Background Notation

Let V be a real 4-dimensional vector space. There is an exceptional isomorphism $\mathfrak{so}(V) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{sl}(S_+) \oplus \mathfrak{sl}(S_-)$, where $S_\pm$ are complex 2-dimensional vector spaces. Under this isomorphism the fundamental representation $V \otimes \mathbb{C}$ is isomorphic to the tensor product $S_+ \otimes S_-$. We write γ for the $\mathbb{R}$-linear map $V \to S_+ \otimes S_-$.

Setup

Consider the double complex

$$C^\infty(V) \otimes \left(\bigoplus_{i+j=\bullet} \Omega^{0,i}(\mathbb{P}(S_+); (\mathcal{O}(-2) \otimes \wedge^j(\mathcal{O}(1) \otimes S_-)) \right)$$

whose vertical (i) differential is the $\overline{\partial}$ operator, and whose horizontal (j) differential we'll write as $d_{\text{hor}} = \partial_a e^a$: the definition will follow shortly. I'd like to compute the cohomology of the total complex here using the spectral sequence of the double complex.

The horizontal differential is defined as follows. Choose a basis $v^1, \dots v^4$ for V, and work in index notation. Write ∂_a for $\frac{\partial}{\partial v^a}$. The global sections of $\mathcal{O}(1) \otimes S_-$ on $\mathbb{P}(S_+)$ are isomorphic to $S_+ \otimes S_-$, so there's a natural map

$$\gamma \colon V \to H^0(\mathbb{P}(S_+); \mathcal{O}(1) \otimes S_-).$$

There's also a natural map

$$\mathrm{Sym}^\bullet(\Gamma(\mathbb{P}(S_+); \mathcal{O}(1) \otimes S_-)) \to \Gamma(\mathbb{P}(S_+); \mathrm{Sym}^\bullet(\mathcal{O}(1) \otimes S_-))$$

so the degree one operator $(-\bullet\gamma(v^a))$ on $\mathrm{Sym}^\bullet(H^0(\mathbb{P}(S_+); \mathcal{O}(1) \otimes S_-))$ yields a degree one operator on $H^0(\mathbb{P}(S_+); \mathrm{Sym}^\bullet(\mathcal{O}(1) \otimes S_-))$ which we denote by e^a. This defines the operator $d_{\text{hor}} = \partial_a e^a$ by the usual summation convention.

Computation

So let's start computing the spectral sequence. The E_1 page is computed by taking the $\overline{\partial}$ differential, which is to say the cohomology groups of our sheaves on $\mathbb{P}(S_+)$. The result is

$$C^\infty(V) \otimes \begin{pmatrix} 0 & 0 & H^0(\mathbb{P}(S_+); \mathcal{O}) \\ H^1(\mathbb{P}(S_+); \mathcal{O}(-2)) & 0 & 0 \end{pmatrix}$$

and both non-zero cohomology groups are isomorphic to $\mathbb{C}$. There is no d_1, but there is an opportunity for a $d_2 \colon C^\infty(V) \otimes H^1(\mathbb{P}(S_+); \mathcal{O}(-2)) \to C^\infty(V) \otimes H^0(\mathbb{P}(S_+); \mathcal{O})$.

What is this differential exactly? Well, fix a volume form on $\mathbb{P}(S_+)$ representing a non-zero cohomology class in $H^1(\mathbb{P}(S_+); \mathcal{O}(-2))$. In other words this is a non-vanishing form $\alpha \in \Omega^{0,1}(\mathbb{P}(S_+); \mathcal{O}(-2)) \cong \Omega^{1,1}(\mathbb{P}(S_+))$. We start with the element $f \otimes \alpha$ in bidegree $(1,0)$ and take its horizontal differential $d_a f \otimes e^a \alpha$. The form $e^a\alpha$ is a global section of

$$\Omega^{1,1}(\mathbb{P}(S_+); \mathcal{O}(1) \otimes S_-) \cong \Omega^{0,1}(\mathbb{P}(S_+); \mathcal{O}(-1) \otimes S_-)$$

so necessarily $\overline{\partial}$-exact because $\overline{\partial} \colon \Omega^{0,0}(\mathbb{P}(S_+); \mathcal{O}(-1)) \to \Omega^{0,1}(\mathbb{P}(S_+); \mathcal{O}(-1))$ is an isomorphism. Therefore we can take the preimage $\overline{\partial}^{-1}(e^a\alpha)$. Now, to conclude we take the horizontal differential again, to end up with

$$\partial_a \partial_b f \otimes e^b(\overline{\partial}^{-1}(e^a\alpha)),$$

and finally we take its cohomology class in $H^0(\mathbb{P}(S_+); \mathcal{O})$.

My expectation (from the context in which this calculation arises) is that the following is true:

Claim. The cohomology class $[e^b(\overline{\partial}^{-1}(e^a\alpha))]$ is $\delta^{ab}[c]$ for some fixed element $[c] \in H^0(\mathbb{P}(S_+);\mathcal{O})$ (it'll be different for different choice of volume form α. Therefore the differential d_2 is the Laplacian operator.

Let's try to check this. Let $\alpha = \omega_{\text{FS}}$, the Fubini-Study Kähler form. Choose coordinate charts z and w for $\mathbb{P}(S_+)$, so that on $\mathbb{C}^\times$ we have $w = z^{-1}$. In terms of the coordinate z, the Fubini-Study form can be written as

$$\omega_{\text{FS}} = f(z)dz \wedge d\overline{z} = \frac{1}{1+z\overline{z}}dz \wedge d\overline{z}.$$

Suppose the coordinates v^a on V are chosen so that their images under γ are monomials in $S_+ \otimes S_-$, say $s^i \otimes t^j$ where $i,j = 1,2$. As sections of the bundle $\mathcal{O}(1) \otimes S_-$ on $\mathbb{P}(S_+)$, in terms of the coordinate z, in the standard local trivialization of $\mathcal{O}(1)$, these sections become $z \otimes t^j$ and $1 \otimes t^j$. In terms of the coordinate w they become $1 \otimes t_j$ and $w \otimes t^j$ respectively. We write a general section in the z chart as $a_i t^i + b_j t^j z$.

$$\begin{aligned}
[(a_i't^i + b_j't^jz)\overline{\partial}^{-1}(a_it^i + b_jt^jz)\alpha] &= [(a_i't^i + b_j't^jz)\overline{\partial}^{-1}\frac{a_it^i + b_jt^jz}{1+z\overline{z}}dz \wedge d\overline{z}] \\
&= [(a_i't^i + b_j't^jz)\left(\int \frac{a_it^i + b_jt^jz}{1+z\overline{z}}d\overline{z}\right)dz] \\
&= [((a_1'a_2 - a_2'a_1) + (a_1'b_2 - a_2'b_1 + b_1'a_2 - b_2'a_1)z + (b_1'b_2 - b_2'b_1)z^2)\frac{\log(1+z\overline{z})}{z}dz] \\
&= \int_{\mathbb{C}}((a_1'a_2 - a_2'a_1) + (a_1'b_2 - a_2'b_1 + b_1'a_2 - b_2'a_1)z + (b_1'b_2 - b_2'b_1)z^2)\frac{\log(1+z\overline{z})}{z(1+z\overline{z})}dz \wedge d\overline{z} \\
&=
\end{aligned}$$

where on the third line we applied the projection $(\mathcal{O}(1) \otimes S_-)^{\otimes 2} \to \wedge^2(\mathcal{O}(1) \otimes S_-)$.