1. $|S_4| = 24$, $\langle(1234)\rangle = 4$, so we're looking for 6 left cosets. We compute that they are

$$\{e, (1234), (13)(24), (1432)\}$$
$$\{(12), (234), (1324), (143)\}$$
$$\{(23), (134), (1423), (142)\}$$
$$\{(34), (124), (1423), (132)\}$$
$$\{(13), (24), (12)(34), (14)(23)\}$$
$$\{(243), (123), (1342), (14)\}.$$

2.
   - True
   - False, e.g. $(12)(34) \in S_4$.
   - True
   - True
   - True
   - False, e.g. $\mathbb{Z}/3\mathbb{Z} \leq D_3$, with quotient $\mathbb{Z}/2\mathbb{Z}$.
   - True, by Lagrange's theorem (g generates a cyclic subgroup of order n.)
   - False, e.g. $x = (14)$, $y = (123)$ $xy = (1234)$ in S_4.
   - False, e.g. consider the trivial homomorphism sending every element to e.
   - False, there is only one group of order 15 up to isomorphism.

3. The map ϕ is a homomorphism if and only if $x^2y^2 = (xy)(xy)$ for all $x, y \in G$. Multiply on the left by x^{-1} and on the right by y^{-1} to see that if this holds then $xy = yx$ for all $x, y \in G$, i.e. G is abelian.

4. Argue by orders of elements. For instance, S_5 has an element $(12)(345)$ of order 6. Elements of $S_4 \times \mathbb{Z}/5\mathbb{Z}$ have the form $(\sigma, \overline{a})$ for $\sigma \in S_4$, $a \in \{0, \ldots, 4\}$. Such elements have order $|\sigma|$ if $a = 0$, or $5|\sigma|$ if $a \neq 0$. But no element of S_4 has order 6, so no element of $S_4 \times \mathbb{Z}/5\mathbb{Z}$ has order 6 either. Thus the groups cannot be isomorphic.

5. First suppose G is cyclic. Then G is generated by a single element g, so g is not contained in any proper subgroup. Conversely, suppose G is not cyclic. Choose $h \in G$. Then the subgroup $H = \langle h \rangle$ must be proper, since G is not cyclic, any hence every element is contained in a proper subgroup.

6. Let $H \leq G$ have index 2. Then G is partitioned into two left cosets, H and $G \backslash H = H'$. Similarly, G is partitioned into two *right* cosets, H and $G \backslash H = H'$. So left cosets and right cosets coincide, and hence H is normal.

7. Choose $x, y \in G$, and look at $x^{-1}y^{-1}xy$, a generator of N. Choose $s \in G$, and conjugate this generator

$$\begin{aligned} s^{-1}(x^{-1}y^{-1}xy)s &= s^{-1}x^{-1}ss^{-1}y^{-1}ss^{-1}xss^{-1}ys \\ &= (s^{-1}xs)^{-1}(s^{-1}ys)^{-1}(s^{-1}xs)(s^{-1}ys) \end{aligned}$$

   which is also in N. Since it suffices to check N is closed under conjugation by checking it on generators, N is normal. Finally,

$$[gN][hN] = [ghN] = [gh(h^{-1}g^{-1}hg)N] = [hgN] = [hN][gN]$$

   so G/N is abelian. (By the way, N here is called the *commutator subgroup* of G. It is the smallest normal subgroup such that the quotient G/N is abelian.)