Fourier Duality in Abelian Quantum Gauge Theory

Chris Elliott

March 24th, 2014

1 Introduction

One often sees mathematical conjectures motivated by dualities in quantum field theory. The usual format is that two mathematical objects are purported to 'arise' from physical quantities in two quantum field theories, so that a physical 'duality' between the theories induces an equivalence between the mathematical objects. While such arguments are only heuristic, one might hope to obtain *proofs* of conjectures arising in this way by providing suitable axiomatic descriptions of the quantum field theories such that the mathematical objects in question arise from the theories by some functorial procedure, and constructing the duality as a morphism between the theories inducing an equivalence on the objects in question.

For the dualities I'm interested in (things like S- and T-dualities) this is still generally inaccessible. The dualities are of a non-perturbative nature, and no suitable non-perturbative axiomatic constructions of general field theories exists. However, in the special case where the theories are *free* we can construct a very general family of theories non-perturbatively, and construct a correspondence of theories by means of a Fourier transform. In this talk I'll explain this construction.

To summarise briefly what we're going to do, I want to start from the data defining a *classical field theory* on a manifold X favoured by physicists: a sheaf of *fields* on X and an *action functional*. The idea is that classical physical states are those that extremise the action. I'll discuss a procedure for *quantising* this data in the case where the theory is free (roughly meaning that the action is quadratic). The object obtained is a *factorisation algebra* describing physically observable quantities on all open sets in X, with a natural *quantum expectation value* map. Duality will then be a correspondence between the local observables (in two different theories) on each open set such that *incident* observables have the same expectation value.

My biggest motivating example is the work of Kapustin and Witten [KW06], which argues that the geometric Langlands conjecture is a consequence of a duality called S-duality. This is a duality that exchanges a maximally supersymmetric gauge theory with gauge group G in four dimensions with a theory with gauge group $G^\vee$, the *Langlands dual group*. Geometric Langlands is supposed to arise as an induced equivalence of the *categories of boundary conditions* on manifolds of form $\Sigma \times \mathbb{R} \times \mathbb{R}_{\geq 0}$ in suitable *topological twists* of these theories. In future work I intend to produce a supersymmetric enhancement of the abelian duality I'll describe today, and to check whether this recovers the abelian case of geometric Langlands, geometric class field theory.

2 Generalised Maxwell Theories

So let's discuss the specific problem I'm concerned with. The main motivating example is that of *abelian Yang-Mills*. Let $T = V/\Lambda$ be a rank r torus (so V is an r-dimensional real vector space and $\Lambda \subseteq V$ is a full rank lattice), and let $T^\vee = V^\vee/\Lambda^\vee$ be its dual torus.

Example 2.1. Let X be a compact oriented Riemannian 4-manifold. Yang-Mills theory with gauge group T has the space of principal T-bundles with connection as its fields. This space admits automorphisms by addition of exact V-valued 1-forms $A \mapsto A + d\phi$ (*gauge symmetry*), and we'd like to work with the quotient space. Note that the fields are defined *locally*, that is we can consider fields on any open set $U \subseteq X$, and they glue to form a *sheaf*. In practice we work with these quotient spaces by defining a sheaf of simplicial groups, or equivalently a sheaf of cochain complexes in non-positive degrees, by the Dold-Kan correspondence. The complex describing our fields here is

$$\begin{aligned} &\bigoplus_P \left(\Omega^0(X;V)[1] \to \Omega^1(X;V)\right) \\ \cong &\Lambda[2] \to \Omega^0(X;V)[1] \to \Omega^1(X;V). \end{aligned}$$

A connection A on a principal T bundle admits a *curvature* form F_A, which is a closed 2-form with *integral periods*, i.e. cohomology class in $H^2(X;\Lambda) \leq H^2_{dR}(X)$. The *action functional* in Yang-Mills theory is the L^2-norm of the curvature

$$S_T(A) = \int_X F_A \wedge *F_A$$

where the wedge pairing includes a pairing between V and $V^\vee$. The space of classical solutions to the equations of motion, i.e. extrema of the action functional, is identified with the space of *harmonic* (closed and coclosed) 2-forms with integral periods. Again, this makes sense locally, even if the integral defining $S_T(A)$ doesn't converge.

Now, we see the first hints at abelian duality. There is an isomorphism between the space of classical solutions for gauge group T and for gauge group $T^\vee$ given by the Hodge star. Abelian duality is supposed to provide a *quantisation* of this simple observation, relating *quantum* observables for gauge group T and gauge group $T^\vee$. This is the simple abelian version of *S-duality*.

There's another example of the same flavour. On a Riemannian 2-manifold we can consider T-valued functions, which admit a derivative which is a 1-form with integral periods, and the action is the L^2-norm of this 1-form. We can ask for the same kind of quantum enhancement of the isomorphism on classical solutions given by the Hodge star, which is the simple abelian version of *T-duality*.

These examples fit into an infinite family of "higher abelian gauge theories", or "generalised Maxwell theories". On an n-manifold, for any $0 < p < n$ we can construct a theory whose fields are "T-(p-1)-bundles with connection"; objects whose curvature is a closed V-valued p-form with cohomology class in $H^p(X;\Lambda)$. Abelian duality will interchange observables in "Hodge dual" theories, that is

$$(p\text{-form theory with gauge group } T) \leftrightarrow ((n-p)\text{-form theory with gauge group } T^\vee)$$

coming from a "quantisation" of the classical isomorphism on solutions to the equations of motion given by the Hodge star. The fields are described by cochain complexes in non-positive degrees of form

$$\begin{aligned} \mathbb{Z}(p)_{\mathcal{D}}(U) = \quad & \mathbb{Z}[p] \hookrightarrow \Omega^0(U)[p-1] \xrightarrow{d} \Omega^1(U)[p-2] \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-1}(U), \\ \cong \quad & C^\infty(U, \mathbb{R}/\mathbb{Z})[p-1] \xrightarrow{id \log} \Omega^1(U)[p-2] \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-1}(U) \end{aligned}$$

where for simplicity I've set the rank r to 1 (I'll do this from now on). So abelian Yang-Mills is the case where $p = 2$, and the sigma model with torus target is the case where $p = 1$. The cochain complexes describe fields with symmetries, symmetries of the symmetries etc (so is a higher-stacky object).

Before we can prove any kind of duality, we'll have to explain how to produce a mathematical description of the local quantum observables in these theories (so we know what kind of objects we need to work with. For this we'll introduce the language of factorisation algebras.

3 Factorisation Algebras and Field Theory

The formalism of *factorisation algebras* gives a description of the *local observables* of a field theory. I'll introduce the basic definitions, and introduce a procedure for producing a factorisation algebra from the Lagrangian descriptions of field theories used by physicists. This procedure is due to Costello and Gwilliam [CG13] [Gwi12].

Definition 3.1. A *prefactorisation algebra* on a topological space X is a precosheaf $\mathcal{F}$ of cochain complexes of vector spaces in non-positive degrees equipped with S_k-equivariant isomorphisms

$$\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_k) \to \mathcal{F}(U_1 \sqcup \cdots \sqcup U_k)$$

for every collection $U_1, \ldots, U_k \subseteq X$ of disjoint open sets.

Roughly speaking, a *factorisation algebra* is a prefactorisation algebra where observables on an open set U are all determined by their values arbitrarily near a finite set of points. The precise definition won't be important for this talk.

Now, we can construct a factorisation algebra associated to a space of fields and an action functional. To an open set U we associate the algebra of functions on the *derived critical locus* of the action functional. The word "derived" is necessary here if we want to get a factorisation algebra rather than just a prefactorisation algebra: if we just take the ordinary critical locus locally we'll find it won't form a sheaf, which prevents functions on it from being a factorisation algebra.

There's a particularly nice way of describing this factorisation algebra which we call the *classical Batalin-Vilkovisky (BV) formalism*. It's a nice point of view because it's easy to figure out how to do deformation quantisation. Specifically, we'll describe a cochain complex of vector spaces which is supposed to be the shifted cotangent bundle $T^*[-1]\Phi(U)$ to the local fields on U (or a linearisation thereof), and add in a new differential so that algebraic functions on this cochain complex are algebraic functions on the derived critical locus. There's a nice story explaining why this works, but I don't have time to tell it now.

For our generalised Maxwell theories, let's skip ahead to the answer. The underlying graded vector space has form

$$\begin{aligned}\mathcal{O}(T^*[-1]\Phi_p(U)) &\cong \mathrm{Sym}(T_0\Phi(U)) \otimes \mathcal{O}(\Phi_p(U)) \\ &\cong \mathrm{Sym}\left(\Omega^*_{\leq p-1}(U)[p-2]\right) \otimes \mathcal{O}(\Phi_p(U))\end{aligned}$$

where $\Phi_p(U)$ is the complex of fields over U, and $T_0\Phi_p(U)$ is its tangent fibre at 0, which we compute to be the shifted truncated de Rham complex $\Omega^*_{lep-1}(U)[p-2]$. The differential is the sum of the internal differentials on $\Phi_p(U)$ and $T_0\Phi_p(U)$ with a new term encoding the action.

What is this term? Well, I'll describe it for general theories. As a dga, functions on the shifted cotangent bundle $T^*[-1]\Phi$ are the same as *polyvector fields* on Φ. Differentiating the action functional S yields a 1-form dS on Φ (and what's more, one can check that this 1-form is defined locally even when the action itself is not). This induces a differential on polyvector fields by contracting with dS

$$\iota_{dS} \colon \wedge^k \mathrm{Vect}(\Phi) \to \wedge^{k-1} \mathrm{Vect}(\Phi).$$

This operator is sometimes called the *classical BV operator*. Including this differential one ends up with a factorisation algebra on X we denote $\mathrm{Obs}^{\mathrm{cl}}(X)$.

Remark 3.2. In the case of the generalised Maxwell theory, the classical BV operator is given by a map $T_0\Phi_p(U)_c \to \mathcal{O}(\Phi_p(U))$, thus $\Omega^p(U)_c \to \mathcal{O}(\Phi_p(U))$. By composing with the curvature map it suffices to define the classical differential as the map $\Omega^{p-1}_c(U) \to \mathcal{O}(\Omega^p_{cl}(U)) \xrightarrow{F^*} \mathcal{O}(\Phi_p(U))$

$$\alpha \mapsto \left(\beta \mapsto \int_U \beta \wedge *d\alpha\right) \mapsto \left(A \mapsto \int_U F_A \wedge *d\alpha\right) = \iota_{dS}(\alpha).$$

Now, how do we quantise from this point of view? The classical observables admit a natural *shifted Poisson (P_0) structure*, since they're given by functions on the shifted cotangent bundle. Deformation quantisation from the BV point of view adds a new term to the differential we've constructed so far, described using this Poisson structure [Gwi12]. From now on we'll restrict attention to *free theories*. Informally, A classical Lagrangian field theory is free if the action functional is *quadratic*, so the derivative of the action functional is *linear*.

Definition 3.3. A classical Lagrangian field theory is called *free* if the classical differential ι_{dS} increases polynomial degrees by one. That is, if we filter $\mathcal{O}(\Phi)$ by polynomial degree and call the k^{th} filtered piece $F^k\mathcal{O}(\Phi)$, the operator ι_{dS} raises degree by one:

$$\iota_{dS}\colon \operatorname{Sym}^i(T_0\Phi[1]) \otimes F^j\mathcal{O}(\Phi) \to \operatorname{Sym}^{i-1}(T_0\Phi[1]) \otimes F^{j+1}\mathcal{O}(\Phi).$$

Define the (quantum) *BV operator* $D\colon \mathcal{O}(T^*[-1]\Phi(U)) \to \mathcal{O}(T^*[-1]\Phi(U))$ by extending an operator built from the Poisson bracket. Set D to be zero on $\mathcal{O}(\Phi(U))$, and to be given by the Poisson bracket in degree 1: $D = \{,\}\colon T_0\Phi(U) \otimes \mathcal{O}(\Phi(U)) \to \mathcal{O}(\Phi(U))$, i.e. the map we described above as "evaluation". We can then extend this to an operator on the whole complex of classical observables according to the formula

$$D(\phi \cdot \psi) = D(\phi) \cdot \psi + (-1)^{|\phi|}\phi \cdot D(\psi) + \{\phi, \psi\}.$$

An algebra with a differential D and Poisson bracket $\{,\}$ satisfying a formula like this is called a *Beilinson-Drinfeld algebra*, or *BD algebra*: Beilinson and Drinfeld constructed in [BD04] a family of operads over the formal disc whose fibre at the origin is the P_0 operad. The BD algebra structure given here is a description of an algebra for a generic fibre of the analogous family defined over all of $\mathbb{C}$ rather than just a formal neighbourhood of the origin.

3.1 Expectation Values

For (sufficiently nice) free quantum field theories we can give a natural description of quantum expectation values in the language we've introduced. These can be computed by means of Feynman path integrals, which are well-behaved because the theory is free (they're infinite-dimensional Gaussian integrals,and can be evaluated by a regularisation procedure). The key property abelian duality will satisfy is that dual observables have the same expectation value.

The expectation value of an observable $\mathcal{O}$ in a free theory (by which I mean a *degree zero* observable: a functional on the fields, which lives in the degree zero piece of the complex of observables), for instance a generalised Maxwell theory, admits three equivalent definitions:

1. It's the result of evaluating a Feynman path integral:
$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int_{H^0(\Phi(X))} e^{-S(\phi)}\mathcal{O}(\phi) D\phi$$
where Z is the path integral $\int_{H^0(\Phi(X))} e^{-S(\phi)} D\phi$, and where the path integral can be rigorously defined by a *regularisation* procedure. For generalised Maxwell theories we've integrating over a space of forms on a compact manifold, and I can define the integral by filtering this space via the Hodge filtration (by sums of eigenspaces of the Laplacian) and taking a limit.
2. It's the result of a *Feynman diagram* calculation. This is a combinatorial procedure encoding the path integral above, computed as a sum over certain types of graphs.
3. It's a natural homological construction coming from the complex of observables. Globally, this complex admits a canonical quasi-isomorphism to the ground field $\mathbb{R}$, and we can take a local observable, extend it to a global observable using the precosheaf structure, then apply this quasi-isomorphism to obtain a number.

All of these rely on a condition being satisfied, which is that the Gaussian $e^{-S(\phi)}$ is *non-degenerate*. There's a slightly technical point here; in general the generalised Maxwell theories *do not* satisfy this condition unless the p^{th} de Rham cohomology of X vanishes. For this (and other) reasons we prefer to compute expectation values in a related theory I call the *closed p-form theory*, which has fields the complex

$$\Omega^0(U)[p] \to \Omega^1(U)[p-1] \to \cdots \to \Omega^p_{cl}(U)$$

and action simply the L^2-norm. Any observable in the closed p-form theory gives an observable in the generalised Maxwell theory by precomposing with the curvature map.

4 Fourier Duality

Now, at last we come to the main point of the talk, the notion of *Fourier duality* for observables in our theories. The Fourier dual is actually described not for the generalised Maxwell theory, or for the closed p-form theory, but for the theory where the fields are *all* p-forms. We can restrict any observables in such a theory to an observable in the closed p-form theory, so a map between observables in the p-form theory gives a *correspondence* between observables in the closed p-form theory, or between observables in the generalised Maxwell theory.

We might try to produce a functional integral definition of the Fourier dual of a local observable, by setting

$$\widetilde{\mathcal{O}}(\widetilde{\alpha})e^{-\widetilde{S}(\widetilde{\alpha})} = \int_{H^0(\Phi(U))} \mathcal{O}(\alpha)e^{-S(\alpha)+i\langle\alpha,\widetilde{\alpha}\rangle} D\alpha$$

using the fact that the Fourier dual of a Gaussian polynomial is also a Gaussian polynomial for the dual Gaussian. This is reasonable when U is compact, but for general U we won't be able to use the regularisation techniques we usually use. As such, we define the Fourier dual using *Feynman diagrams*.

(TODO: Discuss how these are defined.)

- A propagator between linear observables $\mathcal{O}_i$ and $\mathcal{O}_j$ receives weight
$$\frac{1}{2R^2}\langle\mathcal{O}_i,\mathcal{O}_j\rangle = \frac{1}{2R^2}\int_X \mathcal{O}_i \wedge *\mathcal{O}_j.$$
- A source term attached to a linear observable $\mathcal{O}$ receives weight $i/2R^2$.

Now, we can state the main theorem. Here the subscript R refers to the radius of the gauge circle. In general we could just discuss a torus and its dual torus.

Theorem 4.1. Let $\mathcal{O}$ be a local observable in $\mathrm{Obs}^{\mathrm{q}}_{\Omega^p,R}(U)_0$, and let $\widetilde{\mathcal{O}} \in \mathrm{Obs}^{\mathrm{q}}_{\Omega^{n-p},1/2R}(U)_0$ be its Fourier dual observable. Let $r(\mathcal{O})$ and $r(\widetilde{\mathcal{O}})$ be the restrictions to local observables in $\mathrm{Obs}^{\mathrm{q}}_{\Omega^p_{cl}}(U)_0$ and $\mathrm{Obs}^{\mathrm{q}}_{\Omega^{n-p}_{cl}}(U)_0$ respectively. Then, computing the expectation values of $r(\mathcal{O})$ and $r(\widetilde{\mathcal{O}})$, we find

$$\langle r(\mathcal{O})\rangle_R = \langle r(\widetilde{\mathcal{O}})\rangle_{\frac{1}{2R}}.$$

We can expand this into a correspondence of factorisation algebras between dual generalised Maxwell theories, so that incident observables (i.e. the common images of elements from the middle of the correspondence) have the same expectation value. The factorisation algebra at the top is the silly factorisation algebra whose local sections are *just* the degree 0 observables in the free p-form theory, or isomorphically its dual $n-p$ form theory.

References

[BD04] Alexander Beilinson and Vladimir Drinfeld. *Chiral algebras*, volume 51 of *Colloquium Publications*. American Mathematical Society Providence, RI, 2004.

[CG13] Kevin Costello and Owen Gwilliam. *Factorization Algebras in Quantum Field Theory*. 2013. book in progress, available at `http://www.math.northwestern.edu/~costello/factorization.pdf`.

[Gwi12] Owen Gwilliam. *Factorization Algebras and Free Field Theory*. PhD thesis, Northwestern University, 2012.

[KW06] Anton Kapustin and Edward Witten. Electric-magnetic duality and the geometric Langlands program. *arXiv preprint hep-th/0604151*, 2006.

Department of Mathematics, Northwestern University
2033 Sheridan Road, Evanston, IL 60208, USA
celliott@math.northwestern.edu