/G is 1-shifted symplectic, so the mapping stack $\underline{\mathrm{Map}}(E, G/G) = \mathrm{mHiggs}_G(E)$ is equipped with a 0-shifted symplectic structure by [PTVV13, Theorem 2.5]. The role of the Hitchin fibration is played by the Chevalley map $\chi \colon G/G \to T/W$, and therefore

$$\underline{\mathrm{Map}}(E, G/G) \to \underline{\mathrm{Map}}(E, T/W).$$

The fibers of this map over regular points in T/W are given by moduli stacks of the form $\mathrm{Bun}_T(\widetilde{E})^W$ where $\widetilde{E}$ is a W-fold cover of E (the cameral cover). (Chris: Come back to this, at least by dimension counting.)

(Chris: todo: remark on what the base looks like for general D. Probably put it in the last subsection? Include comparison to the story in [DG02].)

Remark 2.11. While the moduli space of multiplicative Higgs bundles makes sense on a general curve it's only after restricting identity to this trichotomy of examples that we'll expect the existence of a Poisson structure. Such a structure arises by the AKSZ construction, i.e. by transgressing the 1-shifted symplectic structure on G/G to the mapping space using a fixed section of the canonical bundle on C (possibly with a boundary condition).

2.2 Stability Conditions

For comparison to results in the literature it is important that we briefly discuss the role of stability conditions for difference connections. In our main example of interest – the rational case – these conditions won't play a role, but they do appear in the comparison results between q-connections and monopoles in the literature for more general curves. For definitions for general G we refer to [Smi15], although see also [AB01] on polystable G-bundles. In what follows we fix a choice of $0 < t_0 < 2\pi R$.

Definition 2.12. Let (P, g) be a q-connection on a curve C, and let χ be a character of G. The χ-*degree* of (P, g) is defined to be

$$\deg_\chi(P, g) = \deg(P \times_\chi \mathbb{C}) - \frac{t_0}{2\pi R} \sum_{i=1}^{k} \deg(\chi \circ \omega^\vee_{z_i}).$$

A q-connection (P, g) on C is *stable* if for every maximal parabolic subgroup $H \subseteq G$ with Levi decomposition $H = LN$ and every reduction of structure group (P_H, g) to H, we have

$$\deg_\chi(P_H, g) < 0$$

for the character $\chi = \det(\mathrm{Ad}_L^{\mathfrak{n}})$ defined to be the determinant of the adjoint representation of L on $\mathfrak{n}$.

The q-connection (P, g) is *polystable* if there exists a (not necessarily maximal) parabolic subgroup H with Levi factor L and a reduction of structure group (P_L, g) to L so that (P_L, g) is a stable q-connection and so that the associated H-bundle is admissible, meaning that for every character χ of H which is trivial on $Z(G)$ the associated line bundle $P_H \times_\chi \mathbb{C}$ has degree zero.

Below we'll write $q\text{-}\mathrm{Conn}_G^{\mathrm{ps}}(C, D, \omega^\vee)$ for the moduli space of polystable q-connections. This moduli space is a smooth algebraic variety of finite type. (Chris: Reference for this other than for q trivial?)

When $C = \mathbb{CP}^1$ every principal G-bundle on C admits an essentially unique (up to the action of the Weyl group) holomorphic reduction of structure group to a maximal torus [Gro57]. Since q-connections for an abelian group are automatically stable, polystability on the sphere is equivalent to admissibility of the torus reduction. As a consequence, for our main example of interest – the rational case – the moduli space of polystable q-connections is equivalent to the moduli space of all q-connections of admissible degree. For instance for $G = \mathrm{SL}_n$ the moduli space of polystable q-connection is equivalent to the moduli space of q-connections on the trivial bundle.

2.3 Poisson Structures from Derived Geometry

As we mentioned above in Remark 2.10, in the case where C is an elliptic curve and there are no punctures there's a symplectic structure on $\mathrm{mHiggs}_G(C)$ given by the AKSZ formalism. More generally, when we do have punctures, we expect the moduli space $\mathrm{mHiggs}_G(C, D)$ to have a Poisson structure with a clear origin story coming from the theory of derived Poisson geometry. In this section we'll explain what this story looks like. However, we emphasise that there are technical obstructions to making this story precise with current technology: this section should be viewed as motivation for the structures we'll discuss in the rest of the paper. We refer the reader to [CPT$^+$17] for the theory of derived Poisson structures and to [MS16, MS17, Spa16] for that of derived coisotropic structures.

Here's the idea. Recall that we can identify the moduli space of singular q-connections on a curve C as a fiber product: $q\text{-}\mathrm{Conn}_G(C, D) \cong \mathrm{Bun}_G^{\mathrm{fr}}(C) \times_{\mathrm{Bun}_G(C \setminus D)^2} \mathrm{Bun}_G(C \setminus D)$ where the map $g\colon \mathrm{Bun}_G^{\mathrm{fr}}(C) \to \mathrm{Bun}_G(C \setminus D)$ is given by $P \mapsto (P|_{C \setminus D}, q^*P|_{C \setminus D})$. Consider the following commutative cube:

$$\begin{tikzcd}
& q\text{-}\mathrm{Conn}_G(C,D) \arrow[rr, "f_1"] \arrow[dl, "f_2"'] \arrow[dd, "\mathrm{res}"] & & \mathrm{Bun}_G(C) \arrow[dl, "g_1"'] \arrow[dd] \\
\mathrm{Bun}_G(C \setminus D) \arrow[rr, "g_2"] \arrow[dd] & & \mathrm{Bun}_G(C \setminus D)^2 \arrow[dd] & \\
& \mathrm{Bun}_G(\mathbb{B})^k \arrow[rr, "r"] \arrow[dl] & & BL^+G^{2k} \arrow[dl] \\
BLG^k \arrow[rr] & & BLG^{2k}. &
\end{tikzcd}$$

Here the top and bottom faces are homotopy Cartesian squares. What does this setup buy us? We'll first answer informally.

Claim. First consider the bottom face of the cube. The stack BLG is 2-shifted symplectic because the Lie algebra $L\mathfrak{g}$ has a non-degenerate invariant pairing: the residue pairing. The Lie subalgebra $L^+\mathfrak{g}$ forms part of a Manin triple $(L\mathfrak{g}, L^+\mathfrak{g}, L^- + 0\mathfrak{g})$ which means that $BL^+G \to BLG$ is 2-shifted Lagrangian. Therefore the bottom face of the cube defines a 2-shifted Lagrangian intersection, which means that the pullback $\mathrm{Bun}_G(\mathbb{B})^k$ is 1-shifted symplectic.

Now consider the top face of the cube. If either C is an elliptic curve, or $C = \mathbb{CP}^1$ and we fix a framing at ∞, then the map $\mathrm{Bun}_G(C \setminus D) \to BLG^k$ is also 2-shifted Lagrangian. In particular $\mathrm{Bun}_G(C \setminus D)$ is 1-shifted Poisson. Finally, the map $\mathrm{Bun}_G(C) \to \mathrm{Bun}_G(C \setminus D)$ is 1-shifted coisotropic, or equivalently the canonical map $\mathrm{Bun}_G(C) \to \mathrm{Bun}_G(C \setminus D) \times_{BLG^k} BL^+G^k$ is 1-shifted Lagrangian. That means that the top face of the cube defines a 1-shifted coisotropic intersection, which means that the pullback $q\text{-}\mathrm{Conn}_G(C, D)$ is 0-shifted Poisson.

The restriction map $q\text{-Conn}_G(C,D) \to \mathrm{Bun}_G(\mathbb{B})^k$ is 1-shifted Lagrangian, which means that if we form the intersection with a k-tuple of Lagrangians in $\mathrm{Bun}_G(\mathbb{B})$ then we obtain a 0-shifted symplectic stack. For example, if $\omega_i^\vee$ is a point in $\mathrm{Bun}_G(\mathbb{B})$ corresponding to a dominant coweight with stabilizer Λ_i then $B\Lambda_i \to \mathrm{Bun}_G(\mathbb{B})$ is 1-shifted Lagrangian, so the moduli stack $q\text{-Conn}_G(C,D,\omega^\vee)$ obtained by taking the derived intersection is ind 0-shifted symplectic.

Now, let's make this claim more precise. The main technical condition that makes this claim subtle comes from the fact that most of the derived stacks appearing in this cube, for instance the stack BLG, are not Artin. As such we need to be careful when we try to, for instance, talk about the tangent complex to such stacks. One can make careful statements using the formalism of "Tate stacks" developed by Hennion [Hen15]. We can therefore make our claim into a more formal conjecture.

Conjecture 2.13. Suppose C is either an elliptic curve or $\mathbb{CP}^1$ with a fixed framing at ∞.

1. The stack BLG is Tate 2-shifted symplectic, and both $BL^+G \to BLG$ and $\mathrm{Bun}_G(C \setminus D) \to BLG^k$ are Tate 2-shifted Lagrangian.
2. The stack $\mathrm{Bun}_G(C \setminus D)$ is ind 1-shifted Poisson, and the map $\mathrm{Bun}_G(C) \to \mathrm{Bun}_G(C \setminus D)$ is ind 1-shifted coisotropic witnessed by the 2-shifted Lagrangian map $BL^+G^k \to BLG^k$.
3. The Lagrangian intersection $\mathrm{Bun}_G(\mathbb{B})$ is Tate 1-shifted symplectic, and the map $B\Lambda_i \to \mathrm{Bun}_G(\mathbb{B})$ associated to the inclusion of the stabilizer of a closed point is 1-shifted Lagrangian.

As a consequence, the moduli stack $q\text{-Conn}_G(C,D)$ is ind 0-shifted Poisson and the moduli stack $q\text{-Conn}_G(C,D,\omega^\vee)$ is 0-shifted symplectic.

Remark 2.14. We should explain heuristically why the Calabi-Yau condition on C is necessary. This is a consequence of the AKSZ formalism in the case where D is empty: for the mapping stack $\underline{\mathrm{Map}}(C, G/G)$ to be 0-shifted symplectic, or for the mapping stack $\underline{\mathrm{Map}}(C, BG)$ to be 1-shifted symplectic, we need C to be compact and 1-oriented. A d-orientation on a smooth complex variety of dimension d is exactly the same as a Calabi-Yau structure.

More generally we can say the following. Let's consider the rational case where $C = \mathbb{CP}^1$. Consider the inclusion $\mathrm{d}r\colon \mathfrak{g}_- = \mathbb{T}_{\mathrm{Bun}_G^{\mathrm{fr}}(\mathbb{CP}^1 \setminus D)}[-1] \to r^*\mathbb{T}_{BLG^k}[-1] = \mathfrak{g}(\!(z)\!)^k$: a map of ind-pro Lie algebras concentrated in degree zero. The residue pairing vanishes after pulling back along r since elements of $\mathfrak{g}_-$ are $\mathfrak{g}$-valued functions on $\mathbb{CP}^1$ with at least a simple pole at every puncture in D. So the map r is isotropic with 0 isotropic structure; this structure is unique for degree reasons. We must check that this structure is non-degenerate. It suffices to check that the sequence

$$\mathbb{T}_{\mathrm{Bun}_G^{\mathrm{fr}}(\mathbb{CP}^1 \setminus D)}[-1] \to r^*\mathbb{T}_{BLG^k}[-1] \to (\mathbb{T}_{\mathrm{Bun}_G^{\mathrm{fr}}(\mathbb{CP}^1 \setminus D)}[-1])^\vee$$

is an exact sequence of ind-pro vector spaces, and therefore an exact sequence of quasi-coherent sheaves on the stack $\mathrm{Bun}_G^{\mathrm{fr}}(\mathbb{CP}^1 \setminus D)$. To do this we identify the pair $(\mathfrak{g}_-, \mathfrak{g}(\!(z)\!)^k)$ as part of a Manin triple, where a complementary isotropic subalgebra to $\mathfrak{g}_-$ is given by $\mathfrak{g}_+ = \mathfrak{g}[[z]]^k$. Using the residue pairing we can identify $\mathfrak{g}_+$ with $(\mathfrak{g}_-)^\vee$ and therefore identify our sequence with the split exact sequence

$$0 \to \mathfrak{g}_- \to \mathfrak{g}(\!(z)\!)^k \to \mathfrak{g}_+ \to 0.$$

Remark 2.15. We can also discuss the analogue of the integrable system discussed in Remark 2.10 with a non-empty divisor D. (Chris: here or above?)

3 Periodic Monopoles

Moduli spaces of q-connections on a Riemann surface C are closely related to moduli spaces of periodic monopoles, i.e. monopoles on 3-manifolds that fiber over the circle (more specifically, with fiber C and monodromy determined by q). Let $G_\mathbb{R}$ be a compact Lie group whose complexification is G. The discussion in this section will mostly follow that of [CH10, Smi15].

Write $M = C \times_q S^1_R$ for the C-bundle over S^1 with monodromy given by the automorphism q. More precisely, M is the Riemannian 3-manifold obtained by gluing the ends of the product $C \times [0, 2\pi R]$ of Riemannian manifolds by the isometry $(x, 2\pi R) \sim (q(x), 0)$.

Definition 3.1. A *monopole* on the Riemannian 3-manifold $M = C \times_q S^1_R$ is a smooth principal $G_\mathbb{R}$-bundle $\boldsymbol{P}$ equipped with a connection A and a section Φ of the associated bundle $\mathfrak{g}_{\boldsymbol{P}}$ satisfying the Bogomolny equation

$$*F_A = \mathrm{d}_A \Phi.$$

Remark 3.2. We should emphasise the difference between the Riemannian 3-manifold $M = C \times_q S^1_R$ appearing in this section and the derived stack $C \times_q S^1_B$ (the mapping torus) appearing in the previous section. These should be thought of as smooth and algebraic realizations of the same object (justified by the comparison Theorem 3.7) but they are a priori defined in different mathematical contexts.

We can rephrase the data of a monopole on M as follows. Let $C_0 = C \times \{0\}$ be the fiber over 0 in S^1, viewed as a Riemann surface. Let P be the restriction of the complexified bundle $\boldsymbol{P}_\mathbb{C}$ to C_0. Consider first the restriction of the complexification of A to a connection A_0 on P over C_0. The $(0, 1)$ part of A_0 automatically defines a holomorphic structure on P. We can introduce an additional piece of structure on this holomorphic G-bundle. In order to do so we can decompose the Bogomolny equation into one real and one complex equation as follows.

$$F_{A_0} - \nabla_t \Phi = 0$$
$$[\overline{\partial}_{A_0}, \nabla_t - i\Phi \mathrm{d}t] = 0$$

where ∇_t is the component of the covariant derivative d_A normal to C_0.

Definition 3.3. From now on we'll use the notation $\mathcal{A}$ for the combination $\nabla_t - i\Phi\mathrm{d}t$: an element of the space $\Omega^0(C_0, \mathfrak{g}_P)\mathrm{d}t$ of sections of the complex vector bundle $\mathfrak{g}_P$ on the complex curve C_0.

Let us now introduce singularities into the story. We'll keep the description brief, referring the reader to [CH10, Smi15] for details.

Definition 3.4. Let $D \subseteq M$ be a finite subset. Let $\omega^\vee$ be a choice of coweight for G. A monopole on $M \setminus D$ has *Dirac singularity* at $z \in D$ with charge $\omega^\vee$ if locally on a neighbourhood of z in M it is obtained by pulling back under $\omega^\vee$ the standard Dirac monopole solution to the Bogomolny equation where Φ is spherically symmetric with a simple pole at z () and the restriction of a connection A to a two-sphere S^2 enclosing the singularity defines a $U(1)$ bundle on this S^2 of degree 1 so that

$$\frac{1}{2\pi} \int_{S^2} F = 1.$$

See e.g. [CH10, Section 2.2] for a more detailed description.

We can also introduce a framing (or a reduction of structure group as in the trigonometric example, though we won't consider the latter in this paper). As usual let $c \in C$ be a point fixed by the automorphism q.

Definition 3.5. A monopole on M with *framing* at the point $c \in C$ is a monopole $(\boldsymbol{P}, A, \Phi)$ on M (possibly with Dirac singularities at D) along with a trivialization of the restriction of $\boldsymbol{P}$ to the circle $\{c\} \times S^1_R$, with the condition that the holonomy of $\mathcal{A}$ around this circle lies in a fixed conjugacy class $f \in G/G$.

The moduli theory of monopoles on 3-manifolds of this form has been studied in the mathematics literature by Foscolo [Fos16], applying the analytic techniques of deformation theory to earlier work on periodic monopoles by Cherkis and Kapustin [CK98, CK01]. In the cases of interest to us it can be obtained as a hyperkähler quotient. Let's focus initially on the rational case, so monopoles on $M = \mathbb{CP}^1 \times_\varepsilon S^1_R$ with Dirac singularities at $D \times \{t_0\}$ and a framing at ∞. Consider the infinite-dimensional vector space $\mathcal{V}$ consisting of pairs (A, Φ) where A is a connection on a fixed principal $G_\mathbb{R}$-bundle $\boldsymbol{P}$ on M, Φ is a section of $\mathfrak{g}_{\boldsymbol{P}}$, and (A, Φ) have a Dirac singularity with charge $\omega^\vee_{z_i}$ at each (z_i, t_0) in $D \times \{t_0\}$. Let $\mathcal{G}$ be the group of gauge transformations of the bundle $\boldsymbol{P}$.

The hyperkähler moment map is given by the Bogomolny functional, namely

$$\mu\colon \mathcal{V} \to \Omega^1(M; (\mathfrak{g}_{\mathbb{R}})_{\boldsymbol{P}})$$
$$(A, \Phi) \mapsto *F_A - \mathrm{d}_A\Phi.$$

Definition 3.6. Let D be a finite subset $\{(z_1, t_1), \ldots, (z_k, t_k)\}$ of points in $M = \mathbb{CP}^1 \times_\varepsilon S^1_R$, and let $\omega_i^\vee$ be a choice of coweight for each point in D. The moduli space $\mathrm{Mon}_G(M, D, \omega^\vee)$ is the hyperkähler quotient

$$\mathrm{Mon}_G(M, D, \omega^\vee) = \mu^{-1}(0)/\mathcal{G}.$$

Now let's address the relationship between periodic monopoles and q-connections. Suppose from now on that q is in the identity component of the group of automorphisms of C (fixing the framing point c if present).

Theorem 3.7. There is an analytic isomorphism between the moduli space of polystable monopoles on $C \times_q S^1$ with Dirac singularities at $D \times \{t_0\}$ (and a possible framing on $\{c\} \times S^1$) and the moduli space of q-connections on C with singularities at D and framing at $\{c\}$. More precisely there is an analytic isomorphism

$$H\colon \mathrm{Mon}_G^{(\mathrm{fr})}(C \times_q S^1, D \times \{t_0\}, \omega^\vee) \to q\text{-}\mathrm{Conn}_G^{\mathrm{ps},(\mathrm{fr})}(C, D, \omega^\vee)$$

given by the holonomy map around S^1, i.e. sending a monopole $(\boldsymbol{P}, \mathcal{A})$ to the holomorphic bundle $P = (\boldsymbol{P}_{\mathbb{C}})|_{C_0}$ with q-connection $g = \mathrm{Hol}_{S^1}(\mathcal{A})\colon P \to q^*(P)$.

Remark 3.8. Note that in this statement we assumed that all the singularities occur in the same location in S^1, i.e. in the same slice $C \times t_0$. This assumption is not necessary, but there is a constraint on the possible locations of the singularities as explained in [CH10, Proposition 3.5].

Proof. (Chris: check) This follows by the same argument as that given by Charbonneau-Hurtubise [CH10] and Smith [Smi15]. More explicitly, first let's think about injectivity, so let $(\boldsymbol{P}, \mathcal{A})$ and $(\boldsymbol{P}', \mathcal{A}')$ be a pair of periodic monopoles on $C \times_q S^1$ with images (P, g) and (P', g') respectively, and choose a bundle isomorphism $\tau\colon P \to P'$ intertwining the q-connections g and g'. One observes first that $\boldsymbol{P}$ and $\boldsymbol{P}'$ are also isomorphic G-bundles since, by intertwining with the q-connections, we have an isomorphism $\boldsymbol{P}|_{C\times\{t\}} \to \boldsymbol{P}'|_{C\times\{t\}}$ for every $t \in S^1$. That the monopole structures also match up follows by the same argument as in [CH10, Proposition 4.7].

For surjectivity, again we'll match the argument in the case where $q = \mathrm{id}$. We begin by extending a holomorphic G-bundle P on C_0 with q-connection g to a G-bundle on $M \setminus (D\times\{t_0\}) = (C\times_q S^1_R) \setminus (D\times\{t_0\})$ Let $\gamma\colon [-2\pi R, 2\pi R] \to \mathrm{Aut}(C)$ be a geodesic with $\gamma(-2\pi R) = q^{-1}$, $\gamma(0) = 1$ and $\gamma(2\pi R) = q$. Let $\widetilde{M}$ be the 3-manifold

$$\widetilde{M} = ((-2\pi R, 2\pi R) \times C) \setminus \bigcup_{j=1}^{k} (A_j^+ \cup A_j^-)$$

where A_j^+ is the arc $\{(t + t_0, \gamma(t)(z_j))\colon t \in (0, 2\pi R - t_0]\}$ and A_j^- is the arc $\{(t + t_0 - 4\pi R, \gamma(t)(z_j))\colon t \in [2\pi R - t_0, 2\pi R)\}$.

Let $\pi\colon \widetilde{M} \to C$ be the projection sending (t, z) to $\gamma(t)(z)$. The bundle P pulls back to a bundle $\pi^*(P)$ on $\widetilde{M}$. We obtain a bundle on $M \setminus (D \times t_0)$ by applying the identification $(t, z) \sim (t - 2\pi R, q(z))$. This bundle extends to an S^1-invariant holomorphic G-bundle on $M \times S^1$. The remainder of the proof – verifying the existence of the monopole structure associated to an appropriate choice of hermitian structure – consists of local analysis which is independent of the value of the parameter q.

It remains to remark on the compatibility of framing data on the two sides. A trivialization of the bundle $\boldsymbol{P}$ along the circle $\{c\} \times S^1$ yields a trivialization of the fiber of the bundle P at c. The condition that the holonomy around the circle at c is f fixes the value of the q-connection at c to be in the conjugacy class f. □

Remark 3.9. Mochizuki [Moc17] proved a stronger result in the rational case for the group $G = \mathrm{GL}_n$. He allows not just a framing at infinity in $\mathbb{CP}^1$ but also a singularity encoded in terms of a B-reduction of the bundle.

3.1 Deformation Theory

In the next section we'll compare symplectic forms on these moduli spaces. In order to do so it will be important to understand the tangent spaces at a point of the source and target. There's a natural description of these tangent spaces in terms of the hypercohomology of certain cochain complexes.

From now on we'll focus on the example we're most interested in. That is, we'll exclusively study the rational situation where $C = \mathbb{CP}^1$ and we fix a framing point $c = \infty$. In this case we can use the description as a hyperkähler reduction. For more general deformation theory calculations we refer to Foscolo [Fos16]. Recall that we can write

$$\mathrm{Mon}_G^{(\mathrm{fr})}(C \times_q S^1, D \times \{t_0\}, \omega^\vee) \cong \mu^{-1}(0)/\mathcal{G}$$

where $\mathcal{G}$ is the group of gauge transformations of $\boldsymbol{P}$ and $\mu\colon \mathcal{V} \to \Omega^1(M; (\mathfrak{g}_R)_{\boldsymbol{P}})$ is the Bogomolny functional $(A, \Phi) \mapsto *F_A - \mathrm{d}_A\Phi$. The tangent complex to this hyperkähler quotient at a point $(\boldsymbol{P}, \mathcal{A})$ can be written as $\Omega^0(M \setminus D; (\mathfrak{g}_\mathbb{R})_{\boldsymbol{P}})[1] \to \mathbb{T}_{\mu^{-1}(0)}$ where $\mathbb{T}_{\mu^{-1}(0)}$ is the tangent complex to the zero locus of the moment map, concentrated in non-negative degrees. Roughly speaking $\mathbb{T}_{\mu^{-1}(0)} = \mathbb{T}_\mathcal{V} \overset{\mathrm{d}\mu}{\to} \Omega^1(M; (\mathfrak{g}_R)_{\boldsymbol{P}})[-1]$.

More explicitly, following [Fos16] let

$$\mathcal{F}^{\mathrm{mon}}_{\boldsymbol{P},\mathcal{A}} = \left(\Omega^0(M \setminus D; (\mathfrak{g}_\mathbb{R})_{\boldsymbol{P}}) \xrightarrow{\mathrm{d}_1} \Omega^1(M \setminus D; (\mathfrak{g}_\mathbb{R})_{\boldsymbol{P}}) \oplus \Omega^0(M \setminus D; (\mathfrak{g}_\mathbb{R})_{\boldsymbol{P}}) \xrightarrow{\mathrm{d}_2} \Omega^1(M \setminus D; (\mathfrak{g}_\mathbb{R})_{\boldsymbol{P}}) \right) \otimes_\mathbb{R} \mathbb{C}$$

placed in degrees -1, 0 and 1 where $\mathrm{d}_1(g) = -(\mathrm{d}_A(g), [\Phi, g])$ and $\mathrm{d}_2(a, \psi) = *\mathrm{d}_A(a) - \mathrm{d}_A(\psi) + [\Phi, a]$. Write $\mathrm{d}_{\mathrm{mon}}$ for the total differential.

Remark 3.10. Here we've chosen a point in the twistor sphere, forgetting the hyperkähler structure and retaining a holomorphic symplectic structure. In other words we've identified the target of the hyperkähler moment map – the space of imaginary quaternions – with $\mathbb{R} \oplus \mathbb{C}$, which is equivalent to choosing a point in the unit sphere of the imaginary quaternions: the twistor sphere.

Remark 3.11. If we restrict $\mathcal{F}^{\mathrm{mon}}_{\boldsymbol{P},\mathcal{A}}$ to a slice $C_t = C \times \{t\}$ in the t-direction we can identify it with a complex of the form

$$\Omega^\bullet(C_t; \mathfrak{g}_P)[1] \overset{[\Phi,-]}{\to} \Omega^\bullet(C_t; \mathfrak{g}_P)$$

with total differential given by d_A on each of the two factors along with the differential $[\Phi, -]$ mixing the two factors. These two summands each split up into the sum of a Dolbeault complex on C with its dual. That is, there's a natural subcomplex of the form

$$\Omega^{0,\bullet}(C_t; \mathfrak{g}_P)[1] \overset{[\Phi,-]}{\to} i\Omega^{0,\bullet}(C_t; \mathfrak{g}_P)\mathrm{d}t$$

where the internal differentials on the two factors are now given by $\overline{\partial}_{A_0}$. This complex is in turn quasi-isomorphic to the complex

$$\Omega^\bullet(S^1; \Omega^{0,\bullet}(C_t; \mathfrak{g}_P))[1]$$

with total differential $\overline{\partial}_{A_0} + \mathrm{d}_\mathcal{A}$.

Remark 3.12. If one introduces a framing at a point $c \in C$ then we must correspondingly twist the complex $\mathcal{F}^{\mathrm{mon}}$ above by the line bundle $\mathcal{O}(c)$ on C. So in that case we define (Chris: check)

$$\mathcal{F}^{\mathrm{mon,fr}}_{(\boldsymbol{P},\mathcal{A})} = \mathcal{F}^{\mathrm{mon}}_{\boldsymbol{P},\mathcal{A}} \otimes (\mathbb{C}_{S^1} \boxtimes \mathcal{O}(c)).$$

The following is proved in [Fos16].

Proposition 3.13. The tangent space of $\mathrm{Mon}_G(S^1 \times C, D \times \{t_0\}, \omega^\vee)$ at the point $(\boldsymbol{P}, \mathcal{A})$ is quasi-isomorphic to the hypercohomology $\mathbb{H}^0(C \times S^1; \mathcal{F}'_{(\boldsymbol{P},\mathcal{A})})$ of a subsheaf $\mathcal{F}' \subseteq \mathcal{F}^{\mathrm{mon}}$ where growth conditions are imposed on the degree 0 part of $\mathcal{F}^{\mathrm{mon}}$ near the singularities.

Now let's consider the tangent complex to the moduli space of q-connections. For the arguments in this article we'll only need to carefully consider the case $q = \mathrm{id}$ of multiplicative Higgs bundles, but we'll include some remarks regarding the more general case. In this case the calculation was performed by Bottacin [Bot95], see also [HM02, Section 4]. Fix a multiplicative Higgs bundle (P, g) on C. We consider the sheaf of cochain complexes on C

$$\mathcal{F}_{(P,g)} = (\mathfrak{g}_P[1] \overset{\mathrm{Ad}_g}{\to} \mathfrak{g}_P(-D))$$

in degrees -1 and 0 with differential given by the adjoint action of g. More precisely let L_g and R_g be the bundle maps $\mathfrak{g}_P \to \mathfrak{g}_P$ obtained as the derivative of left- and right-multiplication. Then $\mathrm{Ad}_g = L_g - R_g$. We can alternatively phrase this, as in [HM02, Section 4], as follows. Define $\mathrm{ad}(g)$ to be the vector bundle

$$\mathrm{ad}(g) = (\mathfrak{g}_P \oplus \mathfrak{g}_P)/\{(X, -gXg^{-1}) : X \in \mathfrak{g}_P\}.$$

Then we can write $\mathcal{F}$ as the sheaf of complexes

$$\mathcal{F}_{(P,g)} = (\mathfrak{g}_P[1] \overset{\mathrm{Ad}_g}{\to} \mathrm{ad}(g))$$

where now Ad_g is just the map $X \mapsto [(X, -X)]$.

Remark 3.14. If one introduces a framing at a point $c \in C$ then we must correspondingly twist the complex $\mathcal{F}$ above by the line bundle $\mathcal{O}(c)$ on C, i.e. we restrict to deformations that preserve the framing and therefore are zero at the point c. So in that case we define

$$\mathcal{F}^{\mathrm{fr}}_{(P,g)} = (\mathfrak{g}_P[1] \overset{\mathrm{Ad}_g}{\to} \mathfrak{g}_P(-D)) \otimes \mathcal{O}(c).$$

Remark 3.15. For more general q we should modify this description by replacing g by a q-connection. Note that one can still define the (q-twisted) adjoint action $X \mapsto gXg^{-1}$ using a q-connection, and so we can still define the complex

$$\mathcal{F}_{(P,g)} = (\mathfrak{g}_P[1] \overset{\mathrm{Ad}_g}{\to} \mathrm{ad}(g))$$

just as in the untwisted case (Chris: check).

This complex defines the deformation theory of the moduli space of multiplicative Higgs bundles.

Proposition 3.16 ([Bot95, Proposition 3.1.3])**.** The tangent space of $\mathrm{mHiggs}_G(C, D, \omega^\vee)$ at the point (P, g) is quasi-isomorphic to the hypercohomology $\mathbb{H}^0(C; \mathcal{F}_{(P,g)})$ of the sheaf $\mathcal{F}$.

Remark 3.17. The remaining hypercohomology of the sheaf $\mathcal{F}_{(P,g)}$ has dimension $\dim \mathfrak{z}_\mathfrak{g}$ in degree -1 and 1. However the moduli space $\mathrm{mHiggs}_G(C, D, \omega^\vee)$ is in fact a smooth algebraic variety. This follows from a result of Hurtubise and Markman [HM02, Theorem 4.13], noting that their argument does not rely on the curve C being of genus 1.

In order to calculate with this hypercohomology group we'll use a Čech resolution. This will be straightforward for the multiplicative Higgs moduli space, and we'll use the isomorphism of Theorem 3.7 to give an analogous description on the monopole side. We define a cover $\mathcal{U} = \{U_0, U_1, \ldots, U_k, U_\infty\}$ of C as follows. Let U_i be a contractible open neighbourhood of the point z_i and let U_∞ be a contractible analytic open neighbourhood of $c \in C$, all chosen to be pairwise disjoint. Let U_0. Finally let $U_0 = C \setminus (D \cup \{c\})$. Since the U_i are contractible and the remaining subset $C \setminus (D \cup \{c\})$ is an affine algebraic curve, which means that for any quasi-coherent sheaf of cochain complexes the higher cohomology groups vanish. Likewise for the intersections: the punctured open sets $U_i^\times$ are analytic open sets of an affine curve.

Specify a representative 0-cocycle $(\alpha_\infty, \{\alpha_i\}, \alpha_0, \beta_\infty, \{\beta_i\})$ for the Čech cohomology group with respect to our chosen cover $\mathcal{U}$. Explicitly a 0-cochain is given by the following data:

$$\begin{aligned}
\alpha_\infty &\in \mathrm{ad}(g)(1)(U_\infty) \\
\alpha_i &\in \mathrm{ad}(g)(-1)(U_i) \text{ for } i = 1, \ldots, k \\
\alpha_0 &\in \mathrm{ad}(g)(C \setminus (D \cup \{\infty\})) \\
\beta_\infty &\in \mathfrak{g}_P(U_\infty^\times) \\
\beta_i &\in \mathfrak{g}_P(U_i^\times) \text{ for } i = 1, \ldots, k
\end{aligned}$$

where the notation (± 1) indicates tensoring by the line bundle $\mathcal{O}(\pm 1)$.

Being a 0-cocycle means that $(\alpha_\infty - \alpha_0)|_{U_\infty^\times} = \mathrm{Ad}_g(\beta_\infty)$ and $(\alpha_i - \alpha_0)|_{U_i^\times} = \mathrm{Ad}_g(\beta_i)$ for each i. We consider 0-cocycles modulo 0-coboundaries of the form $(\mathrm{Ad}_g(f_\infty), \{\mathrm{Ad}_g(f_i)\}, \mathrm{Ad}_g(f_0), (f_\infty - f_0)|_{U_\infty^\times}, \{(f_i - f_0)|_{U_i^\times}\})$. In fact Ad_g is an isomorphism on U_0 for the sections α_0 of $\mathfrak{g}_P$ that occur: those with no poles or zeroes in U_0. That means that we can add a coboundary to force $\alpha_0 = 0$.

Now, rather than describing the tangent space to the moduli space of monopoles we'll define a complex that maps to it which we can define locally with respect to the cover $\mathcal{U}$. Consider the open cover $\mathcal{U} \times S^1 = \{U_i \times S^1\}$ of $M = C \times S^1$. For each element U_i of the cover we can define a map

$$\begin{aligned}\mathcal{F}'(U_i \times S^1) &\to (\Omega^0(S^1; \Omega^{0,\bullet}(U_i; \mathfrak{g}_P)) \to \Omega^1(S^1; \Omega^{0,\bullet}(U_i; \mathfrak{g}_P)(D_{U_i})))[1] \\ &\cong \mathfrak{g}_P(U_i)[1] \to \mathfrak{g}_P(U_i)(D|_{U_i}) \\ &\cong \mathcal{F}(U_i)\end{aligned}$$

where we restrict the sheaf $\mathcal{F}'$ whose hypercohomology calculated the monopole tangent complex to the holomorphic part of the slice at $\{t\} \in S^1$. Altogether this defines a map from the Čech cohomology with respect to this cover to the tangent complex of the moduli space of monopoles. That is, we have an explicit map

$$\check{\mathrm{H}}^\bullet(M, \mathcal{U} \times S^1, \mathcal{F}') \to \mathbb{H}^\bullet(M; \mathcal{F}')$$

that factors through the (isomorphic) tangent complex of the moduli space of multiplicative Higgs bundles. To verify this we need only note that these maps commute with the differentials in the Čech complex, i.e. with the restriction to the intersection of a pair of open sets.

Explicitly a 0-cochain in this Čech complex is given by $(\alpha_\infty, \{\alpha_i\}, \alpha_0, \beta_\infty, \{\beta_i\})$ where now

$$\begin{aligned}\alpha_\infty &\in \mathcal{F}_0^{\mathrm{mon}}(U_\infty \times S^1) \\ \alpha_i &\in \mathcal{F}_0^{\mathrm{mon}}(U_i \times S^1) \text{ for } i = 1, \ldots, k \\ \alpha_0 &\in \mathcal{F}_0^{\mathrm{mon}}(U_0 \times S^1) \\ \beta_\infty &\in \Omega^0_{\mathbb{C}}(U_\infty^\times \times S^1; \mathfrak{g}_{\boldsymbol{P}_{\mathbb{C}}}) \\ \beta_i &\in \Omega^0_{\mathbb{C}}(U_i^\times \times S^1; \mathfrak{g}_{\boldsymbol{P}_{\mathbb{C}}}) \text{ for } i = 1, \ldots, k.\end{aligned}$$

Here we write $\mathcal{F}_0^{\mathrm{mon}}$ to indicate the degree 0 term in the cochain complex. There's a similar condition for being a 0-cocycle involving the differential $\mathrm{d}_{\mathrm{mon}}$.

To conclude this subsection it will also be important to have an explicit description of the derivative of the holonomy map H as a map between tangent spaces. We can describe this map using our Čech resolutions on each contractible open set U_i individually.

Proposition 3.18. The derivative $\mathrm{d}H \colon \mathbb{H}^\bullet(U_i \times S^1; \mathcal{F}'_{(\boldsymbol{P}, \mathcal{A})}) \to \mathbb{H}^\bullet(\mathbb{D}_i; \mathcal{F}_{(P,g)})$ is given on an open patch $U_i \times (0, 2\pi)$ by the formula

$$\mathrm{d}H(\alpha_i) = \mathrm{d}H(\mathrm{d}_{\mathrm{mon}} b_i) = b_i(2\pi)H(\mathcal{A}) - H(\mathcal{A})b_i(0)$$

where $i = 1, \ldots, k$ or ∞. More precisely by $b_i(2\pi)$ and $b_i(0)$ we mean the limit of $b_i(t)$ as $t \to 2\pi$ or 0 respectively.

Proof. Note that the right-hand side is the derivative at $\mathcal{A}$ of the map $B_i \mapsto B_i(2\pi)H(\mathcal{A})B_i(0)^{-1}$ where $B_i \in \Omega^0((U_i \times (0, 2\pi)) \setminus \{(z_i, t_0)\}; \mathfrak{g}_P)$. This is the definition of the action of the group of gauge transformations on the holonomy $H(\mathcal{A})$ from $t = 0$ to 2π. (Chris: say more?) □

4 Symplectic Structures

4.1 The Example of GL_2

(Chris: Ultimately we might move this, but we should probably include a discussion of what happens for GL_2 before or in parallel to discussing the general story.)

(Vasily: Chris, is the idea to put here explicit expressions for Higgs field in terms of some chart Darboux coordinates (p, q) that we have discussed in the past drafts draft?. If all singularities are miniscule, then for GL_2 the leaves could be described quite explicitly as products of regular semi-simple co-adoint orbits in $\mathfrak{gl}_2$. Rouven is finishing a draft with explicit parametrization by Darboux coordinates of a certain family of symplectic leaves for GL_n. Perhaps here we can put a reference on it.)

(Chris: My idea was to include the discussion of, at least, the simple example that you worked out for GL_2 with minimal singularities where you described the geometry of the moduli space and gave an expression for the symplectic form (in Section 2 of the file 2017_11_30_ghiggs). I really only had in mind for GL_2 trying to give a more explicit argument for what will appear below, i.e. that the moduli spaces after fixing residues were naturally symplectic. But maybe we could do what you suggest and described the symplectic leaves in some Darboux coordinates and refer to [FP18] for proof?)

4.2 Symplectic Structures for General G

We begin the more abstract general analysis by briefly discussing the holomorphic symplectic structure on the moduli space of periodic monopoles on $\mathbb{CP}^1$ following the analysis of Cherkis and Kapustin [CK98, CK02]. This structure arises from the description we gave as a hyperkähler quotient. To describe it specifically, let $\delta^{(1)}\mathcal{A}$ and $\delta^{(2)}\mathcal{A}$ be two tangent vectors at $(\boldsymbol{P}, \mathcal{A})$ to the moduli space of monopoles. Recall that $\mathcal{A}$ denotes the combination $\nabla_t - i\Phi \mathrm{d}t$. Choose representatives for these two tangent vectors of the form α_i and α'_i respectively in the Čech resolution we described above. Then we can write the holomorphic symplectic form coming from the hyperkähler reduction in terms of the symplectic pairing on the infinite-dimensional vector space $\mathcal{V}$, which is given by the Killing form on $\mathfrak{g}$ along with the wedge pairing of differential forms. So summing over the local patches in our Čech resolution we can write it as

$$\begin{aligned}\omega_{\mathrm{mon}}(\delta^{(1)}\mathcal{A}, \delta^{(2)}\mathcal{A}) &= \int_M \kappa(\delta^{(1)}\mathcal{A} \wedge \delta^{(2)}\mathcal{A})\mathrm{d}z\mathrm{d}t \\ &= \sum_{i=1}^k \int_{U_i \times S^1} \kappa(\alpha_i \wedge \alpha'_i)\mathrm{d}z\mathrm{d}t\end{aligned}$$

where the contributions to the integral away from the U_i vanish.

Our goal in this section will be describe a symplectic structure on the moduli space of multiplicative Higgs bundles – the rational analogue of Hurtubise and Markman's symplectic structure – and then prove that it's equivalent to this symplectic form under the equivalence between multiplicative Higgs bundles and periodic monopoles.

Remark 4.1. From now on we'll write $\langle -, - \rangle$ to denote the residue pairing between elements of $L\mathfrak{g}$. That is,

$$\langle g_1, g_2 \rangle = \oint_{\mathbb{D}^\times} \kappa(g_1, g_2).$$

Lemma 4.2. There is a natural symplectic structure on the moduli space $\mathrm{mHiggs}^{\mathrm{fr}}_G(\mathbb{CP}^1, D, \omega^\vee)$. In terms of our Čech description it is described by the formula

$$\omega((\{\alpha_i\}, \{\beta_i\}), (\{\alpha'_i\}, \{\beta'_i\})) = \sum_i \langle \rho_g^*(\alpha_i + \alpha_0)|_{U_i^\times}, \rho_g^*(\mathrm{Ad}_g^*)^{-1}(\beta'_i)\rangle - \langle \rho_g^*(\alpha'_i + \alpha'_0)|_{U_i^\times}, \rho_g^*(\mathrm{Ad}_g^*)^{-1}(\beta_i)\rangle,$$

where we use the Killing form to identify β'_i with a $\mathfrak{g}^*$-valued form, and where ρ_g^* is the pullback along the right multiplication by g. In this expression the sum is over $i = 1, \ldots, k$ and $i = \infty$.

Remark 4.3. How should we think about this symplectic structure? There's an intuitive description just as in Hurtubise and Markman's elliptic moduli space. The symplectic structure is induced from the natural equivalence between the tangent and cotangent spaces of the moduli space of multiplicative Higgs bundles as described in Section 3.1. That is, there's a map of complexes of sheaves

$$\begin{tikzcd}
(\mathcal{F}^{\mathrm{fr}}_{(P,g)})^*[2] \arrow[r, equal] \arrow[d] & \Big(\mathfrak{g}_P^*(D)[1] \otimes \mathcal{O}(-c) \arrow[r, "\mathrm{Ad}_g^*"] \arrow[d, "\kappa\circ\mathrm{Ad}_g^*"] & \mathfrak{g}_P^*\mathcal{O}(-c)\Big) \arrow[d, "\mathrm{Ad}_g\circ\kappa^{-1}"] \\
\mathcal{F}^{\mathrm{fr}}_{(P,g)} \arrow[r, equal] & \Big(\mathfrak{g}_P[1]\mathcal{O}(-c) \arrow[r, "\mathrm{Ad}_g"] & \mathfrak{g}_P(-D)\mathcal{O}(-c)\Big)
\end{tikzcd}$$

where here κ denotes the isomorphism from $\mathfrak{g}_P \to \mathfrak{g}_P^*$ induced by the Killing form. The top line is the Serre dual complex to the bottom line; note that the incorporation of the framing was necessary for this to be the case (that is, we're using the relative Calabi-Yau structure on the pair $(\mathbb{CP}^1, c)$). Taking 0th hypercohomology we obtain a map from the cotangent space to the tangent space of our moduli space of multiplicative Higgs bundles.

Proof. (Chris: todo. It should be written so as to be manifestly anti-symmetric, we need to verify non-degeneracy.) □

(Chris: What about the Poisson structure on the total space? These should be leaves. Maybe just remark on this since we haven't calculated the tangent complex of the big space. What about ε-connections?)

Proposition 4.4. The symplectic form on the moduli space of multiplicative Higgs bundles can be written in the form

$$\omega(\delta g, \delta g') = -\sum_{i=1}^{d} \langle b_i^L g^{-1}, b_i'^R g^{-1}\rangle - (b \leftrightarrow b') \tag{1}$$

where δg is represented on U_i by a pair $(b^L, b^R) \in \mathrm{Ad}_g$. Here and throughout the notation "$-(b \leftrightarrow b')$" denotes antisymmetrization.

Proof. Specify a representative cocycle $(\alpha_\infty, \{\alpha_i\}, \alpha_0, \beta_\infty, \{\beta_i\})$ for the first Čech cohomology group. By addition of a coboundary we can assume that $\alpha_0 = 0$, and to force α_∞ to land in $z\mathfrak{g}[[z]]$ and each α_i to land in $z^{-1}\mathfrak{g}[[z]]$. We've now fixed a representative cocycle – there's no further gauge freedom. Ignoring the factor at infinity – since after we add these coboundaries the residue pairing there vanishes – the pairing we end up with looks like

$$\begin{aligned}
\omega(\delta g, \delta g') &= \sum_{i=1}^{d} \langle \rho_g^*(\alpha_i + \alpha_0)|_{U_i^\times}, \rho_g^*(\mathrm{Ad}_g^*)^{-1}(\beta'_i)\rangle - ((\alpha, \beta) \leftrightarrow (\alpha', \beta')) \\
&= \sum_{i=1}^{d} \langle \rho_g^*(\alpha_i + \alpha_0)|_{U_i^\times}, \rho_g^*(\mathrm{Ad}_g^*)^{-1}(\mathrm{Ad}_g^{-1}(\alpha'_i - \alpha'_0))\rangle - (\alpha \leftrightarrow \alpha') \\
&= \sum_{i=1}^{d} \langle \rho_g^*(\alpha_i)|_{U_i^\times}, \rho_g^*(\mathrm{Ad}_g^*)^{-1}(\mathrm{Ad}_g^{-1}(\alpha'_i)|_{U_i^\times})\rangle - (\alpha \leftrightarrow \alpha').
\end{aligned}$$

We can compute the composite operator $(\mathrm{Ad}_g^*)^{-1}\mathrm{Ad}_g^{-1}$ on $\mathrm{ad}(P)(U_i^\times)$. It's given by the inverse of the operator $\mathrm{Ad}_g\mathrm{Ad}_g^*$ which sends a pair $(b^L, b^R) \in \mathrm{ad}(P)(U_i^\times)$ to $(b^L - b^R, b^R - b^L) \sim (b^L + g^{-1}b^L g, b^R - g b^R g^{-1})$. Denote this by $((1 + A_g)b^L, (1 - A_{g^{-1}})b^R)$. We can describe the inverse using the expansion $(1 + A_g)^{-1} = 1 - A_g + A_g^2 + \cdots$. After applying our gauge transformation above the remaining degree of freedom in α_i is its z^{-1} term. Each time we apply A it raises the order in z of this term by one, so only the linear summand $-A_g$ of $(1 + A_g)^{-1}$ contributes to the residue pairing. In the pairing we need to use the invariant pairing on $\mathfrak{g}((z)) \oplus \mathfrak{g}((z))$ that vanishes on the subalgebra spanned by $(X, A_g(X))$, so we take the difference of the residue / Killing form pairings on the two summands (Chris: scaled by 1/2: why this normalization?).

$$\begin{aligned}\omega(\delta g, \delta g') &= \frac{1}{2}\sum_{i=1}^{d} -\langle \rho_g^* A_g b_i^L, \rho_g^* b_i'^R\rangle - \langle \rho_g^* b_i^L, \rho_g^* A_{g^{-1}} b_i'^R\rangle - (b \leftrightarrow b') \\ &= -\sum_{i=1}^{d} \langle b_i^L g^{-1}, b_i'^R g^{-1}\rangle - (b \leftrightarrow b')\end{aligned}$$

using cyclic invariant to identify the two terms. □

We can now establish our main result.

Theorem 4.5. The symplectic structure on $\mathrm{Mon}_G^{\mathrm{fr}}(\mathbb{CP}^1 \times S^1, D \times \{0\}, \omega^\vee)$ and the pullback of the symplectic structure on $\mathrm{mHiggs}_G^{\mathrm{ps,fr}}(\mathbb{CP}^1, D, \omega^\vee)$ under H coincide.

Proof. This is straightforward now, by combining the two local descriptions of the symplectic structure on monopoles and multiplicative Higgs bundles on a neighbourhood of a puncture with the description in Proposition 3.18. So, let's begin by taking the symplectic form ω_{mHiggs} on the moduli space $\mathrm{mHiggs}_G^{\mathrm{ps,fr}}(\mathbb{CP}^1, D, \omega^\vee)$ and evaluating it at the image under $\mathrm{d}H$ of two elements $(\alpha_i, \alpha_i') = (\mathrm{d}_{\mathrm{mon}}(b_i), \mathrm{d}_{\mathrm{mon}}(b_i'))$. Let us write $b_i(0) = b_i^L$ and $b_i(2\pi) = b_i^R$ for brevity. Likewise for consistency with the calculations above let us denote the image $H(\mathcal{A})$ under the holonomy map by g.

According to Proposition 3.18 we have

$$\begin{aligned}\omega_{\mathrm{mHiggs}}(\mathrm{d}H(\alpha_i), \mathrm{d}H(\alpha_i')) &= \omega_{\mathrm{mHiggs}}(\mathrm{d}H(\mathrm{d}_{\mathrm{mon}}(b_i)), \mathrm{d}H(\mathrm{d}_{\mathrm{mon}}(b_i'))) \\ &= \omega_{\mathrm{mHiggs}}(b_i^R g - g b_i^L, b_i'^R g - g b_i'^L).\end{aligned}$$

We can write this in terms of a pairing on the bundle $\mathrm{ad}(g)$. Represent the class $b_i^R g - g b_i^L$ by the pair $(b_i^R g, -g b_i^L)$. Then applying the description of ω_{mHiggs} provided by Proposition 4.4 we have

$$\begin{aligned}\omega_{\mathrm{mHiggs}}(\mathrm{d}H(\alpha_i), \mathrm{d}H(\alpha_i')) &= \omega_{\mathrm{mHiggs}}((b_i^R g, -g b_i^L), (b_i'^R g, -g b_i'^L)) \\ &= \langle b_i^R, g b_i'^L g^{-1}\rangle - (b \leftrightarrow b')\end{aligned}$$

where the remaining terms are killed by the antisymmetrization.

On the other hand, we can evaluate the pairing $\omega_{\mathrm{Mon}}(\alpha_i, \alpha_i') = \omega_{\mathrm{Mon}}(\mathrm{d}_{\mathrm{mon}}(b_i), \mathrm{d}_{\mathrm{mon}}(b_i'))$ via integration by parts. We'll compute the symplectic pairing for the full Čech complex, but it will split into a sum over open sets U_i. The result is that

$$\begin{aligned}\omega_{\mathrm{Mon}}(\{\alpha_i\}, \{\alpha_i'\}) &= \sum_{i=0,\dots,k,\infty} \int_{U_i} \kappa(\mathrm{d}_{\mathrm{mon}}(b_i) \wedge \mathrm{d}_{\mathrm{mon}}(b_i'))\mathrm{d}z - (b \leftrightarrow b') \\ &= \sum_{i=1,\dots,k,\infty} \int_{\partial \mathbb{D}_i \times [0,2\pi]} \kappa(b_i - b_0, \mathrm{d}_{\mathrm{mon}}(b_i'))\mathrm{d}z - (b \leftrightarrow b')\end{aligned}$$

using here the fact that $\mathrm{d}_{\mathrm{mon}} b_i = \mathrm{d}_{\mathrm{mon}} b_0$ on $U_i \cap U_0$ and that $\mathrm{d}_{\mathrm{mon}} b_i(t) = 0$ when $t = 0$ or 2π. By Stokes' theorem we then have

$$\begin{aligned}\omega_{\mathrm{Mon}}(\{\alpha_i\}, \{\alpha_i'\}) &= \sum_{i=1,\dots,k,\infty} \oint_{\partial \mathbb{D}_i} \kappa(b_i^R - b_0^R, b_i'^R) - \kappa(b_i^L - b_0^L, b_i'^L) - (b \leftrightarrow b') \\ &= \sum_{i=1,\dots,k,\infty} \oint_{\partial \mathbb{D}_i} -\kappa(b_0^R, b_i'^R) + \kappa(b_0^L, b_i'^L) - (b \leftrightarrow b').\end{aligned}$$

Pick out the summand corresponding to U_i. Choose our potentials so that on the boundary $\partial\mathbb{D}_i$ we have $b_0^L = b_0'^L = 0$. We can do this by setting $b^R = \delta g g^{-1}$ since g is non-singular on U_0. This choice means that on $U_0 \cap U_i$, since $\delta g = b_i^R g - g b_i^L = b_0^R g$ we can make the identification $b_0^R = b_i^R - g b_i^L g^{-1}$. Therefore

$$\omega_{\mathrm{Mon}}(\{\alpha_i\}, \{\alpha'_i\}) = \sum_{i=1,\dots,k,\infty} \langle (g b_i^L g^{-1}, b_i'^R \rangle \quad - \quad (b \leftrightarrow b')$$

agreeing with the expression coming from mHiggs. (Chris: indeed there's a sign error somewhere which I haven't tracked down yet. On the other hand maybe it isn't correct to say that the two symplectic forms are equal, maybe they're just proportional?) ☐

Remark 4.6. (Chris: connection to Shapiro's symplectic leaves in [Sha16]. The claim is that taking the Taylor expansion at infinity defines a Poisson map to $\widetilde{G} \subseteq G_1[[z^{-1}]]$ sending symplectic leaves to (discrete) unions of symplectic leaves. One can do the classification more completely in the type A case.)

5 Hyperkähler Structures

(Vasily: As of July 19, 2018 I have not yet passed this and following sections)

The results of the previous section imply that the symplectic structure on $\mathrm{mHiggs}_G^{\mathrm{ps,fr}}(\mathbb{CP}^1, D, \omega^\vee)$ extends canonically to a hyperkähler structure. In this section we'll compare the twistor rotation with the deformation to the moduli space of ε-connections. (Chris: todo: expand on this)

Let us begin by describing the hyperkähler structure on the moduli space of periodic monopoles and how the holomorphic symplectic structure varies under rotation in the twistor sphere. We'll begin by observing that the moduli space of periodic monopoles can be described as a hyperkähler quotient as in the work of Atiyah and Hitchin [AH88]. This was demonstrated by Cherkis and Kapustin [CK02] for the group SU(2), see also Foscolo [Fos16][Theorem 7.12] (Chris: is the Cherkis-Kapustin result sufficiently rigorous for our purposes? Foscolo only talks about SO(3). Reference for more general G? Maybe [NP12]?).

(Chris: Key observation to include: the action of twistor rotation is by SO(3)-action on a space of lattices. More precisely the fibers in the twistor sphere are obtained, with their holomorphic symplectic structures by performing this rotation.)

(Chris: define what I mean by ε-deformation for monopoles.)

Proposition 5.1. (Chris: Monopoles in the large R (Gaiotto) limit – twistor rotation and ε-deformation coincide.)

Proof. ☐

Corollary 5.2. If we equip $\mathrm{mHiggs}_G^{\mathrm{ps,fr}}(\mathbb{CP}^1, D, \omega^\vee)$ with the complex structure at the twistor parameter $\eta \in \mathbb{C}$, the resulting complex manifold is isomorphic to $\varepsilon\text{-}\mathrm{Conn}_G^{\mathrm{ps,fr}}(\mathbb{CP}^1, D, \omega^\vee)$.

Proof. ☐

(Chris: comment on the holomorphic symplectic structure vs the one discussed above.)

(Chris: comment on the example of the Nahm transform.)

6 Quantization and Yangians

(Chris: We're still discussing how this story goes.)

7 Duality

(Chris: For a first draft I'm inserting some notes I prepared for a talk. Todo: talk about motivation and the content we discussed regarding non-simply-laced groups)

Pseudo-Conjecture 7.1 (Multiplicative Geometric Langlands)**.** Let G be a Langlands self-dual group. There is an equivalence of categories

$$\text{A-Branes}_{q^{-1}}(\text{mHiggs}_G(C, D, \omega^\vee)) \cong \text{B-Branes}(q\text{-Conn}_G(C, D, \omega^\vee))$$

where the category on the right-hand side depends on the value q.

What does this mean, and are there situations in which we can make it precise? We'll discuss three examples where we can say something more concrete. In each case, by "B-branes" we'll just mean the category $\text{Coh}(q\text{-Conn}_G(C, D, \omega^\vee))$ of coherent sheaves. By "A-branes" we'll mean some version of $q^\vee$*-difference connections* on the stack $\text{Bun}_G(C)$.

Remark 7.2. This equivalence is supposed to interchange objects corresponding to branes of opers on the two sides, and introduce an analogue of the Feigin-Frenkel isomorphism between deformed W-algebras (see [FRSTS98, STSS98]. This isomorphism only holds for self-dual groups, which motivates the restriction to the self-dual case here.

7.1 The Abelian Case

Suppose $G = \text{GL}(1)$ (more generally we could consider a higher rank abelian gauge group). In general for an abelian group the moduli spaces we've defined are trivial – for instance the rational and trigonometric spaces are always discrete. However there is one interesting non-trivial example: the elliptic case. For simplicity let's consider the abelian situation with $D = \varnothing$: the case with no punctures.

Definition 7.3. A *q-difference module* on a variety X with automorphism q is a module for the sheaf $\Delta_{q,X}$ of non-commutative rings generated by $\mathcal{O}_X$ and an invertible generator Φ with the relation $\Phi \cdot f = q^*(f) \cdot \Phi$. Write $\text{Diff}_q(X)$ for the category of q-difference modules on X.

In the abelian case the space $q\text{-Conn}_{\text{GL}(1)}(E)$ is actually a stack, but one can split off the stacky part to define difference modules on it. Indeed, for any q one can write

$$\text{Bun}_{\text{GL}(1)}(E) =\cong B\text{GL}(1) \times \mathbb{Z} \times E^\vee$$

and so

$$q\text{-Conn}_{\text{GL}(1)}(E) \cong B\text{GL}(1) \times \mathbb{Z} \times (E^\vee \times_q \mathbb{C}^\times)$$

which means one can define difference modules on these stacks associated to an automorphism of $E^\vee$ or $E^\vee \times_q \mathbb{C}^\times$ respectively.

Conjecture 7.4. There is an equivalence of categories for any $q \in \mathbb{CP}^1$

$$\text{Diff}_q(\text{Bun}_{\text{GL}(1)}(E)) \cong \text{Coh}(q^{-1}\text{-Conn}_{\text{GL}(1)}(E)).$$

In this abelian case we can go even farther and make a more sensitive 2-parameter version of the conjecture.

Conjecture 7.5. There is an equivalence of categories for any $q_1, q_2 \in \mathbb{CP}^1$

$$\mathrm{Diff}_{q_1}(q_2\text{-Conn}_{\mathrm{GL}(1)}(E) \cong \mathrm{Diff}_{q_2^{-1}}(q_1^{-1}\text{-Conn}_{GL(1)}(E)$$

where q_1 is the automorphism of $E^\vee \times_{q_2} \mathbb{C}^\times$ acting fiberwise over each point of $\mathbb{C}^\times$.

This conjecture should be provable using the same techniques as the ordinary geometric Langlands correspondence in the abelian case, i.e. by a (quantum) twisted Fourier-Mukai transform (as constructed by Polishchuk and Rothstein [PR01]).

7.2 The Classical Case

Now, let's consider the limit $q \to 0$. This will give a conjectural statement involving coherent sheaves on both sides analogous to the classical limit of the geometric Langlands conjecture as conjectured by Donagi and Pantev [DP10]. The existence of an equivalence isn't so interesting in the self-dual case (where both sides are the same), but the classical multiplicative Langlands functor should be an *interesting* non-trivial equivalence. For example we can make the following conjecture

Conjecture 7.6. Let G be a Langlands self-dual group and let E be an elliptic curve. There is an automorphism of categories (for the rational, trigonometric and elliptic moduli spaces)

$$F\colon \mathrm{Coh}(\mathrm{mHiggs}_G(E) \cong \mathrm{Coh}(\mathrm{mHiggs}_G(E)$$

so that the following square commutes:

$$\begin{tikzcd}
\mathrm{Coh}(\mathrm{mHiggs}_T(E)) \arrow[r, "\mathrm{FM}"] \arrow[d, "p_* q^!"'] & \mathrm{Coh}(\mathrm{mHiggs}_T(E)) \arrow[d, "p_* q^!"] \\
\mathrm{Coh}(\mathrm{mHiggs}_G(E)) \arrow[r, "F"] & \mathrm{Coh}(\mathrm{mHiggs}_G(E)).
\end{tikzcd}$$

Here we're using the natural morphisms $p\colon \mathrm{mHiggs}_B(E) \to \mathrm{mHiggs}_G(E)$ and $q\colon \mathrm{mHiggs}_B(E) \to \mathrm{mHiggs}_T(E)$, and FM is the Fourier-Mukai transform.

It ought to be possible to state something a bit stronger that includes singularities in this auto-duality.

7.3 The Rational Type A Case

There's one more example where we can say something precise, and even draw a connection to the ordinary geometric Langlands correspondence. We already mentioned the Nahm transform in the previous section: in the case where $G = \mathrm{GL}(n)$ and C is $\mathbb{C}$ (where as usual we fix framing data at infinity) the Nahm transform identifies multiplicative Higgs bundles of degree k with *ordinary* Higgs bundles on $\mathbb{CP}^1$ for the group $\mathrm{GL}(k)$ with $n+2$ tame singularities (with appropriate fixed locations and residues).

Claim. Under the Nahm transform, Pseudo-Conjecture 7.1 in the rational case for the group $\mathrm{GL}(n)$ becomes the ordinary geometric Langlands conjecture on $\mathbb{CP}^1$ with tame ramification.

References

[AB01] Boudjemaa Anchouche and Indranil Biswas. Einstein-Hermitian connections on polystable principal bundles over a compact Kähler manifold. *Amer. J. Math.*, 123(2):207–228, 2001.

[AH88] Michael Atiyah and Nigel Hitchin. *The geometry and dynamics of magnetic monopoles.* M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1988.

[Bot95] Francesco Bottacin. Symplectic geometry on moduli spaces of stable pairs. *Ann. Sci. École Norm. Sup. (4)*, 28(4):391–433, 1995.

[CH10] Benoît Charbonneau and Jacques Hurtubise. Singular Hermitian-Einstein monopoles on the product of a circle and a Riemann surface. *Int. Math. Res. Not.*, 2011(1):175–216, 2010.

[CK98] Sergey Cherkis and Anton Kapustin. Singular monopoles and supersymmetric gauge theories in three dimensions. *Nuclear Physics B*, 525(1-2):215–234, 1998.

[CK01] Sergey Cherkis and Anton Kapustin. Nahm transform for periodic monopoles and $\mathcal{N}=2$ super Yang-Mills theory. *Communications in Mathematical Physics*, 218(2):333–371, 2001.

[CK02] Sergey Cherkis and Anton Kapustin. Hyper-Kähler metrics from periodic monopoles. *Physical Review D*, 65(8):084015, 2002.

[CPT$^+$17] Damien Calaque, Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi. Shifted Poisson structures and deformation quantization. *Journal of Topology*, 10(2):483–584, 2017.

[DG02] Ron Donagi and Dennis Gaitsgory. The gerbe of Higgs bundles. *Transform. Groups*, 7(2):109–153, 2002.

[DP10] Ron Donagi and Tony Pantev. Lectures on the geometric Langlands conjecture and non-abelian Hodge theory. *Second International School on Geometry and Physics Geometric Langlands and Gauge Theory*, page 129, 2010.

[EV98] Pavel Etingof and Alexander Varchenko. Geometry and classification of solutions of the classical dynamical Yang-Baxter equation. *Comm. Math. Phys.*, 192(1):77–120, 1998.

[Fos16] Lorenzo Foscolo. Deformation theory of periodic monopoles (with singularities). *Comm. Math. Phys.*, 341(1):351–390, 2016.

[FP18] Rouven Frassek and Vasily Pestun. A family of GL(r) multiplicative Higgs bundles on rational base. *arXiv preprint arXiv:1808.00799*, 2018.

[FRSTS98] Edward Frenkel, Nicolai Reshetikhin, and Michel Semenov-Tian-Shansky. Drinfeld–Sokolov reduction for difference operators and deformations of-$\mathcal{W}$algebras i. the case of Virasoro algebra. *Communications in mathematical physics*, 192(3):605–629, 1998.

[Gro57] Alexander Grothendieck. Sur la classification des fibrés holomorphes sur la sphère de Riemann. *Amer. J. Math.*, 79:121–138, 1957.

[Hen15] Benjamin Hennion. *Formal loops and tangent Lie algebras.* PhD thesis, Université de Montpellier, 2015.

[HM02] Jacques Hurtubise and Eyal Markman. Elliptic Sklyanin integrable systems for arbitrary reductive groups. *Advances in Theoretical and Mathematical Physics*, 6(5):873–978, 2002.

[Moc17] Takuro Mochizuki. Periodic monopoles and difference modules. *arXiv preprint arXiv:1712.08981*, 2017.

[MS16] Valerio Melani and Pavel Safronov. Derived coisotropic structures. *arXiv preprint arXiv:1608.01482*, 2016.

[MS17] Valerio Melani and Pavel Safronov. Derived coisotropic structures II: stacks and quantization. *arXiv preprint arXiv:1704.03201*, 2017.

[NP12] Nikita Nekrasov and Vasily Pestun. Seiberg-Witten geometry of four dimensional $\mathcal{N}=2$ quiver gauge theories. *arXiv preprint arXiv:1211.2240*, 2012.

[PR01] Alexander Polishchuk and Mitchell Rothstein. Fourier transform for D-algebras, I. *Duke Mathematical Journal*, 109(1):123–146, 2001.

[PTVV13] Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi. Shifted symplectic structures. *Publications mathématiques de l'IHÉS*, 117(1):271–328, 2013.

[Sha16] Alexander Shapiro. Poisson geometry of monic matrix polynomials. *Int. Math. Res. Not.*, (17):5427–5453, 2016.

[Smi15] Benjamin H Smith. Singular G-monopoles on $S^1 \times \Sigma$. *arXiv preprint arXiv:1501.03337*, 2015.

[Spa16] Theodore Spaide. Shifted symplectic and Poisson structures on spaces of framed maps. *arXiv preprint arXiv:1607.03807*, 2016.

[STSS98] Michel Semenov-Tian-Shansky and Alexey Sevostyanov. Drinfeld–Sokolov reduction for difference operators and deformations of $\mathcal{W}$-algebras II. the general semisimple case. *Communications in mathematical physics*, 192(3):631–647, 1998.

[Zhu17] Xinwen Zhu. An introduction to affine Grassmannians and the geometric Satake equivalence. In *Geometry of moduli spaces and representation theory*, volume 24 of *IAS/Park City Math. Ser.*, pages 59–154. Amer. Math. Soc., Providence, RI, 2017.

Institut des Hautes Études Scientifiques
35 Route de Chartres, Bures-sur-Yvette, 91440, France
celliott@ihes.fr
pestun@ihes.fr