## Pinsky 1.4.10 (d)

Note that

$$\sum_{n=1}^{\infty} \frac{1}{(a^2-n^2)^2} = \frac{1}{2}\left(\sum_{n=-\infty}^{\infty} \frac{1}{(a^2-n^2)^2} - \frac{1}{a^4}\right)$$

so to compute $\sum_{n=1}^{\infty} \frac{1}{n^4}$ it suffices to compute the limit of the right hand side above as $a \to 0$. We do this using the formula from the previous part of the problem:

$$\begin{aligned}
\frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{1}{(a^2-n^2)^2} - \frac{1}{a^4}\right) &= \frac{1}{2}\left(\frac{\pi}{2a^2\sin^2(\pi a)}\int_{-\pi}^{\pi}\cos^2(ax)dx - \frac{1}{a^4}\right)\\
&= \frac{1}{2}\left(\frac{\pi}{2a^2\sin^2(\pi a)}\left(\frac{\sin(2\pi a)}{2a} + \pi\right) - \frac{1}{a^4}\right)\\
&= \frac{\pi a\sin(2\pi a) + 2\pi^2 a^2 - 4\sin^2(\pi a)}{8a^4\sin^2(\pi a)}.
\end{aligned}$$

To compute the limit of this expression as $a \to 0$ we expand the numerator as a Taylor series in $a$. Since $\frac{a^k}{a^4\sin^2(\pi a)} \to 0$ as $a \to 0$ if $k > 6$, we can ignore the terms in the Taylor series of the numerator of degree greater than 6.

$$\begin{aligned}
\pi a\sin(2\pi a) + 2\pi^2a^2 - 4\sin^2(\pi a) &= \pi a\left(2\pi a - \frac{8\pi^3a^3}{3!} + \frac{32\pi^5a^5}{5!}\right) + 2\pi^2a^2 - 4\left(\pi a - \frac{\pi^3a^3}{3!} + \frac{\pi^5a^5}{5!}\right)^2 + O(a^7)\\
&= (2\pi^2 + 2\pi^2 - 4\pi^2)a^2 + \left(\frac{-8\pi^4}{3!} + \frac{8\pi^4}{3!}\right)a^4 + \left(\frac{32\pi^6}{5!} - \frac{8\pi^6}{5!} - \frac{4\pi^6}{(3!)^2}\right)a^6 + O(a^7)\\
&= \frac{8}{90}\pi^6a^6 + O(a^7).
\end{aligned}$$

So we're left with only an $a^6$ term.[1] Plugging this back into our fraction, the sum we want is given by the limit

$$\begin{aligned}
\lim_{a\to 0}\frac{\frac{8}{90}\pi^6a^6}{8a^4\sin^2(\pi a)} &= \frac{\pi^4}{90}\lim_{a\to 0}\left(\frac{(\pi a)}{\sin(\pi a)}\right)^2\\
&= \frac{\pi^4}{90}\times(1)^2 = \frac{\pi^4}{90}
\end{aligned}$$

as required.

[1] Since the expression we're trying to compute converges to the sum $\sum_{n=1}^{\infty}\frac{1}{n^4}$, which is finite by – for instance – the ratio test, the potentially divergent terms in the numerator of degree less than 6 all had to cancel: we didn't really need to compute them.