TWISTS OF SUPERCONFORMAL ALGEBRAS (PRELIMINARY VERSION)

CHRIS ELLIOTT, OWEN GWILLIAM, AND MATTEO LOTITO

ABSTRACT. We give a systematic approach to the study of twists of superconformal classical and quantum field theories by an odd nilpotent element of the superconformal algebra, applicable in all dimensions and signatures. In particular we explain how, by twisting, we can extract vertex algebras and $\mathbb{E}_n$ algebras from the observables of such a field theory. We discuss the possible twists in three and four dimensions in some detail.

CONTENTS

1. INTRODUCTION

In this paper we analyze the notion of supersymmetric twisting in the context of a superconformal classical or quantum field theory. The idea of twisting goes back to Witten [Wit88b]: given a field theory with an action of a super Lie algebra $\mathfrak{g}$ and an odd element $\mathcal{Q} \in \mathfrak{g}$ satisfying $[\mathcal{Q}, \mathcal{Q}] = 0$, we may consider the deformation of the theory by the action of $\mathcal{Q}$. For example, if $H_{\mathcal{Q}}$ is the Hamiltonian associated to $\mathcal{Q}$ we may replace the algebra $\mathrm{Op}_{\mathrm{loc}}$ of local operators by the cohomology

$$\mathrm{Op}_{\mathrm{loc}}^{\mathcal{Q}} = \mathrm{H}^\bullet(\mathrm{Op}_{\mathrm{loc}}, \{H_{\mathcal{Q}}, -\}),$$

where the condition $[\mathcal{Q}, \mathcal{Q}] = 0$ ensures that the differential does indeed square to zero.

Witten originated this idea as a recipe for the construction of topological quantum field theories. If the stress-energy tensor $T_{\mu\nu}$ is exact for the action of $\mathcal{Q}$, then it vanishes in cohomology, and the associated twisted theory is topological. In [ES19; ESW22] the first author together with Safronov and Williams provided an analysis of the possible topological twists associated to the action of super Poincaré algebras, and described the twists of super Yang–Mills theories in the BV formalism. In the present work our goal is to extend this construction to the context of *superconformal* field theories.

Working in the superconformal case is not simply a matter of reproducing the calculations from this earlier work in another example, there are substantially different features that we may take advantage of to construct interesting and novel algebras of operators upon twisting. One way of realizing these is to begin with a field theory defined on the conformal compactification $C(\mathbb{R}^{p,q})$ of a pseudo-Riemannian affine space. We review this idea in Section 2.1, in brief the conformal compactification is a pseudo-Riemannian manifold with a dense submanifold conformally equivalent to $\mathbb{R}^{p,q}$, on which the group $\mathrm{Conf}(p,q)$ of all conformal transformations acts transitively. The choice of embedding $\mathbb{R}^{p,q} \hookrightarrow C(\mathbb{R}^{p,q})$ provides a subgroup of $\mathrm{Conf}(p,q)$ isomorphic to the group of isometries of $\mathbb{R}^{p,q}$, but this embedding is not canonical, there are many possible choices. In the next section we will outline a technique by which we may construct interesting algebras of operators from this set-up.

The ideas in this paper, in particular the origin of vertex algebras from superconformal twists, originate in work of Beem, Lemos, Liendo, Peelaers, Rastelli and van Rees [Bee+15b], extended in a series of subsequent papers by these authors [Bee+15a; BRR15; BPR17; BR18] that address a collection of further examples and explore their consequences. In this paper we do not focus on specific theories, but rather analyze formal aspects of superconformal actions on algebras of observables and the structures that may arise upon twisting.

1.1. **What Twisting Does to Superconformal Theories.** Let Obs denote the algebra of (classical or quantum) observables of a superconformal field theory. The most natural place for a superconformal field theory to live is the conformal compactification $C(\mathbb{R}^{p,q})$, so our algebra Obs can be thought of as living over the conformal compactification. That is, for any open subset $U \subseteq C(\mathbb{R}^{p,q})$ we can study the collection $\mathrm{Obs}(U)$ of observables supported on the subset U.

Remark 1.1. Notice that, while we could choose an embedding $\mathbb{R}^{p,q} \hookrightarrow C(\mathbb{R}^{p,q})$ if we wanted to, and study the restriction of the observables to this affine subspace, such a choice is unnatural: it breaks the conformal symmetry.

If $\mathcal{Q}$ is an odd element of the complexified superconformal Lie algebra such that $[\mathcal{Q},\mathcal{Q}] = 0$, we may study the algebra of observables $\mathrm{Obs}^{\mathcal{Q}}$ in the theory *twisted* by $\mathcal{Q}$. This twisted algebra still lives over $C(\mathbb{R}^{p,q})$, and our goal in this paper is to develop a systematic understanding of the properties of this algebra purely in terms of the algebra of the superconformal group. To this end, let us describe a few natural computations (completely in terms of $\mathcal{Q}$, independent of the choice of theory) that we can perform, and what they mean for the twisted theory.

- **Closed elements of the Lie algebra**: Let
$$\mathfrak{z}_{\mathcal{Q}}^{\mathbb{C}} = \{A \in \mathfrak{sconf}(n|\mathcal{N},\mathbb{C})_0 \colon [A,\mathcal{Q}] = 0\}.$$
The Lie subalgebra $\mathfrak{z}_{\mathcal{Q}}^{\mathbb{C}}$ consists of even symmetries that commute with the supercharge $\mathcal{Q}$. These symmetries continue to act on the twisted theory $\mathrm{Obs}^{\mathcal{Q}}$ by infinitesimal conformal transformations. The action of non-exact symmetries is broken when we pass to the twist.
- **Closed elements of the Lie group**: Let
$$Z_{\mathcal{Q}}^{\mathbb{C}} \subseteq \mathrm{SConf}(n|\mathcal{N},\mathbb{C})_0$$

denote the Lie group exponentiating $\mathfrak{z}_{\mathcal{Q}}^{\mathbb{C}}$, and let $Z_{\mathcal{Q}}$ denote its intersection with the real form $\mathrm{SConf}(p,q|\mathcal{N})_0$. This Lie subgroup consists of the even symmetries that continue to act on the twisted theory $\mathrm{Obs}^{\mathcal{Q}}$ at the group level, by genuine conformal transformations.

Example 1.2. If $Z_{\mathcal{Q}}$ contains the group of isometries of an affine subspace $\mathbb{R}^{a,b} \subseteq C(\mathbb{R}^{p,q})$ then we may study the observables in $\mathrm{Obs}^{\mathcal{Q}}$ supported along this subspace: this algebra of restricted observables is now isometry invariant.

- **Exact elements of the Lie algebra**: Let

$$\mathfrak{b}_{\mathcal{Q}}^{\mathbb{C}} = \{A \in \mathfrak{sconf}(n|\mathcal{N},\mathbb{C})_0 \colon A = [\mathcal{Q},\mathcal{Q}'] \text{ for some } \mathcal{Q}'\}.$$

  These even symmetries now not only *act* on the twisted theory, but act in a manner that is manifestly trivialized. That is to say, the conserved currents for symmetries $\mathfrak{b}_{\mathcal{Q}}^{\mathbb{C}}$ are exact elements of $\mathrm{Obs}^{\mathcal{Q}}$, so vanish in cohomology.

- **Exact elements of the Lie group**: Let

$$B_{\mathcal{Q}}^{\mathbb{C}} \subseteq \mathrm{SConf}(n|\mathcal{N},\mathbb{C})_0$$

  denote the Lie group exponentiating $\mathfrak{b}_{\mathcal{Q}}^{\mathbb{C}}$, and let $B_{\mathcal{Q}}$ denote its intersection with the real form $\mathrm{SConf}(p,q|\mathcal{N})_0$. This Lie subgroup consists of the even symmetries that actually act on spacetime by genuine conformal transformations, that are additionally trivialized on the twisted theory.

We can draw conclusions from this analysis most readily for specific types of symmetry occurring in the groups $Z_{\mathcal{Q}}$ and $B_{\mathcal{Q}}$.

Example 1.3 ($\mathbb{E}_k$-algebras). Suppose there is an affine subspace $\mathbb{R}^k \subseteq C(\mathbb{R}^{p,q})$ with definite signature so that the group of isometries of the subspace lies in $B_{\mathcal{Q}}$. Then if we restrict observables of $\mathrm{Obs}^{\mathcal{Q}}$ to those supported along the subspace, we have an isometry action that is trivialized up to homotopy. If the dilation of $\mathbb{R}^k$ also lies in $B_{\mathcal{Q}}$ then these restricted observables have the structure of a *framed* $\mathbb{E}_k$*-algebra*: an algebra over the operad of framed little k-disks. This structure encodes a lot of the properties of a k-dimensional topological field theory that can be coherently defined on all oriented k-manifolds.

Remark 1.4. When doing this calculation carefully, we will not only need the group of isometries of $\mathbb{R}^k$ to be exact, but we will need to be able to choose the potentials coherently: we will need a potential for the full algebra of isometries as in Definition 7.8.

Example 1.5 (Vertex Algebras). Suppose that there is an affine subspace $\mathbb{R}^2 \subseteq C(\mathbb{R}^{p,q})$ now, so that the full group of isometries of $\mathbb{R}^2$ lies in $Z_{\mathcal{Q}}$, but only the complexified translation $\overline{d}_z$ lie in $\mathfrak{b}_{\mathcal{Q}}^{\mathbb{C}}$, where z is a complex coordinate on $\mathbb{R}^2 \cong \mathbb{C}$. Then if we restrict observables of $\mathrm{Obs}^{\mathcal{Q}}$ to those supported along the subspace, we have an isometry action so that the complexified antiholomorphic translation is trivialized up to homotopy. In this case (given some additional technical assumptions), the restricted observables can be equipped with the structure of a *vertex algebra*. This is the original construction considered by Beem et al [Bee+15b] in their derivation of vertex algebras from $\mathcal{N} = 2$ 4d superconformal field theory.

Our analysis also allows for a situation that lies in between these two extremes, where we extract a field theory on $\mathbb{R}^k \times \mathbb{C}^\ell$ whose observables are topological along the k real directions and holomorphic along the ℓ complex directions.

Remark 1.6. We can act by the Lie group $\mathrm{SConf}(p,q|\mathcal{N})_0$ simultaneously on the space of nilpotent supercharges $\mathcal{Q}$ by the adjoint action and on the spacetime $C(\mathbb{R}^{p,q})$ by conformal transformations. The resulting twisted theories are equivalent. We observe, however, that if we study the algebra of operators after restricting to an affine subspace, the conformal transformations will move this subspace around. So for instance, the intersection with a choice of open subspace $\mathbb{R}^{p,q} \subseteq C(\mathbb{R}^{p,q})$ may look quite different for different points in the same orbit.

In the main body of this paper, we will – therefore – be investigating the following questions.

(1) For a given superconformal group $\mathrm{SConf}(p,q|\mathcal{N})$, what is the locus $\mathcal{N}\mathrm{ilp}$ of square-zero odd elements of the complexified Lie algebra? What are the orbits for the adjoint action of the even part of the superconformal group?
(2) For each orbit, what do the Lie groups and Lie algebras $\mathfrak{z}^{\mathbb{C}}_{\mathcal{Q}}, Z_{\mathcal{Q}}, \mathfrak{b}^{\mathbb{C}}_{\mathcal{Q}}, B_{\mathcal{Q}}$ look like?
(3) Can we locate affine subspaces whose isometry groups are $\mathcal{Q}$-closed? If so we can obtain nice isometry invariant algebras of observables on $\mathbb{R}^k$ from the $\mathcal{Q}$-twisted theory.
(4) What does the subalgebra of exact isometries of such a subspace look like – both real and complexified – and can we find potentials for these subalgebras? If so the algebra of twisted observables will have nice invariance properties that we can analyze.

Notice that all of these questions are totally independent from the specific choice of superconformal theory, and can be applied to draw conclusions in a completely uniform way.

2. Superconformal Algebras and Twists

We rapidly review the conformal and superconformal groups and their associated (super) Lie algebras. For the conformal situation, we follow Schottenloher's [Sch08] conventions (see Chapter 2).

2.1. **The conformal case.**

Throughout we will work with a pair (p,q) of natural numbers such that $p+q>2$. The case $p+q=2$ is strikingly different and so deserves a separate examination.

Let $\mathbb{R}^{p,q}$ denote the vector space $\mathbb{R}^{p+q}$ equipped with the nondegenerate symmetric bilinear pairing of signature (p,q). We say it has dimension $D=p+q$. We denote the pairing by

$$\langle v,w\rangle = v_1w_1+\cdots+v_pw_p - v_{p+1}w_{p+1} - \cdots - v_{p+q}w_{p+q},$$

and use $|v|^2=\langle v,v\rangle$ for the associated "norm." A crucial role for us is played by a closely related space, as it leads to a clean definition of the conformal group and — later — will be the spacetime relevant to twisted superconformal field theories. Let $\mathbb{RP}^{p+q+1}$ denote the $p+q+1$-dimensional real projective space.

Definition 2.1. *Let $C(\mathbb{R}^{p,q})$ denote the* conformal compactification *of $\mathbb{R}^{p,q}$. It is the projectivization of the real variety given by the vectors $v \in \mathbb{R}^{p+1,q+1}$ such that $|v|^2=0$. In other words, it is the hypersurface in the projective space $\mathbb{RP}^{p+q+1}$ cut out by the quadratic equation*

$$0 = x_0^2+\cdots+x_p^2 - x_{p+1}^2 - \cdots - x_{p+q+1}^2$$

where $(x_0 : \cdots : x_{p+q+1})$ *is a point given in homogeneous coordinates.*

In concrete terms, there is an embedding $\iota\colon \mathbb{R}^{p,q} \hookrightarrow \mathbb{R}^{p+1,q+1}$ by

$$\iota(v) = \left(\frac{1-|v|}{2}, v, \frac{1+|v|}{2}\right).$$

In even more explicit terms, $\iota(v) = \left(\frac{1-|v|}{2}, v_1, \ldots, v_{p+q}, \frac{1+|v|}{2}\right)$. Note that $\iota(v)$ is always a vector of norm zero, and that it is never the zero vector. We can thus map $\mathbb{R}^{p,q}$ into the projective space $\mathbb{RP}^{p+q+1}$ by sending v to $\left(\frac{1-|v|}{2} : v_1 : \ldots : v_{p+q} : \frac{1+|v|}{2}\right)$, using homogeneous coordinates. Then $C(\mathbb{R}^{p,q})$ is the natural completion of this copy of $\mathbb{R}^{p,q}$ in projective space.

It is straightforward to describe the symmetries of this compactification: we take the linear automorphisms of $\mathbb{R}^{p+1,q+1}$ that preserve the subspace of norm-zero vectors and then projectivize.

Definition 2.2. *The* conformal group $\mathrm{Conf}(p,q)$ *is the connected component of the identity in* $\mathrm{O}(p+1, q+1)/\{\pm 1\}$.

There are familiar classes of transformations within this group:

- the orthogonal transformations $\mathrm{SO}(p,q)$,
- the translations $\mathbb{R}^{p+q}$,
- the dilations $\mathbb{R}_{>0}$, and
- the famed "special conformal transformations," the subgroup of which is isomorphic to a second copy of the additive group $\mathbb{R}^{p+q}$.

In fact, these subgroups collectively generate the full conformal group; in other words, we can always factor a conformal transformation as some composition of transformations of those four types.

Remark 2.3. The identification of these classes of generators of conformal transformations is associated to the chosen conformal embedding $\iota\colon \mathbb{R}^{p,q} \to C(\mathbb{R}^{p,q})$. For instance the orthogonal transformations, translations and dilations generate the subgroup of conformal transformations that preserve the image of ι. We obtain distinct but conjugate subgroups by choosing alternative conformal embeddings – from our construction of the conformal compactification we obtain such an embedding from every codimension two linear subspace of the form $\mathbb{R}^{p,q} \subseteq \mathbb{R}^{p+1,q+1}$, hence an embedding $\mathrm{SO}(p,q) \hookrightarrow \mathrm{SO}(p+1, q+1)$.

The Lie algebra $\mathfrak{conf}(p,q)$ of conformal symmetries is thus isomorphic to $\mathfrak{so}(p+1, q+1)$.

Lemma 2.4. *As a vector space, there is a direct sum decomposition*

$$\mathfrak{conf}(p,q) \cong \mathfrak{so}(p,q) \oplus \mathbb{R}^{p+q} \oplus \mathbb{R} \oplus \mathbb{R}^{p+q},$$

where the summands correspond, in order, to the Lie algebras of the four subgroups listed above.

This Lie algebra has a representation in vector fields on $\mathbb{R}^{p,q}$, by differentiating the action of the group on the compactification, and the infinitesimal translations, orthogonal transformations, and dilations act in the familiar way, which we write out below in explicit detail. This infinitesimal action plays a central role in characterizing the superconformal Lie algebra below.

We now recall a standard notation in physics for generators of $\mathfrak{conf}(p,q)$ and write the commutation relations. It explicitly realizes $\mathfrak{conf}(p,q)$ as a Lie subalgebra of the vector fields on $\mathbb{R}^{p+q}$.

Notation 2.5. Let μ, ν denote indices running from 1 to $p+q$, associated to the standard basis for $\mathbb{R}^{p+q}$. The generators of $\mathfrak{conf}(p,q)$ are

$$\begin{array}{rcc} M_{\mu\nu} & = & x_\mu \partial_\nu - x_\nu \partial_\mu \\ P_\mu & = & -\partial_\mu \\ D & = & -x_\mu \partial^\mu \\ K_\mu & = & x^2 \partial_\mu - 2 x_\mu x_\nu \partial^\nu . \end{array}$$

Remark 2.6. As in Remark 2.3 the choice of decomposition of the conformal Lie algebra into these four subspaces is non-canonical: it depends on a choice of linear subspace $\mathbb{R}^{p,q} \subseteq \mathbb{R}^{p+1,q+1}$.

2.2. **The superconformal case.** We will focus on superconformal Lie *algebras*, as they play the key role in this paper and because they are simpler to describe in technical terms (In particular, we will not discuss super analogs of the conformal compactification.) Our approach is borrowed from Shnider's elegant and efficient discussion [Shn88]. We begin by recalling how one formulates the super versions of the spacetimes $\mathbb{R}^{p,q}$ and of their isometries. That is, we review what super Minkowski space is and what the super Poincaré group is (in arbitrary signature) to elucidate the logic that motivates a formulation of superconformal algebras.

Definition 2.7. *The* Poincaré group *in signature* (p,q) *is the group*

$$\mathrm{ISO}(p,q) = \mathbb{R}^{p+q} \rtimes \mathrm{SO}(p,q)$$

or oriented isometries of $\mathbb{R}^{p,q}$. *The* Lorentz group *is the subgroup* $\mathrm{SO}(p,q) \subseteq \mathrm{ISO}(p,q)$ *that fixes the origin in* $\mathbb{R}^{p,q}$.

We can (and should) view $\mathbb{R}^{p,q}$ as the homogeneous space given by quotient of the Poincaré group by the Lorentz group. There is a natural spin cover of the Poincaré group where we replace $\mathrm{SO}(p,q)$ by its spin cover. We will typically focus on the common Lie algebra $\mathfrak{so}(p,q)$ of these groups, which does not care about the spin cover. The Lie algebra $\mathfrak{so}(p,q)$ has an infinitesimal action on $\mathbb{R}^{p,q}$.

Super versions of these objects arise as extensions that depend on spinors as follows. Pick a real (Majorana) spinorial representation S of $\mathrm{Spin}(p,q)$ and a spin-equivariant symmetric bilinear map $\Gamma \colon S \times S \to \mathbb{R}^{p+q}$. (Here S need not be irreducible; it can be a direct sum of spinor representations. The pairing is unique up to rescaling if S is irreducible.)

Definition 2.8. *The* super Poincaré group *is the super Lie group whose odd component is S and whose reduced (bosonic) Lie group is the Poincaré spin group; the group structure depends on the pairing Γ on spinors. We denote the super Poincaré group associated to this data by* $\mathrm{ISO}(p,q|S)$.

There is a super manifold that is the quotient of this super group by the Lorentz spin group; we denote it by $\mathbb{R}^{p,q|s}$, where s will denote the dimension of S. The super Poincaré Lie algebra $\mathfrak{iso}(p,q|S)$ acts infinitesimally on this superspace. We will often wish to refer to the symmetries of S that commute with the spin action. We make the following definition.

Definition 2.9. *The* R-symmetry group G_R *of the super Poincaré group* $\mathrm{ISO}(p,q|S)$ *is the group of outer automorphisms of* $\mathrm{ISO}(p,q|S)$ *that act trivially on the even part.*

In looking for a super version of the conformal algebra, we want to construct a super Lie algebra $\mathfrak{sconf}(p,q|S)$ that depends on this choice of a Majorana spinor representation S and the bilinear form. It should include the super Poincaré Lie algebra as a subalgebra, just as the Poincaré Lie algebra is a subalgebra of $\mathfrak{conf}(p,q)$. Moreover, the reduced Lie algebra of $\mathfrak{sconf}(p,q|S)$ should be $\mathfrak{conf}(p,q)$. In other words, $\mathfrak{sconf}(p,q|S)$ should be a common generalization of usual supersymmetry algebra and usual conformal algebra.

It is useful to ask for something stronger: a common generalization *in how they act as symmetries* of the supersymmetry algebra and the conformal algebra. We thus request super Lie algebras satisfying two properties:

Definition 2.10 (c.f [Shn88])**.** *A super Lie algebra* $\mathfrak{g}$ *is a* superconformal algebra *of signature* p,q *if* $\mathfrak{g}$ *acts on super Minkowski space* $\mathbb{R}^{p,q|s}$ *by infinitesimal derivations, and the following conditions hold:*

(1) There is an embedding $i\colon \mathfrak{iso}(p,q|S) \hookrightarrow \mathfrak{g}$ *of super Lie algebras, and the restriction of the action along i coincides with the standard action of infinitesimal super isometries.*
(2) There is an embedding $j\colon \mathfrak{conf}(p,q) \hookrightarrow \mathfrak{g}$*, and the restriction of the action along j coincides with the standard action of infinitesimal conformal transformations on* $\mathbb{R}^{p,q}$*, acting trivially on the odd coordinates.*

The remarkable theorem is that such a super Lie algebra only exists when $p+q<7$, as discovered by Nahm [Nah78], although we emphasize here the clean mathematical approach of Shnider [Shn88], who articulates these properties as explicit hypotheses. Shnider begins by studying the complexification of the situation above, as any such real super Lie algebras has a complexifiction and there are no subtleties of signature in this setting. He then shows that if such a complex super Lie algebra exists, there is a simple super Lie algebra that is a quotient or subalgebra and that satisfies the same properties. Moreover, its bosonic part splits as the sum of $\mathfrak{so}(p+q,\mathbb{C})$ and a complementary ideal. He then uses Kac's classification of simple super Lie algebras to show that such complex super Lie algebras do *not* exist above dimension 6. Identifying the relevant super Lie algebras in dimensions less than or equal to 6 is possible and already accomplished.

We will follow some of this strategy below, typically analyzing a problem in the complexification before turning to real forms. As in the bulk of the literature, we restrict our attention to the simple cases. Let us briefly summarize these simple examples, corresponding to complexified superconformal algebras in dimensions 3 to 6.

- In dimension 3, the superconformal algebras are $\mathfrak{osp}(k|4,\mathbb{C})$ for $k\geq 1$. The even part is $\mathfrak{so}(k,\mathbb{C})\oplus\mathfrak{sp}(4,\mathbb{C})\cong\mathfrak{so}(k,\mathbb{C})\oplus\mathfrak{so}(5,\mathbb{C})$, containing the conformal algebra $\mathfrak{so}(5,\mathbb{C})\cong\mathfrak{conf}(3,\mathbb{C})$.
- In dimension 4, the superconformal algebras are $\mathfrak{sl}(4|k,\mathbb{C})$ for $k\geq 1, k\neq 4$, and $\mathfrak{psl}(4|4,\mathbb{C})$ in the special case $k=4$. The even part is $\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(k,\mathbb{C})\oplus\mathbb{C}$ if $k\neq 4$ and $\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$ in the $k=4$ case, containing the conformal algebra $\mathfrak{so}(6,\mathbb{C})\cong\mathfrak{conf}(4,\mathbb{C})$.
- In dimension 5, there is a unique superconformal algebra, the exceptional super Lie algebra $\mathfrak{f}(4,\mathbb{C})$. The even part is $\mathfrak{so}(7,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$, containing the conformal algebra $\mathfrak{so}(7,\mathbb{C})\cong\mathfrak{conf}(5,\mathbb{C})$.
- In dimension 6, the superconformal algebras are $\mathfrak{osp}(8|2k,\mathbb{C})$ for $k\geq 1$. The even part is $\mathfrak{so}(8,\mathbb{C})\oplus\mathfrak{sp}(2k,\mathbb{C})$, containing the conformal algebra $\mathfrak{so}(8,\mathbb{C})\cong\mathfrak{conf}(6,\mathbb{C})$.

Remark 2.11. In each of these examples, the even part of the superconformal algebra includes $\mathfrak{conf}(n, \mathbb{C}) \cong \mathfrak{so}(n+2, \mathbb{C})$ as a summand. We will denote the quotient algebra by $\mathfrak{g}_R$, and refer to it as the algebra of R-symmetries. In each case – as we can see directly from the classification – $\mathfrak{g}_R$ coincides with the Lie algebra of the group G_R of outer automorphisms of the superconformal algebra that act trivially on the even part (the *group* of R-symmetries).

2.3. **The question of classifying twists.** Topological field theories have had an enormous impact in both theoretical physics and mathematics, and the most important examples arise from supersymmetric theories by a twisting procedure, introduced by Witten [Wit88a]. We recall here a less sophisticated version of that procedure before we turn to superconformal field theories.

2.3.1. *How twists appear in the translation case.* Loosely speaking, a supersymmetric field theory is a field theory on $\mathbb{R}^{p,q}$ with an action of a super Poincaré Lie algebra. (One typically asks for more, but we will focus on this kind of symmetry for now.) It is reasonable to explore how elements of the supertranslations act on the theory. Let $\mathfrak{stran}(p, q|S)$ be this super Lie algebra, where S denotes the direct sum of spinor representations that provides the odd component and the non-trivial Lie bracket is given by a vector-valued pairing Γ on S.

Definition 2.12. *A* translation twist *is an odd element $\mathcal{Q} \in \Pi S$ such that $[\mathcal{Q}, \mathcal{Q}] = \Gamma(\mathcal{Q}, \mathcal{Q}) = 0$.*

A choice of twist leads to some interesting structures:

(1) A subspace $\mathfrak{b}_\mathcal{Q} \subset \mathbb{R}^{p,q}$ defined by
$$\mathfrak{b}_\mathcal{Q} = \mathrm{Im}(\mathcal{Q})$$
where we view the twist as a map $\mathcal{Q}\colon \Pi S \to \mathbb{R}^{p,q}$ sending s to $[Q, s]$. Such a subspace determines a foliation $\mathcal{F}^\mathcal{Q}$ of $\mathbb{R}^{p,q}$ into affine leaves.

(2) A deformation of a supersymmetric theory by adding $\mathcal{Q}$ to its BRST differential (i.e., by adding a term to its Lagrangian that encodes how $\mathcal{Q}$ acts on the theory). This deformed theory is "trivial" along leaves of the foliation $\mathcal{F}^\mathcal{Q}$ in the sense that BRST cohomology classes of observables are isomorphic under translation along these leaves.

(3) A differential $\mathbb{Z}/2$-graded Lie algebra $(\mathfrak{stran}(p, q|S), [\mathcal{Q}, -])$, which can be seen as acting on the deformed field theory.

We call such a $\mathcal{Q}$-deformation of a supersymmetric field theory a *twisted theory*, and we see that it admits a symmetry by this dg Lie algebra.

When $\mathfrak{b}_\mathcal{Q}$ is all of $\mathbb{R}^{p,q}$, we call $\mathcal{Q}$ a *topological* twist as the associated twisted theories must be "topological" in the sense that we can move insertion points around without changing expected values. But there are cases where $\mathfrak{b}_\mathcal{Q}$ is a proper subspace, leading to more subtle changes in the theory. For instance, in the Euclidean setting, twists lead to theories that can be holomorphic in some directions and topological in others.

Recently, there has been a systematic classification of the space of twists, motivated by the desire to organize and analyze the behavior of twisted supersymmetric field theories. See work of the first author with Safronov and Williams [ES19; ESW22] and of Eager, Saberi and Walcher [ESW21] for details of this classification.

The following definition when applied to the super Lie algebra $\mathfrak{g} = \mathfrak{strans}(p, q|S)$ characterizes the space of possible supertranslation twists.

Definition 2.13. *Let $\mathfrak{g}$ be a super Lie algebra. The* affine nilpotence variety *of $\mathfrak{g}$ is the affine quadric subvariety of $\mathfrak{g}_1$ defined by*

$$\mathcal{N}\mathrm{ilp}_{\mathfrak{g}} = \{\mathcal{Q} \in \mathfrak{g}_1 \colon [\mathcal{Q}, \mathcal{Q}] = 0\}.$$

This affine variety is preserved by the rescaling action, and the projective nilpotence variety $\mathbb{P}\mathcal{N}\mathrm{ilp}_{\mathfrak{g}} \subseteq \mathbb{P}(\mathfrak{g}_1)$ *of $\mathfrak{g}$ is its projectivization.*

Remark 2.14. The affine and projective nilpotence varieties both carry an action of the Lie algebra $\mathfrak{g}_0$ inherited from the adjoint action. If $\mathfrak{g}$ is the super Lie algebra of a super Lie group G then the even part G_0 of G also acts on the nilpotence varieties by the group adjoint action.

In the case where $\mathfrak{g} = \mathfrak{strans}(p, q|S)$ the nilpotence variety carries a natural action of the group $\mathrm{Spin}(p, q) \times G_R$ where G_R is the group of R-symmetries.

Definition 2.15. *The* moduli stack of twists *for the supertranslation algebra is the quotient stack*

$$\mathcal{T}\mathrm{wist}_{\mathfrak{g}} = \left[\mathcal{N}\mathrm{ilp}_{\mathfrak{g}}/(\mathrm{Spin}(p, q) \times G_R)\right].$$

2.3.2. *How twists appear in the conformal case.* We raise here the question of studying the superconformal analog of twists.

Let $\mathfrak{sconf}(p, q|S)$ be the super Lie algebra, where S denotes the spinor representation that provides the odd component and Γ is the pairing on S.

Definition 2.16. *A* conformal twist *is an odd element $\mathcal{Q} \in \Pi S$ such that $[\mathcal{Q}, \mathcal{Q}] = \Gamma(\mathcal{Q}, \mathcal{Q}) = 0$.*

A twist $\mathcal{Q}$ determines a subspace of $\mathfrak{conf}(p, q)$ and hence a foliation of the conformal compactification $C(\mathbb{R}^{p,q})$. If we view a superconformal field theory as living on $C(\mathbb{R}^{p,q})$, a twist then determines a deformation to a theory that is "trivial" along the leaves of this foliation. We call it the *$\mathcal{Q}$-twisted* superconformal theory. Moreover, the dg Lie algebra $(\mathfrak{sconf}(p, q|S), [\mathcal{Q}, -])$ acts on this twisted theory. It is to be hoped that such twisted theories admit phenomena at least as special and interesting as twists of supersymmetric theories.

Hence we address the following question in this paper:

For $p + q > 2$, what are the twists of the superconformal Lie algebras?

Below we begin to explore this question by doing a few explicit examples and then state the result in all cases. Finally, we discuss how twisting affects theories in qualitative terms.

Following [ES19; ESW21], we phrase our problem as follows.

Definition 2.17. *The* conformal nilpotence variety *for $\mathfrak{sconf}(p, q|S)$ is the nilpotence variety associated to the superconformal Lie algebra under the construction of Definition 2.13.*

Our goal is to describe the geometric properties of these varieties.

3. The Observables of a Twisted Superconformal Theory

Given a superconformal theory with $\mathfrak{sconf}(p, q|S)$ as a symmetry algebra, a nilpotent supercharge $\mathcal{Q}$ can added to the action as a BRST operator, leading to a new theory that has the $\mathcal{Q}$-cohomology of $\mathfrak{sconf}(p, q|S)$ as a symmetry algebra. For supersymmetric theories, this process (often dubbed "twisting") is well-known and produces new theories with remarkable properties, such as the topological field theories encoding Donaldson invariants of 4-manifolds. A crucial step is to understand the $\mathcal{Q}$-cohomology of $\mathfrak{spoin}(p,q|S)$ and its geometric meaning. For superconformal theories, the twists can have more exotic geometric consequences and the associated twisted theories have surprising behavior, including the localization discovered by Beem et al [Bee+15b]. In this section we will give a physical description of howf superconformal twists affect a theory, and in section 7 we offer detailed mathematical formulations using the language of disc-algebras and factorization algebras.

3.1. **Subalgebras of $\mathcal{Q}$-closed and $\mathcal{Q}$-exact elements.** Let $\mathcal{Q}$ be an odd square-zero element of $\mathfrak{sconf}(p,q|S)$. As we hinted above, the properties of the $\mathcal{Q}$-twist of a superconformal field theory defined on $C(\mathbb{R}^{p,q})$ — e.g., how close this twisted theory is to being topological — are determined by the cohomology of $\mathfrak{sconf}(p,q|S)$ under the differential $[\mathcal{Q},-]$. We will make this idea more precise here.

Definition 3.1. *Let $Z_{\mathcal{Q}} \subseteq \mathrm{Conf}(p,q) \times G_R$ denote the stabilizer group of the element $\mathcal{Q}$ under the adjoint action. Let $\mathfrak{z}_{\mathcal{Q}}$ denote the (real) Lie algebra of $Z_{\mathcal{Q}}$. In other words, it is the even part of the centralizer of $\mathcal{Q}$ in the superconformal algebra.*

Definition 3.2. *Let $\mathfrak{b}^{\mathbb{C}}_{\mathcal{Q}}$ denote the image of the operator*

$$[\mathcal{Q},-] \colon \mathfrak{sconf}(n|S,\mathbb{C})_1 \to \mathfrak{sconf}(n|S,\mathbb{C})_0,$$

arising from the adjoint action of $\mathcal{Q}$.

These two Lie algebras are related, as the notation — drawn from homological algebra — suggests.

Proposition 3.3. *The subspace $\mathfrak{b}^{\mathbb{C}}_{\mathcal{Q}} \subseteq \mathfrak{z}_{\mathcal{Q}} \otimes_{\mathbb{R}} \mathbb{C}$ is a (complex) Lie subalgebra. In fact, it is an ideal, and so the quotient $\mathfrak{z}^{\mathbb{C}}_{\mathcal{Q}}/\mathfrak{b}^{\mathbb{C}}_{\mathcal{Q}}$ is a Lie algebra.*

As a result, we can identify $\mathfrak{b}^{\mathbb{C}}_{\mathcal{Q}}$ as the Lie algebra of a Lie subgroup $B^{\mathbb{C}}_{\mathcal{Q}} \subseteq Z^{\mathbb{C}}_{\mathcal{Q}}$ inside the complexification of $Z_{\mathcal{Q}}$. Write $B_{\mathcal{Q}}$ for the real Lie subgroup of $Z_{\mathcal{Q}}$ obtained with Lie algebra $\mathfrak{b}_{\mathcal{Q}} = \mathfrak{b}^{\mathbb{C}}_{\mathcal{Q}} \cap \mathfrak{z}_{\mathcal{Q}}$.

Proof. Let $X \in \mathfrak{z}_{\mathcal{Q}}$ and $Y = [\mathcal{Q},\mathcal{Q}']$ in $\mathfrak{b}_{\mathcal{Q}}$. Then

$$[\mathcal{Q},[X,\mathcal{Q}']] = [[\mathcal{Q},X],\mathcal{Q}'] + [X,[\mathcal{Q},\mathcal{Q}']] = [X,Y]$$

so $[X,Y]$ is also in $\mathfrak{b}_{\mathcal{Q}}$. This computation shows $\mathfrak{b}_{\mathcal{Q}}$ is an ideal. $\square$

3.2. **Observables and BRST operators.** There are two major types of manifolds on which a superconformal field theory might live. It might be defined on $\mathbb{R}^{p,q}$ (as is usually implicit when a Lagrangian is given in coordinates), or it might be defined on the conformal compactification $C(\mathbb{R}^{p,q})$.

3.2.1. *The maximal case.* To start let us suppose the theory $\mathcal{T}$ lives on the biggest possible space, namely $C(\mathbb{R}^{p,q})$. In that case, we may suppose that the superconformal group acts as symmetries of the theory; in particular, its even part provides a real Lie group acting as symmetries of the theory. Let $\mathcal{T}^{\mathcal{Q}}$ denote the twisted theory, in which case the groups $Z_{\mathcal{Q}}$ and its subgroup $B_{\mathcal{Q}}$ continue to act as symmetries of the twisted theory. Similarly, there are the infinitesimal symmetries given by actions of the Lie algebras $\mathfrak{z}_{\mathcal{Q}}$ and $\mathfrak{b}_{\mathcal{Q}}$, which act on operators and hence also on operator products.

Note that group elements of SConf_0 that do *not* fix $\mathcal{Q}$ are *not* symmetries. Nor do non-$\mathcal{Q}$-closed elements of $\mathfrak{sconf}_0$ act as infinitesimal symmetries. Twisting the theory changes the symmetries.

In the twisted theory, we focus on $\mathcal{Q}$-closed operators (i.e., we only consider an operator $\mathcal{O}$ satisfying $\mathcal{Q}\mathcal{O} = 0$). Moreover, $\mathcal{Q}$-exact operators (i.e., $\mathcal{O} = \mathcal{Q}\mathcal{O}'$ for some operator $\mathcal{O}'$) have trivial expected value. These properties have strong consequences for how the symmetries act on operators. Recall that in a topological twist of a supersymmetric theory, an n-point function

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle$$

is independent of the insertion points $\{x_i\}$ because translation is $\mathcal{Q}$-exact. The same argument implies that for a conformal twist, the operator product is constant if you move an insertion point along a $B_{\mathcal{Q}}$-orbits:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle = \langle \mathcal{O}_1(bx_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle$$

for any $b \in B_{\mathcal{Q}}$. In formulas, we see this claim infinitesimally: let $\beta = [\mathcal{Q}, \mathcal{Q}']$ be a $\mathcal{Q}$-exact element of the superconformal algebra and compute

$$\begin{aligned}\langle (\beta\mathcal{O}_1(x_1))\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle &= \langle \mathcal{Q}(\mathcal{Q}'\mathcal{O}_1(x_1))\cdots\mathcal{O}_n(x_n)\rangle + \langle \mathcal{Q}'(\mathcal{Q}\mathcal{O}_1(x_1))\cdots\mathcal{O}_n(x_n)\rangle\\ &= 0 + 0 = 0.\end{aligned}$$

Thus, the twisted theory $\mathcal{T}^{\mathcal{Q}}$ will behave like it is "topological" along $B_{\mathcal{Q}}$-orbits, in the sense that moving insertions along those orbits does not affect operator products.

By contrast, the action of $Z_{\mathcal{Q}}$ is more like the action of translation in a supersymmetric theory that is not twisted. For instance, let $\mathcal{O}_1$ and $\mathcal{O}_2$ be $\mathcal{Q}$-closed operators and $z \in Z_{\mathcal{Q}}$. Then the operator product satisfies

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \langle \mathcal{O}_1(zx_1)\mathcal{O}_2(zx_2)\rangle,$$

so that it is preserved by "translation" in the z-direction of *both* insertion points x_1 and x_2. On the other hand, if we translate just one insertion, the expected value changes: $\langle \mathcal{O}_1(zx_1)\mathcal{O}_2(x_2)\rangle$ varies with z, much as a 2-point function typically depends on the relative position of the insertion.

To summarize, we should analyze $C(\mathbb{R}^{p,q})$ (and configurations of multiple points in it) into $Z_{\mathcal{Q}}$-orbits and, further, into $B_{\mathcal{Q}}$-orbits. The twisted theory $\mathcal{T}^{\mathcal{Q}}$ will behave like it is "topological" along $B_{\mathcal{Q}}$-orbits and equivariant along $Z_{\mathcal{Q}}$-orbits.

Here we have emphasized the geometry of how these groups act on spacetime, but these groups also act, in some sense, "internally," as part of the superconformal group involves R-symmetries and not conformal symmetries.

3.2.2. *Working on an affine patch.* Now suppose we pick an "affine patch" $\mathbb{R}^{p,q} \subset C(\mathbb{R}^{p,q})$. We mean here that we pick the image $g\mathbb{R}^{p,q}$ of the defining $\mathbb{R}^{p,q}$ under the action of some element g of the conformal group. On such a patch, a theory has a nice coordinate description, and so it can be analyzed very explicitly. Note, however, that orbits of the groups $Z_{\mathcal{Q}}$ and $B_{\mathcal{Q}}$ are not contained in the affine patch.

Nonetheless, our above arguments about operators and their products carry over to this situation, with small modifications. For instance if $\mathcal{O}_1$ and $\mathcal{O}_2$ are $\mathcal{Q}$-closed operators and $z \in Z_{\mathcal{Q}}$, then the operator product satisfies

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \langle \mathcal{O}_1(zx_1)\mathcal{O}_2(zx_2)\rangle,$$

so long as the points x_1, x_2 and their translates zx_1, zx_2 all live in the affine patch. Similarly, the operator product is constant if you move an insertion point along a $B_{\mathcal{Q}}$-orbit intersected with the affine patch.

4. 4D SUPERCONFORMAL TWISTS

4.1. **Complex Nilpotence Variety.** Over $\mathbb{C}$, let us consider the 4d $\mathcal{N} = k$ superconformal algebra $\mathfrak{A}_k$ for $k \neq 4$ (we will discuss the $k = 4$ case in the next subsection). The superconformal algebra can be identified with the simple super Lie algebra $\mathfrak{sl}(4|k)$. The even part of $\mathfrak{A}_k$ is isomorphic to

$$(\mathfrak{A}_k)_0 \cong \mathfrak{sl}(4,\mathbb{C}) \oplus \mathfrak{sl}(k,\mathbb{C}) \oplus \mathbb{C} \cong \mathfrak{so}(6,\mathbb{C}) \oplus \mathfrak{sl}(k,\mathbb{C}) \oplus \mathbb{C}$$

where we use the exceptional isomorphism between $\mathfrak{sl}(4,\mathbb{C})$ and $\mathfrak{so}(6,\mathbb{C})$. The odd part of $\mathfrak{A}_k$ can be identified with

$$(\mathfrak{A}_k)_1 \cong V_4 \otimes W_k \oplus V_4^* \otimes W_k^* \cong S_+ \otimes W_k \oplus S_- \otimes W_k^*$$

where V_4 is the four-dimensional defining representation of $\mathfrak{sl}(4,\mathbb{C})$, which is isomorphic to the positive semispin (or Weyl spinor) representation of $\mathfrak{so}(6,\mathbb{C})$, and W_k is the defining k-dimensional representation of the factor $\mathfrak{sl}(k,\mathbb{C})$. The one-dimensional factor in the center of $(\mathfrak{A}_k)_0$ acts trivially.

The Lie bracket between two odd homogeneous elements $Q \otimes w, Q' \otimes w'$ is given as follows. For any k, there is a $\mathfrak{sl}(k,\mathbb{C})$-invariant decomposition $W_k \otimes W_k^* \cong \mathfrak{sl}(k,\mathbb{C}) \oplus \mathbb{C}$. Denote the linear mprojections onto the two factors by $\mathrm{Tr}\colon W_k \otimes W_k^* \to \mathbb{C}$ and $\mathrm{red}\colon W_k \otimes W_k^* \to \mathfrak{sl}(k,\mathbb{C})$. We identify the bracket as follows. Pairs of elements in the same summand of $(\mathfrak{A}_k)_1$ commute, whereas the bracket of a pair of elements in opposite summands is given by

$$[Q_+ \otimes w_+, Q'_- \otimes w'_-] = \\ \big(\mathrm{red}(Q_+ \otimes Q'_-) \cdot \mathrm{Tr}(w_+ \otimes w'-), \mathrm{Tr}(Q_+ \otimes Q'_-) \cdot \mathrm{red}(w_+ \otimes w'_-), \mathrm{Tr}(Q_+ \otimes Q'_-) \cdot \mathrm{Tr}(w_+ \otimes w'_-)\big).$$

Let us compute the associated superconformal nilpotence variety. In this calculation I will make the identification

$$V_4 \otimes W_k \oplus V_4^* \otimes W_k^* = \mathrm{hom}(V_4^*, W_k) \oplus \mathrm{hom}(W_k, V_4^*).$$

Proposition 4.1. *The complex nilpotence variety for the 4d $\mathcal{N} = k$ superconformal algebra with $k \neq 4$ superconformal algebra is*

$$\mathcal{N}_{\mathbb{C},k} = \{(\mathcal{Q}_+, \mathcal{Q}_-) \in \mathrm{hom}(V_4^*, W_k) \oplus \mathrm{hom}(W_k, V_4^*)\colon \mathcal{Q}_+ \circ \mathcal{Q}_- = 0, \mathcal{Q}_- \circ \mathcal{Q}_+ = 0\}.$$

Proof. This follows from the description of the bracket. In terms of the linear maps $\mathcal{Q}_+, \mathcal{Q}_-$ we have

$$[\mathcal{Q}, \mathcal{Q}] = 2[\mathcal{Q}_+, \mathcal{Q}_-],$$

which equals zero if

$$\begin{aligned}\mathrm{red}(\mathcal{Q}_- \circ \mathcal{Q}_+) &= 0 \in \mathfrak{sl}(V_4^*)\\ \mathrm{red}(\mathcal{Q}_+ \circ \mathcal{Q}_-) &= 0 \in \mathfrak{sl}(W_k)\\ \text{and } \mathrm{Tr}(\mathcal{Q}_- \circ \mathcal{Q}_+) + \mathrm{Tr}(\mathcal{Q}_+ \circ \mathcal{Q}_-) &= 0 \in \mathbb{C}.\end{aligned}$$

In terms of linear maps, trace refers the the literal trace, and $\mathrm{red}(M) = M - \mathrm{Tr}(M)\mathrm{Id}$. So the conditions are saying that $\mathcal{Q}_- \circ \mathcal{Q}_+$ and $\mathcal{Q}_+ \circ \mathcal{Q}_-$ are both diagonal, and their traces sum to zero.

Now we use the condition that $k \neq 4$. This tells us that either $\mathcal{Q}_- \circ \mathcal{Q}_+$ has rank less than 4 (if $k < 4$) or $\mathcal{Q}_+ \circ \mathcal{Q}_-$ has rank less than k (if $k > 4$). In either case, the only diagonal matrix with less than full rank is the zero matrix, so we must have $\mathrm{Tr}(\mathcal{Q}_- \circ \mathcal{Q}_+) = -\mathrm{Tr}(\mathcal{Q}_+ \circ \mathcal{Q}_-) = 0$, and hence $\mathcal{Q}_-\mathcal{Q}_+$ and $\mathcal{Q}_+\mathcal{Q}_-$ both equal zero.

□

4.2. **The $\mathcal{N} = 4$ case.** The 4d $\mathcal{N} = 4$ superconformal algebra is defined slightly differently. The super Lie algebra $\mathfrak{sl}(4|k)$ is not simple if $k = 4$: it has a one-dimensional center generated by the element $\mathrm{diag}(1, 1, 1, 1|1, 1, 1, 1)$. In order to obtain a simple super Lie algebra we should take the quotient by this element to obtain the super Lie algebra $\mathfrak{psl}(4|4)$. So, we identify the 4d $\mathcal{N} = 4$ complex superconformal algebra $\mathfrak{A}_4$ with $\mathfrak{psl}(4|4)$. The even part of $\mathfrak{A}_4$ can be identified with

$$(\mathfrak{A}_4)_0 \cong \mathfrak{sl}(4, \mathbb{C}) \oplus \mathfrak{sl}(4, \mathbb{C}) \cong \mathfrak{so}(6, \mathbb{C}) \oplus \mathfrak{sl}(4, \mathbb{C}).$$

The odd part of $\mathfrak{A}_4$ can be identified with

$$(\mathfrak{A}_k)_1 \cong V_4 \otimes W_4 \oplus V_4^* \otimes W_4^* \cong S_+ \otimes W_4 \oplus S_- \otimes W_4^*$$

identically to the description from the previous subsection.

The Lie bracket between two odd homogeneous elements $Q \otimes w, Q' \otimes w'$ is given, again, similarly to the $\mathcal{N} \neq 4$ case. Pairs of elements in the same summand of $(\mathfrak{A}_4)_1$ commute, whereas the bracket of a pair of elements in opposite summands is given by

$$[Q_+ \otimes w_+, Q'_- \otimes w'_-] = \big(\mathrm{red}(Q_+ \otimes Q'_-) \cdot \mathrm{Tr}(w_+ \otimes w'-), \mathrm{Tr}(Q_+ \otimes Q'_-) \cdot \mathrm{red}(w_+ \otimes w'_-)\big).$$

Proposition 4.2. *The complex nilpotence variety for the 4d $\mathcal{N} = 4$ superconformal algebra is*

$$\begin{aligned}\mathcal{N}_{\mathbb{C},4} = \{Q_+ \otimes w_+ + Q_- \otimes w_- \colon \mathrm{Tr}(Q_+ \otimes Q_-) = \mathrm{Tr}(w_+ \otimes w_-) = 0\} \cup_{\{0\}}\\ \{Q_+ \otimes w_+ + Q_- \otimes w_- \colon \mathrm{red}(Q_+ \otimes Q_-) = \mathrm{red}(w_+ \otimes w_-) = 0\}.\end{aligned}$$

Proof. The right-hand side is contained in the left-hand side, this is immediate from the description of the Lie bracket. If $Q = Q_+ \otimes w_+ + Q_- \otimes w_-$ is a square zero element then $\mathrm{Tr}(Q_+ \otimes Q_-)$ if and only if $\mathrm{Tr}(w_+ \otimes w_-) = 0$ by the same argument as Proposition 4.1. If neither trace term equals zero and $[Q, Q] = 0$ then we must have $\mathrm{red}(Q_+ \otimes Q_-) = \mathrm{red}(w_+ \otimes w_-) = 0$. □

4.3. **Complex Group Orbit.** Let us discuss the orbits in $\mathcal{N}_{\mathbb{C},k}$ with respect to the action of the even part of the complex superconformal group: $\mathrm{Spin}(6, \mathbb{C}) \times \mathrm{SL}(k, \mathbb{C}) \times \mathbb{C}^\times \cong \mathrm{SL}(4, CC) \times \mathrm{SL}(k, \mathbb{C}) \times \mathbb{C}^\times$. These orbits are precisely distinguished by their *ranks*.

Proposition 4.3. *The orbits of $\mathcal{N}_{\mathbb{C},k}$ are in bijection with pairs (r_+, r_-) of non-negative integers where $r_+ + r_- \leq \min(4, k)$.*

Proof. Given a square-zero supercharge $(\mathcal{Q}_+, \mathcal{Q}_-)$, we may assign to it a pair of non-negative integers by letting $r_\pm = \mathrm{rk}(\mathcal{Q}_\pm)$. Since $\mathrm{Im}(\mathcal{Q}_+) \subseteq \ker(\mathcal{Q}_-)$ we have $r_+ + r_- \leq k$, and since $\mathrm{Im}(\mathcal{Q}_-) \subseteq \ker(\mathcal{Q}_+)$ we have $r_+ + r_- \leq 4$.

Now, we may choose bases for V_4^*, W_k so that $\mathcal{Q}_+, \mathcal{Q}_-$ are represented by matrices of the following form:

$$\mathcal{Q}_+ = \begin{pmatrix} \mathrm{id}_{r_+} & 0 \\ 0 & 0 \end{pmatrix}, \ \mathcal{Q}_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{id}_{r_-} \end{pmatrix}$$

(note that these are not square matrices: we write 0 to refer to rectangular zero matrices of appropriate sizes). In other words, the action of $\mathrm{GL}(4, \mathbb{C}) \times \mathrm{GL}(k, \mathbb{C})$ on the locus of square-zero supercharges of ranks (r_+, r_-) is transitive. To complete the proof, we just need to ensure that we can choose our two change of basis matrices to have the same determinant. Alternatively, it is enough to find an element $(X, Y) \in \mathrm{GL}(4, \mathbb{C}) \times \mathrm{GL}(k, \mathbb{C})$ stabilizing the diagonal matrices 4.3 where $\det(X)/\det(Y) = \lambda \in \mathbb{C}^\times$ for all values of λ.

To do this, again use the fact that $k \neq 4$. If $k < 4$ then $r_+ + r_- < 4$, and we let $X = \mathrm{diag}(1, \ldots, 1, \lambda, 1, \ldots, 1)$ with λ in the $r_+ + 1^{\text{st}}$ position and $Y = \mathrm{id}_k$. Likewise if $k > 4$ then $r_+ + r_- < k$, and we let $Y = \mathrm{diag}(1, \ldots, 1, \lambda^{-1}, 1, \ldots, 1)$ with λ^{-1} in the $r_+ + 1^{\text{st}}$ position, and $X = \mathrm{id}_4$. □

4.4. **Real Group Orbits.** We can now analyze the orbit structure of superconformal twisted theories under the superconformal action in various signatures. The complex superconformal group, acting on the super Lie algebra $\mathfrak{A}_k$ by the adjoint action, is isomorphic to $\mathrm{SL}(4|k, \mathbb{C})$ if $k \neq 4$, with even part isomorphic to

$$\begin{aligned} \mathrm{SConf}(4|k, \mathbb{C})_0 &\cong \{(A, B) \in \mathrm{GL}(4, \mathbb{C}) \times \mathrm{GL}(k, \mathbb{C}) \colon \det(A)\det(B) = 1\} \\ &\cong \mathrm{SL}(4, \mathbb{C}) \times \mathrm{SL}(k, \mathbb{C}) \times \mathbb{C}^\times \end{aligned}$$

where the latter isomorphism sends (A, B) to $(\det(A)^{-1}A, \det(B)^{-1}B, \det(A))$. In the $k = 4$ case the superconformal group is identified with $\mathrm{PSL}(4|4, \mathbb{C})$ with even part isomorphic to

$$\mathrm{SL}(4, \mathbb{C}) \times \mathrm{SL}(4, \mathbb{C}).$$

An element $(A, B, \lambda) \in \mathrm{SL}(4, \mathbb{C}) \times \mathrm{SL}(k, \mathbb{C}) \times \mathbb{C}^\times$ in such a Lie group acts on

$$(\mathfrak{A}_k)_1 \cong \hom(W_k^*, V_4) \oplus \hom(W_k, V_4^*)$$

by

$$(\mathcal{Q}_+, \mathcal{Q}_-) \mapsto (\lambda^{-2} B \mathcal{Q}_+ A^{-1}, \lambda^2 A \mathcal{Q}_- B^{-1}).$$

Let $\mathcal{O}_{r_+, r_-}$ denote the complex group orbit associated to a pair (r_+, r_-) of ranks. The following observation is immediate.

Proposition 4.4. *There is a smooth map*

$$\pi \colon \mathcal{O}_{r_+, r_-} \to \mathrm{Gr}(r_+, 4) \times \mathrm{Gr}(r_-, 4),$$

and the fiber over a point (X_+, X_-) in the image of π is identified with

$$\pi^{-1}(X_+, X_-) \cong \mathrm{GL}(r_+, \mathbb{C}) \times \mathrm{GL}(r_-, \mathbb{C}).$$

In terms of π, the group $\mathrm{SL}(4,\mathbb{C})$ *acts purely on the base, and the group* $\mathrm{SL}(k,\mathbb{C})\times\mathbb{C}^\times$ *acts on the two factors of the fiber by left and right multiplication respectively, viewing* $\mathrm{GL}(r,\mathbb{C})\subseteq\mathrm{GL}(k,\mathbb{C})$ *as the subspace of matrices with zeroes outside the upper-left block.*

So, let us choose a real form of the supergroup $\mathrm{SConf}(4|k,\mathbb{C})$ containing the real form $\mathrm{SO}(p+1,5-p)$ of the complex 4d conformal group $\mathrm{SO}(6,\mathbb{C})$. We will write $G_\mathbb{R}$ for the even part of this real supergroup, so $G_\mathbb{R}$ is a real form of $\mathrm{SL}(4,\mathbb{C})\times\mathrm{SL}(k,\mathbb{C})\times\mathbb{C}^\times$, or of $\mathrm{SL}(4,\mathbb{C})\times\mathrm{SL}(4,\mathbb{C})$ in the $k=4$ case. We will discuss the $G_\mathbb{R}$-orbits in the complex nilpotence variety for the possible choices of signature.

Our strategy is to use the observation of Proposition 4.4. We can split the orbits up in terms of a quotient of $GL(r_+,\mathbb{C})\times\mathrm{GL}(r_-,\mathbb{C})$ by a real subgroup, fibered over the set of orbits of a real form of $\mathrm{SL}(4,\mathbb{C})$ in a subspace of the product $\mathrm{Gr}(r_+,4)\times\mathrm{Gr}(r_-,4)$ of Grassmannians.

Example 4.5. For example we may consider the case of chiral supercharges, where $r_-=0$ (or identically the case $r_+=0$). So in this case we are studying the orbits for a real form of $\mathrm{SL}(4,\mathbb{C})$ on the complex Grassmannian $\mathrm{Gr}(r_+,4,\mathbb{C})$.

Example 4.6. For a more general pair of ranks (r_+,r_-), there is an embedding

$$\phi\colon \pi(\mathcal{O}_{r_+,r_-})\to \mathrm{Fl}(r_+,k-r_-,4,\mathbb{C}),$$

where $\mathrm{Fl}(r_+,k-r_-,4,\mathbb{C})$ denotes the partial flag variety of subspaces of dimension r_+ and $k-{}$ $+r_-$. The embedding is given by

$$\phi(\mathcal{Q}_+,\mathcal{Q}_-)=(\mathrm{Im}(Q_+)\subseteq\mathrm{Ker}(\mathcal{Q}_-^*)\subseteq V_4)\,.$$

So we are now considering the orbits for a real form of $\mathrm{SL}(4,\mathbb{C})$ on a subspace of the partial flag variety.

It is possible to work these orbit decompositions out by hand, but we may instead apply quite general results of Fels–Huckleberry–Wolf (going back to work of Wolf from the 1960's [Wol69])to analyze these orbits.

Theorem 4.7 ([FHW06, Theorem 3.2.1])**.** *Let P be a parabolic subgroup of* $\mathrm{SL}(n,\mathbb{C})$*, and let G be a real form of* $\mathrm{SL}(n,\mathbb{C})$ *of real rank ρ. Then the number of G-orbits in the partial flag variety* $\mathrm{SL}(n,\mathbb{C})/P$ *is precisely* $\binom{\rho+2}{2}$*. There is a unique closed orbit of real dimension equal to the complex dimension of the partial flag variety.*

In particular, the orbit space is finite, so for instance in the chiral case as in Example 4.5 our orbit space takes the form

$$(\mathrm{GL}(r_+,\mathbb{C})/H)^{\sqcup m}$$

for $m=\binom{\rho+2}{2}$, where $H=G_\mathbb{R}\cap\mathrm{GL}(k,\mathbb{C})\times\mathbb{C}^\times$, and where ρ is the real rank of $G_\mathbb{R}\cap\mathrm{SL}(4,\mathbb{C})$.

4.4.1. *Signature* $(4,0)$. We can identify

$$\mathrm{SConf}(4,0|2\ell,\mathbb{R}) \cong \mathrm{SL}(2|\ell,\mathbb{H}).$$

In particular, the even part can be identified with

$$\mathrm{SConf}(4,0|2\ell,\mathbb{R})_0 \cong \mathrm{SL}(2,\mathbb{H}) \times \mathrm{GL}(\ell,\mathbb{H}).$$

Remark 4.8. When we write $\mathrm{SL}(n|m,\mathbb{H})$, this refers to the set of quaternionic matrices such that

$$\mathrm{Nm}(\mathrm{sdet}(A)) = 1$$

where $\mathrm{Nm}(q) \in \mathbb{R}_{\geq 0}$ denotes the norm of a quaternion q. In particular $\dim_{\mathbb{R}} \mathrm{SL}(n,\mathbb{H}) = 4n^2 - 1$.

The real rank of $\mathrm{SL}(2,\mathbb{H})$ is one, so in a partial flag variety of the form $\mathrm{SL}(4,\mathbb{C})/P$ there are always precisely three orbits for the left-action of $\mathrm{SL}(2,\mathbb{H})$.

Example 4.9. For the $\mathcal{N} = 2\ell$ superconformal algebra with $2\ell \neq 4$, in the space of chiral supercharges of rank $(r_+,0)$ the space of orbits in signature $(4,0)$ can be identified with

$$(\mathrm{GL}(2\ell,\mathbb{C})/(\mathrm{GL}(\ell,\mathbb{H}))^{\sqcup 3}.$$

Example 4.10. If $2\ell = 2$, we can identify

$$\mathrm{GL}(2,\mathbb{C})/\mathrm{GL}(1,\mathbb{H}) \cong \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2) \times \mathbb{C}^\times/\mathbb{R}^\times \cong \mathfrak{h}^3 \times S^1$$

where $\mathfrak{h}^3$ refers to hyperbolic 3-space. So topologically, the space of orbits of rank $(2,0)$ chiral supercharges in the $\mathcal{N} = 2$ superconformal algebra, in Euclidean signature, can be identified with

$$(\mathbb{R}^3 \times S^1)^{\sqcup 3}.$$

4.4.2. *Signature* $(3,1)$. We can identify

$$\mathrm{SConf}(3,1|k,\mathbb{R}) \cong \mathrm{SU}(2,2|k).$$

In particular, the even part can be identified with

$$\mathrm{SConf}(3,1|k,\mathbb{R})_0 \cong \mathrm{SU}(2,2) \times \mathrm{SU}(k) \times \mathrm{U}(1).$$

Example 4.11. In the chiral case with $\mathcal{N} = 2$ the $\mathrm{SU}(2,2)$-action on the complex Grassmannian $\mathrm{Gr}(2,4;\mathbb{C})$ has orbits that we can describe quite concretely. They are given by the possible restrictions of the split signature Hermitian metric on $\mathbb{C}^{2,2}$ to a plane. There are six orbits: the restricted (possibly degenerate) Hermitian space can look like

$$\mathbb{C}^{2,0}, \mathbb{C}^{1,1}, \mathbb{C}^{0,2}, \mathbb{C}_{\mathrm{null}} \times \mathbb{C}^{1,0}, \mathbb{C}_{\mathrm{null}} \times \mathbb{C}^{0,1}, (\mathbb{C}_{\mathrm{null}})^2,$$

where $\mathbb{C}_{\mathrm{null}}$ refers to $\mathbb{C}$ with a trivial, completely degenerate metric.

4.4.3. *Signature* $(2,2)$. We can identify

$$\mathrm{SConf}(2,2|k,\mathbb{R}) \cong \mathrm{SL}(4|k,\mathbb{R}).$$

In particular, the even part can be identified with

$$\mathrm{SConf}(2,2|k,\mathbb{R})_0 \cong \mathrm{SL}(4,\mathbb{R}) \times \mathrm{SL}(k,\mathbb{R}) \times \mathbb{R}^\times.$$

The real rank of $\mathrm{SL}(4,\mathbb{R})$ is three, so in a partial flag variety of the form $\mathrm{SL}(4,\mathbb{C})/P$ there are always precisely ten orbits for the left-action of $\mathrm{SL}(4,\mathbb{R})$.

Example 4.12. In the chiral rank 2 case we can again identify the space of orbits. We find ten disjoint copies of the homogeneous space

$$\mathrm{GL}(2,\mathbb{C})/(\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^\times) \cong (\mathbb{R}^2 \times (S^1)^2,$$

where now $\mathrm{SL}(2,\mathbb{C})/\mathrm{SL}(2,\mathbb{R}) \cong \mathbb{R}^2 \times S^1$ is a 3d hyperboloid of one sheet $\{x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1\} \subseteq \mathbb{R}^4$.

So, overall,the set of orbits in this split signature case is identified with $(\mathbb{R}^2 \times (S^1)^2)^{\sqcup 10}$.

4.5. **Closed and Exact Transformations.** Let us discuss some specific properties of the complex orbits.

(1) **Rank** $(0,0)$: As usual, the zero supercharge always squares to zero. If $k < 2$ this is the only point in $\mathcal{N}_k^{(3)}$. The kernel of $[\mathcal{Q}, -]$ consists of the entirety of $\mathfrak{so}(6,\mathbb{C}) \oplus \mathfrak{sl}(k,\mathbb{C}) \oplus \mathbb{C}$, and the image of $[\mathcal{Q}, -]$ is just $\{0\}$.

(2) **Rank** $(r_+,0)$ **and** $(0,r_-)$: We may refer to these as *chiral* supercharges, where either $\mathcal{Q}_+$ or $\mathcal{Q}_-$ vanishes. Such supercharges automatically square to zero, so these strata in $\mathcal{N}_{\mathbb{C},k}$ are isomorphic to $V_4 \otimes W_4 \backslash \{0\}$ and $V_4^* \otimes W_4^* \backslash \{0\}$ respectively.

The stabilizer $Z_\mathcal{Q} \subseteq \mathrm{SL}(4,\mathbb{C}) \times \mathrm{SL}(k,\mathbb{C}) \times \mathbb{C}^\times$ can be described in the following way. Let $P_{r,k} \subseteq \mathrm{SL}(k,\mathbb{C})$ be the parabolic subgroup with block diagonal Levi subgroup $\mathrm{SL}(r,\mathbb{C}) \times \mathrm{SL}(k-r,\mathbb{C})$. Write

$$F_r \colon P_{r,4} \times P_{r,k} \to \mathrm{SL}(r,\mathbb{C})$$

be the homomorphism given by projection onto $\mathrm{SL}(r,\mathbb{C}) \times \mathrm{SL}(r,\mathbb{C})$ (projection onto the upper-left block in each factor) composed with the product map $\mathrm{SL}(r,\mathbb{C}) \times \mathrm{SL}(r,\mathbb{C}) \to \mathrm{SL}(r,\mathbb{C})$.

Proposition 4.13. *If $\mathcal{Q}$ is a chiral supercharge of rank $(r,0)$, the stabilizer $Z_\mathcal{Q}$ is isomorphic to $F_r^{-1}(I) \times \mathbb{C}^\times$.*

Proof. We can check this by choosing a basis in which $\mathcal{Q}$ is represented by a block diagonal matrix of rank r. The stabilizer $Z_\mathcal{Q}$ is then readily computed by solving

$$\left(\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right)\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)\left(\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right) = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right).$$

Here $A = \left(\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right)$ is an element of $\mathrm{SL}(4,\mathbb{C})$ and $B = \left(\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right)$ is an element of $\mathrm{SL}(k,\mathbb{C})$ (so, for instance, B_1 is an $r \times r$ block, B_2 is an $r \times (k-r)$ block, and so on). We find $A_1 = B_1^{-1}$ and $A_3 = B_2 = 0$. In other words (A,B) lies in F_r^{-1}. $\square$

In particular, the dimension of $Z_{\mathcal{Q}}$ is given as

$$\dim Z_{\mathcal{Q}} = k^2 - kr + r^2 - 4r + 14$$

if $r \neq k$ and

$$\dim Z_{\mathcal{Q}} = k^2 - kr + r^2 - 4r + 15$$

if $r = k$.

Similarly, we can describe the Lie algebra $\mathfrak{b}_{\mathcal{Q}} = \mathrm{Im}([\mathcal{Q}, -])$ as a subalgebra of $\mathfrak{z}_{\mathcal{Q}} = \mathrm{Lie}(Z_{\mathcal{Q}})$. This is a similar direct computation. We find the following.

Proposition 4.14. *Let $\mathcal{Q}$ be a chiral supercharge of rank r. The Lie algebra $\mathfrak{b}_{\mathcal{Q}} \subseteq \mathfrak{sl}(4,\mathbb{C}) \oplus \mathfrak{sl}(k,\mathbb{C}) \subseteq \mathfrak{sl}(4,\mathbb{C}) \oplus \mathfrak{sl}(k,\mathbb{C}) \oplus \mathbb{C}$ is isomorphic to the Lig algebra of block matrices of the following form:*

$$\mathfrak{b}_r = \left\{ \left(\left(\begin{array}{c|c} \alpha & \beta \\ \hline 0 & 0 \end{array} \right), \left(\begin{array}{c|c} \alpha & 0 \\ \hline \gamma & 0 \end{array} \right) \right) \in \mathfrak{sl}(4,\mathbb{C}) \oplus \mathfrak{sl}(k,\mathbb{C}) : \right.$$

$$\left. \alpha \in \mathfrak{sl}(r,\mathbb{C}), \beta \in \mathrm{Mat}_{r\times(4-r)}(\mathbb{C}), \gamma \in \mathrm{Mat}_{(k-r)\times r}(\mathbb{C}) \right\}.$$

In particular, we have

$$\dim \mathfrak{b}_{\mathcal{Q}} = kr - r^2 + 4r.$$

For reference, let's give a tabular description of the dependence of the dimension of $\mathfrak{z}_{\mathcal{Q}}$ and $\mathfrak{b}_{\mathcal{Q}}$ on r. Recall that $r \leq \min(4,k)$. So the dimensions for each k, in terms of r are given in Table 1.

r	$\dim \mathfrak{z}_{\mathcal{Q}}$	$\dim \mathfrak{b}_{\mathcal{Q}}$	$\dim \mathfrak{z}_{\mathcal{Q}}/\mathfrak{b}_{\mathcal{Q}}$
1	$k^2 - k + 11$	$k + 3$	$k^2 - 2k + 8$
2	$k^2 - 2k + 10$	$2k + 4$	$k^2 - 4k + 6$
3	$k^2 - 3k + 11$	$3k + 3$	$k^2 - 6k + 8$
4	$k^2 - 4k + 14$	$4k$	$k^2 - 8k + 14$

TABLE 1. For chiral supercharges $\mathcal{Q}$ of rank r we give the dimension of the kernel and image of $[\mathcal{Q}, -]$ as well as the dimension of the quotient. If $k = r$ the values in the second and fourth columns must be increased by one.

We note from this computation that the inclusion $\mathfrak{z}_{\mathcal{Q}} \subseteq \mathfrak{b}_{\mathcal{Q}}$ if always proper, at least for $k \neq 4$ (we will address the $k = \mathcal{N} = 4$ in Section 4.2 below).

(3) **Rank** $(1,1)$: Suppose $k > 1$ and consider a square zero supercharge of the form $Q_+ \otimes w_+ + Q_- \otimes w_-$. The square-zero condition in this case is equivalent to the conditions $\mathrm{Tr}(w_+, w_-) = \mathrm{Tr}(Q_+, Q_-) = 0$. We can again analyze the stabilizer and the image of a supercharge of this form.

Proposition 4.15. *Let $\mathcal{Q} = Q_+ \otimes w_+ + Q_- \otimes w_-$ be a rank $(1,1)$ square zero supercharge. The image $\mathfrak{b}_{\mathcal{Q}} \subseteq \mathfrak{sl}(4,\mathbb{C}) \times \mathfrak{sl}(k,\mathbb{C}) \times \mathbb{C}$ of the linear map $[\mathcal{Q}, -]$ is isomorphic to*

$$Z_{\mathcal{Q}} = \{(A, B) \in M_{21} \times M_{12} : \mathrm{Tr}(A) + \mathrm{Tr}(B) = 0\},$$

where for $i \neq j$, $M_{ij} \subseteq \mathfrak{gl}(n,\mathbb{C})$ is the Lie subalgebra of matrices whose non-zero entries all lie in row i or column j.

Proposition 4.16. *Let $\mathcal{Q}$ be as above. The kernel $\mathfrak{z}_{\mathcal{Q}}$ is the Lie algebra of the stabilizer subgroup $Z_{\mathcal{Q}} \subseteq \mathrm{SL}(4,\mathbb{C}) \times \mathrm{SL}(k,\mathbb{C}) \times \mathbb{C}^\times$, whose elements are given in block form as*

$$\left\{ \left(\begin{pmatrix} a & 0 & 0 & 0 \\ * & b & * & * \\ * & 0 & * & * \\ * & 0 & * & * \end{pmatrix}, \begin{pmatrix} a^{-1} & * & * & \cdots & * \\ 0 & b^{-1} & 0 & \cdots & 0 \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & & & \vdots \\ 0 & * & * & \cdots & * \end{pmatrix}, \lambda \right) \in \mathrm{SL}(4,\mathbb{C}) \times \mathrm{SL}(k,\mathbb{C}) \times \mathbb{C}^\times : a, b \in \mathbb{C}^\times \right\}.$$

5. 3D Superconformal Twists

5.1. **Complex Nilpotence Variety.** Over $\mathbb{C}$, the 3d $\mathcal{N} = k$ superconformal algebra $\mathfrak{sconf}(3|k,\mathbb{C})$ can be identified with the super Lie algebra $\mathfrak{osp}(k|4,\mathbb{C})$. The even part of $\mathfrak{sconf}(3|k,\mathbb{C})$ is isomorphic to

$$(\mathfrak{sconf}(3|k,\mathbb{C})_0 \cong \mathfrak{so}(k,\mathbb{C}) \oplus \mathfrak{sp}(4,\mathbb{C}) \cong \mathfrak{so}(k,\mathbb{C}) \oplus \mathfrak{so}(5,\mathbb{C}).$$

The odd part of $\mathfrak{sconf}(3|k,\mathbb{C})$ can be identified with

$$(\mathfrak{sconf}(3|k,\mathbb{C}))_1 \cong W_k \otimes V_4 \cong W_k \otimes \mathbb{S}_4,$$

where W_k is the k-dimensional fundamental representation of $\mathfrak{so}(k,\mathbb{C})$, and V_4 is the 4-dimensional fundamental representation of $\mathfrak{sp}(4,\mathbb{C})$, or equivalently the (Dirac) spinor representation of $\mathfrak{so}(5,\mathbb{C})$. The Lie bracket between two odd homogeneous elements $w \otimes Q, w' \otimes Q'$ is given as follows. Write $\omega(,)$ for the symplectic pairing on V_4, and write $g(,)$ for the symmetric pairing on W_{2k}. We have

$$[w \otimes Q, w' \otimes Q'] = (w \wedge w')\omega(Q,Q') + g(w,w')(Q \cdot Q')$$

using the isomorphism of $\mathfrak{so}(5,\mathbb{C})$ representations $\mathrm{Sym}^2(\mathbb{S}_4) \to \mathfrak{so}(5,\mathbb{C})$ on the second factor.

Let us compute the associated superconformal nilpotence variety.

Proposition 5.1. *The complex nilpotence variety for the 3d $\mathcal{N} = k$ superconformal algebra is*

$$\begin{aligned} \mathcal{N}_k^{(3)} \cong (\{(w,w') \in W_k^2 \colon g(w,w) = g(w',w') = g(w,w') = 0\} \times (\mathbb{CP}^1)^2)/ \\ (((0,w'),(c,c')) \sim ((0,w'),(0,c')), ((w,0),(c,c')) \sim ((w,0),(c,0))). \end{aligned}$$

Proof. Let Q_1, Q_2, Q_3, Q_4 be a Darboux basis for $\mathbb{S}_4$, such that $\omega(Q_1,Q_2) = \omega(Q_3,Q_4) = 1$. Let

$$\mathcal{Q} = (w_1,w_2,w_3,w_4) = w_1 \otimes Q_1 + w_2 \otimes Q_2 + w_3 \otimes Q_3 + w_4 \otimes Q_4$$

be an arbitrary element of $(\mathfrak{sconf}(3|k,\mathbb{C}))_1$. Suppose this element squares to zero. If we first consider the $\mathfrak{so}(5,\mathbb{C})$ summand of $[\boldsymbol{Q},\boldsymbol{Q}]$, this means that

$$\begin{aligned} \sum_{i,j=1}^4 g(w_i,w_j)(Q_i \cdot Q_j) = 0 \\ \implies g(w_i,w_j) = 0 \text{ for all } i,j. \end{aligned}$$

Now, considering the $\mathfrak{so}(2k,\mathbb{C})$ summand of $[\boldsymbol{Q},\boldsymbol{Q}]$, we have

$$\begin{aligned}\sum_{i,j=1}^{4}(w_i\wedge w_j)\omega(Q_i,Q_j)&=0\\ \implies w_1\wedge w_2&=0\\ \text{and } w_3\wedge w_4&=0.\end{aligned}$$

So let $w_1=w$, $w_2=cw$, $w_3=w'$ and $w_4=c'w'$ for complex constants c,c'. □

Remark 5.2. How does the usual supersymmetric nilpotence variety embed inside the superconformal nilpotence variety? Two copies of the 3d $\mathcal{N}=k$ supersymmetry algebra embed inside $\mathfrak{sconf}(3|k,\mathbb{C})$. We can understand the embedding by considering the restriction of the adjoint representation of $\mathfrak{sconf}(3|k,\mathbb{C})$ to a representation of the subalgebra $\mathfrak{so}(k,\mathbb{C})\oplus\mathfrak{so}(3,\mathbb{C})$. This restricted representation takes the following form (at the level of super vector spaces)

$$(\mathfrak{so}(k,\mathbb{C})\oplus\mathfrak{so}(3,\mathbb{C})\oplus(\mathbb{C}^3)^2\oplus\mathbb{C})\oplus\Pi(W_k\otimes(\mathbb{S}_2)^2),$$

where $\mathbb{S}_2$ is the (Dirac) spinor representation of $\mathfrak{so}(3,\mathbb{C})$. From this point of view, the usual supersymmetric nilpotence variety embeds either by considering the locus $w'=0$ or the locus $w=0$ inside the superconformal nilpotence variety.

Let us break down the orbits in $\mathcal{N}_k^{(3)}$ under the action of the even part $\mathrm{SConf}(3|k,\mathbb{C})_0\cong\mathrm{Spin}(5,\mathbb{C})\oplus\mathrm{SO}(k,\mathbb{C})$ of the superconformal group. We will index the orbits by *rank*: that is, the rank of a square-zero supercharge $\mathcal{Q}\in W_k\otimes V_4$ viewed as a linear map $V_4\to W_k^*$.

- **Rank 0**: The zero supercharge always squares to zero. If $k<2$ this is the only point in $\mathcal{N}_k^{(3)}$. The kernel of $[\mathcal{Q},-]$ consists of the entirety of $\mathfrak{so}(5,\mathbb{C})\oplus\mathfrak{so}(k,\mathbb{C})$, and the image of $[\mathcal{Q},-]$ is just $\{0\}$.
- **Rank 1**: Rank one nilpotent supercharges $\mathcal{Q}=Q\otimes w$ exist for all $k\geq 2$. The action of $\mathrm{SConf}(3|k,\mathbb{C})_0$ on the space of rank one nilpotent supercharges is transitive.

  Let us write $\mathfrak{p}_k\subseteq\mathfrak{so}(k,\mathbb{C})$ for the maximal parabolic subalgebra with block diagonal Levi quotient $\mathfrak{so}(2,\mathbb{C})\oplus\mathfrak{so}(k-2,\mathbb{C})$. Write $f_k\colon\mathfrak{p}_k\to\mathbb{C}\cong\mathfrak{so}(2,\mathbb{C})$ for the projection onto the $\mathfrak{so}(2,\mathbb{C})$ summand in this quotient.

  Proposition 5.3. *The kernel $\mathfrak{z}_{\mathcal{Q}}$ of the operator $[\mathcal{Q},-]$ is isomorphic to the kernel of the map*

  $$f_5-f_k\colon\mathfrak{p}_5\oplus\mathfrak{p}_k\to\mathbb{C}$$

  as a subalgebra of $\mathfrak{so}(5,\mathbb{C})\oplus\mathfrak{so}(k,\mathbb{C})\cong\mathfrak{sp}(4,\mathbb{C})\oplus\mathfrak{so}(k,\mathbb{C})$.

  Proof. Without loss of generality let $\mathcal{Q}=Q_1\otimes w$ where $w=(1,i,0,\dots,0)\in W_k$. The subalgebra $\mathfrak{z}_{\mathcal{Q}}$ is given as

  $$\mathfrak{z}_{\mathcal{Q}}=\{(A,B)\in\mathfrak{sp}(4,\mathbb{C})\oplus\mathfrak{so}(k,\mathbb{C})\colon AQ_1=\lambda Q_1, Bw=-\lambda w \text{ for some } \lambda\in\mathbb{C}\}.$$

  We first check that the reductive quotient $\mathfrak{l}$ of $\ker(f_5-f_k)$ is contained in $\mathfrak{z}_{\mathcal{Q}}$. Here $\mathfrak{l}$ is the subalgebra

  $$\mathfrak{l}=\mathfrak{so}(2,\mathbb{C})\oplus\mathfrak{so}(3,\mathbb{C})\oplus\mathfrak{so}(k-2,\mathbb{C})\subseteq\mathfrak{so}(5,\mathbb{C})\oplus\mathfrak{so}(k,\mathbb{C})$$

embedded via the map $(A, X, Y) \mapsto (\mathrm{diag}(A, X), \mathrm{diag}(-A, Y))$. If $\lambda \mapsto A_\lambda$ under the natural isomorphism $\mathbb{C}^\times \cong \mathrm{SO}(2, \mathbb{C})$, certainly

$$\mathrm{diag}(-A_\lambda, Y)w = -\lambda w.$$

Let

$$X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{so}(3, \mathbb{C}).$$

Under the exceptional isomorphism $\mathfrak{so}(5, \mathbb{C}) \to \mathfrak{sp}(4, \mathbb{C})$, the matrix $\mathrm{diag}(A_\lambda, X)$ maps to

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & -\lambda & 0 \\ 0 & c & 0 & -a \end{pmatrix} \in \mathfrak{sp}(4, \mathbb{C}).$$

So in particular $\mathrm{diag}(A_\lambda, X)Q_1 = \lambda Q_1$ as required.

The parabolic subalgebra $\mathfrak{p}_k \subseteq \mathfrak{so}(k, \mathbb{C})$ has nilradical of dimension $k-2$. To complete the argument we must show that the nilradical of $\mathfrak{p}_5 \oplus \mathfrak{p}_k$ is contained in $\mathfrak{z}_Q$. In other words, it is sufficient to show that the quotient $\mathfrak{z}_Q/\mathfrak{l}$ is an abelian subgroup of the same dimension as the nilradical of $\mathfrak{p}$, which is exactly $(5-2) + (k-2) = k+1$. We can compute this directly. The quotient $\mathfrak{z}_Q/\mathfrak{l}$ is spanned by matrices of the form

$$\begin{pmatrix} 0 & \alpha & \beta & \gamma \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \end{pmatrix} \in \mathfrak{sp}(4, \mathbb{C})$$

together with matrices of the form

$$\begin{pmatrix} 0 & 0 & x_1 & \cdots & x_{k-2} \\ 0 & 0 & ix_1 & \cdots & ix_{k-2} \\ -x_1 & -ix_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x_{k-2} & -ix_{k-2} & 0 & \cdots & 0 \end{pmatrix} \in \mathfrak{so}(k, \mathbb{C}).$$

The ideal generated by these elements is indeed an abelian subalgebra of dimension $k+1$ as required. □

The image $\mathfrak{b}_Q \subseteq \mathfrak{z}_Q$ of $[Q, -]$ is four-dimensional, spanned by a three-dimensional subspace of $\mathfrak{sp}(4, \mathbb{C})$ and a fourth generator contained in the diagonally embedded copy of $\mathfrak{so}(2, \mathbb{C})$ in the kernel described above.

- **Rank 2**: Rank two nilpotent supercharges exist if $k \geq 4$, and this is the maximal possible rank. There are two orbits in the space of rank two nilpotent supercharges. We can use the $\mathfrak{so}(5, \mathbb{C})$ action to scale the values of c, c' in Proposition 5.1 to zero, so we are studying the orbits in the space of pairs of orthogonal non-zero null vectors in W_k.

Proposition 5.4. *If $k \geq 4$, there is a one parameter family of* $\mathrm{SO}(k,\mathbb{C})$ *orbits in the space of pairs of orthogonal non-zero null vectors in W_k:*

$$\mathrm{Orb}_x = \{A \cdot ((1,i,0,\dots,0),(x,ix,1,i,0,\dots,0))\colon A \in \mathrm{SO}(k,\mathbb{C})\}$$

where $x \in \mathbb{C}$.

Proof. Given a pair (w_1, w_2) of orthogonal null-vectors, we may use the transitive $\mathrm{SO}(k,\mathbb{C})$ action to pass to a point in the orbit of the form $((1,i,0,\dots,0),w)$, where $w \neq 0$ is orthogonal to $(1,i,0,\dots,0)$. So, this means that $w = x(1,i,0,\dots,0) + w'$, where $x \in \mathbb{C}$ and $w' = (0,0,z_3,z_4,\dots,z_k)$, where not all the z_j are equal to zero.

Let us now consider the $\mathrm{SO}(k-2,\mathbb{C})$ action on the last $k-2$ coordinates. This action is transitive on the space of null vectors of the form $(0,0,z_3,z_4,\dots,z_k)$, so we may pass to a point in the same orbit where $z_3 = 1, z_4 = i$ and $z_j = 0$ for $j > 4$.

The subgroup $\mathrm{SO}(k-2,\mathbb{C})$ is precisely the stabilizer of the element $(1,i,0,\dots,0) \in W_k$, so these elements lie in different orbits for different values of $x \in \mathbb{C}$. $\square$

Example 5.5. In the $\mathcal{N} = 2$ case there is a single non-zero orbit of nilpotent supercharges given by elements $\mathcal{Q} = Q \otimes w$ where Q is a non-zero element of V_4 and w is a non-zero null vector in W_2. A representative point is given by $Q = (1,0,0,0)$ and $w = (1,i)$. The kernel $\mathfrak{z}_{\mathcal{Q}}$ is given by the subalgebra

$$\mathfrak{p} = \left\{ \begin{pmatrix} a & b & c & d \\ 0 & e & d & f \\ 0 & 0 & -a & 0 \\ 0 & g & -b & -e \end{pmatrix} \right\} \subseteq \mathfrak{sp}(4,\mathbb{C}) \oplus \mathfrak{so}(2,\mathbb{C}),$$

embedded via the map $X \mapsto (X, -a) \in \mathfrak{sp}(4,\mathbb{C}) \oplus \mathfrak{so}(2,\mathbb{C})$. The image $\mathfrak{b}_{\mathcal{Q}} \subseteq \mathfrak{z}_{\mathcal{Q}}$ is spanned by the four elements

$$\left\{ \left(\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, 0 \right), \left(\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, 0 \right), \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, -1 \right), \left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, 0 \right) \right\}$$

in $\mathfrak{sp}(4,\mathbb{C}) \oplus \mathfrak{so}(2,\mathbb{C})$.

Example 5.6. Let us describe the superconformal nilpotence variety in the case where $k = 2$. So the auxiliary space W_2 is two-dimensional, and the null cone Null_2 can be identified with the the singular quadric $\{x^2 + y^2 = 0\} \subseteq \mathbb{C}^2$. According to Proposition 5.1, the complex nilpotence variety consists of points $[(w,w'),(c,c')]$ in $\mathrm{Null}_2^2 \times (\mathbb{CP}^1)^2$ modulo the relation identifying c with 0 if $w = 0$ and c' with 0 if $w' = 0$, subject to the condition that w and w' are orthogonal. The following claim follows immediately.

Proposition 5.7. *We can identify*

$$\begin{aligned} \mathcal{N}_{\mathbb{C},2} &\cong \Big(\big(\{(x,ix,x',ix') \in \mathbb{C}^4\} \cup_{\{0\}} \{(x,-ix,x',-ix') \in \mathbb{C}^4\} \big) \times (\mathbb{CP}^1)^2 \Big) / \sim \\ &\cong ((\mathbb{C}^2 \cup_{\{0\}} \mathbb{C}^2) \times (\mathbb{CP}^1)^2)/\sim, \end{aligned}$$

where $\sim$ collapses the first factor of $(\mathbb{CP}^1)^2$ over $\{0\} \cup_{\{0\}} \mathbb{C}^2$ and the second factor of $(\mathbb{CP}^1)^2$ over $\mathbb{C}^2 \cup_{\{0\}}$ $\{0\}$.

6. 5D AND 6D SUPERCONFORMAL TWISTS

6.1. **5d Twists.** In dimension 5 there is a unique complex superconformal algebra, namely the exceptional super Lie algebra $\mathfrak{A} = \mathfrak{f}(4)$. This algebra contains the 5d $\mathcal{N} = 1$ supersymmetry algebra as a subalgebra. Indeed, the even part of $\mathfrak{A}$ is given by

$$(\mathfrak{A})_0 = \mathfrak{so}(7, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}),$$

and the odd part of $\mathfrak{A}$ is given by

$$(\mathfrak{A})_1 = S \otimes W$$

where S is the 8-dimensional (Dirac) spinor representation of $\mathfrak{so}(7, \mathbb{C})$ and W is the 2-dimensional defining representation of $\mathfrak{sl}(2, \mathbb{C})$.

The bracket between two odd elements is given as follows. The 64-dimensional representation $S \otimes S$ of $\mathfrak{so}(7, \mathbb{C})$ decomposes as

$$\begin{aligned} S \otimes S &\cong \wedge^2(S) \oplus \mathrm{Sym}^2(S) \\ &\cong (V_7 \oplus \mathfrak{so}(7, \mathbb{C})) \oplus (\mathbb{C} \oplus U_{35}), \end{aligned}$$

where U_{35} is a 35-dimensional irreducible representation. In particular, there is an anti-symmetric equivariant linear map

$$F\colon S \otimes S \to \mathfrak{so}(7, \mathbb{C}),$$

and a symmetric equivariant linear map

$$\langle , \rangle \colon S \otimes S \to \mathbb{C}.$$

Using the notation Tr, red from Section 4.1, the bracket is given by

$$[Q \otimes w, Q' \otimes w'] = \big(F(Q \otimes Q') \cdot \mathrm{Tr}(w \otimes w'), \langle Q, Q' \rangle \cdot \mathrm{red}(w \otimes w')\big).$$

When we restrict $(\mathfrak{A})_1$ to a representation of $\mathfrak{so}(5, \mathbb{C}) \subseteq \mathfrak{so}(7, \mathbb{C})$ we do indeed obtain two copies of $S_{(5)} \otimes W$, where $S_{(5)}$ is the four-dimensional (Dirac) spinor representation of $\mathfrak{so}(5, \mathbb{C})$. That is, two copies of the spinors in the 5d $\mathcal{N} = 1$ supersymmetry algebra.

Let us compute the associated superconformal nilpotence variety.

Proposition 6.1. *The complex nilpotence variety for the 5d superconformal algebra is isomorphic to*

$$\mathcal{N}_{\mathbb{C}} \cong \{(Q_1, Q_2) \in S^2 \colon F(Q_1 \otimes Q_2) = 0 \text{ and } \langle Q_i, Q_j \rangle = 0, \text{ for any } i, j = 1, 2\}.$$

Proof. Let $\{w_1, w_2\}$ be a Darboux basis for W with respect to the trace pairing. A general element of $(\mathfrak{A})_1$ can be written as $Q = Q_1 \otimes w_1 + Q_2 \otimes w_2$ for some $(Q_1, Q_2) \in S^2$. Let us compute the bracket $[Q, Q]$. According to our description of the bracket, the $\mathfrak{so}(7, \mathbb{C})$ summand of $[Q, Q]$ is

$$\begin{aligned} [Q, Q]_{\mathfrak{so}(7,\mathbb{C})} &= \sum_{i,j=1}^{2} F(Q_i \otimes Q_j) \mathrm{Tr}(w_i \otimes w_j) \\ &= 2F(Q_1 \otimes Q_2). \end{aligned}$$

The $\mathfrak{sl}(2,\mathbb{C})$ summand of $[Q,Q]$ is

$$\begin{aligned}[Q,Q]_{\mathfrak{sl}(2,\mathbb{C})} &= \sum_{i,j=1}^{2} \langle Q_i, Q_j\rangle \mathrm{red}(w_i \otimes w_j) \\ &= \langle Q_1, Q_1\rangle e + \langle Q_2, Q_2\rangle f + \langle Q_1, Q_2\rangle h\end{aligned}$$

Where $\{e,f,h\}$ is the standard basis for $\mathfrak{sl}(2,\mathbb{C})$. Therefore $[Q,Q]=0$ if and only if

$$F(Q_1 \otimes Q_2) = \langle Q_1, Q_1\rangle = \langle Q_2, Q_2\rangle = \langle Q_1, Q_2\rangle = 0.$$

□

6.2. **6d Twists.** In dimension 6 there exist $\mathcal{N}=(k,0)$ superconformal algebras for all positive integers k. We can identify superconformal algebras in dimension 6 with the simple super Lie algebras $\mathfrak{osp}(8|2k)$ using the triality isomorphism to identify the defining representation of $\mathfrak{so}(8,\mathbb{C})$ with a (Weyl) spinor representation. So, writing $\mathfrak{A}_k = \mathfrak{osp}(8|2k)$ for the $\mathcal{N}=k$ superconformal algebra we can identify the even part with

$$(\mathfrak{A}_k)_0 = \mathfrak{so}(8,\mathbb{C}) \oplus \mathfrak{sp}(2k,\mathbb{C}),$$

and the odd part with

$$(\mathfrak{A}_k)_1 = V_8 \otimes W_{2k} \cong S_+ \otimes W_{2k},$$

where V_8 is the defining representation of $\mathfrak{so}(8,\mathbb{C})$, which is transformed into the Weyl spinor representation S_+ by the triality automorphism of $\mathfrak{so}(8,\mathbb{C})$. As before, W_{2k} denotes the $2k$-dimensional defining representation of $\mathfrak{sp}(2k,\mathbb{C})$.

The Lie bracket between two odd elements of $\mathfrak{A}_k$ is defined as follows, in a similar way to the definitions in other dimensions given above.

$$[v \otimes w, v' \otimes w'] = (v \wedge v' \omega(w,w'), g(v,v')F(w \otimes w')),$$

where $F\colon W_{2k} \otimes W_{2k} \to \mathfrak{sp}(2k,\mathbb{C})$ is the canonical projection.

Remark 6.2. It is interesting to note that the $\mathcal{N}=(2,0)$ complex superconformal algebra in six dimensions and the $\mathcal{N}=k$ complex superconformal algebra in three dimensions are actually isomorphic; the roles of the conformal and R-symmetry transformations are interchanged. Both are isomorphic to the simple super Lie algebra $\mathfrak{osp}(8|4,\mathbb{C})$. In particular in the $\mathcal{N}=(2,0)$ case the classification of complex orbits for the even part of the complexified superconformal group is identical to the classification from Section 5.

7. Twisted Observables and Factorization Algebras

There are many ways to approach QFT mathematically, and here we will use a framework that captures the spacetime-dependence of operators and operator products via operads. In fact, we will use two approaches drawn from [CG17; CG21], freely referencing those books below. The first approach uses prefactorization algebras and is highly flexible, allowing one to work on arbitrary manifolds and with operators whose support is an arbitrary open subspace. A key result of [CG21] is that a Lagrangian field theory on the manifold M naturally produces a prefactorization algebra on M, so we have a wealth of examples. (That book also examines the perturbative quantization of such theories in Euclidean signature and shows these also yield prefactorization algebras.) The second approach restricts to $M = \mathbb{R}^n$ and to open disks therein. It matches closely with standard manipulations in the physics literature and has the additional virtue that there are convenient comparison results letting one extract more familiar algebraic structures — like vertex algebras or associative algebras — efficiently. It may be more intuitive for most readers.

Both approaches let us make mathematical statements that are sharp versions of the discussion in Section 3. Our discussion here amounts, primarily, to mimicking ideas and results from [CG17; ES19], which examine chiral conformal and supersymmetric theories, respectively. The arguments carry over straightforwardly to superconformal theories, so we often state the result and merely indicate what needs to be changed.

7.1. **Superconformal prefactorization algebras.** We will follow the conventions of [CG17; CG21], so our prefactorization algebras take values in cochain complexes of differentiable vector spaces. This setting allows us to talk about smooth action of a Lie group on a prefactorization algebra, with a Lie algebra acting by differential operators. The experienced reader, however, will see how the idea of such symmetry could be ported to other settings.

Definition 7.1. *A* conformal (pre)factorization algebra *is a (pre)factorization algebra on a compactification $C(\mathbb{R}^{p,q})$ that is smoothly equivariant for the action of the conformal group* $\mathrm{Conf}(p,q)$*.*

Similarly, we can work in $\mathbb{Z}/2$-graded cochain complexes of differentiable vector spaces for the superconformal case.

Definition 7.2. *A* superconformal (pre)factorization algebra *is a conformal (pre)factorization algebra on a compactification $C(\mathbb{R}^{p,q})$ for which the Lie algebra action of $\mathfrak{conf}(p,q)$ is extended to an action of a superconformal algebra $\mathfrak{sconf}(p,q,|S)$.*

A conformal twist $\mathcal{Q}$ for $\mathfrak{sconf}(p,q|S)$ then determines a deformation of such a superconformal factorization algebra.

Definition 7.3. *Given a $\mathfrak{sconf}(p,q|S)$-superconformal prefactorization algebra $\mathcal{F}$ on $C(\mathbb{R}^{p,q})$ and a conformal twist $\mathcal{Q}$, let $\mathcal{F}^{\mathcal{Q}}$ denote the $\mathcal{Q}$-*twisted *prefactorization algebra with values in $\mathbb{Z}/2$-graded cochain complexes of differentiable vector spaces whose underlying super vector space are those of $\mathcal{F}$ but whose differential is $\mathrm{d}_{\mathcal{F}} + \mathcal{Q}$.*

This construction is a version of "adding $\mathcal{Q}$ as a BRST operator."

We note the following, which follows by inspection (at least for the reader of sections 3.7 and 4.8 of [CG17] or of section 2 of [ES19]).

Lemma 7.4. *The groups $Z_{\mathcal{Q}}$ and $B_{\mathcal{Q}}$ act smoothly-equivariantly on $\mathcal{F}^{\mathcal{Q}}$ by restricting the action of* $\mathrm{SConf}(p,q)_0$ *on $\mathcal{F}$. Moreover, the dg Lie algebra $(\mathfrak{sconf}(p,q|S), \mathcal{Q})$ acts by derivations on $\mathcal{F}^{\mathcal{Q}}$.*

This lemma already provides a precise encoding of much of the discussion in section 3. It says, for instance, that for any element $z \in Z_{\mathcal{Q}}$ and for any disjoint open sets U_1, U_2 inside a larger open set $V \subset M = C(\mathbb{R}^{p,q})$, there is a commuting diagram

$$\begin{tikzcd} \mathcal{F}^{\mathcal{Q}}(U_1) \otimes \mathcal{F}^{\mathcal{Q}}(U_2) \arrow[r] \arrow[d, "\cong"] & \mathcal{F}^{\mathcal{Q}}(V) \arrow[d, "\cong"] \\ \mathcal{F}^{\mathcal{Q}}(zU_1) \otimes \mathcal{F}^{\mathcal{Q}}(zU_2) \arrow[r] & \mathcal{F}^{\mathcal{Q}}(zV) \end{tikzcd}$$

where zU_1, for instance, denotes the translation of U_1 by z. (See definition 3.7.1 of [CG17].) If we take V to be the whole manifold and pick a conformal state (or expected value map) $\langle - \rangle : \mathcal{F}^{\mathcal{Q}}(M) \to \mathbb{C}$, this equivariance implies

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \langle \mathcal{O}_1(zx_1)\mathcal{O}_2(zx_2)\rangle$$

where $x_i \in U_i$ and $\mathcal{O}_i$ is a point-supported observable.

We will discuss the possibility of restricting to observables on a submanifold of $C(\mathbb{R}^{p,q})$ in section 7.3 below.

7.2. **Algebras of operator products.** We overview one way to encode the most important operators and their products for a field theory on the manifold $\mathbb{R}^n$. The essential idea is that one specifies the operators supported in any disk

$$D_r(x) = \{y \in \mathbb{R}^n : \ |y - x| < r\}$$

and how operators from disjoint disks multiply to give operators on a bigger disk. We assume that the spaces of operators varies smoothly as one adjusts the location and radius of the disks, and we assume that operator product varies smoothly in those parameters as well. More explicitly, let $\mathrm{Disk}_{r_1,\ldots,r_k;R}$ denote the open subset of $\mathbb{R}^{(k+1)n}$ given by points $(x_1, \ldots, x_k; y)$ such that the disks $D_{r_i}(x_i)$ are pairwise disjoint but each sits inside $D_R(y)$. Let $\mathcal{A}$, which we will call a *n-disks algebra*, denote the following data:

- For each positive real number r, there is a cochain complex A_r, which we view as the operators supported in the disk $D_r(0)$.
- For each point $p \in \mathrm{Disk}_{r_1,\ldots,r_k;R}$, there is a cochain map

  $$m_p \colon A_{r_1} \otimes \cdots \otimes A_{r_k} \to A_R,$$

  which we view as encoding the operator product. We require that m_p vary smoothly with the configuration p, i.e., there is a smooth function

  $$m \colon \mathrm{Disk}_{r_1,\ldots,r_k;R} \to \mathrm{Hom}(A_{r_1} \otimes \cdots \otimes A_{r_k}, A_R)$$

  from the space of configurations into the space of "multiplications."
- These maps compose in a natural way based on embedding disks into bigger disks. To be precise, suppose we pick a point $p \in \mathrm{Disk}_{r_1,\ldots,r_k;R}$ and points $q_i \in \mathrm{Disk}_{s^i_1,\ldots,s^i_{j_i};r_i}$ for every i such that "outgoing" disk of radius r_i from q_i equals the incoming disk of radius $_i$ from p. Then we have the equation

  $$m_p \circ (m_{q_1} \otimes \cdots \otimes m_{q_k}) = m_P$$

  where P denotes the point in $\mathrm{Disk}_{s^1_1,\ldots,s^1_{j_1},s^2_1,\ldots s^k_{j_k};R}$ arising by "composing" the disks.

A detailed treatment of this notion appears in section 4.8 of [CG17] and in section 2 of [ES19].

Remark 7.5. A technical issue is to formulate in what sense m is a smooth function, which requires finding a setting where the codomain Hom has the necessary structure. In [CG17] it is shown that differentiable vector spaces provide a sufficient setting and it is shown that most topological vector spaces are differentiable vector spaces.

When a Lie group G acts smoothly by isometries on $\mathbb{R}^n$, it induces an action on the configuration spaces $\mathrm{Disk}_{r_1,\ldots,r_k;R}$, so one can talk about a n-disks algebra $\mathcal{A}$ being G-equivariant. Similarly, one can formulate how a dg Lie algebra can act by derivations on $\mathcal{A}$.

Definition 7.6. *An n-disks algebra $\mathcal{A}$ with values in $\mathbb{Z}/2$-graded cochain complexes of differentiable vector spaces is* superconformal *if $\mathfrak{sconf}(p,q|S)$ acts by derivations on $\mathcal{A}$, where $n = p + q$.*

A conformal twist $\mathcal{Q}$ for $\mathfrak{sconf}(p,q|S)$ then determines a deformation of such a superconformal disks-algebra.

Definition 7.7. *Given a $\mathfrak{sconf}(p,q|S)$-superconformal $p+q$-disks algebra $\mathcal{A}$ and a conformal twist $\mathcal{Q}$, let $\mathcal{A}^{\mathcal{Q}}$ denote the $\mathcal{Q}$*-twisted *n-disks algebra whose underlying super vector space are those of $\mathcal{A}$ but whose differential is $\mathrm{d}_{\mathcal{A}} + \mathcal{Q}$, where $n = p + q$.*

A natural source of examples is to take a (well-behaved) superconformal prefactorization algebra (or twist thereof) and ask what it assigns to disks inside $\mathbb{R}^{p,q}$. Lagrangian theories with superconformal symmetry, whether classical or quantum, thus give many examples.

7.3. **Obtaining vertex algebras or $\mathbb{E}_k$ algebras.** Under some reasonable hypotheses, one can obtain a vertex algebra or an $\mathbb{E}_k$ algebra (i.e., algebra over the little k-disks operad) from the observables of a field theory, whether encoded as a prefactorization algebra or as a disk-algebra. Precise results appear in Chapter 5 of [CG17] and Section 2 of [ES19]. Here we will state their consequences.

We will explain the essential format by an example. Let $\mathcal{Q}$ be a square-zero element of $\mathfrak{sconf}(p,q|S)$. Suppose that the stabilizer Z_Q contains a subgroup of $\mathrm{ISO}(p,q)$ that is isomorphic to $\mathrm{ISO}(p',q')$. For simplicity, suppose this subgroup is the isometries of an isometric embedding $j\colon \mathbb{R}^{p',q'} \hookrightarrow \mathbb{R}^{p,q}$. We can consider observables supported along this submanifold: define $j^*\mathcal{F}^{\mathcal{Q}}$ on $\mathbb{R}^{p',q'}$ by

$$j^*\mathcal{F}^{\mathcal{Q}}(U) = \lim_{U \subset V \subset \mathbb{R}^{p,q}} \mathcal{F}^{\mathcal{Q}}(V),$$

where V runs over open subsets of $\mathbb{R}^{p,q}$ and lim denotes the limit in the appropriate category (when working with cochain complexes, we mean an ∞-categorical limit and hence, in practice, a homotopy limit).

Corollary 2.30 of [ES19] gives two conditions that guarantee that $j^*\mathcal{F}^{\mathcal{Q}}$ determines an algebra over the little $p'+q'$-disks operad — or more colloquially, that the twist looks like a topological field theory along $\mathbb{R}^{p',q'}$. These conditions are

- the sub-Lie algebra of translations inside the Lie algebra of $\mathrm{ISO}(p',q')$ admit a $\mathcal{Q}$-potential and
- any inclusion of disks in $\mathbb{R}^{p',q'}$ determines a quasi-isomorphism of observables on the disks.

This second condition follows if dilation also admits a $\mathcal{Q}$-potential, in the following sense.

Definition 7.8. *Let $\mathfrak{h} \subseteq \mathfrak{sconf}(p+q|S,\mathbb{C})_0$ be a Lie subalgebra. A $\mathcal{Q}$-*potential *for $\mathfrak{h}$ is a linear subspace $\mathfrak{h}' \subseteq \mathfrak{sconf}(p+q|S,\mathbb{C})_1$ such that*

- $[\mathcal{Q}, -]$ *defines a linear isomorphism $\mathfrak{h}' \to \mathfrak{h}$.*
- $[\mathfrak{h}', \mathfrak{h}'] = 0$.
- $[\mathfrak{h}, \mathfrak{h}'] \subseteq \mathfrak{h}'$*, and as an $\mathfrak{h}$-representation $\mathfrak{h}'$ is isomorphic to the adjoint representation.*

Corollary 2.39 applies to the case that $q' = 0$. It says that one gets a *framed* $\mathbb{E}_{p'}$-algebra if, furthermore, the subalgebra $\mathfrak{so}(p')$ admits a $\mathcal{Q}$-potential. Such a situation appears not infrequently in Euclidean signature. (An analog of this result should hold for mixed signature as well, using the techniques of [ES19].)

References

[Bee+15a] Christopher Beem, Wolfger Peelaers, Leonardo Rastelli, and Balt C. van Rees. "Chiral algebras of class S". *Journal of High Energy Physics* 2015.5 (May 2015). URL: `https://doi.org/10.1007%2Fjhep05%282015%29020`.

[Bee+15b] Christopher Beem et al. "Infinite Chiral Symmetry in Four Dimensions". *Communications in Mathematical Physics* 336.3 (Jan. 2015), pp. 1359–1433. URL: `https://doi.org/10.1007%2Fs00220-014-2272-x`.

[BPR17] Christopher Beem, Wolfger Peelaers, and Leonardo Rastelli. "Deformation Quantization and Superconformal Symmetry in Three Dimensions". *Communications in Mathematical Physics* 354.1 (Feb. 2017), pp. 345–392. URL: `https://doi.org/10.1007%2Fs00220-017-2845-6`.

[BR18] Christopher Beem and Leonardo Rastelli. "Vertex operator algebras, Higgs branches, and modular differential equations". *Journal of High Energy Physics* 2018.8 (Aug. 2018). URL: `https://doi.org/10.1007%2Fjhep08%282018%29114`.

[BRR15] Christopher Beem, Leonardo Rastelli, and Balt C. van Rees. "$\mathcal{W}$ symmetry in six dimensions". *JHEP* 05 (2015), p. 017. arXiv: `1404.1079 [hep-th]`.

[CG17] Kevin Costello and Owen Gwilliam. *Factorization algebras in quantum field theory. Vol. 1*. Vol. 31. New Mathematical Monographs. Cambridge University Press, Cambridge, 2017, pp. ix+387.

[CG21] Kevin Costello and Owen Gwilliam. *Factorization algebras in quantum field theory. Vol. 2*. Vol. 41. New Mathematical Monographs. Cambridge University Press, Cambridge, 2021, pp. xiii+402. URL: `https://doi.org/10.1017/9781316678664`.

[ES19] Chris Elliott and Pavel Safronov. "Topological twists of supersymmetric algebras of observables". *Comm. Math. Phys.* 371.2 (2019), pp. 727–786. arXiv: `1805.10806 [math-ph]`.

[ESW21] Richard Eager, Ingmar Saberi, and Johannes Walcher. "Nilpotence varieties". *Annales Henri Poincare* 22.4 (2021), pp. 1319–1376. arXiv: `1807.03766 [hep-th]`.

[ESW22] Chris Elliott, Pavel Safronov, and Brian R. Williams. "A taxonomy of twists of supersymmetric Yang–Mills theory". *Selecta Math.* 28.4 (2022), p. 73. arXiv: `2002.10517 [math-ph]`.

[FHW06] Gregor Fels, Alan Huckleberry, and Joseph A. Wolf. *Cycle spaces of flag domains*. Vol. 245. Progress in Mathematics. A complex geometric viewpoint. Birkhäuser Boston, Inc., Boston, MA, 2006, pp. xx+339. URL: `https://doi.org/10.1007/0-8176-4479-2`.

[Nah78] W. Nahm. "Supersymmetries and their Representations". *Nucl. Phys. B* 135 (1978), p. 149.

[Sch08] M. Schottenloher. *A mathematical introduction to conformal field theory*. Second. Vol. 759. Lecture Notes in Physics. Springer-Verlag, Berlin, 2008, pp. xvi+249.

[Shn88] Steven Shnider. "The superconformal algebra in higher dimensions". *Lett. Math. Phys.* 16.4 (1988), pp. 377–383. URL: `https://doi-org.amherst.idm.oclc.org/10.1007/BF00402046`.

[Wit88a] Edward Witten. "Topological Quantum Field Theory". *Commun. Math. Phys.* 117 (1988), p. 353.

[Wit88b] Edward Witten. "Topological quantum field theory". *Communications in Mathematical Physics* 117.3 (1988), pp. 353–386.

[Wol69] Joseph A Wolf. "The action of a real semisimple group on a complex flag manifold. I: Orbit structure and holomorphic arc components". *Bulletin of the American Mathematical Society* 75.6 (1969), pp. 1121–1237.