The Multiplicative Hitchin System in Supersymmetric Gauge Theory

Chris Elliott

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Abstract

Multiplicative Higgs bundles are an analogue of ordinary Higgs bundles where the Higgs field takes values in a Lie group instead of its Lie algebra. In this talk I'll discuss two contexts where multiplicative Higgs bundles appear in supersymmetric gauge theory. I'll explain how hyperkähler structures on these moduli spaces arise physically and mathematically and relate to the theory of Poisson Lie groups, and finally I'll introduce a speculative multiplicative analogue of the geometric Langlands conjecture. This is based on joint work in progress with Vasily Pestun.

1 Introduction

My goal in this talk is to introduce you to an algebro-geometric object – the *multiplicative Hitchin system* – and explain how it shows up in quantum field theory in a few different contexts. I'll also talk about a speculative version of the geometric Langlands conjecture for this object. This is joint work in progress with Vasily Pestun.

I'll begin by defining the moduli space of multiplicative Higgs bundles on a curve, then explain our main motivation for considering it: it is isomorphic to the Seiberg-Witten integrable system of a certain $\mathcal{N} = 2$ quiver gauge theory, and gives a natural alternative description of the integrable system structure on that space.

2 Multiplicative Higgs Bundles

Let's explain the main object of study: the moduli space of multiplicative Higgs bundles. The moduli space we'll consider has been discussed in various guises before: we refer to Frenkel-Ngô [FN11], Bouthier [Bou15a, Bou14, Bou15b], and Hurtubise-Markman [HM02] – it is sometimes referred to as the moduli space of "G-pairs". Throughout this talk I'll work over the complex numbers. I'll write C for a smooth complex curve and G for a complex reductive group.

Here's the idea. Recall that a *Higgs bundle* on C is a principal G-bundle P along with a section ϕ of the coadjoint bundle $\mathrm{ad}(P)^*$ twisted by the canonical bundle. The set of Higgs bundles can be promoted to the closed points of a stack: the moduli stack of Higgs bundles.

Remark 2.1. I'm going to ignore that tricky twist by the canonical bundle. For this talk I'll only be interested in the Calabi-Yau case, where C is either an elliptic curve or an object modelling $\mathbb{C}$ or $\mathbb{C}^\times$ with appropriate boundary conditions. While one can make sense of ordinary Higgs bundles on any curve this won't be true anymore for the multiplicative version I'm going to define in a moment, at least not along with all the structure that the moduli space of Higgs bundles usually includes.

Let's give a concise definition of the moduli space of Higgs bundles (without the canonical bundle twist).

Definition 2.2. The moduli space $\mathrm{Higgs}_G^{\mathcal{O}}(C)$ of $\mathcal{O}$-Higgs bundles on C is the moduli space $\underline{\mathrm{Map}}(C, \mathfrak{g}^*/G)$ of maps into the coadjoint quotient stack $\mathfrak{g}^*/G$.

The *multiplicative* version of this moduli space replaces the coadjoint bundle $\mathrm{ad}(P)^*$ by the *group* adjoint bundle $\mathrm{Ad}(P)$. So a closed point in the moduli space corresponds to a principal G-bundle P on C along with a section of $\mathrm{Ad}(P)$: i.e. an automorphism of P. Let's give the analogous concise definition as a mapping space.

Definition 2.3. The moduli space $\mathrm{GpHiggs}_G(C)$ of multiplicative Higgs bundles on C is the moduli space $\underline{\mathrm{Map}}(C, G/G)$ of maps into the group adjoint quotient stack G/G.

2.1 Including Poles

So far this moduli space won't be very interesting, especially in the important example where $C = \mathbb{A}^1$. We get something more interesting by introducing simple poles for our multiplicative Higgs fields. Let $D \subseteq C$ be a finite set of points in C.

Definition 2.4. The moduli space $\mathrm{GpHiggs}_G(C, D)$ of multiplicative Higgs bundles on C with poles at the subset D is the moduli space modelling a G-bundle P on C equipped with a section of $\mathrm{Ad}(P)|_{C \setminus D}$. Globally we define the moduli space as the fiber product

$$\mathrm{GpHiggs}_G(C, D) := \mathrm{GpHiggs}_G(C \setminus D) \times_{\mathrm{Bun}_G(C \setminus D)} \mathrm{Bun}_G(C).$$

We'd like to prescribe the behaviour of the multiplicative Higgs field near the punctures. In the neighbourhood of a point $z \in D$ the multiplicative Higgs field is described by an element of $\mathrm{Maps}(\mathbb{D}^\times, G) = G((z))$, where $\mathbb{D}^\times$ is the formal punctured disk. This element is only well-defined up to gauge transformations which extend across the puncture.

We'll actually only specify the local behaviour up to the action of $\mathrm{Maps}(\mathbb{D}, G)^2 = G[\![z]\!]^2$ on the left and right. The set of $G[\![z]\!]$ double cosets in $G((z))$ is in canonical bijection with the set of dominant coweights of the group G, so at each puncture $z \in D$ we fix a dominant coweight $\omega_z^\vee$. I'll write $\omega^\vee$ for short for the set $\{\omega_z^\vee : z \in D\}$.

To specify the moduli space, we can think of the space of double cosets as a stack

$$G[\![z]\!]\backslash G((z))/G[\![z]\!] = G[\![z]\!]\backslash \mathrm{Gr}_G$$

where Gr_G is the affine Grassmannian. For each $z \in D$ fix a left $G[\![z]\!]$-orbit in Gr_G corresponding to a coweight $\omega_z^\vee$, and let $\Lambda_z \subseteq G[\![z]\!]$ be the corresponding stabilizer.

Definition 2.5. The moduli space $\mathrm{GpHiggs}_G(C, D, \omega^\vee)$ of multiplicative Higgs bundles on C with poles at the subset $D = \{z_1, \ldots, z_k\}$ and fixed residue $\omega_z^\vee$ at each $z \in D$ is defined to be the fiber product

$$\mathrm{GpHiggs}(C, D, \omega^\vee) = \mathrm{GpHiggs}(C, D) \times_{(G[\![z]\!]\backslash \mathrm{Gr}_G)^k} (B\Lambda_{z_1} \times \cdots \times B\Lambda_{z_k}).$$

Remark 2.6. These moduli spaces are empty unless the coweights at each puncture are chosen appropriately. Specifically one needs to assume that the sum $\sum_{z \in D} \mathrm{ord}\langle \rho, \omega_z^\vee \rangle$ is equal to zero, where ρ is the Weyl vector, and ord denotes the order of the pole or zero of a representative element of $\mathbb{C}((z))$.

Examples 2.7.

1. Rational: Let $C = \mathbb{CP}^1$ and consider the moduli space of multiplicative Higgs bundles with a fixed framing at infinity. In other words, consider the fiber product $\mathrm{GpHiggs}(\mathbb{CP}^1, D, \omega^\vee) \times_{G/G} g_\infty$, where we view G/G as $\underline{\mathrm{Map}}(\{\infty\}, G/G)$. This is a finite-dimensional smooth variety whose points are G-valued rational functions with fixed simple poles and zeroes in $\mathbb{C}$ and asymptotic to $g_\infty z^d$ near $z = \infty$.

2. Trigonometric: We still let $C = \mathbb{CP}^1$, but now instead of fixing a framing at ∞ we fix the following data. Choose a pair B_+, B_- of opposite Borel subgroups of G with unipotent radicals N_+ and N_-. We consider G-bundles on $\mathbb{CP}^1$ with a meromorphic Higgs field $g(z)$ with fixed poles, and with $g(0) \in B_+$ and $g(\infty) \in N_-$. With fixed residues this again defines a finite-dimensional smooth variety.

3. Elliptic: Now let $C = E$, an elliptic curve. In this case we really can just use the above definition and consider the moduli space $\mathrm{GpHiggs}_G(E, D, \omega^\vee)$. This is now a stack with a smooth map down to the stack $\mathrm{Bun}_G(E)$ of principal G-bundles on the elliptic curve. In particular in this case one can consider the non-trivial moduli space where $D = \varnothing$: $\mathrm{GpHiggs}_G(E) = \underline{\mathrm{Map}}(E, G/G)$.

Before we discuss the integrable system structure of this moduli space we'll motivate its appearance via gauge theory.

3 Physical Background

In order to motivate this construction let me tell you a story from physics which leads us to an interesting question. I'll try and tell the story in a way that doesn't assume you already know much QFT.

The main character of this story is an $\mathcal{N} = 2$ supersymmetric gauge theory in dimension 4. To specify such a gauge theory one need to fix some data:

1. A compact semisimple group G_{gauge}: the *gauge group* (I'm saving the notation G for something else shortly).
2. A representation V of G_{gauge}: the *matter representation*.
3. A complex number τ_i for each simple factor of G_{gauge}: the *coupling constants*.
4. A complex number m_j for each irreducible summand of V: the *masses*.

Most of these theories are pretty badly behaved when you try to quantize them, but there's an especially nice family of "quiver gauge theories". One chooses the group G_{gauge} to be a product of $\mathrm{SU}(n_i)$, where we think of the factors as associated to the vertices of an ADE quiver. One then chooses the representation V to have a summand V_{ij} for each edge of the quiver isomorphic to the bifundamental representation of $\mathrm{SU}(n_i) \times \mathrm{SU}(n_j)$, plus a summand for each vertex that looks like k_i copies of the fundamental representation of $\mathrm{SU}(n_i)$. The masses associated to the bifundamental representations are fixed, but the masses of the fundamental representations are free: we label them as $m_{i,f}$.

These theories are "superconformal" when $k_i = \sum_j C_{ij} n_j$, where C_{ij} is the Cartan matrix of the ADE quiver. This is why we used an ADE quiver specifically: you can build a theory like the above out of any quiver but these are almost the only superconformal examples (you can also use an affine ADE quiver with $k_i = 0$).

3.1 Moduli of Vacua

One can build an algebro-geometric object out of an $\mathcal{N} = 2$ supersymmetric field theory.

Construction 3.1. The *moduli space of vacua* is the spectrum of the "chiral ring" of the $\mathcal{N} = 2$ theory. This affine scheme $\mathcal{B}$ comes equipped with the structure of a special Kähler metric. That is, a Kähler structure along with a flat torsion-free symplectic connection compatible with the complex structure (i.e. with $d_\nabla J = 0$). We can additionally cook up a full rank ∇-flat lattice Λ in $T\mathcal{B}$ whose dual in $T^*\mathcal{B}$ is Lagrangian.

This data is equivalent to the data of an *algebraic integrable system* with base $\mathcal{B}$ (see Freed [Fre99, Theorem 3.4]). That is, a holomorphic symplectic manifold $X \to \mathcal{B}$ whose fibers are generically Lagrangian abelian varieties.

In the example of an ADE quiver gauge theory, Nekrasov and Pestun [NP12] calculated the total space X of this Seiberg-Witten integrable system.

Theorem 3.2 (Nekrasov-Pestun)**.** The Seiberg-Witten integrable system for the $\mathcal{N} = 2$ quiver gauge theory associated to a complex simple group G of ADE type is isomorphic to the moduli space of *periodic monopoles*, i.e. periodic monopoles on $\mathbb{R}^2 \times S^1$ for the group G with Dirac singularities at the points $(m_{i,f}, 1)$ with charge given by the fundamental coweight $\lambda_i^\vee$ and with a fixed framing at ∞.

A *monopole* on a Riemannian 3-manifold M for the group G is a G-bundle P on M with connection A along with a section Φ of $\mathrm{ad}(P)$ that satisfy the Bogomolny equation

$$*F_A - \mathrm{d}_A \Phi = 0.$$

I won't explain exactly what a Dirac singularity is, except to note that they are well-behaved local singularities indexed by coweights of G. By a framing we mean a fixed limit for the holonomy of $A + i\Phi$ around S^1 at ∞.

Now we can explain what this has to do with the moduli space of multiplicative Higgs bundles that I introduced earlier!

Theorem 3.3 (Charbonneau-Hurtubise [CH10] (for $\mathrm{GL}(n)$), Smith [Smi15] (for general G))**.** Let $D \subseteq C \times S^1$ be a finite subset, write π for the projection $C \times S^1 \to C$ and assume that D contains at most one point in each fiber of π. Fix a dominant coweight $\omega_z^\vee$ at each point $z \in D$. There is an analytic equivalence

$$\mathrm{Mon}_G(C \times S^1, D, \omega^\vee) \xrightarrow{\sim} \mathrm{GpHiggs}_G(C, \pi(D), \omega^\vee)$$

between the moduli space of monopoles on $C \times S^1$ and the moduli space of (poly-stable) multiplicative Higgs bundles on C with compatible singularities, given by taking the holonomy of $A + i\Phi$ around the circle S^1.

Remark 3.4. The theorem of Charbonneau-Hurtubise and Smith includes a poly-stability condition for the multiplicative Higgs bundles that appear. This condition is vacuous in our key (rational and trigonometric) examples, since every vector bundle on $\mathbb{CP}^1$ splits as a sum of line bundles.

This equivalence commutes with natural maps down to the base $\mathcal{B}$ on the two sides. Here one can identify

$$\mathcal{B} = \Gamma\left(C; \bigoplus_{i=1}^{r} \mathcal{O}_C\left(\omega_D^\vee(\lambda_i) \cdot D\right)\right)$$

where $\omega_D^\vee(\lambda_i) \cdot D$ denotes the divisor $\{\omega_z^\vee(\lambda_i) \cdot z : z \in D\}$. This space of global sections is the same as the space of maps $C \setminus D \to T/W$ with simple poles with prescribed residues at the divisor D. The space of multiplicative Higgs bundles maps down to $\mathcal{B}$ by composing with the characteristic polynomial (Chevalley) map $\chi \colon G/G \to T/W$. On the other hand the projection down to $\mathcal{B}$ in the integrable system as calculated by Nekrasov and Pestun is indeed given by the map $\chi(\oint_{S^1} A + i\Phi)$.

Remark 3.5. I'd like to mention another origin of the moduli space of multiplicative Higgs bundles from supersymmetric gauge theory. One can compute the *holomorphic twist* of $\mathcal{N} = 2$ supersymmetric 5d gauge theory. It makes sense to consider this twisted theory on manifolds of the form $\mathbb{D} \times C \times S^1$ where C is a Calabi-Yau curve and $\mathbb{D}$ is a formal disk. One can also define the twisted theory in the presence of Gukov-Witten surface defects at a finite set D of points in $C \times S^1$. If one compute the space of solutions to the equations of motion in this twisted theory one recovers the space $\mathrm{GpHiggs}_G(C, \pi(D), \omega^\vee)$ of multiplicative Higgs bundles.

4 Integrable System Structures

So, the upshot of this discussion is that according to the theorem of Charbonneau, Hurtubise and Smith the Seiberg-Witten integrable system associated to our ADE gauge theory has an *algebraic* description – i.e. the map $X \to \mathcal{B}$ can be given the structure of an algebraic map between algebraic varieties. In fact the holomorphic symplectic structure on the phase space also admits a description from this multiplicative Higgs point of view, and even better it can be naturally connected to the theory of Poisson Lie groups. The result combines some deep results from the literature with some new analysis and interpretation: let me summarise the story.

Theorem 4.1 (In progress)**.** In the rational, trigonometric and elliptic cases we discussed in example 2.7 the multiplicative Hitchin system with fixed residues $\mathrm{GpHiggs}_G(C, D, \omega^\vee)$ has the structure of an algebraic completely integrable system with base $\mathcal{B}$. If we don't fix residues, the moduli space $\mathrm{GpHiggs}_G(C, D)$ has a Poisson structure with bracket coming from the rational, trigonometric or (for type A only) elliptic R-matrix, and the space $\mathrm{GpHiggs}_G(C, D, \omega^\vee)$ is a symplectic leaf.

In the elliptic case, this is a theorem of Hurtubise and Markman [HM02]. In the rational case we can also prove the claim using their techniques. In the trigonometric case this is work in progress.

Remark 4.2. In light of the Charbonneau-Hurtubise-Smith result identifying the multiplicative Higgs moduli space with a space of periodic monopoles there's actually already an integrable system (and indeed hyperkähler) structure on our moduli space. The space of periodic monopoles can be described explicitly as a hyperkähler quotient. We'll discuss what it means for these structures to coincide in a moment.

In particular, this claim implies that the moduli spaces $\mathrm{GpHiggs}(C, D, \omega^\vee)$ are hyperkähler (although not necessarily canonically: fixing a canonical structure requires fixing a polarization, i.e. a positive integral $1,1$-form on each generic fiber of the integrable system). We can actually go further and explain what happens when we vary the complex structure in the twistor sphere. The deformed moduli spaces have a natural description in terms of *q-difference connections.*

Definition 4.3. Let q be an automorphism of a curve C – as usual we'll think of the three examples where $C = \mathbb{C}, \mathbb{C}^\times$ or E, in which case we can think of C as its own group of automorphisms. A *q-difference connection* on C is a G-bundle P along with an isomorphism $A\colon P \to q^*P$ of G-bundles. One can consider q-difference connections with poles at a finite subset $D \subseteq C$: just as for multiplicative Higgs fields we can fix the behaviour of a q-difference connection near a pole by fixing a closed point in $G[\![z]\!] \setminus \mathrm{Gr}_G$, or equivalently a dominant coweight.

The *moduli space* of q-difference connections on C with singularities at D and residues $\{\omega^\vee\}$ is defined to be

$$q\text{-}\mathrm{Conn}_G(C, D, \omega^\vee) := \underline{\mathrm{Map}}(C \times_q S^1_B \setminus D, BG) \times_{(G[\![z]\!] \setminus \mathrm{Gr}_G)^k} (B\Lambda_1 \times \cdots \times B\Lambda_k)$$

where by $C \times_q S^1_B$ we mean the mapping torus of the automorphism q, viewed as a derived stack, and by $C \times_q S^1_B \setminus D$ we mean the complement of the subset D of the fiber over $1 \in S^1$. When $q \to 0$ this recovers the moduli space of multiplicative Higgs fields.

Remark 4.4. It actually makes sense to construct holomorphic symplectic structures on $q\text{-}\mathrm{Conn}_G(C, D, \omega^\vee)$ for every q uniformly using ideas from derived Poisson geometry (as in [CPT$^+$17, MS16]). We don't have time to talk about this construction today, but the idea is to exhibit a 1-shifted Lagrangian structure on the 1-shifted symplectic stack $G[\![z]\!]\backslash G((z))/G[\![z]\!]$ representing G-bundles on the formal bubble $\mathbb{B} = \mathbb{D} \sqcup_{\mathbb{D}^\times} \mathbb{D}$.

We have the following expectation.

Conjecture 4.5. The Hurtubise-Smith equivalence is a map of holomorphic symplectic spaces.

If we knew this held, there would be a number of interesting consequences.

1. If we choose a radius r, the moduli space of periodic G-monopoles (so monopoles on $\mathbb{R}^2 \times S^1_r$) is not just holomorphic symplectic but hyperkähler. The conjecture then gives a canonically associated hyperkähler structure to the multiplicative Hitchin system (depending on r).

2. In the limit $r \to \infty$ one can explicitly describe the twistor family of holomorphic symplectic spaces on the monopole side. Consider the space $\mathrm{Mon}_G(C \times S^1_r, D, \omega^\vee)_{J_\zeta}$, i.e. the space considered in the complex structure at ζ in the twistor sphere. Take the limit $r \to \infty$ and $\zeta \to 0$, keeping the product $r\zeta = q$ fixed. In this limit we can identify $\mathrm{Mon}_G(C \times S^1_r, D, \omega^\vee)_{J_\zeta}$ with monopoles on the twisted product $\mathrm{Mon}_G(C \times_q S^1_1, D, \omega^\vee)_{J_0}$ in the untwisted complex structure. In fact the argument of Charbonneau and Hurtubise works equally well for this twisted product, providing an equivalence

$$\mathrm{Mon}_G(C \times_q S^1_1, D, \omega^\vee) \to q\text{-}\mathrm{Conn}_G(C, D, \omega^\vee).$$

Therefore the conjecture implies the following.

Corollary 4.6. If $\mathrm{GpHiggs}_G(C, D, \omega^\vee)$ is equipped with the hyperkähler metric in the $r \to \infty$ limit, in the complex structure at q in a neighbourhood of 0 in the twistor sphere it becomes algebraically isomorphic to $q\text{-}\mathrm{Conn}_G(C, D, \omega^\vee)$.

3. We can establish this corollary in the rational type A case by means of the Nahm transform, which reduces the claim to an analogous claim about the Hitchin system where we know that the motion in the twistor family deforms the space of Higgs bundles to the space of flat λ-connections.

5 Multiplicative Geometric Langlands

Either the construction of the multiplicative Hitchin system as a twist of 5d $\mathcal{N} = 2$ super Yang-Mills, or the SYZ perspective on mirror symmetry leads us to attend to formulate the following imprecise conjecture.

Pseudo-Conjecture 5.1 (Multiplicative Geometric Langlands)**.** There is an equivalence of categories

$$\text{A-Branes}_{q^{-1}}(\mathrm{GpHiggs}_G(C, D, \omega^\vee)) \cong \text{B-Branes}(q\text{-}\mathrm{Conn}_G(C, D, \omega^\vee))$$

where the category on the right-hand side depends on the value q.

What does this mean, and are there situations in which we can make it precise? We'll discuss three examples where we can say something more concrete. In each case, by "B-branes" we'll just mean the category $\mathrm{Coh}(q\text{-}\mathrm{Conn}_G(C, D, \omega^\vee))$ of coherent sheaves. By "A-branes" we'll mean some version of $q^\vee$*-difference connections* on the stack $\mathrm{Bun}_G(C)$.

5.1 The Abelian Case

Suppose $G = \mathrm{GL}(1)$ (more generally we could consider a higher rank abelian gauge group). In general for an abelian group the moduli spaces we've defined are trivial – for instance the rational and trigonometric spaces are always discrete. However there is one interesting non-trivial example: the elliptic case. For simplicity let's consider the abelian situation with $D = \varnothing$: the case with no punctures.

Definition 5.2. A *q-difference module* on a variety X with automorphism q is a module for the sheaf $\Delta_{q,X}$ of non-commutative rings generated by $\mathcal{O}_X$ and an invertible generator Φ with the relation $\Phi \cdot f = q^*(f) \cdot \Phi$. Write $\mathrm{Diff}_q(X)$ for the category of q-difference modules on X.

In the abelian case the space $q\text{-}\mathrm{Conn}_{\mathrm{GL}(1)}(E)$ is actually a stack, but one can split off the stacky part to define difference modules on it. Indeed, for any q one can write

$$\mathrm{Bun}_{\mathrm{GL}(1)}(E) =\cong B\mathrm{GL}(1) \times \mathbb{Z} \times E^\vee$$

and so

$$q\text{-}\mathrm{Conn}_{\mathrm{GL}(1)}(E) \cong B\mathrm{GL}(1) \times \mathbb{Z} \times (E^\vee \times_q \mathbb{C}^\times)$$

which means one can define difference modules on these stacks associated to an automorphism of $E^\vee$ or $E^\vee \times_q \mathbb{C}^\times$ respectively.

Conjecture 5.3. There is an equivalence of categories for any $q \in \mathbb{CP}^1$

$$\mathrm{Diff}_q(\mathrm{Bun}_{\mathrm{GL}(1)}(E)) \cong \mathrm{Coh}(q^{-1}\text{-}\mathrm{Conn}_{\mathrm{GL}(1)}(E)).$$

In this abelian case we can go even farther and make a more sensitive 2-parameter version of the conjecture.

Conjecture 5.4. There is an equivalence of categories for any $q_1, q_2 \in \mathbb{CP}^1$

$$\mathrm{Diff}_{q_1}(q_2\text{-}\mathrm{Conn}_{\mathrm{GL}(1)}(E) \cong \mathrm{Diff}_{q_2^{-1}}(q_1^{-1}\text{-}\mathrm{Conn}_{GL(1)}(E)$$

where q_1 is the automorphism of $E^\vee \times_{q_2} \mathbb{C}^\times$ acting fiberwise over each point of $\mathbb{C}^\times$.

This conjecture should be provable using the same techniques as the ordinary geometric Langlands correspondence in the abelian case, i.e. by a (quantum) twisted Fourier-Mukai transform (as constructed by Polishchuk and Rothstein [PR01]).

5.2 The Classical Case

Now, let's consider the limit $q \to 0$. This will give a conjectural statement involving coherent sheaves on both sides analogous to the classical limit of the geometric Langlands conjecture as conjectured by Donagi and Pantev [DP10].

Conjecture 5.5. Let G be a reductive group of ADE type with Langlands dual $G^\vee$. There is an equivalence of categories (for the rational, trigonometric and elliptic moduli spaces)

$$\mathrm{Coh}(\mathrm{GpHiggs}_G(E, D, \omega^\vee)) \cong \mathrm{Coh}(\mathrm{GpHiggs}_{G^\vee}(E, D, \omega))$$

where ω_z is the weight of G, or coweight of $G^\vee$, dual to $\omega_z^\vee$.

In the abelian case this conjectural equivalence should be given by the Fourier-Mukai transform.

5.3 The Rational Type A Case

There's one more example where we can say something precise, and even draw a connection to the ordinary geometric Langlands correspondence. We already mentioned the Nahm transform in the previous section: in the case where $G = \mathrm{GL}(n)$ and C is $\mathbb{C}$ (where as usual we fix framing data at infinity) the Nahm transform identifies multiplicative Higgs bundles of degree k with *ordinary* Higgs bundles on $\mathbb{CP}^1$ for the group $\mathrm{GL}(k)$ with $n+2$ tame singularities (with appropriate fixed locations and residues).

Claim. Under the Nahm transform, Pseudo-Conjecture 5.1 in the rational case for the group $\mathrm{GL}(n)$ becomes the ordinary geometric Langlands conjecture on $\mathbb{CP}^1$ with tame ramification.

References

[Bou14] Alexis Bouthier. La fibration de Hitchin-Frenkel-Ngô et son complexe d'intersection. *arXiv preprint arXiv:1409.1275*, 2014.

[Bou15a] Alexis Bouthier. Dimension des fibres de Springer affines pour les groupes. *Transformation Groups*, 20(3):615–663, 2015.

[Bou15b] Alexis Bouthier. Géométrisation du lemme fondamental pour l'algébre de Hecke. *arXiv preprint arXiv:1502.07148*, 2015.

[CH10] Benoît Charbonneau and Jacques Hurtubise. Singular Hermitian-Einstein monopoles on the product of a circle and a Riemann surface. *International Mathematics Research Notices*, 2011(1):175–216, 2010.

[CPT+17] Damien Calaque, Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi. Shifted Poisson structures and deformation quantization. *Journal of Topology*, 10(2):483–584, 2017.

[DP10] Ron Donagi and Tony Pantev. Lectures on the geometric Langlands conjecture and non-abelian Hodge theory. *Second International School on Geometry and Physics Geometric Langlands and Gauge Theory*, page 129, 2010.

[FN11] Edward Frenkel and Bao Châu Ngô. Geometrization of trace formulas. *Bulletin of Mathematical Sciences*, 1(1):129–199, 2011.

[Fre99] Daniel S Freed. Special Kähler manifolds. *Communications in Mathematical Physics*, 203(1):31–52, 1999.

[HM02] Jacques Hurtubise and Eyal Markman. Elliptic Sklyanin integrable systems for arbitrary reductive groups. *Advances in Theoretical and Mathematical Physics*, 6(5):873–978, 2002.

[MS16] Valerio Melani and Pavel Safronov. Derived coisotropic structures. *arXiv preprint arXiv:1608.01482*, 2016.

[NP12] Nikita Nekrasov and Vasily Pestun. Seiberg-Witten geometry of four dimensional $\mathcal{N} = 2$ quiver gauge theories. *arXiv preprint arXiv:1211.2240*, 2012.

[PR01] Alexander Polishchuk and Mitchell Rothstein. Fourier transform for D-algebras, I. *Duke Mathematical Journal*, 109(1):123–146, 2001.

[Smi15] Benjamin H Smith. Singular G-monopoles on $S^1 \times \Sigma$. *arXiv preprint arXiv:1501.03337*, 2015.

Institut des Hautes Études Scientifiques
35 Route de Chartres, Bures-sur-Yvette, 91440, France
celliott@ihes.fr