

Multiplicative Hitchin Systems and Supersymmetric Gauge Theory

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Abstract

1 Introduction

2 Multiplicative Higgs Bundles and q -Connections

We'll begin with an abstract definition of the moduli spaces we'll investigate in this paper using the language of derived algebraic geometry.

Let G be a reductive complex algebraic group, let C be a smooth complex algebraic curve and fix a finite set $D = \{z_i, \dots, z_k\}$ of closed points in C . We write $\mathrm{Bun}_G(C)$ for the moduli stack of G -bundles on C , which we view as a mapping stack $\underline{\mathrm{Map}}(C, BG)$ into the classifying stack of G .

Definition 2.1. The moduli stack of *multiplicative G -Higgs fields* on C with singularities at D is the fiber product

$$\mathrm{mHiggs}_G(C, D) = \mathrm{Bun}_G(C) \times_{\mathrm{Bun}_G(C \setminus D)} \underline{\mathrm{Map}}(C \setminus D, G/G)$$

where G/G is the adjoint quotient stack.

Remark 2.2. A closed point of $\mathrm{mHiggs}_G(C, D)$ consists of a principal G -bundle P on C along with an automorphism of the restriction $P|_{C \setminus D}$, i.e. a section of $\mathrm{ad}(P)$ away from D .

The adjoint quotient stack can also be described as the derived loop space $\underline{\mathrm{Map}}(S_B^1, BG)$ of the classifying stack, where S_B^1 is the “Betti stack” of S^1 , i.e. the constant derived stack at the simplicial set S^1 . We can therefore view $\underline{\mathrm{Map}}(C \setminus D, G/G)$ instead as the mapping stack $\underline{\mathrm{Map}}((C \setminus D) \times S_B^1, BG)$, and the moduli stack of multiplicative Higgs bundles instead as

$$\mathrm{mHiggs}_G(C, D) = \underline{\mathrm{Map}}((C \times S_B^1) \setminus (D \times \{0\}), BG).$$

The source of this mapping stack can be q -deformed. Indeed, let q denote an automorphism of the curve C . Write $C \times_q S_B^1$ for the *mapping torus* of q , i.e the derived fiber product

$$C \times_q S_B^1 = C \times_{C \times C} C$$

where the two maps $C \rightarrow C \times C$ are given by the diagonal and the q -twisted diagonal $(1, q)$ respectively.

Definition 2.3. The moduli stack of q *difference connections* for the group G on C with singularities at D is the mapping space

$$q\text{-Conn}_G(C, D) = \underline{\mathrm{Map}}((C \times_q S_B^1) \setminus (D \times \{0\}), BG).$$

In particular when $q = 1$ this recovers the moduli stack of multiplicative Higgs bundles.

Remark 2.4. A closed point of $q\text{-Conn}_G(C, D)$ consists of a principal G -bundle P on C along with a q *difference connection*: an isomorphism of G -bundles $P|_{C \setminus D} \rightarrow q^*P|_{C \setminus D}$ away from the divisor D . For an introduction to the classical theory of q -difference connections we refer the reader to [STSS98]. (Chris: I'm not sure what to say about q -difference equations. You suggested citing Sauloy but I'm not sure exactly what. He has articles on q -difference modules which don't seem so relevant here?)

2.1 Residue Conditions

These moduli stacks are typically of infinite type. In order to obtain finite type stacks, and later in order to define symplectic rather than only Poisson structure, we can fix the behaviour of our multiplicative Higgs fields and q difference connections near the punctures $D \subseteq C$.

We'll write \mathbb{D} to denote the *formal disk* $\text{Spec } \mathbb{C}[[z]]$. Likewise we'll write \mathbb{D}^\times for the *formal punctured disk* $\text{Spec } \mathbb{C}((z))$. We'll then write \mathbb{B} for the derived pushout $\mathbb{D} \sqcup_{\mathbb{D}^\times} \mathbb{D}$. Let $LG = \underline{\text{Map}}(\mathbb{D}^\times, G)$ and let $L^+G = \underline{\text{Map}}(\mathbb{D}, G)$.

There is a canonical inclusion $\mathbb{B}^{\sqcup k} \rightarrow (C \times_q S_B^1) \setminus (D \times \{0\})$, the inclusion of the formal punctured neighbourhood of $D \times \{0\}$. This induces a restriction map on mapping spaces

$$\text{res}_D: q\text{-Conn}_G(C, D) \rightarrow \text{Bun}_G(\mathbb{B})^k.$$

The following is well-known (see e.g. the expository article [Zhu17]).

Lemma 2.5. The set of closed points of $\text{Bun}_G(\mathbb{B})$ is in canonical bijection with the set of dominant coweights of G .

Definition 2.6. Choose a map from D to the set of dominant coweights and denote it by $\omega^\vee: z_i \mapsto \omega_{z_i}^\vee$. Write Λ_i for the isotropy group of the point $\omega_{z_i}^\vee$ in $\text{Bun}_G(\mathbb{B})$. The moduli stack of q difference connections on C with singularities at D and fixed residues given by ω^\vee is defined to be the fiber product

$$q\text{-Conn}_G(C, D, \omega^\vee) = q\text{-Conn}_G(C, D) \times_{\text{Bun}_G(\mathbb{B})^k} (B\Lambda_1 \times \cdots \times B\Lambda_k).$$

Examples 2.7. The most important examples for our purposes are given by the following rational/trigonometric/elliptic trichotomy.

- **Rational:** We can enhance the definition of our moduli space by including a framing at a point $c \in C$ not contained in D . We always assume that such framed points are fixed by the automorphism q .

Definition 2.8. The moduli space of q -difference connections on C with a framing at c is defined to be the relative mapping space

$$q\text{-Conn}_G^{\text{fr}}(C) = \underline{\text{Map}}(C \times_q S_B^1, BG; f)$$

where $f: \{c\} \times S_B^1 \rightarrow BG$ (or equivalently $f: \{c\} \rightarrow G/G$) is a choice of adjoint orbit. We can define the framed mapping space with singularities and fixed residues in exactly the same way as above.

In this paper we'll be most interested in the following example. Let $C = \mathbb{CP}^1$ with framing point $c = \infty$, and consider automorphisms of the form $z \mapsto z + \varepsilon$ for $\varepsilon \in \mathbb{C}$. Choose a finite subset $D \subseteq \mathbb{A}^1$ and label the points $z_i \in D$ by dominant coweights $\omega_{z_i}^\vee$. We can then study the moduli space $\varepsilon\text{-Conn}_G^{\text{fr}}(\mathbb{CP}^1, D, \omega^\vee)$. The main object of study in this paper will be the holomorphic symplectic structure on this moduli space.

Note that the motivation for this definition comes in part from Spaide's formalism [Spa16] of AKSZ symplectic structures on relative mapping spaces – in this formalism \mathbb{CP}^1 with a single framing point is relatively 1-oriented, so mapping spaces out of it with 1-shifted symplectic targets have AKSZ 0-shifted symplectic structures.

- **Trigonometric:** Alternatively, we can enhance our definition by including a reduction of structure group at a point $c \in C$ not contained in D , again fixed by the automorphism q .

Definition 2.9. The moduli space of q -difference connections on C with an H -reduction at c for a subgroup $H \subseteq G$ is defined to be the fiber product

$$q\text{-Conn}_G^{H,c}(C) = \underline{\text{Map}}(C \times_q S_B^1, BG) \times_{G/G} H/H$$

associated to the evaluation at c map $\underline{\text{Map}}(C \times_q S_B^1, BG) \rightarrow G/G$. We can define the moduli space with H -reduction with singularities and fixed residues in the same way as above.

Again let $C = \mathbb{CP}^1$. Fix a pair of opposite Borel subgroups B_+ and $B_- \subseteq G$ with unipotent radicals N_\pm and consider the moduli space of q -connections with B_+ -reduction at 0 and N_- -reduction at ∞ . We'll now take q to be an automorphism of the form $z \mapsto qz$ for $q \in \mathbb{C}^\times$. We'll defer in depth analysis of this example to future work.

- **Elliptic:** Finally, let $C = E$ be a smooth curve of genus one. In this case we won't fix any additional boundary data, but just consider the moduli space $q\text{-Conn}_G(E, D, \omega^\vee)$. In the case $q = 1$ this space – or rather its polystable locus – was studied by Hurtubise and Markman [HM02], who proved that it can be given the structure of an algebraic integrable system with symplectic structure related to the elliptic R-matrix of Etingof and Varchenko [EV98].

Remark 2.10. In the elliptic case it's natural to ask to what extent Hurtubise and Markman's integrable system structure can be extended from the variety of polystable multiplicative Higgs bundles to the full moduli stack. If D is empty then it's easy to see that we have a symplectic structure given by the AKSZ construction of Pantev-Toën-Vaquié-Vezzosi [PTVV13]. Indeed, E is compact 1-oriented and the quotient stack G/G is 1-shifted symplectic, so the mapping stack $\underline{\text{Map}}(E, G/G) = \text{mHiggs}_G(E)$ is equipped with a 0-shifted symplectic structure by [PTVV13, Theorem 2.5]. The role of the Hitchin fibration is played by the Chevalley map $\chi: G/G \rightarrow T/W$, and therefore

$$\underline{\text{Map}}(E, G/G) \rightarrow \underline{\text{Map}}(E, T/W).$$

The fibers of this map over regular points in T/W are given by moduli stacks of the form $\text{Bun}_T(\tilde{E})^W$ where \tilde{E} is a W -fold cover of E (the cameral cover). Note that in this unramified case the curve \tilde{E} also has genus 1; counting dimensions we see that the base has dimension $r = \text{rk}(G)$ and the generic fibers are r -dimensional (Lagrangian) tori.

Remark 2.11. While the moduli space of multiplicative Higgs bundles makes sense on a general curve it's only after restricting identity to this trichotomy of examples that we'll expect the existence of a Poisson structure. Such a structure arises by the AKSZ construction, i.e. by transgressing the 1-shifted symplectic structure on G/G to the mapping space using a fixed section of the canonical bundle on C (possibly with a boundary condition).

We can define the global Chevalley map as in Remark 2.10 in the case of non-empty D as well. We'll show that in the rational case this defines a completely integrable system structure.

Definition 2.12. Fix a curve C , a divisor D and a dominant coweight $\omega_{z_i}^\vee$ at each point z_i in D . The *multiplicative Hitchin base* is the stack

$$\mathcal{B}(C, D, \omega^\vee) = \text{Sect}(C, X(D, \omega^\vee)/W)$$

of sections of $X(D, \omega^\vee)/W$, the T/W -bundle on C where $X(D, \omega^\vee)$ is the T -bundle characterized by the condition that the associated line bundle $X(D, \omega^\vee) \times_T \lambda$ corresponding to a weight λ is given by $\mathcal{O}(\sum \omega_{z_i}^\vee(\lambda) \cdot z_i)$ (c.f. [HM02, Section 3.3]).

The *multiplicative Hitchin fibration* is the map

$$\pi: \text{mHiggs}_G(C, D, \omega^\vee) \rightarrow \mathcal{B}(C, D, \omega^\vee)$$

given by post-composing a map $C \setminus D \rightarrow G/G$ with the Chevalley map $\chi: G/G \rightarrow T/W$.

Proposition 2.13. The multiplicative Hitchin fibration described above is well-defined.

Proof. We need to verify that the image of a point in $\mathrm{mHiggs}_G(C, D, \omega^\vee)$ under π , viewed as a section of the trivial T/W -bundle on $C \setminus D$, extends to a section of $X(D, \omega^\vee)/W$, look locally near a singularity $z_i \in D$. Let $\phi \in G((z_i))$ be a local representative for a multiplicative Higgs field in $\mathrm{mHiggs}_G(C, D, \omega^\vee)$, i.e. an element of the associated $G[[z_i]]$ -adjoint orbit. Since ϕ is equivalent to $z_i^{-\omega_{z_i}^\vee}$ under the action of $G[[z_i]]^2$ by left and right multiplication, without loss of generality we can say that $\phi = z_i^{-\omega_{z_i}^\vee} \phi_0$ for some $\phi_0 \in G[[z_i]]$. Consider $\chi(\phi) \in T((z_i))$ – the singular part of this element is the same as $\chi(z_i^{-\omega_{z_i}^\vee} g)$ for some $g \in G$, which implies the section extends to a meromorphic T/W -valued function of the required type. \square

Remark 2.14. (Chris: Compare to the base in [HM02]. Variety versus stack? Difference in processing the W -action? They are birational.)

We'll show that this fibration defines a completely integrable system in the rational case where $C = \mathbb{CP}^1$ with a framing at ∞ . We should first observe that the generic fibers are half-dimensional tori. Computing the fibers of the Hitchin fibration works similarly to the non-singular case: a point in the base is, in particular, a map $C \setminus D \rightarrow T/W$. Suppose this map lands in the regular locus T^{reg}/W , then an element of the fiber over this point defines, in particular, a map $C \setminus D \rightarrow BT/W$. We would like to argue that the fiber consists of T -bundles on the cameral cover \tilde{C} : a W -fold cover of C ramified at the divisor D .

In order to justify this we'll compare our moduli space with the space of abstract Higgs bundles introduced by Donagi and Gaitsgory [DG02]. Our argument will follow the same ideas as the arguments of [HM02, Section 6].

Definition 2.15. An *abstract G -Higgs bundle* on a curve C is a principal G -bundle P along with a sub-bundle \mathfrak{c} of \mathfrak{g}_P of *regular centralizers*, meaning that the fibers are subalgebras of \mathfrak{g} which arise as the centralizer of a regular element of \mathfrak{g} . Write $\mathrm{Higgs}_G^{\mathrm{abs}}(C)$ for the moduli stack of abstract G -Higgs bundles on C .

There's an algebraic map from the regular part of our moduli space $\mathrm{mHiggs}_G(C, D, \omega^\vee)_{\mathrm{reg}}$ (where the Higgs field is required to take regular values) into $\mathrm{Higgs}_G^{\mathrm{abs}}(C)$ that sends a multiplicative Higgs bundle (P, g) to the abstract Higgs bundle (P, \mathfrak{c}_g) , where \mathfrak{c}_g is the sub-bundle of \mathfrak{g}_P fixed by the adjoint action of the multiplicative Higgs field g (Chris: check). What's more, there is a commutative square relating the Hitchin fibrations for the multiplicative and generalized moduli spaces:

$$\begin{array}{ccc} \mathrm{mHiggs}_G(C, D, \omega^\vee)_{\mathrm{reg}} & \longrightarrow & \mathrm{Higgs}_G^{\mathrm{abs}}(C) \\ \downarrow & & \downarrow \\ \mathcal{B}(C, D, \omega^\vee)_{\mathrm{reg}} & \longrightarrow & \mathrm{Cam}_G(C), \end{array}$$

where $\mathrm{Cam}_G(C)$ is the stack of cameral covers of C , as defined in [DG02, Section 2.8]. Indeed, (Chris: relating projections, which means there's a map on the fibers. If we check the map on total spaces is an embedding so is the map on fibers, so we at least have an embedding into torus of the appropriate dimension (up to subtleties about the W -equivariance). Dimension count would say it's an open embedding.)

Remark 2.16. (Chris: remark on the existence of the Hitchin section. Mention for later how the definition of q -opers will go.)

(Chris: Include comparison to the story in [DG02]. As Hurtubise and Markman explain there's a map from a multiplicative Higgs bundle to a generalized Higgs bundle in their sense by taking the associated bundle of regular centralizers.)

2.2 Stability Conditions

For comparison to results in the literature it is important that we briefly discuss the role of stability conditions for difference connections. In our main example of interest – the rational case – these conditions won't play a role, but

they do appear in the comparison results between q -connections and monopoles in the literature for more general curves. For definitions for general G we refer to [Smi15], although see also [AB01] on polystable G -bundles. In what follows we fix a choice of $0 < t_0 < 2\pi R$.

Definition 2.17. Let (P, g) be a q -connection on a curve C , and let χ be a character of G . The χ -degree of (P, g) is defined to be

$$\deg_\chi(P, g) = \deg(P \times_\chi \mathbb{C}) - \frac{t_0}{2\pi R} \sum_{i=1}^k \deg(\chi \circ \omega_{z_i}^\vee).$$

A q -connection (P, g) on C is *stable* if for every maximal parabolic subgroup $H \subseteq G$ with Levi decomposition $H = LN$ and every reduction of structure group (P_H, g) to H , we have

$$\deg_\chi(P_H, g) < 0$$

for the character $\chi = \det(\text{Ad}_L^n)$ defined to be the determinant of the adjoint representation of L on \mathfrak{n} .

The q -connection (P, g) is *polystable* if there exists a (not necessarily maximal) parabolic subgroup H with Levi factor L and a reduction of structure group (P_L, g) to L so that (P_L, g) is a stable q -connection and so that the associated H -bundle is admissible, meaning that for every character χ of H which is trivial on $Z(G)$ the associated line bundle $P_H \times_\chi \mathbb{C}$ has degree zero.

Below we'll write $q\text{-Conn}_G^{\text{ps}}(C, D, \omega^\vee)$ for the moduli space of polystable q -connections. This moduli space is a smooth algebraic variety of finite type. If q is the identity this is a theorem of Charbonneau, Hurtubise and Smith. For more general q , as a smooth manifold these moduli spaces are actually independent when one varies q – we'll see this when we describe an equivalent description of these moduli spaces as spaces of periodic monopoles on diffeomorphic manifolds.

When $C = \mathbb{CP}^1$ every principal G -bundle on C admits an essentially unique (up to the action of the Weyl group) holomorphic reduction of structure group to a maximal torus [Gro57]. Since q -connections for an abelian group are automatically stable, polystability on the sphere is equivalent to admissibility of the torus reduction. As a consequence, for our main example of interest – the rational case – the moduli space of polystable q -connections is equivalent to the moduli space of all q -connections of admissible degree. For instance for $G = \text{SL}_n$ the moduli space of polystable q -connection is equivalent to the moduli space of q -connections on the trivial bundle.

2.3 Poisson Structures from Derived Geometry

As we mentioned above in Remark 2.10, in the case where C is an elliptic curve and there are no punctures there's a symplectic structure on $\text{mHiggs}_G(C)$ given by the AKSZ formalism. More generally, when we do have punctures, we expect the moduli space $\text{mHiggs}_G(C, D)$ to have a Poisson structure with a clear origin story coming from the theory of derived Poisson geometry. In this section we'll explain what this story looks like. However, we emphasise that there are technical obstructions to making this story precise with current technology: this section should be viewed as motivation for the structures we'll discuss in the rest of the paper. We refer the reader to [CPT⁺17] for the theory of derived Poisson structures and to [MS16, MS17, Spa16] for that of derived coisotropic structures.

Here's the idea. Recall that we can identify the moduli space of singular q -connections on a curve C as a fiber product: $q\text{-Conn}_G(C, D) \cong \text{Bun}_G^{\text{fr}}(C) \times_{\text{Bun}_G(C \setminus D)^2} \text{Bun}_G(C \setminus D)$ where the map $g: \text{Bun}_G^{\text{fr}}(C) \rightarrow \text{Bun}_G(C \setminus D)$ is given by $P \mapsto (P|_{C \setminus D}, q^*P|_{C \setminus D})$. Consider the following commutative cube:

$$\begin{array}{ccccc}
q\text{-Conn}_G(C, D) & \xrightarrow{f_1} & \text{Bun}_G(C) & & \\
\swarrow f_2 & \downarrow & \swarrow g_1 & & \\
\text{Bun}_G(C \setminus D) & \xrightarrow{g_2} & \text{Bun}_G(C \setminus D)^2 & & \\
\downarrow & \downarrow \text{res} & \downarrow r & & \\
& \text{Bun}_G(\mathbb{B})^k & \xrightarrow{\quad} & BL^+G^{2k} & \\
\downarrow & \downarrow & \downarrow & \swarrow & \\
BLG^k & \xrightarrow{\quad} & BLG^{2k} & &
\end{array}$$

Here the top and bottom faces are homotopy Cartesian squares. What does this setup buy us? We'll first answer informally.

Claim. First consider the bottom face of the cube. The stack BLG is 2-shifted symplectic because the Lie algebra $L\mathfrak{g}$ has a non-degenerate invariant pairing: the residue pairing. The Lie subalgebra $L^+\mathfrak{g}$ forms part of a Manin triple $(L\mathfrak{g}, L^+\mathfrak{g}, L^-\mathfrak{g})$ which means that $BL^+G \rightarrow BLG$ is 2-shifted Lagrangian. Therefore the bottom face of the cube defines a 2-shifted Lagrangian intersection, which means that the pullback $\text{Bun}_G(\mathbb{B})^k$ is 1-shifted symplectic.

Now consider the top face of the cube. If either C is an elliptic curve, or $C = \mathbb{CP}^1$ and we fix a framing at ∞ , then the map $\text{Bun}_G(C \setminus D) \rightarrow BLG^k$ is also 2-shifted Lagrangian. In particular $\text{Bun}_G(C \setminus D)$ is 1-shifted Poisson. Finally, the map $\text{Bun}_G(C) \rightarrow \text{Bun}_G(C \setminus D)$ is 1-shifted coisotropic, or equivalently the canonical map $\text{Bun}_G(C) \rightarrow \text{Bun}_G(C \setminus D) \times_{BLG^k} BL^+G^k$ is 1-shifted Lagrangian. That means that the top face of the cube defines a 1-shifted coisotropic intersection, which means that the pullback $q\text{-Conn}_G(C, D)$ is 0-shifted Poisson.

The restriction map $q\text{-Conn}_G(C, D) \rightarrow \text{Bun}_G(\mathbb{B})^k$ is 1-shifted Lagrangian, which means that if we form the intersection with a k -tuple of Lagrangians in $\text{Bun}_G(\mathbb{B})$ then we obtain a 0-shifted symplectic stack. For example, if ω_i^\vee is a point in $\text{Bun}_G(\mathbb{B})$ corresponding to a dominant coweight with stabilizer Λ_i then $B\Lambda_i \rightarrow \text{Bun}_G(\mathbb{B})$ is 1-shifted Lagrangian, so the moduli stack $q\text{-Conn}_G(C, D, \omega^\vee)$ obtained by taking the derived intersection is ind 0-shifted symplectic.

Now, let's make this claim more precise. The main technical condition that makes this claim subtle comes from the fact that most of the derived stacks appearing in this cube, for instance the stack BLG , are not Artin. As such we need to be careful when we try to, for instance, talk about the tangent complex to such stacks. One can make careful statements using the formalism of ‘‘Tate stacks’’ developed by Hennion [Hen15]. We can therefore make our claim into a more formal conjecture.

Conjecture 2.18. Suppose C is either an elliptic curve or \mathbb{CP}^1 with a fixed framing at ∞ .

1. The stack BLG is Tate 2-shifted symplectic, and both $BL^+G \rightarrow BLG$ and $\text{Bun}_G(C \setminus D) \rightarrow BLG^k$ are Tate 2-shifted Lagrangian.
2. The stack $\text{Bun}_G(C \setminus D)$ is ind 1-shifted Poisson, and the map $\text{Bun}_G(C) \rightarrow \text{Bun}_G(C \setminus D)$ is ind 1-shifted coisotropic witnessed by the 2-shifted Lagrangian map $BL^+G^k \rightarrow BLG^k$.
3. The Lagrangian intersection $\text{Bun}_G(\mathbb{B})$ is Tate 1-shifted symplectic, and the map $B\Lambda_i \rightarrow \text{Bun}_G(\mathbb{B})$ associated to the inclusion of the stabilizer of a closed point is 1-shifted Lagrangian.

As a consequence, the moduli stack $q\text{-Conn}_G(C, D)$ is ind 0-shifted Poisson and the moduli stack $q\text{-Conn}_G(C, D, \omega^\vee)$ is 0-shifted symplectic.

Remark 2.19. We should explain heuristically why the Calabi-Yau condition on C is necessary. This is a consequence of the AKSZ formalism in the case where D is empty: for the mapping stack $\underline{\text{Map}}(C, G/G)$ to be 0-shifted symplectic, or for the mapping stack $\underline{\text{Map}}(C, BG)$ to be 1-shifted symplectic, we need C to be compact and 1-oriented. A d -orientation on a smooth complex variety of dimension d is exactly the same as a Calabi-Yau structure.

More generally we can say the following. Let's consider the rational case where $C = \mathbb{CP}^1$. Consider the inclusion $dr: \mathfrak{g}_- = \mathbb{T}_{\text{Bun}_G^{\text{fr}}(\mathbb{CP}^1 \setminus D)}[-1] \rightarrow r^*\mathbb{T}_{BLG^k}[-1] = \mathfrak{g}((z))^k$: a map of ind-pro Lie algebras concentrated in degree zero.

The residue pairing vanishes after pulling back along r since elements of \mathfrak{g}_- are \mathfrak{g} -valued functions on \mathbb{CP}^1 with at least a simple pole at every puncture in D . So the map r is isotropic with 0 isotropic structure; this structure is unique for degree reasons. We must check that this structure is non-degenerate. It suffices to check that the sequence

$$\mathbb{T}_{\mathrm{Bun}_G^{\mathrm{fr}}(\mathbb{CP}^1 \setminus D)}[-1] \rightarrow r^* \mathbb{T}_{BLG^k}[-1] \rightarrow (\mathbb{T}_{\mathrm{Bun}_G^{\mathrm{fr}}(\mathbb{CP}^1 \setminus D)}[-1])^\vee$$

is an exact sequence of ind-pro vector spaces, and therefore an exact sequence of quasi-coherent sheaves on the stack $\mathrm{Bun}_G^{\mathrm{fr}}(\mathbb{CP}^1 \setminus D)$. To do this we identify the pair $(\mathfrak{g}_-, \mathfrak{g}((z))^k)$ as part of a Manin triple, where a complementary isotropic subalgebra to \mathfrak{g}_- is given by $\mathfrak{g}_+ = \mathfrak{g}[[z]]^k$. Using the residue pairing we can identify \mathfrak{g}_+ with $(\mathfrak{g}_-)^\vee$ and therefore identify our sequence with the split exact sequence

$$0 \rightarrow \mathfrak{g}_- \rightarrow \mathfrak{g}((z))^k \rightarrow \mathfrak{g}_+ \rightarrow 0.$$

3 Periodic Monopoles

Moduli spaces of q -connections on a Riemann surface C are closely related to moduli spaces of periodic monopoles, i.e. monopoles on 3-manifolds that fiber over the circle (more specifically, with fiber C and monodromy determined by q). Let $G_{\mathbb{R}}$ be a compact Lie group whose complexification is G . The discussion in this section will mostly follow that of [CH10, Smi15].

Write $M = C \times_q S_R^1$ for the C -bundle over S^1 with monodromy given by the automorphism q . More precisely, M is the Riemannian 3-manifold obtained by gluing the ends of the product $C \times [0, 2\pi R]$ of Riemannian manifolds by the isometry $(x, 2\pi R) \sim (q(x), 0)$.

Definition 3.1. A *monopole* on the Riemannian 3-manifold $M = C \times_q S_R^1$ is a smooth principal $G_{\mathbb{R}}$ -bundle \mathbf{P} equipped with a connection A and a section Φ of the associated bundle $\mathfrak{g}_{\mathbf{P}}$ satisfying the Bogomolny equation

$$*F_A = d_A \Phi.$$

Remark 3.2. We should emphasise the difference between the Riemannian 3-manifold $M = C \times_q S_R^1$ appearing in this section and the derived stack $C \times_q S_B^1$ (the mapping torus) appearing in the previous section. These should be thought of as smooth and algebraic realizations of the same object (justified by the comparison Theorem 3.8) but they are a priori defined in different mathematical contexts.

We can rephrase the data of a monopole on M as follows. Let $C_0 = C \times \{0\}$ be the fiber over 0 in S^1 , viewed as a Riemann surface. Let P be the restriction of the complexified bundle $\mathbf{P}_{\mathbb{C}}$ to C_0 . Consider first the restriction of the complexification of A to a connection A_0 on P over C_0 . The $(0, 1)$ part of A_0 automatically defines a holomorphic structure on P . We can introduce an additional piece of structure on this holomorphic G -bundle. In order to do so we can decompose the Bogomolny equation into one real and one complex equation as follows.

$$\begin{aligned} F_{A_0} - \nabla_t \Phi &= 0 \\ [\bar{\partial}_{A_0}, \nabla_t - i\Phi dt] &= 0 \end{aligned}$$

where ∇_t is the component of the covariant derivative d_A normal to C_0 .

Definition 3.3. From now on we'll use the notation \mathcal{A} for the combination $\nabla_t - i\Phi dt$: an element of the space $\Omega^0(C_0, \mathfrak{g}_P)dt$ of sections of the complex vector bundle \mathfrak{g}_P on the complex curve C_0 .

Let us now introduce singularities into the story. We'll keep the description brief, referring the reader to [CH10, Smi15] for details.

Definition 3.4. Let $D \subseteq M$ be a finite subset. Let ω^\vee be a choice of coweight for G . A monopole on $M \setminus D$ has *Dirac singularity* at $z \in D$ with charge ω^\vee if locally on a neighbourhood of z in M it is obtained by pulling back under ω^\vee the standard Dirac monopole solution to the Bogomolny equation where Φ is spherically symmetric with

a simple pole at z () and the restriction of a connection A to a two-sphere S^2 enclosing the singularity defines a $U(1)$ bundle on this S^2 of degree 1 so that

$$\frac{1}{2\pi} \int_{S^2} F = 1.$$

See e.g. [CH10, Section 2.2] for a more detailed description.

We can also introduce a framing (or a reduction of structure group as in the trigonometric example, though we won't consider the latter in this paper). As usual let $c \in C$ be a point fixed by the automorphism q .

Definition 3.5. A monopole on M with *framing* at the point $c \in C$ is a monopole (\mathbf{P}, A, Φ) on M (possibly with Dirac singularities at D) along with a trivialization of the restriction of \mathbf{P} to the circle $\{c\} \times S_R^1$, with the condition that the holonomy of A around this circle lies in a fixed conjugacy class $f \in G/G$.

The moduli theory of monopoles on general compact 3-manifolds was described by Pauly [Pau98]. In this paper we'll be interested in moduli spaces of monopoles on 3-manifolds of the form $C \times S^1$, possibly with a fixed framing at a point in C .

Remark 3.6. The moduli space of periodic monopoles on $\mathbb{R}^2 \times S^1$ specifically has been studied in the mathematics literature by Foscolo [Fos16], applying the analytic techniques of deformation theory to earlier work on periodic monopoles by Cherkis and Kapustin [CK98, CK01]. This analysis considers a less restrictive boundary condition at infinity in \mathbb{R}^2 than a framing, and therefore requires more sophisticated analysis than we'll need to consider in the present paper.

In the cases of interest to us the moduli space of periodic monopoles can be obtained as a hyperkähler quotient. Let's focus initially on the rational case, meaning monopoles on $M = \mathbb{CP}^1 \times_\varepsilon S_R^1$ with Dirac singularities at $D \times \{t_0\}$ and a framing at ∞ . Consider the infinite-dimensional vector space \mathcal{V} consisting of pairs (A, Φ) where A is a connection on a fixed principal $G_{\mathbb{R}}$ -bundle \mathbf{P} on M , Φ is a section of $\mathfrak{g}_{\mathbf{P}}$, and (A, Φ) have a Dirac singularity with charge $\omega_{z_i}^\vee$ at each (z_i, t_0) in $D \times \{t_0\}$. Let \mathcal{G} be the group of gauge transformations of the bundle \mathbf{P} .

The hyperkähler moment map is given by the Bogomolny functional, namely

$$\begin{aligned} \mu: \mathcal{V} &\rightarrow \Omega^1(M; (\mathfrak{g}_{\mathbb{R}})_{\mathbf{P}}) \\ (A, \Phi) &\mapsto *F_A - d_A \Phi. \end{aligned}$$

Definition 3.7. Let D be a finite subset $\{(z_1, t_1), \dots, (z_k, t_k)\}$ of points in $M = \mathbb{CP}^1 \times_\varepsilon S_R^1$, and let ω_i^\vee be a choice of coweight for each point in D . The moduli space $\text{Mon}_G(M, D, \omega^\vee)$ is the hyperkähler quotient

$$\text{Mon}_G(M, D, \omega^\vee) = \mu^{-1}(0)/\mathcal{G}.$$

Now let's address the relationship between periodic monopoles and q -connections. Suppose from now on that q is in the identity component of the group of automorphisms of C (fixing the framing point c if present).

Theorem 3.8. There is an analytic isomorphism between the moduli space of polystable monopoles on $C \times_q S^1$ with Dirac singularities at $D \times \{t_0\}$ (and a possible framing on $\{c\} \times S^1$) and the moduli space of q -connections on C with singularities at D and framing at $\{c\}$. More precisely there is an analytic isomorphism

$$H: \text{Mon}_G^{(\text{fr})}(C \times_q S^1, D \times \{t_0\}, \omega^\vee) \rightarrow q\text{-Conn}_G^{\text{ps}, (\text{fr})}(C, D, \omega^\vee)$$

given by the holonomy map around S^1 , i.e. sending a monopole (\mathbf{P}, A) to the holomorphic bundle $P = (\mathbf{P}_C)|_{C_0}$ with q -connection $g = \text{Hol}_{S^1}(A): P \rightarrow q^*(P)$.

Remark 3.9. Note that in this statement we assumed that all the singularities occur in the same location in S^1 , i.e. in the same slice $C \times t_0$. This assumption is not necessary, but there is a constraint on the possible locations of the singularities as explained in [CH10, Proposition 3.5].

Proof. (**Chris: check**) This follows by the same argument as that given by Charbonneau-Hurtubise [CH10] and Smith [Smi15]. More explicitly, first let's think about injectivity, so let $(\mathbf{P}, \mathcal{A})$ and $(\mathbf{P}', \mathcal{A}')$ be a pair of periodic monopoles on $C \times_q S^1$ with images (P, g) and (P', g') respectively, and choose a bundle isomorphism $\tau: P \rightarrow P'$ intertwining the q -connections g and g' . One observes first that \mathbf{P} and \mathbf{P}' are also isomorphic G -bundles since, by intertwining with the q -connections, we have an isomorphism $\mathbf{P}|_{C \times \{t\}} \rightarrow \mathbf{P}'|_{C \times \{t\}}$ for every $t \in S^1$. That the monopole structures also match up follows by the same argument as in [CH10, Proposition 4.7].

For surjectivity, again we'll match the argument in the case where $q = \text{id}$. We begin by extending a holomorphic G -bundle P on C_0 with q -connection g to a G -bundle on $M \setminus (D \times \{t_0\}) = (C \times_q S_R^1) \setminus (D \times \{t_0\})$. Let $\gamma: [-2\pi R, 2\pi R] \rightarrow \text{Aut}(C)$ be a geodesic with $\gamma(-2\pi R) = q^{-1}$, $\gamma(0) = 1$ and $\gamma(2\pi R) = q$. Let \widetilde{M} be the 3-manifold

$$\widetilde{M} = ((-2\pi R, 2\pi R) \times C) \setminus \bigcup_{j=1}^k (A_j^+ \cup A_j^-)$$

where A_j^+ is the arc $\{(t + t_0, \gamma(t)(z_j)) : t \in (0, 2\pi R - t_0)\}$ and A_j^- is the arc $\{(t + t_0 - 4\pi R, \gamma(t)(z_j)) : t \in [2\pi R - t_0, 2\pi R)\}$.

Let $\pi: \widetilde{M} \rightarrow C$ be the projection sending (t, z) to $\gamma(t)(z)$. The bundle P pulls back to a bundle $\pi^*(P)$ on \widetilde{M} . We obtain a bundle on $M \setminus (D \times t_0)$ by applying the identification $(t, z) \sim (t - 2\pi R, q(z))$. This bundle extends to an S^1 -invariant holomorphic G -bundle on $M \times S^1$. The remainder of the proof – verifying the existence of the monopole structure associated to an appropriate choice of hermitian structure – consists of local analysis which is independent of the value of the parameter q .

It remains to remark on the compatibility of framing data on the two sides. A trivialization of the bundle \mathbf{P} along the circle $\{c\} \times S^1$ yields a trivialization of the fiber of the bundle P at c . The condition that the holonomy around the circle at c is f fixes the value of the q -connection at c to be in the conjugacy class f . \square

Remark 3.10. Mochizuki [Moc17] proved a stronger result in the rational case for the group $G = \text{GL}_n$. He allows not just a framing at infinity in \mathbb{CP}^1 but also a singularity encoded in terms of a B -reduction of the bundle.

3.1 Deformation Theory

In the next section we'll compare symplectic forms on these moduli spaces. In order to do so it will be important to understand the tangent spaces at a point of the source and target. There's a natural description of these tangent spaces in terms of the hypercohomology of certain cochain complexes.

From now on we'll focus on the example we're most interested in. That is, we'll exclusively study the rational situation where $C = \mathbb{CP}^1$ and we fix a framing point $c = \infty$. In this case we can use the description as a hyperkähler reduction. For more general deformation theory calculations we refer to Foscolo [Fos16]. Recall that we can write

$$\text{Mon}_G^{(\text{fr})}(C \times_q S^1, D \times \{t_0\}, \omega^\vee) \cong \mu^{-1}(0)/\mathcal{G}$$

where \mathcal{G} is the group of gauge transformations of \mathbf{P} and $\mu: \mathcal{V} \rightarrow \Omega^1(M; (\mathfrak{g}_R)_{\mathbf{P}})$ is the Bogomolny functional $(A, \Phi) \mapsto *F_A - d_A \Phi$. The tangent complex to this hyperkähler quotient at a point $(\mathbf{P}, \mathcal{A})$ can be written as $\Omega^0(M \setminus D; (\mathfrak{g}_R)_{\mathbf{P}})[1] \rightarrow \mathbb{T}_{\mu^{-1}(0)}$ where $\mathbb{T}_{\mu^{-1}(0)}$ is the tangent complex to the zero locus of the moment map, concentrated in non-negative degrees. Roughly speaking $\mathbb{T}_{\mu^{-1}(0)} = \mathbb{T}_{\mathcal{V}} \xrightarrow{d\mu} \Omega^1(M; (\mathfrak{g}_R)_{\mathbf{P}})[-1]$.

More explicitly, following [Fos16] let

$$\mathcal{F}_{\mathbf{P}, \mathcal{A}}^{\text{mon}} = \left(\Omega^0(M \setminus D; (\mathfrak{g}_R)_{\mathbf{P}}) \xrightarrow{d_1} \Omega^1(M \setminus D; (\mathfrak{g}_R)_{\mathbf{P}}) \oplus \Omega^0(M \setminus D; (\mathfrak{g}_R)_{\mathbf{P}}) \xrightarrow{d_2} \Omega^1(M \setminus D; (\mathfrak{g}_R)_{\mathbf{P}}) \right) \otimes_{\mathbb{R}} \mathbb{C}$$

placed in degrees $-1, 0$ and 1 where $d_1(g) = -(d_A(g), [\Phi, g])$ and $d_2(a, \psi) = *d_A(a) - d_A(\psi) + [\Phi, a]$. Write d_{mon} for the total differential.

Remark 3.11. Here we've chosen a point in the twistor sphere, forgetting the hyperkähler structure and retaining a holomorphic symplectic structure. In other words we've identified the target of the hyperkähler moment map – the space of imaginary quaternions – with $\mathbb{R} \oplus \mathbb{C}$, which is equivalent to choosing a point in the unit sphere of the imaginary quaternions: the twistor sphere.

Remark 3.12. If we restrict $\mathcal{F}_{\mathcal{P}, \mathcal{A}}^{\text{mon}}$ to a slice $C_t = C \times \{t\}$ in the t -direction we can identify it with a complex of the form

$$\Omega^\bullet(C_t; \mathfrak{g}_P)[1] \xrightarrow{[\Phi, -]} \Omega^\bullet(C_t; \mathfrak{g}_P)$$

with total differential given by d_A on each of the two factors along with the differential $[\Phi, -]$ mixing the two factors. These two summands each split up into the sum of a Dolbeault complex on C with its dual. That is, there's a natural subcomplex of the form

$$\Omega^{0,\bullet}(C_t; \mathfrak{g}_P)[1] \xrightarrow{[\Phi, -]} i\Omega^{0,\bullet}(C_t; \mathfrak{g}_P)dt$$

where the internal differentials on the two factors are now given by $\bar{\partial}_{A_0}$. This complex is in turn quasi-isomorphic to the complex

$$\Omega^\bullet(S^1; \Omega^{0,\bullet}(C_t; \mathfrak{g}_P))[1]$$

with total differential $\bar{\partial}_{A_0} + d_A$.

Remark 3.13. If one introduces a framing at a point $c \in C$ then we must correspondingly twist the complex \mathcal{F}^{mon} above by the line bundle $\mathcal{O}(c)$ on C . So in that case we define (Chris: check)

$$\mathcal{F}_{(\mathcal{P}, \mathcal{A})}^{\text{mon, fr}} = \mathcal{F}_{\mathcal{P}, \mathcal{A}}^{\text{mon}} \otimes (\mathbb{C}_{S^1} \boxtimes \mathcal{O}(c)).$$

The following is proved in [Fos16].

Proposition 3.14. The tangent space of $\text{Mon}_G(S^1 \times C, D \times \{t_0\}, \omega^\vee)$ at the point $(\mathcal{P}, \mathcal{A})$ is quasi-isomorphic to the hypercohomology $\mathbb{H}^0(C \times S^1; \mathcal{F}'_{(\mathcal{P}, \mathcal{A})})$ of a subsheaf $\mathcal{F}' \subseteq \mathcal{F}^{\text{mon}}$ where growth conditions are imposed on the degree 0 part of \mathcal{F}^{mon} near the singularities.

Now let's consider the tangent complex to the moduli space of q -connections. For the arguments in this article we'll only need to carefully consider the case $q = \text{id}$ of multiplicative Higgs bundles, but we'll include some remarks regarding the more general case. In this case the calculation was performed by Bottacin [Bot95], see also [HM02, Section 4]. Fix a multiplicative Higgs bundle (P, g) on C . We consider the sheaf of cochain complexes on C

$$\mathcal{F}_{(P, g)} = (\mathfrak{g}_P[1] \xrightarrow{\text{Ad}_g} \mathfrak{g}_P(-D))$$

in degrees -1 and 0 with differential given by the adjoint action of g . More precisely let L_g and R_g be the bundle maps $\mathfrak{g}_P \rightarrow \mathfrak{g}_P$ obtained as the derivative of left- and right-multiplication. Then $\text{Ad}_g = L_g - R_g$. We can alternatively phrase this, as in [HM02, Section 4], as follows. Define $\text{ad}(g)$ to be the vector bundle

$$\text{ad}(g) = (\mathfrak{g}_P \oplus \mathfrak{g}_P) / \{(X, -gXg^{-1}) : X \in \mathfrak{g}_P\}.$$

Then we can write \mathcal{F} as the sheaf of complexes

$$\mathcal{F}_{(P, g)} = (\mathfrak{g}_P[1] \xrightarrow{\text{Ad}_g} \text{ad}(g))$$

where now Ad_g is just the map $X \mapsto [(X, -X)]$.

Remark 3.15. If one introduces a framing at a point $c \in C$ then we must correspondingly twist the complex \mathcal{F} above by the line bundle $\mathcal{O}(c)$ on C , i.e. we restrict to deformations that preserve the framing and therefore are zero at the point c . So in that case we define

$$\mathcal{F}_{(P, g)}^{\text{fr}} = (\mathfrak{g}_P[1] \xrightarrow{\text{Ad}_g} \mathfrak{g}_P(-D)) \otimes \mathcal{O}(c).$$

Remark 3.16. For more general q we should modify this description by replacing g by a q -connection. Note that one can still define the (q -twisted) adjoint action $X \mapsto gXg^{-1}$ using a q -connection, and so we can still define the complex

$$\mathcal{F}_{(P,g)} = (\mathfrak{g}_P[1] \xrightarrow{\text{Ad}_g} \text{ad}(g))$$

just as in the untwisted case ([Chris: check](#)).

This complex defines the deformation theory of the moduli space of multiplicative Higgs bundles.

Proposition 3.17 ([Bot95, Proposition 3.1.3]). The tangent space of $\text{mHiggs}_G(C, D, \omega^\vee)$ at the point (P, g) is quasi-isomorphic to the hypercohomology $\mathbb{H}^0(C; \mathcal{F}_{(P,g)})$ of the sheaf \mathcal{F} .

Remark 3.18. The remaining hypercohomology of the sheaf $\mathcal{F}_{(P,g)}$ has dimension $\dim \mathfrak{z}_{\mathfrak{g}}$ in degree -1 and 1 . However the moduli space $\text{mHiggs}_G(C, D, \omega^\vee)$ is in fact a smooth algebraic variety. This follows from a result of Hurtubise and Markman [HM02, Theorem 4.13], noting that their argument does not rely on the curve C being of genus 1.

In order to calculate with this hypercohomology group we'll use a Čech resolution. This will be straightforward for the multiplicative Higgs moduli space, and we'll use the isomorphism of Theorem 3.8 to give an analogous description on the monopole side. We define a cover $\mathcal{U} = \{U_0, U_1, \dots, U_k, U_\infty\}$ of C as follows. Let U_i be a contractible open neighbourhood of the point z_i and let U_∞ be a contractible analytic open neighbourhood of $c \in C$, all chosen to be pairwise disjoint. Let U_0 . Finally let $U_0 = C \setminus (D \cup \{c\})$. Since the U_i are contractible and the remaining subset $C \setminus (D \cup \{c\})$ is an affine algebraic curve, which means that for any quasi-coherent sheaf of cochain complexes the higher cohomology groups vanish. Likewise for the intersections: the punctured open sets U_i^\times are analytic open sets of an affine curve.

Specify a representative 0-cocycle $(\alpha_\infty, \{\alpha_i\}, \alpha_0, \beta_\infty, \{\beta_i\})$ for the Čech cohomology group with respect to our chosen cover \mathcal{U} . Explicitly a 0-cochain is given by the following data:

$$\begin{aligned} \alpha_\infty &\in \text{ad}(g)(1)(U_\infty) \\ \alpha_i &\in \text{ad}(g)(-1)(U_i) \text{ for } i = 1, \dots, k \\ \alpha_0 &\in \text{ad}(g)(C \setminus (D \cup \{\infty\})) \\ \beta_\infty &\in \mathfrak{g}_P(U_\infty^\times) \\ \beta_i &\in \mathfrak{g}_P(U_i^\times) \text{ for } i = 1, \dots, k \end{aligned}$$

where the notation (± 1) indicates tensoring by the line bundle $\mathcal{O}(\pm 1)$.

Being a 0-cocycle means that $(\alpha_\infty - \alpha_0)|_{U_\infty^\times} = \text{Ad}_g(\beta_\infty)$ and $(\alpha_i - \alpha_0)|_{U_i^\times} = \text{Ad}_g(\beta_i)$ for each i . We consider 0-cocycles modulo 0-coboundaries of the form $(\text{Ad}_g(f_\infty), \{\text{Ad}_g(f_i)\}, \text{Ad}_g(f_0), (f_\infty - f_0)|_{U_\infty^\times}, \{(f_i - f_0)|_{U_i^\times}\})$. In fact Ad_g is an isomorphism on U_0 for the sections α_0 of \mathfrak{g}_P that occur: those with no poles or zeroes in U_0 . That means that we can add a coboundary to force $\alpha_0 = 0$.

Now, rather than describing the tangent space to the moduli space of monopoles we'll define a complex that maps to it which we can define locally with respect to the cover \mathcal{U} . Consider the open cover $\mathcal{U} \times S^1 = \{U_i \times S^1\}$ of $M = C \times S^1$. For each element U_i of the cover we can define a map

$$\begin{aligned} \mathcal{F}'(U_i \times S^1) &\rightarrow (\Omega^0(S^1; \Omega^{0,\bullet}(U_i; \mathfrak{g}_P)) \rightarrow \Omega^1(S^1; \Omega^{0,\bullet}(U_i; \mathfrak{g}_P)(D_{U_i}))) [1] \\ &\cong \mathfrak{g}_P(U_i)[1] \rightarrow \mathfrak{g}_P(U_i)(D|_{U_i}) \\ &\cong \mathcal{F}(U_i) \end{aligned}$$

where we restrict the sheaf \mathcal{F}' whose hypercohomology calculated the monopole tangent complex to the holomorphic part of the slice at $\{t\} \in S^1$. Altogether this defines a map from the Čech cohomology with respect to this cover to the tangent complex of the moduli space of monopoles. That is, we have an explicit map

$$\check{H}^\bullet(M, \mathcal{U} \times S^1, \mathcal{F}') \rightarrow \mathbb{H}^\bullet(M; \mathcal{F}')$$

that factors through the (isomorphic) tangent complex of the moduli space of multiplicative Higgs bundles. To verify this we need only note that these maps commute with the differentials in the Čech complex, i.e. with the restriction to the intersection of a pair of open sets.

Explicitly a 0-cochain in this Čech complex is given by $(\alpha_\infty, \{\alpha_i\}, \alpha_0, \beta_\infty, \{\beta_i\})$ where now

$$\begin{aligned}\alpha_\infty &\in \mathcal{F}_0^{\text{mon}}(U_\infty \times S^1) \\ \alpha_i &\in \mathcal{F}_0^{\text{mon}}(U_i \times S^1) \text{ for } i = 1, \dots, k \\ \alpha_0 &\in \mathcal{F}_0^{\text{mon}}(U_0 \times S^1) \\ \beta_\infty &\in \Omega_{\mathbb{C}}^0(U_\infty^\times \times S^1; \mathfrak{g}_{P_{\mathbb{C}}}) \\ \beta_i &\in \Omega_{\mathbb{C}}^0(U_i^\times \times S^1; \mathfrak{g}_{P_{\mathbb{C}}}) \text{ for } i = 1, \dots, k.\end{aligned}$$

Here we write $\mathcal{F}_0^{\text{mon}}$ to indicate the degree 0 term in the cochain complex. There's a similar condition for being a 0-cocycle involving the differential d_{mon} .

To conclude this subsection it will also be important to have an explicit description of the derivative of the holonomy map H as a map between tangent spaces. We can describe this map using our Čech resolutions on each contractible open set U_i individually.

Proposition 3.19. The derivative $dH: \mathbb{H}^\bullet(U_i \times S^1; \mathcal{F}'_{(P, \mathcal{A})}) \rightarrow \mathbb{H}^\bullet(\mathbb{D}_i; \mathcal{F}_{(P, g)})$ is given on an open patch $U_i \times (0, 2\pi)$ by the formula

$$dH(\alpha_i) = dH(d_{\text{mon}} b_i) = b_i(2\pi)H(\mathcal{A}) - H(\mathcal{A})b_i(0)$$

where $i = 1, \dots, k$ or ∞ . More precisely by $b_i(2\pi)$ and $b_i(0)$ we mean the limit of $b_i(t)$ as $t \rightarrow 2\pi$ or 0 respectively.

Proof. Note that the right-hand side is the derivative at \mathcal{A} of the map $B_i \mapsto B_i(2\pi)H(\mathcal{A})B_i(0)^{-1}$ where $B_i \in \Omega^0((U_i \times (0, 2\pi)) \setminus \{(z_i, t_0)\}; \mathfrak{g}_P)$. This is the definition of the action of the group of gauge transformations on the holonomy $H(\mathcal{A})$ from $t = 0$ to 2π . (Chris: say more?) \square

4 Symplectic Structures

4.1 The Example of GL_2

(Chris: Ultimately we might move this, but we should probably include a discussion of what happens for GL_2 before or in parallel to discussing the general story.)

(Vasily: Chris, is the idea to put here explicit expressions for Higgs field in terms of some chart Darboux coordinates (p, q) that we have discussed in the past drafts draft?. If all singularities are miniscule, then for GL_2 the leaves could be described quite explicitly as products of regular semi-simple co-adjoint orbits in \mathfrak{gl}_2 . Rouven is finishing a draft with explicit parametrization by Darboux coordinates of a certain family of symplectic leaves for GL_n . Perhaps here we can put a reference on it.)

(Chris: My idea was to include the discussion of, at least, the simple example that you worked out for GL_2 with minimal singularities where you described the geometry of the moduli space and gave an expression for the symplectic form (in Section 2 of the file 2017_11_30_gHiggs). I really only had in mind for GL_2 trying to give a more explicit argument for what will appear below, i.e. that the moduli spaces after fixing residues were naturally symplectic. But maybe we could do what you suggest and described the symplectic leaves in some Darboux coordinates and refer to [FP18] for proof?)

4.2 Symplectic Structures for General G

We begin the more abstract general analysis by briefly discussing the holomorphic symplectic structure on the moduli space of periodic monopoles on \mathbb{CP}^1 following the analysis of Cherkis and Kapustin [CK98, CK02]. This structure arises from the description we gave as a hyperkähler quotient. To describe it specifically, let $\delta^{(1)}\mathcal{A}$ and $\delta^{(2)}\mathcal{A}$ be two tangent vectors at (P, \mathcal{A}) to the moduli space of monopoles. Recall that \mathcal{A} denotes the combination $\nabla_t - i\Phi dt$. Choose representatives for these two tangent vectors of the form α_i and α'_i respectively in the Čech resolution we described above. Then we can write the holomorphic symplectic form coming from the hyperkähler reduction in terms of the symplectic pairing on the infinite-dimensional vector space \mathcal{V} , which is given by the Killing form on \mathfrak{g} along with the wedge pairing of differential forms. So summing over the local patches in our Čech resolution we can write it as

$$\begin{aligned}\omega_{\text{mon}}(\delta^{(1)}\mathcal{A}, \delta^{(2)}\mathcal{A}) &= \int_M \kappa(\delta^{(1)}\mathcal{A}, \delta^{(2)}\mathcal{A}) \\ &= \sum_{i=1}^k \int_{U_i \times S^1} \kappa(\alpha_i, \alpha'_i) dz dt\end{aligned}$$

where the contributions to the integral away from the U_i vanish.

Our goal in this section will be describe a symplectic structure on the moduli space of multiplicative Higgs bundles – the rational analogue of Hurtubise and Markman’s symplectic structure – and then prove that it’s equivalent to this symplectic form under the equivalence between multiplicative Higgs bundles and periodic monopoles.

Remark 4.1. From now on we’ll write $\langle -, - \rangle$ to denote the residue pairing between elements of $L\mathfrak{g}$. That is,

$$\langle g_1, g_2 \rangle = \oint_{\mathbb{D}^\times} \kappa(g_1, g_2).$$

Lemma 4.2. There is a natural non-degenerate anti-symmetric pairing on the moduli space $\text{mHiggs}_G^{\text{fr}}(\mathbb{CP}^1, D, \omega^\vee)$. In terms of our Čech description it is described by the formula

$$\omega((\{\alpha_i\}, \{\beta_i\}), (\{\alpha'_i\}, \{\beta'_i\})) = \frac{1}{2} \sum_i \langle \rho_g^*(\alpha'_i + \alpha'_0)|_{U_i^\times}, \rho_g^*(\text{Ad}_g^*)^{-1}(\beta_i) \rangle - \langle \rho_g^*(\alpha_i + \alpha_0)|_{U_i^\times}, \rho_g^*(\text{Ad}_g^*)^{-1}(\beta'_i) \rangle,$$

where we use the Killing form to identify β'_i with a \mathfrak{g}^* -valued form, and where ρ_g^* is the pullback along the right multiplication by g . In this expression the sum is over $i = 1, \dots, k$ and $i = \infty$.

Remark 4.3. How should we think about this structure? There’s an intuitive description just as in Hurtubise and Markman’s elliptic moduli space. The pairing is induced from the natural equivalence between the tangent and cotangent spaces of the moduli space of multiplicative Higgs bundles as described in Section 3.1. That is, there’s a map of complexes of sheaves

$$\begin{array}{ccccc}(\mathcal{F}_{(P,g)}^{\text{fr}})^*[2] & \xlongequal{\quad} & \left(\mathfrak{g}_P^*(D)[1] \otimes \mathcal{O}(-c) \xrightarrow{\text{Ad}_g^*} \mathfrak{g}_P^*\mathcal{O}(-c) \right) \\ \downarrow & & \downarrow \kappa \circ \text{Ad}_g^* & & \downarrow \text{Ad}_g \circ \kappa^{-1} \\ \mathcal{F}_{(P,g)}^{\text{fr}} & \xlongequal{\quad} & \left(\mathfrak{g}_P[1]\mathcal{O}(-c) \xrightarrow{\text{Ad}_g} \mathfrak{g}_P(-D)\mathcal{O}(-c) \right)\end{array}$$

where here κ denotes the isomorphism from $\mathfrak{g}_P \rightarrow \mathfrak{g}_P^*$ induced by the Killing form. The top line is the Serre dual complex to the bottom line; note that the incorporation of the framing was necessary for this to be the case (that is, we’re using the relative Calabi-Yau structure on the pair (\mathbb{CP}^1, c)). Taking 0th hypercohomology we obtain a map from the cotangent space to the tangent space of our moduli space of multiplicative Higgs bundles.

Remark 4.4. At this point we will not prove that the pairing is a symplectic structure: we will not verify the Jacobi identity. While we expect that it is possible to prove this directly using the techniques of [HM02, Section 5], instead we’ll see this below by proving that the pairing coincides with the symplectic pairing on the moduli space of periodic monopoles.

Proof. We need only verify that the pairing described is non-degenerate. It suffices to check non-degeneracy for each individual summand, i.e. that for fixed i , given α_i if the expression $\langle \rho_g^*(\alpha_i)|_{U_i^\times}, \rho_g^*(\text{Ad}_g^*)^{-1}(\beta'_i) \rangle$ vanishes for all β'_i then $\alpha_i = 0$ (and the same with α and β interchanged). This is immediate because the residue pairing $\langle -, - \rangle$ is non-degenerate and the map $(\text{Ad}_g^*)^{-1}$ is an isomorphism on the open set U_i^\times . \square

Remark 4.5. (Chris: Remark on the Poisson structure on the total space (ind-Poisson structure really) These should be leaves. Maybe just remark on this since we haven't calculated the tangent complex of the big space. Likewise, what about ε -connections?)

In fact the moduli space is not only symplectic, but the total space of a completely integrable system – the multiplicative Hitchin system described in Definition 2.12. We already know that the generic fibers are half-dimensional tori, it remains to verify that they're isotropic.

Proposition 4.6. The fibers of the multiplicative Hitchin fibration, in the rational case, are isotropic.

Proof. In order to check this we need to describe the subspace of the tangent space to $\text{mHiggs}_G^{\text{fr}}(C, D, \omega^\vee)$ at a point (P, g) tangent to the Hitchin fiber in our Čech description. (Chris: Todo: this needs further thought. I think the tangent complex to the fiber has β_i, β_∞ able to take arbitrary values in \mathfrak{c}_g the centralizer of g (fix g to be rrs and this is a Cartan), and the α_i determined by the cocycle condition. When the forms are valued in \mathfrak{c}_g the operators Ad_g and $(\text{Ad}_g^*)^{-1}$ act trivially, which makes our pairing symmetric. Since it's also anti-symmetric it vanishes, which implies isotropy. So if we verify that those are indeed the required tangent vectors then we're done. Note that if the tangent complex of the fiber is given by the hypercohomology of a subsheaf concentrated in degree -1 then for dimension reasons I think it has to be this.) \square

We'll now begin the comparison between the pairing described above and the symplectic structure on the moduli space of periodic monopoles.

Proposition 4.7. The non-degenerate pairing on the moduli space of multiplicative Higgs bundles can be written in the form

$$\omega(\delta g, \delta g') = \sum_{i=1}^d \langle b_i^L g^{-1}, b_i'^R g^{-1} \rangle - (b \leftrightarrow b') \quad (1)$$

where δg is represented on U_i by a pair $(b^L, b^R) \in \text{Ad}_g$. Here and throughout the notation “ $-(b \leftrightarrow b')$ ” denotes antisymmetrization.

Proof. Specify a representative cocycle $(\alpha_\infty, \{\alpha_i\}, \alpha_0, \beta_\infty, \{\beta_i\})$ for the first Čech cohomology group. By addition of a coboundary we can assume that $\alpha_0 = 0$, and to force α_∞ to land in $z\mathfrak{g}[[z]]$ and each α_i to land in $z^{-1}\mathfrak{g}[[z]]$. We've now fixed a representative cocycle – there's no further gauge freedom. Ignoring the factor at infinity – since after we add these coboundaries the residue pairing there vanishes – the pairing we end up with looks like

$$\begin{aligned} \omega(\delta g, \delta g') &= \frac{1}{2} \sum_{i=1}^d \langle \rho_g^*(\alpha'_i + \alpha'_0)|_{U_i^\times}, \rho_g^*(\text{Ad}_g^*)^{-1}(\beta_i) \rangle - ((\alpha, \beta) \leftrightarrow (\alpha', \beta')) \\ &= \frac{1}{2} \sum_{i=1}^d \langle \rho_g^*(\alpha'_i + \alpha'_0)|_{U_i^\times}, \rho_g^*(\text{Ad}_g^*)^{-1}(\text{Ad}_g^{-1}(\alpha_i - \alpha_0)) \rangle - (\alpha \leftrightarrow \alpha') \\ &= \frac{1}{2} \sum_{i=1}^d \langle \rho_g^*(\alpha'_i)|_{U_i^\times}, \rho_g^*(\text{Ad}_g^*)^{-1}(\text{Ad}_g^{-1}(\alpha_i)|_{U_i^\times}) \rangle - (\alpha \leftrightarrow \alpha'). \end{aligned}$$

We can compute the composite operator $(\text{Ad}_g^*)^{-1}\text{Ad}_g^{-1}$ on $\text{ad}(P)(U_i^\times)$. It's given by the inverse of the operator Ad_gAd_g^* which sends a pair $(b^L, b^R) \in \text{ad}(P)(U_i^\times)$ to $(b^L - b^R, b^R - b^L) \sim (b^L + g^{-1}b^L g, b^R - gb^R g^{-1})$. Denote this by $((1 + A_g)b^L, (1 - A_{g^{-1}})b^R)$. We can describe the inverse using the expansion $(1 + A_g)^{-1} = 1 - A_g + A_g^2 + \dots$. After applying our gauge transformation above the remaining degree of freedom in α_i is its z^{-1} term. Each time

we apply A it raises the order in z of this term by one, so only the linear summand $-A_g$ of $(1 + A_g)^{-1}$ contributes to the residue pairing. In the pairing we need to use the invariant pairing on $\mathfrak{g}((z)) \oplus \mathfrak{g}((z))$ that vanishes on the subalgebra spanned by $(X, A_g(X))$, so we take the difference of the residue / Killing form pairings on the two summands.

$$\begin{aligned}\omega(\delta g, \delta g') &= \frac{1}{2} \sum_{i=1}^d -\langle \rho_g^* A_g b_i'^L, \rho_g^* b_i^R \rangle - \langle \rho_g^* b_i'^L, \rho_g^* A_{g^{-1}} b_i^R \rangle - (b \leftrightarrow b') \\ &= \sum_{i=1}^d \langle b_i^L g^{-1}, b_i'^R g^{-1} \rangle - (b \leftrightarrow b')\end{aligned}$$

using cyclic invariant to identify the two terms. \square

(Chris: We haven't included an explicit comparison with the rational Poisson Lie group. Perhaps include it here as a corollary.)

We can now establish our main result.

Theorem 4.8. The symplectic structure on $\text{Mon}_G^{\text{fr}}(\mathbb{CP}^1 \times S^1, D \times \{0\}, \omega^\vee)$ and the pullback of the non-degenerate pairing on $\text{mHiggs}_G^{\text{ps, fr}}(\mathbb{CP}^1, D, \omega^\vee)$ under H coincide.

Proof. This is straightforward now, by combining the two local descriptions of the symplectic structure on monopoles and multiplicative Higgs bundles on a neighbourhood of a puncture with the description in Proposition 3.19. So, let's begin by taking the symplectic form ω_{mHiggs} on the moduli space $\text{mHiggs}_G^{\text{ps, fr}}(\mathbb{CP}^1, D, \omega^\vee)$ and evaluating it at the image under dH of two elements $(\alpha_i, \alpha'_i) = (d_{\text{mon}}(b_i), d_{\text{mon}}(b'_i))$. Let us write $b_i(0) = b_i^L$ and $b_i(2\pi) = b_i^R$ for brevity. Likewise for consistency with the calculations above let us denote the image $H(\mathcal{A})$ under the holonomy map by g .

According to Proposition 3.19 we have

$$\begin{aligned}\omega_{\text{mHiggs}}(dH(\alpha_i), dH(\alpha'_i)) &= \omega_{\text{mHiggs}}(dH(d_{\text{mon}}(b_i)), dH(d_{\text{mon}}(b'_i))) \\ &= \omega_{\text{mHiggs}}(b_i^R g - g b_i^L, b_i'^R g - g b_i'^L).\end{aligned}$$

We can write this in terms of a pairing on the bundle $\text{ad}(g)$. Represent the class $b_i^R g - g b_i^L$ by the pair $(b_i^R g, -g b_i^L)$. Then applying the description of ω_{mHiggs} provided by Proposition 4.7 we have

$$\begin{aligned}\omega_{\text{mHiggs}}(dH(\alpha_i), dH(\alpha'_i)) &= \omega_{\text{mHiggs}}((b_i^R g, -g b_i^L), (b_i'^R g, -g b_i'^L)) \\ &= -\langle b_i^R, g b_i'^L g^{-1} \rangle - (b \leftrightarrow b')\end{aligned}$$

where the remaining terms are killed by the antisymmetrization.

On the other hand, we can evaluate the pairing $\omega_{\text{Mon}}(\alpha_i, \alpha'_i) = \omega_{\text{Mon}}(d_{\text{mon}}(b_i), d_{\text{mon}}(b'_i))$ via integration by parts. We'll compute the symplectic pairing for the full Čech complex, but it will split into a sum over open sets U_i . The result is that

$$\begin{aligned}\omega_{\text{Mon}}(\{\alpha_i\}, \{\alpha'_i\}) &= \sum_{i=0, \dots, k, \infty} \int_{U_i} \kappa(d_{\text{mon}}(b_i) \wedge d_{\text{mon}}(b'_i)) dz - (b \leftrightarrow b') \\ &= \sum_{i=1, \dots, k, \infty} \int_{\partial \mathbb{D}_i \times [0, 2\pi]} \kappa(b_i - b_0, d_{\text{mon}}(b'_i)) dz - (b \leftrightarrow b')\end{aligned}$$

using here the fact that $d_{\text{mon}} b_i = d_{\text{mon}} b_0$ on $U_i \cap U_0$ and that $d_{\text{mon}} b_i(t) = 0$ when $t = 0$ or 2π . By Stokes' theorem we then have

$$\begin{aligned}\omega_{\text{Mon}}(\{\alpha_i\}, \{\alpha'_i\}) &= \sum_{i=1, \dots, k, \infty} \oint_{\partial \mathbb{D}_i} \kappa(b_i^R - b_0^R, b_i'^R) - \kappa(b_i^L - b_0^L, b_i'^L) - (b \leftrightarrow b') \\ &= \sum_{i=1, \dots, k, \infty} \oint_{\partial \mathbb{D}_i} -\kappa(b_0^R, b_i'^R) + \kappa(b_0^L, b_i'^L) - (b \leftrightarrow b').\end{aligned}$$

Pick out the summand corresponding to U_i . Choose our potentials so that on the boundary $\partial\mathbb{D}_i$ we have $b_0^L = b_0'^L = 0$. We can do this by setting $b^R = \delta g g^{-1}$ since g is non-singular on U_0 . This choice means that on $U_0 \cap U_i$, since $\delta g = b_i^R g - g b_i^L = b_0^R g$ we can make the identification $b_0^R = b_i^R - g b_i^L g^{-1}$. Therefore

$$\omega_{\text{Mon}}(\{\alpha_i\}, \{\alpha'_i\}) = \sum_{i=1, \dots, k, \infty} \langle (g b_i^L g^{-1}, b_i'^R) \rangle - (b \leftrightarrow b')$$

agreeing with the expression coming from mHiggs. \square

In particular the non-degenerate pairing on the multiplicative Higgs moduli space is determined by a closed 2-form, and the multiplicative Hitchin fibers are Lagrangian.

Corollary 4.9. The moduli space $\text{mHiggs}_G^{\text{fr}}(\mathbb{CP}^1, D, \omega^\vee)$ has the structure of a completely integrable system.

Remark 4.10. (Chris: connection to Shapiro's symplectic leaves in [Sha16]. The claim is that taking the Taylor expansion at infinity defines a Poisson map to $\tilde{G} \subseteq G_1[[z^{-1}]]$ sending symplectic leaves to (discrete) unions of symplectic leaves. One can do the classification more completely in the type A case.)

5 Hyperkähler Structures

The results of the previous section imply that the symplectic structure on $\text{mHiggs}_G^{\text{ps, fr}}(\mathbb{CP}^1, D, \omega^\vee)$ extends canonically to a hyperkähler structure. In this section we'll compare the twistor rotation with the deformation to the moduli space of ε -connections. We'll show that in an appropriate limit passing to a point ε in the twistor sphere we obtain a complex manifold equivalent to the moduli space of ε -connections as a deformation of the moduli space of multiplicative Higgs bundles.

Let us begin by describing the hyperkähler structure on the moduli space of periodic monopoles and how the holomorphic symplectic structure varies under rotation in the twistor sphere. We'll begin by recalling that the moduli space of periodic monopoles can be described as a hyperkähler quotient as in the work of Atiyah and Hitchin [AH88]. In a context with more general boundary data at ∞ this was demonstrated by Cherkis and Kapustin [CK02] for the group $\text{SU}(2)$, see also Foscolo [Fos16][Theorem 7.12]. In the case of \mathbb{CP}^1 with a fixed framing the analysis is much easier.

Recall from Definition 3.7 that the moduli space of periodic monopoles on \mathbb{CP}^1 with a fixed framing at ∞ and fixed Dirac singularities can be described as the quotient

$$\text{Mon}_G^{\text{fr}}(\mathbb{CP}^1 \times_\varepsilon S_R^1, D, \omega^\vee) = \mu^{-1}(0)/\mathcal{G},$$

where

- The space \mathcal{V} of fields is the space of pairs (A, Φ) consisting of framed connections A on the trivial principal $G_{\mathbb{R}}$ -bundle \mathbf{P} on $\mathbb{CP}^1 \times_\varepsilon S_R^1$, Φ is a section of $\mathfrak{g}_{\mathbf{P}}$ vanishing at $\{\infty\} \times S_R^1$, and the pair (A, Φ) has a Dirac singularity with charge $\omega_{z_i}^\vee$ at each point (z_i, t_0) in $D \times \{t_0\}$.
- $\mu: \mathcal{V} \rightarrow \Omega^1(\mathbb{CP}^1 \times_\varepsilon S_R^1; (\mathfrak{g}_{\mathbb{R}})_{\mathbf{P}})/\mathfrak{g}_{\mathbb{R}}$ is the map sending a monopole (A, Φ) to the $\mathfrak{g}_{\mathbb{R}}$ -valued 1-form $*F_A - d_A \Phi$ vanishing at $\{\infty\} \times S_R^1$.
- The group \mathcal{G} of gauge transformations consists of automorphisms of the trivial bundle \mathbf{P} fixing the framing.

This is an instance of hyperkähler reduction. Indeed the infinite-dimensional space \mathcal{V} admits a hyperkähler structure with symplectic forms defined by

$$\omega_i((A, \Phi), (A', \Phi')) = \int_{\mathbb{CP}^1 \times_\varepsilon S_R^1} \left(\kappa(A_i, \Phi') - \kappa(\Phi, A'_i) + \sum_{j,k=1,2,3} \varepsilon_{ijk} \kappa(A_j, A'_k) \right) dx dy dt,$$

where $i = 1, 2, 3$, and where A can be written as $A_1 dt + A_2 dx + A_3 dy$ with respect to a choice $dz = d(x + iy)$ of meromorphic volume form on \mathbb{CP}^1 with a second order pole at ∞ . This descends to a hyperkähler structure on the quotient since (Chris: ... probably I want to find a good reference here. Could refer to [Fos13, Section 1.4.2].)

Let's restrict to the case where $\varepsilon = 0$ for now. Consider the holomorphic symplectic structure $\omega_2 + i\omega_3$. We can alternatively write the formula above for this choice of holomorphic symplectic in the form

$$\omega(\mathcal{A}, \mathcal{A}') = \int_{\mathbb{CP}^1 \times S_R^1} \kappa(\mathcal{A}, \mathcal{A}') dz dt,$$

as described at the beginning of Section 4.2. (Chris: This needs justification. What happens to the xy term? At face value you get something more like the antisymmetrization of $\int \kappa(\mathcal{A}, A'_2 + iA'_3) dz$.)

This description for the hyperkähler structure also tells us clearly what happens when we perform a rotation in the twistor sphere. The following statement follows by identifying the holomorphic symplectic structure at ε in the twistor sphere by applying the corresponding rotation in $\text{SO}(3)$ to the coordinates x, y and t on \mathbb{R}^3 in the expression for the holomorphic symplectic structure at 0. Note that when we calculate the symplectic form as an integral as above, it's enough to take the integral over just $\mathbb{R}^2 \times S_R^1$ instead of $\mathbb{CP}^1 \times S_R^1$.

Proposition 5.1. Let ζ be a point in the twistor sphere. The holomorphic symplectic form on the moduli space $\text{Mon}_G^{\text{fr}}(\mathbb{CP}^1 \times S_R^1, D, \omega^\vee)$ of periodic monopoles is given by the formula

$$\omega_\varepsilon(\delta^{(1)}\mathcal{A}, \delta^{(2)}\mathcal{A}) = \int_{\mathbb{R}^2 \times S_R^1} \kappa(\delta^{(1)}\mathcal{A}, \delta^{(2)}\mathcal{A}) \zeta(dz \wedge dt),$$

where we identify ζ with an element of $\text{SO}(3)$ and apply this rotation to the volume form $dz \wedge dt$. Equivalently we can identify $\mathbb{R}^2 \times \mathbb{CP}^1$ with the quotient $\mathbb{R}^3 \times L$ where L is the rank one lattice $\{0\}^2 \times 2\pi R\mathbb{Z}$ and identify the rotated holomorphic symplectic form with

$$\omega_\zeta(\delta^{(1)}\mathcal{A}, \delta^{(2)}\mathcal{A}) = \int_{\mathbb{R}^3/(\zeta(L))} \kappa(\delta^{(1)}\mathcal{A}, \delta^{(2)}\mathcal{A}) dz \wedge dt.$$

Now, we'll perform our identification of the twistor deformation on the multiplicative Higgs side by rewriting this twistor rotation in a somewhat different way in the large R limit. In this limit we can identify the twistor rotation with the deformation obtained by replacing the product $\mathbb{CP}^1 \times S_R^1$ with the product twisted by an automorphism of \mathbb{R}^2 . More precisely, the rotated symplectic structure can then be described in the following way.

Theorem 5.2. Let $\left(\text{mHiggs}_G^{\text{fr}}(\mathbb{CP}^1, D, \omega^\vee)\right)_{\zeta, R}$ be the family of complex manifolds obtained by pulling back the holomorphic symplectic structure $\omega_{\zeta, R}$ on $\text{Mon}_G^{\text{ps, fr}}(\mathbb{CP}^1 \times S_R^1, D, \omega^\vee)$ at ζ in the twistor sphere. In the limit where $R \rightarrow \infty$ with $2\pi\zeta R = \varepsilon$ fixed this complex structure coincides with the complex structure on $\varepsilon\text{-Conn}_G^{\text{fr}}(\mathbb{CP}^1, D, \omega^\vee)$. (Chris: be more precise about what the limit means here and in the proof)

Proof. According to Theorem 3.8 it's enough to show that the complex structure on $\text{Mon}_G^{\text{fr}}(\mathbb{CP}^1 \times S_R^1, D, \omega^\vee)$ at the point ζ converges to the complex structure on the ε -deformation $\text{Mon}_G^{\text{ps, fr}}(\mathbb{CP}^1 \times_\varepsilon S_R^1, D, \omega^\vee)$. The key observation that we'll use is that, fixing $2\pi\zeta R = \varepsilon$ we can identify the rotated lattice with a *sheared* lattice. Namely, when we rotate the rank one lattice $L_{R,0} = (0, 0, 2\pi R)\mathbb{Z}$ by ζ we obtain the lattice

$$\zeta(L_{R,0}) = \frac{1}{1 + |\zeta|^2} (\text{Re}(\varepsilon), \text{Im}(\varepsilon), 2\pi R(1 - |\zeta|^2))\mathbb{Z}.$$

We can view the quotient of \mathbb{R}^3 by the rotated lattice as a sheared lattice with different radius and shear. Specifically, we identify

$$\mathbb{R}^3/\zeta(L_{R,0}) \cong \mathbb{R}^2 \times_{\frac{\varepsilon}{1+|\zeta|^2}} S_{R \frac{1-|\zeta|^2}{1+|\zeta|^2}}^1.$$

In order to identify the desired complex structures it's enough to identify the full family of symplectic structures on the ε -deformation with the appropriately rotated family of symplectic structures without deforming. So, if we

apply the twistor rotation by ζ to the holomorphic symplectic structure $\omega_2 + i\omega_3$ on $\text{Mon}_G^{\text{fr}}(\mathbb{CP}^1 \times S_R^1, D, \omega^\vee)$ (the argument will be identical for the other points in the \mathbb{CP}^1 family of holomorphic symplectic structures) we obtain the pairing

$$\omega_\zeta(\delta^{(1)}\mathcal{A}, \delta^{(2)}\mathcal{A}) = \int_{\mathbb{R}^3/(\zeta(L_{R,0}))} \kappa(\delta^{(1)}\mathcal{A}, \delta^{(2)}\mathcal{A}) dz \wedge dt.$$

On the other hand we can identify the holomorphic symplectic structure on $\text{Mon}_G^{\text{ps,fr}}(\mathbb{CP}^1 \times_\varepsilon S_R^1, D, \omega^\vee)$ as

$$\omega_\zeta(\delta^{(1)}\mathcal{A}, \delta^{(2)}\mathcal{A}) = \int_{\mathbb{R}^3/(L_{R,\varepsilon})} \kappa(\delta^{(1)}\mathcal{A}, \delta^{(2)}\mathcal{A}) dz \wedge dt.$$

In particular, by tuning the radius and the shear parameter the above calculation means there's a hyperkähler equivalence between the moduli spaces $\text{Mon}_G^{\text{fr}}(\mathbb{CP}^1 \times S_R^1, D, \omega^\vee)$ and $\text{Mon}_G^{\text{ps,fr}}(\mathbb{CP}^1 \times_{\frac{\varepsilon}{1+|\zeta|^2}} S_{R\frac{1-|\zeta|^2}{1+|\zeta|^2}}^1, D, \omega^\vee)$ after rotating the twistor sphere by ζ . (Chris: need to say something about the moduli spaces as smooth manifolds before we compare the hyperkähler structures. There's a diffeomorphism coming from the diffeomorphism between sheared and unsheared versions of the 3-manifold.)

Now, let's consider the large R limit. (Chris: ... absolute convergence of complex structures / symplectic structures?) \square

Remark 5.3. (Chris: comment on the example of the Nahm transform.)

6 Quantization and Yangians

(Chris: In this section, begin by recalling the claim that the map to the Poisson Lie group is a Poisson map. Mention the map the other direction, which should also be Poisson. Consider the corresponding maps on the algebras of functions: need to be clear about what this means for infinite-type or ind-schemes. Then talk about quantization and recall the definition of the (completed) Yangian. State as a theorem that quantized functions on our symplectic leaves are modules for the Yangian.)

7 Twisted Gauge Theory

(Chris: calculations of the twist of 5d $\mathcal{N} = 2$ super Yang-Mills with monopole operators yielding out moduli spaces)

8 Langlands Duality

(Chris: For a first draft I'm inserting some notes I prepared for a talk. Todo: talk about motivation and the content we discussed regarding non-simply-laced groups)

Pseudo-Conjecture 8.1 (Multiplicative Geometric Langlands). Let G be a Langlands self-dual group. There is an equivalence of categories

$$\text{A-Branes}_{q^{-1}}(\text{mHiggs}_G(C, D, \omega^\vee)) \cong \text{B-Branes}(q\text{-Conn}_G(C, D, \omega^\vee))$$

where the category on the right-hand side depends on the value q .

What does this mean, and are there situations in which we can make it precise? We'll discuss three examples where we can say something more concrete. In each case, by "B-branes" we'll just mean the category $\text{Coh}(q\text{-Conn}_G(C, D, \omega^\vee))$ of coherent sheaves. By "A-branes" we'll mean some version of q^\vee -difference connections on the stack $\text{Bun}_G(C)$.

Remark 8.2. This equivalence is supposed to interchange objects corresponding to branes of opers on the two sides, and introduce an analogue of the Feigin-Frenkel isomorphism between deformed W-algebras (see [FRSTS98, STSS98]). This isomorphism only holds for self-dual groups, which motivates the restriction to the self-dual case here.

8.1 The Abelian Case

Suppose $G = \mathrm{GL}(1)$ (more generally we could consider a higher rank abelian gauge group). In general for an abelian group the moduli spaces we've defined are trivial – for instance the rational and trigonometric spaces are always discrete. However there is one interesting non-trivial example: the elliptic case. For simplicity let's consider the abelian situation with $D = \emptyset$: the case with no punctures.

Definition 8.3. A q -difference module on a variety X with automorphism q is a module for the sheaf $\Delta_{q,X}$ of non-commutative rings generated by \mathcal{O}_X and an invertible generator Φ with the relation $\Phi \cdot f = q^*(f) \cdot \Phi$. Write $\mathrm{Diff}_q(X)$ for the category of q -difference modules on X .

In the abelian case the space $q\text{-Conn}_{\mathrm{GL}(1)}(E)$ is actually a stack, but one can split off the stacky part to define difference modules on it. Indeed, for any q one can write

$$\mathrm{Bun}_{\mathrm{GL}(1)}(E) \cong \mathrm{BGL}(1) \times \mathbb{Z} \times E^\vee$$

and so

$$q\text{-Conn}_{\mathrm{GL}(1)}(E) \cong \mathrm{BGL}(1) \times \mathbb{Z} \times (E^\vee \times_q \mathbb{C}^\times)$$

which means one can define difference modules on these stacks associated to an automorphism of E^\vee or $E^\vee \times_q \mathbb{C}^\times$ respectively.

Conjecture 8.4. There is an equivalence of categories for any $q \in \mathbb{CP}^1$

$$\mathrm{Diff}_q(\mathrm{Bun}_{\mathrm{GL}(1)}(E)) \cong \mathrm{Coh}(q^{-1}\text{-Conn}_{\mathrm{GL}(1)}(E)).$$

In this abelian case we can go even farther and make a more sensitive 2-parameter version of the conjecture.

Conjecture 8.5. There is an equivalence of categories for any $q_1, q_2 \in \mathbb{CP}^1$

$$\mathrm{Diff}_{q_1}(q_2\text{-Conn}_{\mathrm{GL}(1)}(E)) \cong \mathrm{Diff}_{q_2^{-1}}(q_1^{-1}\text{-Conn}_{\mathrm{GL}(1)}(E))$$

where q_1 is the automorphism of $E^\vee \times_{q_2} \mathbb{C}^\times$ acting fiberwise over each point of \mathbb{C}^\times .

This conjecture should be provable using the same techniques as the ordinary geometric Langlands correspondence in the abelian case, i.e. by a (quantum) twisted Fourier-Mukai transform (as constructed by Polishchuk and Rothstein [PR01]).

8.2 The Classical Case

Now, let's consider the limit $q \rightarrow 0$. This will give a conjectural statement involving coherent sheaves on both sides analogous to the classical limit of the geometric Langlands conjecture as conjectured by Donagi and Pantev [DP10]. The existence of an equivalence isn't so interesting in the self-dual case (where both sides are the same), but the classical multiplicative Langlands functor should be an *interesting* non-trivial equivalence. For example we can make the following conjecture

Conjecture 8.6. Let G be a Langlands self-dual group and let E be an elliptic curve. There is an automorphism of categories (for the rational, trigonometric and elliptic moduli spaces)

$$F: \text{Coh}(\text{mHiggs}_G(E)) \cong \text{Coh}(\text{mHiggs}_G(E))$$

so that the following square commutes:

$$\begin{array}{ccc} \text{Coh}(\text{mHiggs}_T(E)) & \xrightarrow{\text{FM}} & \text{Coh}(\text{mHiggs}_T(E)) \\ p_* q^! \downarrow & & \downarrow p_* q^! \\ \text{Coh}(\text{mHiggs}_G(E)) & \xrightarrow{F} & \text{Coh}(\text{mHiggs}_G(E)). \end{array}$$

Here we're using the natural morphisms $p: \text{mHiggs}_B(E) \rightarrow \text{mHiggs}_G(E)$ and $q: \text{mHiggs}_B(E) \rightarrow \text{mHiggs}_T(E)$, and FM is the Fourier-Mukai transform.

It ought to be possible to state something a bit stronger that includes singularities in this auto-duality.

8.3 The Rational Type A Case

There's one more example where we can say something precise, and even draw a connection to the ordinary geometric Langlands correspondence. We already mentioned the Nahm transform in the previous section: in the case where $G = \text{GL}(n)$ and C is \mathbb{C} (where as usual we fix framing data at infinity) the Nahm transform identifies multiplicative Higgs bundles of degree k with *ordinary* Higgs bundles on \mathbb{CP}^1 for the group $\text{GL}(k)$ with $n + 2$ tame singularities (with appropriate fixed locations and residues).

Claim. Under the Nahm transform, Pseudo-Conjecture 8.1 in the rational case for the group $\text{GL}(n)$ becomes the ordinary geometric Langlands conjecture on \mathbb{CP}^1 with tame ramification.

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