

Abelian Duality for Generalised Maxwell Theories

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Abstract

We describe a construction of generalised Maxwell theories – higher analogues of abelian gauge theories – in the factorization algebra formalism of Costello and Gwilliam, allowing for analysis of the structure of local observables. We describe the phenomenon of abelian duality for local observables in these theories as a Fourier dual, relating observables in theories with dual abelian gauge groups and inverted coupling constants in a way compatible with the local structure. We give a description of expectation values in this theory and prove that duality preserves expectation values. Duality is shown to, for instance, interchange higher analogues of Wilson and 't Hooft operators.

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1 Introduction and Motivation

The aim of this paper is to give a detailed account of the phenomenon of *S-duality* in a very simple situation, as a duality between families of *free* quantum field theories, in a way allowing explicit understanding and computation of the local structure of the duality. This free version of S-duality was called *abelian duality* by Witten in his 1995 paper on the subject [Wit95], in which he addresses a relationship between the partition functions of abelian pure Yang-Mills theories on a compact 4-manifold. This relationship, and generalisations to higher degree, were also studied by Verlinde in [Ver95]. Analogues of this duality in lower dimensions were also known, for instance the *T-duality* between sigma models with dual torus targets on a 2-manifold, and a duality between sigma models and abelian gauge theories on a 3-manifold (these are described in [Wit99], explicit calculations of duality for the partition functions in three dimensions have been performed by Prodanov and Sen [PS00] and by Broda and Duniec [BD04b], and a detailed analysis of duality for more general observables was recently performed by Beasley [Bea14b, Bea14a]). These theories fit into an infinite family of theories whose fields model connections on higher torus bundles, models for which have been described by Freed [Fre00] using *ordinary differential cochains* to model the fields. The quantisations of these theories were further studied by Barbón [Bar95], who discussed abelian duality for the partition functions in higher degree theories, and Kelnhofer [Kel09], who explained how to compute the vacuum expectation values of gauge invariant observables in these theories using the language of ordinary differential cochains.

In the context of these generalised Maxwell theories, abelian duality refers to the following phenomenon for generalised Maxwell theories on an n -manifold X :

Theorem 1.1. To every gauge invariant observable \mathcal{O} in a degree k generalised Maxwell theory with gauge group T , we can produce a gauge invariant *dual* observable $\tilde{\mathcal{O}}$ in the theory of degree $n - k$ and with gauge group T^\vee such that the vacuum expectation values agree:

$$\langle \mathcal{O} \rangle_{R,T} = \langle \tilde{\mathcal{O}} \rangle_{\frac{1}{2R}, T^\vee}.$$

We do not require X to be compact, so abelian duality makes sense for *local* observables. As such we describe abelian duality as a relationship between a pair of *factorization algebras* modelling the local quantum observables in the quantum field theories. The duality is compatible with the structure maps in the factorization algebras, but does not extend to a morphism of factorization algebras because of an obstruction to defining a dual observable on non-contractible open sets, where observables may not be determined purely by the curvature of a field. Instead duality arises as a *correspondence* of factorization algebras.

Despite the theories being free, duality of observables is still a non-trivial phenomenon to investigate. The dual of a gauge-invariant observable can have a qualitatively different nature to the original observable. For instance, in

abelian Yang-Mills we verify that the dual of an abelian Wilson operator (a holonomy operator around a loop) is an 't Hooft operator (corresponding to imposing a singularity condition on the fields around a loop).

Now, we can outline the structure of the paper. In section 2 we begin by describing the general formalism we use to construct the factorization algebra of quantum observables, starting from a sheaf of fields and an action functional. This formalism (based on that of Batalin-Vilkovisky) was developed by Costello and Gwilliam [CG15] [Gwi12] as a formulation of quantisation techniques common in the physics literature (described for instance in Witten's expository note [Wit90]) as homological algebra. While we only use the theory for free theories here (their methods can also be used to study the perturbative parts of interacting theories), we do need to allow for spaces of fields that are neither linear nor connected, so we go over the formalism with a certain amount of care.

Having set up the abstract theory we construct the main objects of study in section 3: the factorization algebras of observables in *generalised Maxwell theories*. These are free quantum field theories whose fields model connections on higher principal torus bundles, whose action functional generalises the Yang-Mills action. These theories are closely related to simpler free theories – where the fields are just p -forms and the action is just the L^2 norm – by mapping a connection to its curvature. Observables of interest (such as Wilson and 't Hooft operators in abelian Yang-Mills) factor through the curvature, so in a sense “come from” this simpler theory. As such we can prove results about the expectation values of observables purely in the world of curvatures.

The theory of expectation values arises naturally from the factorization algebra formalism. In section 4 we describe how to abstractly define expectation values of gauge invariant observables by viewing the observables as living in a cochain complex with canonically trivialisable cohomology. To compute expectation values we use classical physical techniques: *Feynman diagrams* and *regularisation*. Since the theory is free these methods are very well-behaved, and encode results about convergent sequences of finite-dimensional Gaussian integrals.

Finally, in section 5 we introduce abelian duality for observables in our theories as a *Fourier dual*. This also admits a diagrammatic description, but we prove that duality preserves expectation values using a Plancherel's theorem at each regularised level. It is worth remarking that there are three different “levels” of factorization algebra necessary to make sense of Fourier duality for observables in the generalised Maxwell theory. The dual itself is defined for a theory where the fields consist of all p -forms, but at this level duality doesn't preserve expectation values. An observable in this theory restricts to an observable in a theory where the fields consist of only *closed* p -forms and at this level duality *does* preserve expectation values. However in order to define a dual we now need to *choose* an extension from an observable acting on closed p -forms to an observable acting on all p -forms. We can phrase this in terms of a correspondence of factorization algebras: observables are called *incident* if they are the images of the same observable under a pair of restriction maps (to the closed p -form theory and its dual). The third level is that of the generalised Maxwell theory we're really interested in. On an open set we can construct a map from observables in the closed p -form theory to observables in the generalised Maxwell theory, which is an isomorphism of local sections of the factorization algebra if the open set is contractible, for instance. This gives us a way of defining a dual of a local observable in the original generalised Maxwell theory.

In future work I hope to investigate extensions to twisted supersymmetric abelian (higher) gauge theories. In particular an understanding of abelian duality for topological twists of maximally supersymmetric abelian gauge theories, which is sufficiently complete to include an understanding of boundary conditions, should – according to the approach of Kapustin and Witten in [KW06] – recover the theory of *geometric class field theory*. Specifically one expects an equivalence of the categories of branes which should recover the abelian version of the geometric Langlands correspondence, as realised independently by Laumon [Lau96] and Rothstein [Rot96] by a twisted Fourier-Mukai transformation.

Acknowledgements

I would like to thank Kevin Costello for many helpful ideas and discussions throughout this project, and Thel Seraphim for some of the initial ideas on how to view abelian duality for expectation values as a version of Plancherel's theorem. I would also like to thank Saul Glasman, Sam Gunningham, Owen Gwilliam, Boris Hanin, Theo Johnson-Freyd, David Nadler, Toly Preygel, Nick Rozenblyum and Jesse Wolfson for helpful conversations.

Finally, I'd like to thank Theo Johnson-Freyd, Aron Heleodoro and Philsang Yoo for carefully reading an earlier draft and offering many helpful comments and corrections. Figures were created using the diagramming software *Dia*.

2 The BV Formalism for Free Field Theories

2.1 The Idea of the BV Formalism

The *Batalin-Vilkovisky formalism* (hereafter referred to as the BV formalism) gives a description of the moduli space of solutions to the equations of motion in a classical field theory that is particularly amenable to quantisation. When we quantise following the BV recipe, we will see – in the case of a free theory – that the Feynman path integral description of the expectation values of observables naturally falls out. This quantisation procedure admits an extension to interacting theories: see [Cos11b] and [CG15] for details.

We start with a rough description and motivation of the classical BV formalism. In its simplest form, a classical field theory consists of a space Φ of *fields* (often the global sections of a sheaf over a manifold X which we call *spacetime*), and a map $S: \Phi \rightarrow \mathbb{R}$: the *action functional*. The physical states in this classical system are supposed to be those states which extremise the action, i.e. the *critical locus* of S : the locus in Φ where $dS = 0$. This can be written as an intersection, specifically as

$$\text{Crit}(S) = \Gamma_{dS} \cap X$$

where Γ_{dS} is the graph of dS in the cotangent bundle T^*X , and X is the zero section. The classical BV formalism gives a model for functions on the *derived* critical locus of S : that is, more than just forming the pullback in spaces given by this intersection, one forms a derived pullback in a homotopy category of spaces, and considers its ring of functions.

We can describe the ring of functions on the derived critical locus explicitly as a derived tensor product by resolving $\mathcal{O}(\Gamma_{dS})$ as an $\mathcal{O}(T^*X)$ -module. We choose the Koszul resolution. Explicitly this says:

$$\mathcal{O}(\Gamma_{dS}) \sim \left(\cdots \longrightarrow \mathcal{O}(T^*X) \otimes_{\mathcal{O}(X)} \wedge^2 T_X \longrightarrow \mathcal{O}(T^*X) \otimes_{\mathcal{O}(X)} T_X \longrightarrow \mathcal{O}(T^*X) \right)$$

where T_X denote the module of vector fields, and the differential is extended from the map $\mathcal{O}(T^*X) \otimes_{\mathcal{O}(X)} T_X \rightarrow \mathcal{O}(T^*X)$ sending $f \otimes v$ to $fv - f\iota_{dS}(v)$ as a derivation with respect to the wedge product. Taking this complex and tensoring with $\mathcal{O}(X)$ we find the complex $PV(X)$ of *polyvector fields* on X , i.e. exterior powers of the ring of vector fields placed in non-positive degrees, with the differential $-\iota_{dS}$ from vector fields to functions extended to a differential on the whole complex as a derivation for the wedge product. This model for functions on the derived critical locus is the BV model for the algebra of *classical observables* in the Lagrangian field theory.

Now, we motivate the *quantum* BV formalism by means of a toy example. Let Φ be a finite-dimensional vector space, and let S be a quadratic form on this vector space. In this toy example, quantum field theory (in Euclidean signature) boils down to the computation of the Gaussian integrals

$$\begin{aligned} \langle \mathcal{O} \rangle &= \frac{\int_{\Phi} \mathcal{O}(\phi) e^{-S(\phi)/\hbar} d\phi}{\int_{\Phi} e^{-S(\phi)/\hbar} d\phi} \\ &= \frac{1}{Z} \int_{\Phi} \mathcal{O}(\phi) e^{-S(\phi)/\hbar} d\phi \end{aligned}$$

for polynomial functions \mathcal{O} on Φ (writing Z for the normalising factor $\int e^{-S(\phi)/\hbar} d\phi$). Here \hbar is a positive real number and $d\phi$ is a volume form on Φ . Equivalently, we can think of this as computing the cohomology class of a top degree element $\mathcal{O}d\phi$ in a twisted de Rham complex: the complex of polynomial differential forms $\Omega_{\text{poly}}^*(\Phi)$ with differential $d - \frac{1}{\hbar}(\wedge dS)$. Finally, contracting with the top form $d\phi$ gives an isomorphism of graded vector spaces $PV(\Phi)[- \dim \Phi] \rightarrow \Omega^*(\Phi)$, which becomes an isomorphism of *complexes* when one gives the space of polyvector

fields the differential $D - \frac{1}{\hbar}\iota_{dS}$, where D is the *BV operator* given by transferring the exterior derivative along the map $\iota_{d\phi}$. Concretely, let x_1, \dots, x_n form a basis for Φ , and let $\partial_1, \dots, \partial_n$ be the corresponding basis on $T_0\Phi$. Say $d\phi = dx_1 \wedge \dots \wedge dx_n$. Then

$$D = \sum_{i=1}^n \frac{\partial}{\partial x_i} \frac{\partial}{\partial(\partial_i)}.$$

If Φ is infinite-dimensional then we can no longer immediately make sense of the original Gaussian integral (though we can compute it as a suitable limit), nor of “top” degree forms in the twisted de Rham complex. But the complex $PV(\Phi)$ in degrees ≤ 0 and the differential $D - \frac{1}{\hbar}\iota_{dS}$ still makes sense, and we can still compute the cohomology class of a degree zero element, thus defining its *expectation value* directly. What’s more, we see that, considering instead the isomorphic complex with differential $\hbar D - \iota_{dS}$, in the “classical” limit as $\hbar \rightarrow 0$ we recover the BV description of the algebra of classical observables. So this explicitly gives a *quantisation* of that algebra. This quantisation is no longer a dg-algebra: the Leibniz identity receives a correction term proportional to \hbar coming from the classical Poisson bracket.

The general BV formalism therefore gives a model for the classical and quantum observables in a free Lagrangian field theory (i.e. a theory with quadratic action) following this outline. The classical observables are constructed as an algebra of polyvector fields with the differential ι_{dS} , and a quantisation is produced by deforming this differential with a BV operator analogous to the one above. One builds this operator by – approximately – identifying Darboux co-ordinates on the (shifted) cotangent bundle to the fields and defining an operator using a formula like the one given above. An easier way to describe this is to use the Poisson bracket on functions on the shifted cotangent bundle (the so-called *antibracket*), and to extend this to a BV operator by an inductive formula on the degree.

2.2 Derived Spaces from Cochain Complexes

When defining a classical field theory on a manifold X , it’s not completely clear what kind of object one should use to define the “sheaf of fields” of a Lagrangian field theory on X . One can build a classical field theory starting from a cochain complex of vector spaces (this is the approach used by Costello and Gwilliam [CG15]), but this excludes the most natural treatment of many interesting non-linear examples, such as sigma models. It also doesn’t allow for discrete data in the space of fields, for instance the choice of G -bundle for fields in Yang-Mills theory. Witten’s work on abelian duality [Wit95] shows that this discrete invariant is necessary for the existence of duality phenomena: one sees theta functions in the partition function of an abelian gauge theory only after summing over all topological sectors.

We’ll use the following definition of a classical Lagrangian field theory which, while not the most general definition possible, allows for discrete and non-linear pieces in the fields suitable for the free theories we will consider.

Definition 2.1. A *classical Lagrangian field theory* on a manifold X consists of a sheaf Φ of simplicial abelian algebraic groups on X (the *fields*), and a morphism of sheaves of simplicial schemes

$$\mathcal{L}: \Phi \rightarrow \text{Dens}$$

(the *action functional*) where Dens denotes the sheaf of densities on X , thought of as a sheaf of abelian groups.

If X is compact then we can integrate global densities. The resulting map $S: \Phi(X) \rightarrow \mathbb{R}$ is called the *action functional*. Similarly, we can integrate local *compactly supported* sections of Φ to define a local action functional. In the next section I’ll explain how to produce the factorization space of classical observables from this data, but first I’ll explain some ways in which one might produce such a simplicial abelian group from more naïve data.

Remark 2.2. Note that while definition 2.1 mixes the world of manifolds with the world of schemes, the two types of geometry play very different roles. For other applications we could relax the condition that X is a smooth manifold, and instead allow it to be any site. In contrast, the condition that the local sections $\Phi(U)$ on an open set U form a (simplicial) scheme will be essential.

Remark 2.3. It's worth noting that we could also define a theory with fermions by allowing Φ to be instead a sheaf of simplicial abelian super algebraic groups. We won't need this generality for the examples of this paper.

Suppose that instead of a sheaf of simplicial algebraic groups, Φ is a sheaf of simplicial abelian groups on the site

$$\text{Open}(X) \times \mathcal{Zar}/\mathbb{R}$$

where \mathcal{Zar}/\mathbb{R} denotes the big Zariski site of \mathbb{R} -algebras (equivalently, Φ is a sheaf on X taking values in a category of sheaves of simplicial abelian groups). One can build a sheaf $\tilde{\Phi}$ of simplicial algebraic groups on X by the following procedure. Fix an open set $U \subseteq X$. Consider all maps of simplicial sheaves from a scheme \mathcal{X} to $\Phi(U)$, and take the homotopy colimit of this diagram in the category of simplicial schemes. Call the result $\tilde{\Phi}(U)$, and observe that varying U defines a sheaf of simplicial abelian algebraic groups on X as required. We could make the same construction over \mathbb{C} instead of \mathbb{R} (or for that matter over a more general field), and indeed many of the examples we consider will be complex-valued theories.

Example 2.4. Let V be a sheaf of cochain complexes of vector spaces on X concentrated in non-positive degrees. Applying the Dold-Kan correspondence, we can think of this as a sheaf of simplicial vector spaces $\text{DK}(V)$. This induces a sheaf of simplicial vector spaces on the product site $\text{Open}(X) \times \mathcal{Zar}/\mathbb{R}$ by setting

$$\Phi(U, R) = \text{DK}(V(U) \otimes_{\mathbb{R}} R).$$

We observe that, for fixed U , this sheaf is already representable by the dg-scheme $\text{Spec}(\text{Sym}(V^\vee))$, and it's unnecessary to take a further homotopy colimit. That is to say the homotopy colimit over all dg-schemes mapping to Φ is trivial.

Example 2.5 (Yang-Mills Theory). Let G be a compact connected Lie group. Define a sheaf Φ of simplicial groups on $\text{Open}(X) \times \mathcal{Zar}/\mathbb{R}$ modelling the stack of connections on principal G -bundles: set

$$\Phi(U, R) = \bigoplus_P (\Omega^0(U; \mathfrak{g}_P) \otimes_{\mathbb{R}} R[1] \rightarrow \Omega^1(U; \mathfrak{g}_P) \otimes_{\mathbb{R}} R)$$

where the sum is over principal G bundles $P \rightarrow X$ up to isomorphism. This gives, under Dold-Kan, a simplicial sheaf for each U , thus a simplicial scheme upon forming a homotopy colimit. This simplicial scheme then gives a model for a moduli stack of connections on principal G -bundles, and allows one to define classical Yang-Mills theory. We'll generalise this construction in the case where G is abelian.

Example 2.6 (Higher Maxwell Theory). The main objects of study in this paper are a family of theories generalising Yang-Mills theory with gauge group $U(1)$, following the description of generalised Maxwell theory through ordinary differential cochains described in [Fre00]. The fields in this theory should describe “circle $(p-1)$ -bundles with connection”, for p some positive integer (the reason we use $p-1$ will become clear later: the “curvature” of a field in such a theory will be a p -form). The starting point for the description of these theories is the *smooth Deligne complex*: a sheaf of cochain complexes of abelian groups on a manifold X given by

$$\begin{aligned} \mathbb{Z}(p)_{\mathcal{D}}(U) &= \mathbb{Z}[p] \hookrightarrow \Omega^0(U)[p-1] \xrightarrow{d} \Omega^1(U)[p-2] \xrightarrow{d} \dots \xrightarrow{d} \Omega^{p-1}(U) . \\ &\cong C^\infty(U, \mathbb{R}/\mathbb{Z})[p-1] \xrightarrow{-id \log} \Omega^1(U)[p-2] \xrightarrow{d} \dots \xrightarrow{d} \Omega^{p-1}(U) \end{aligned}$$

or its complexified version

$$\mathbb{Z}(p)_{\mathcal{D}, \mathbb{C}}(U) = C^\infty(U, \mathbb{C}^\times)[p-1] \xrightarrow{-id \log} \Omega^1(U; \mathbb{C})[p-2] \xrightarrow{d} \dots \xrightarrow{d} \Omega^{p-1}(U; \mathbb{C}) .$$

We can extend this complex to a sheaf of cochain complexes of abelian groups (or, by Dold-Kan, of simplicial abelian groups) whose \mathbb{C} -points are the complexified Deligne complex. Indeed, for an \mathbb{C} -algebra R , define a cochain complex

$$C^\infty(U; R^\times)[p-1] \longrightarrow (\Omega^1(U; \mathbb{C}) \otimes_{\mathbb{C}} R)[p-2] \longrightarrow \dots \longrightarrow \Omega^{p-1}(U; \mathbb{C}) \otimes_{\mathbb{C}} R$$

where the first map is given by $-id \log$, and the latter maps simply by the de Rham differential in the first variable. The colimit procedure described above produces a sheaf of simplicial algebraic groups on U .

We might think of these complexified fields as higher principal \mathbb{C}^\times bundles with connection, or as higher complex line bundles with connection.

2.3 The Action Functional and the Classical Factorisation Space

Once we have the sheaf of fields Φ we can apply the classical BV procedure to build a model for the derived critical locus of the action that is amenable to quantisation. The first step is to describe the shifted cotangent bundle $T^*[-1]\Phi$ as a derived stack, or – to avoid requiring too much formalism from derived algebraic geometry – describing the algebra of functions $\mathcal{O}(T^*[-1]\Phi)$ as a cochain complex. Fixing an open set $U \subseteq X$, the local fields $\Phi = \Phi(U)$ form a simplicial abelian algebraic group, so in particular the (shifted) cotangent bundle should be trivialisable. That is, we define

$$\begin{aligned} T^*[-1]\Phi &\cong T_0^*\Phi[-1] \times \Phi \\ \text{so } \mathcal{O}(T^*[-1]\Phi) &\cong \mathcal{O}(T_0^*\Phi[-1]) \otimes \mathcal{O}(\Phi) \end{aligned}$$

where the first line is only heuristic (though it should be possible to make it precise with the machinery of derived algebraic geometry). The functions on the shifted cotangent fibre are easy to describe, because the cotangent fibre is a (dg-) *vector space*. For any Φ we can define $\mathcal{O}(T_0^*\Phi[-1]) \cong \text{Sym}((T_0\Phi)[1])$, so it is only necessary to describe the complex $T_0\Phi$. For instance, if Φ is an abelian variety we have

$$\mathcal{O}(T^*[-1]\Phi) \cong \text{Sym}((\Phi \otimes_{\mathbb{Z}} \mathbb{R})[1]) \otimes \mathcal{O}(\Phi).$$

We'll describe $T_0\Phi$ in the examples that we're interested in this paper in section 3. This calculation gives the ring $\mathcal{O}(T^*[-1]\Phi)$ a natural interpretation as a ring of *polyvector fields* on Φ . Indeed, the dg-vector space $T_0\Phi \otimes \mathcal{O}(\Phi)$ precisely describes vector fields on Φ . Placing this space in degree -1 and taking graded symmetric powers (i.e. alternating powers, by the usual sign rule), we produce the algebra of polyvector fields on Φ .

Remark 2.7. All algebraic constructions with topological vector spaces in this paper take place in the context of nuclear Frechét spaces (or cochain complexes thereof). For instance, the dual space V^\vee of a vector space V is always the continuous dual equipped with the strong topology, and the tensor product is the completed projective tensor product. Likewise, while we identify the ring of algebraic functions on a space V with the symmetric algebra $\text{Sym}(V^\vee)$, our constructions will all extend to the *completed* symmetric algebra.

Now we introduce the action. Recall that as well as the fields, our Lagrangian field theory data included a map of sheaves from Φ to the sheaf of densities on X . While it is not, in general, possible to integrate the resulting local densities, it *is* possible to define the first variation of this “local action functional”. One defines a compactly supported 1-form dS on Φ , i.e. an element of $T_0^*\Phi_c \otimes \mathcal{O}(\Phi) \cong \text{Hom}(T_0\Phi_c, \mathcal{O}(\Phi))$ where $T_0\Phi_c$ denotes the *compactly supported* tangent vectors: that is, $T_0\Phi$ describes a sheaf of cochain complexes on X , and we consider the compactly supported sections on the set U we have in mind. Then dS is the linear map sending a compactly supported tangent direction v to the functional

$$\phi \mapsto \int_U \mathcal{L}(L_v(\phi))$$

where $L_v(\phi)$ denotes the Lie derivative of ϕ along the vector field v . The compact support condition ensures that $\mathcal{L}(L_v(\phi))$ is a compactly supported density, so the integral is well-defined.

The action functional S describes a modified version of the shifted cotangent bundle by modifying the internal differential on the functions. After identifying the functions on the shifted cotangent bundle with polyvector fields the 1-form dS naturally defines a degree one linear operator, namely the interior product

$$\iota_{dS}: \mathcal{O}(T^*[-1]\Phi) \rightarrow \mathcal{O}(T^*[-1]\Phi).$$

More explicitly, the operator is extended as a derivation from the operator $T_0\Phi_c \otimes \mathcal{O}(\Phi) \rightarrow \mathcal{O}(\Phi)$ given by pairing a vector field with the 1-form $dS \in T_0^*\Phi_c \otimes \mathcal{O}(\Phi)$.

Remark 2.8. If the fields Φ are described by a cochain complex rather than a more general simplicial or dg-scheme then we can describe the complex $\mathcal{O}(T^*[-1]\Phi)$ of polyvector fields and the classical differential ι_{dS} even more directly. The global functions now form a symmetric algebra

$$\mathcal{O}(T^*[-1]\Phi) \cong \text{Sym}(\Phi^\vee \oplus \Phi[1])$$

generated by linear functions and linear vector fields on Φ . We'll see this later in some examples of free theories, where the action is encoded by a linear operator $\Phi \rightarrow \Phi^\vee$ of degree one.

The above discussion took place for a fixed open set $U \subseteq X$. Let's now describe the relationship between the classical observables on different open sets. We describe locality using the machinery of *factorization algebras*, as developed in [CG15]. We recall the basic definitions.

Definition 2.9. A *prefactorization algebra* \mathcal{F} on a space X taking values in a symmetric monoidal category \mathcal{C} with small colimits is a \mathcal{C} -valued presheaf on X equipped with S_k -equivariant isomorphisms

$$\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_k) \rightarrow \mathcal{F}(U_1 \sqcup \cdots \sqcup U_k)$$

for every collection $U_1, \dots, U_k \subseteq X$ of disjoint open sets.

An open cover $\{U_i\}$ of a space X is called *factorising* if for every finite subset $\{x_i, \dots, x_\ell\}$ of U there is a collection $U_{i_1}, \dots, U_{i_\ell}$ of *pairwise disjoint* sets in the cover such that $\{x_i, \dots, x_\ell\} \subseteq U_{i_1} \cup \cdots \cup U_{i_\ell}$.

Given an open cover of X and a presheaf \mathcal{F} on X we can construct a simplicial object in \mathcal{C} called the *Čech complex* of \mathcal{F} , defined as

$$\check{C}(U, \mathcal{F}) = \bigoplus_{k=1}^{\infty} \left(\bigoplus_{U_{i_1}, \dots, U_{i_k}} \mathcal{F}(U_{i_1} \cap \cdots \cap U_{i_k})[k-1] \right)$$

with the usual Čech maps

A prefactorization algebra is a *factorization algebra* if for every open set $U \subseteq M$ and every factorising cover $\{U_i\}$ of U , the natural map $\text{colim } \check{C}(U, \mathcal{F}) \rightarrow \mathcal{F}(U)$ is an isomorphism in \mathcal{C} .

Remark 2.10. The examples we'll discuss will all be prefactorization algebras taking values in the *homotopy category* of cochain complexes, so isomorphisms are quasi-isomorphisms of complexes.

Definition 2.11. The *factorization algebra of classical observables* associated to the classical Lagrangian theory (Φ, \mathcal{L}) is the factorization algebra $\text{Obs}_{\Phi}^{\text{cl}}(U)$ valued in cochain complexes whose sections on U are given by the complex $\mathcal{O}(T^*[-1]\Phi)$ with differential given by the internal differential d_{Φ} on $\mathcal{O}(\Phi)$ plus the classical differential $-\iota_{dS}$.

The fact that this forms a factorization, rather than just a prefactorization, is theorem 4.5.1 in [Gwi12]. It follows from the fact that Φ forms a sheaf, so the global functions $\mathcal{O}(\Phi)$ forms a cosheaf.

Finally, we need to address the *Poisson structure* on the classical observables. This is definition 2.1.3 in [Gwi12].

Definition 2.12. A P_0 -*factorization algebra* is a factorization algebra \mathcal{F} valued in cochain complexes, such that each $\mathcal{F}(U)$ is equipped with a commutative product and a degree 1 antisymmetric map $\{, \}: \mathcal{F}(U) \otimes \mathcal{F}(U) \rightarrow \mathcal{F}(U)$ which is a biderivation for the product, satisfies the identity

$$d\{x, y\} = \{dx, y\} + (-1)^{|x|}\{x, dy\}$$

and is compatible with the prefactorization structure.

We expect such a structure on the classical observables coming from the *shifted symplectic* structure on the shifted cotangent bundle, but we can recall a familiar concrete description (the well-known Schouten bracket on polyvector fields). Firstly, there's an evaluation map

$$T_0\Phi \otimes \mathcal{O}(\Phi) \rightarrow \mathcal{O}(\Phi)$$

taking an element $v \otimes f$ to $df(v) \in \mathcal{O}(\Phi)$ (thinking of the tangent vector $v \in T_0\Phi$ as a constant vector field on Φ). We use this to define the Schouten bracket in low polyvector field degrees:

$$\begin{aligned} \{1 \otimes f_1, 1 \otimes f_2\} &= 0 \\ \{v_1 \otimes f_1, 1 \otimes f_2\} &= 1 \otimes df_2(v_1) \cdot f_1 \\ \{v_1 \otimes f_1, v_2 \otimes f_2\} &= v_2 \otimes df_2(v_1) \cdot f_1 - v_1 \otimes df_1(v_2) \cdot f_2 \end{aligned}$$

This extends uniquely to an antisymmetric degree 1 pairing on the whole algebra of polyvector fields as a biderivation with respect to, as usual, the wedge product of polyvector fields.

2.4 Quantisation of Free Factorisation Algebras

From now on we will restrict attention to *free* field theories, where we can use the intuitive, non-perturbative notion of BV quantisation described in section 2.1. Informally, A classical Lagrangian field theory is free if the action functional is *quadratic*, so the derivative of the action functional is *linear*.

Definition 2.13. A classical Lagrangian field theory is called *free* if the classical differential ι_{dS} increases polynomial degrees by one. That is, if we filter $\mathcal{O}(\Phi)$ by polynomial degree and call the k^{th} filtered piece $F^k\mathcal{O}(\Phi)$, the operator ι_{dS} raises degree by one:

$$\iota_{dS}: \text{Sym}^i(T_0\Phi[1]) \otimes F^j\mathcal{O}(\Phi) \rightarrow \text{Sym}^{i-1}(T_0\Phi[1]) \otimes F^{j+1}\mathcal{O}(\Phi).$$

Now, Let (Φ, \mathcal{L}) be a free classical theory, and let $\text{Obs}^{\text{cl}}(\Phi(U))$ be the complex of classical observables on an open set U . We'll quantise the local observables by adding a new term to the differential on this complex: the *BV operator*, which we'll denote by D . This operator is built from the P_0 -algebra structure on the classical observables, following the method of deformation quantisation for free theories described in [Gwi12].

Define the *BV operator* $D: \mathcal{O}(T^*[-1]\Phi(U)) \rightarrow \mathcal{O}(T^*[-1]\Phi(U))$ by extending an operator built from the Poisson bracket. Set D to be zero on $\mathcal{O}(\Phi(U))$, and to be given by the Poisson bracket in degree 1: $D = \{, \}: T_0\Phi(U) \otimes \mathcal{O}(\Phi(U)) \rightarrow \mathcal{O}(\Phi(U))$, i.e. the map we described above as “evaluation”. We can then extend this to an operator on the whole complex of classical observables according to the formula

$$D(\phi \cdot \psi) = D(\phi) \cdot \psi + (-1)^{|\phi|} \phi \cdot D(\psi) + \{\phi, \psi\}.$$

An algebra with a differential D and Poisson bracket $\{, \}$ satisfying a formula like this is called a *Beilinson-Drinfeld algebra*, or *BD algebra*: Beilinson and Drinfeld constructed in [BD04a] a family of operads over the formal disc whose fibre at the origin is the P_0 operad. The BD algebra structure given here is a description of an algebra for a generic fibre of the analogous family defined over all of \mathbb{C} rather than just a formal neighbourhood of the origin.

Example 2.14. If Φ is a complex of vector spaces, so the classical observables are given by $\text{Sym}(\Phi[1] \oplus \Phi^\vee)$, then we can construct the BV operator even more directly. The Poisson bracket, restricted to $\text{Sym}^{\leq 2}$ is given by the evaluation map $\Phi \otimes \Phi^\vee \rightarrow \mathbb{C}$ from Sym^2 to Sym^0 , and zero otherwise. This extends uniquely to a degree 1 operator on the whole complex of classical observables, lowering Sym degree by 2 as a BD structure, as above.

Equipped with this operator we can now define the quantum observables.

Definition 2.15. The factorization algebra of *quantum observables* for the free theory (Φ, \mathcal{L}) is the factorization algebra with local sections on U given by the cochain complex

$$\text{Obs}^q(U) = (\mathcal{O}(T^*[-1]\Phi(U)), d_\Phi - \iota_{dS} + D)$$

where d_Φ is the differential coming from the internal differential on Φ , ι_{dS} is the classical BV differential, and D is the quantum BV differential as defined above.

That this procedure really does define a factorization algebra is proved in [Gwi12].

Remark 2.16. A more standard thing to write would consider a differential $d_\Phi - \iota_{dS} + \hbar D$, and have the BV operator D defined by a formula like

$$D(\phi \cdot \psi) = D(\phi) \cdot \psi + (-1)^{|\phi|} \phi \cdot D(\psi) + \hbar \{\phi, \psi\}$$

where we'd adjoined a formal parameter \hbar to the algebra of classical observables. We'd then obtain a module flat over $\mathbb{R}[[\hbar]]$ which recovered the classical observables upon setting \hbar to zero. I haven't done this because when working exclusively with free theories it's possible to work completely *non-perturbatively*, i.e. to evaluate at a non-zero value of \hbar . The duality phenomena I'm investigating are only visible non-perturbatively (taking into account all topological sectors) so this is necessary, however we lose the ability to consider the quantum observables in any theory which is not free.

2.5 Smearing Observables

In order to describe expectation values in section 4 we need to identify a dense subfactorization algebra of the classical observables which are well-behaved. To do so, we restrict to a special setting: where the algebra of functions $\mathcal{O}(\Phi(U))$ on the fields is a *free commutative dg algebra* on a cochain complex of vector spaces. This includes for instance examples where $\Phi(U)$ is modelled by a complex of vector spaces, or the product of a complex of vector spaces and a compact abelian variety. So we can write

$$\mathcal{O}(\Phi(U)) = \text{Sym}(V)$$

for some complex V , hence

$$\mathcal{O}(T^*[-1]\Phi(U)) = \text{Sym}(T_0\Phi(U)[1] \oplus V).$$

Observe that in fact $V \leq T_0^*\Phi$. Indeed, $\text{Sym}(T_0^*\Phi) = \mathcal{O}(T_0\Phi)$ which admits an injective map from $\mathcal{O}(\Phi)$ (taking the ∞ -jet of a global algebraic function).

Now, suppose further that the theory is free. Then as an operator on this symmetric algebra the classical differential ι_{dS} is non-increasing in Sym-degree. It can be made to *preserve* Sym-degree by completing the square (up to a constant factor), and therefore it can be described as the extension of its *linear* part, which we'll denote

$$Q: T_0\Phi(U) \rightarrow V$$

to the whole complex, as a derivation. Thus the classical observables are themselves given by the free cdga on a complex, namely the complex $T_0\Phi(U)[1] \xrightarrow{Q} V$. Let's analyse this complex.

Example 2.17. If $\Phi(U)$ is a cochain complex of vector spaces (in degrees ≤ 0), then the complex of classical observables is free on the cochain complex

$$\Phi(U)[1] \xrightarrow{Q} \Phi(U)^\vee.$$

The *smeared* or *smooth* classical observables form a dense subalgebra of $\text{Obs}^{\text{cl}}(U)$, defined using the data of an *invariant pairing* on the fields. That is, we require an antisymmetric map of sheaves of cohomological degree -1

$$\langle -, - \rangle: \Phi \otimes \Phi \rightarrow \text{Dens}$$

which is non-degenerate as a pairing on the stalks. This pairing defines a non-degenerate pairing $\Phi_c(U) \otimes \Phi_c(U) \rightarrow \mathbb{C}$ by integration over U , and hence an embedding $T_0\Phi(U)_c \hookrightarrow T_0^*\Phi(U)$, sending a compactly supported vector field to the functional “pair with that vector field”. Since $V \subseteq T_0^*\Phi(U)$ naturally, we can define V^{sm} to be the intersection $V \cap T_0\Phi(U)_c \leq T_0^*\Phi(U)$ (or more precisely, we form the pullback in the category of cochain complexes of vector spaces).

Definition 2.18. The classical *smeared* or *smooth* observables on an open set U are defined to be the free cdga

$$\text{Obs}_\Phi^{\text{sm}, \text{cl}}(U) = \text{Sym}(T_0\Phi_c(U)[1] \xrightarrow{Q} V^{\text{sm}}).$$

If $\mathcal{O}(\Phi(U))$ is a free cdga for every open set $U \subseteq X$ then the smooth observables define a subfactorization algebra of $\text{Obs}_\Phi^{\text{cl}}$, dense in every degree.

Remark 2.19. Again, consider the situation where $\Phi(U)$ is a cochain complex of vector spaces. Then the smooth observables include into all classical observables induced by the inclusion of complexes coming from the pairing

$$\left(\Phi(U)_c[1] \xrightarrow{Q} \Phi(U) \right) \hookrightarrow \left(\Phi_c(U)[1] \xrightarrow{Q} \Phi(U)^\vee \right).$$

Definition 2.20. We call a free classical field theory equipped with an invariant pairing *elliptic* if $\text{Obs}^{\text{cl}}(U)$ is a free cdga for every U , and the resulting complex of *linear* smeared observables

$$\mathcal{E}(X) = T_0\Phi_c(X)[1] \xrightarrow{Q} V^{\text{sm}}$$

on the total space of the manifold X is an elliptic complex.

This is a fairly mild assumption that is satisfied in most realistic free physical theories on compact orientable manifolds (and the theory admits an extension to describe classical observables in interacting theories also, as described in [CG15] and [Cos11a]). We describe some free examples (which all admit interacting extensions described by Costello and Gwilliam).

Example 2.21. 1. **Scalar field theories**

Let $\Phi(U) = C^\infty(U)$, the sheaf of smooth functions on a compact manifold, and let \mathcal{L} be the Lagrangian density for a free scalar field of mass m , namely

$$\mathcal{L}(\phi) = d\phi \wedge *d\phi - m^2\phi \wedge *\phi.$$

The smeared classical observables in this theory are generated by the elliptic complex $C^\infty(X)[1] \xrightarrow{Q} C^\infty(X)$ where $Q = \Delta - m^2$.

2. **Abelian Chern-Simons theory**

Let T be a torus, let $P \rightarrow X^3$ be a principal T bundle on a compact 3-manifold, and let $\Phi(U)$ be the sheaf describing connections on P (where we trivialise the torsor by choosing a fixed reference connection). Chern-Simons theory on this fixed bundle is described by the complex $\Omega^*(U; \mathfrak{t})$, where \mathfrak{t} is the (abelian) Lie algebra of T . Taking a Dolbeault complex on a complex manifold instead of a de Rham complex describes instead *holomorphic* Chern-Simons theory (and indeed, in this language it makes sense to define Chern-Simons theory in any dimension as the theory whose algebra of classical observables are built from this complex).

3. **Abelian Yang-Mills theory**

Let T be a torus, let $P \rightarrow X^4$ be a principal T bundle on a compact Riemannian 4-manifold, and let $\Phi(U)$ be the sheaf describing connections on P as above. Yang-Mills theory on this fixed bundle is described by the shifted cotangent to the Atiyah-Singer-Donaldson complex describing anti-self-dual connections on P , that is, the complex

$$\begin{array}{ccccccc} \Omega^0(U; \mathfrak{t}) & \xrightarrow{d} & \Omega^1(U; \mathfrak{t}) & \xrightarrow{d_+} & \Omega_+^2(U; \mathfrak{t}) & & \\ & & & \nearrow 1 & & & \\ & & \Omega_+^2(U; \mathfrak{t}) & \xrightarrow{d} & \Omega^3(U; \mathfrak{t}) & \xrightarrow{d} & \Omega^4(U; \mathfrak{t}) \end{array}$$

in degrees -1 to 2. Here $\Omega_+^2(U; \mathfrak{t})$ denotes the space of self-dual 2-forms and d_+ is the composition of d with projection onto this space.

We'll see shortly examples of p -form theories that also fit into this framework.

Definition 2.22. The *quantum* smeared observables on an open set U are given by the cochain complex with the same underlying graded abelian group as the classical smeared observables, but with differential $\text{Sym}(Q) + D$, where D is the *smeared BV operator* extended from the operator $\text{Sym}^2(\mathcal{E}(U)) \rightarrow \text{Sym}^0(\mathcal{E}(U))$ given by the invariant pairing restricted to $\mathcal{E}(U)$ according to the BD product formula

$$D(\phi \cdot \psi) = D(\phi) \cdot \psi + (-1)^{|\phi|} \phi \cdot D(\psi) + \{\phi, \psi\}.$$

The quantum smeared observables embed into the whole complex of quantum observables as a subcomplex, dense in each degree. To see this one just needs to check that the quantum BV operators commute with the inclusion, which follows directly from the definitions.

3 Generalised Maxwell Theories as Factorisation Algebras

Having described the general formalism, I'll explain the specific theories which we'll be studying: the *generalised Maxwell theories*, a family of theories including as its simplest two examples sigma models with target a torus and abelian pure Yang-Mills theories.

3.1 Generalised Maxwell Theories

3.1.1 The Classical Factorisation Algebra

We already discussed the fields in generalised Maxwell theories in example 2.6 of the previous section: we built a sheaf of simplicial abelian algebraic groups Φ_p from an ordinary differential cohomology complex $\mathbb{Z}(p)_{\mathcal{D}}$. These are the fields in the theories we'll be interested in. The *action* on a compact manifold is defined as the L^2 -norm of the *curvature* of a field.

Definition 3.1. The *curvature map* is the map of sheaves of simplicial algebraic groups $F: \Phi_p \rightarrow \Omega_{cl}^p$ induced from the exterior derivative as a map of sheaves of cochain complexes $\mathbb{Z}(p)_{\mathcal{D}} \rightarrow \Omega_{cl}^p$ by the universal property of the homotopy colimit.

With this in mind, we define the Lagrangian density of a local field $\phi \in \Phi_p(U)$ to be

$$\mathcal{L}(\phi) = R^2 F(\phi) \wedge *F(\phi)$$

for R a positive real number, using the Riemannian metric on U . If this density is integrable, the resulting integral is just the L^2 -norm of $F(\phi)$. We call the number R here a *coupling constant*, and think of it as the radius of the gauge group circle, pre complexification. We could also have produced this scaling by redefining the fields: in our Deligne complex we might have included the lattice $2\pi R\mathbb{Z}$ instead of \mathbb{Z} , yielding a circle of radius R in the cohomology.

Remark 3.2. We can generalise this setting from a circle (or, in the complexified story, \mathbb{C}^\times) to a higher rank torus $T \cong V/L$, where L is a full rank lattice in a real normed vector space V . Let $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ and $T_{\mathbb{C}}$ denote the complexifications of V and T . The fields in the theory described above generalise immediately by taking algebraic $T_{\mathbb{C}}$ valued functions mapping into the algebraic de Rham complex with values in $V_{\mathbb{C}}$. There is now a curvature map taking values in $\Omega_{cl}^p \otimes V_{\mathbb{C}}$, and we define a Lagrangian density functional by

$$\mathcal{L}(\phi) = \|F(\phi) \wedge *F(\phi)\|^2$$

where $\| - \|$ is the norm on $V_{\mathbb{C}}$.

In this description, the coupling constants arise from the choice of lattice $L \leq V$: for instance we obtain the theory with coupling constant R above by choosing $2\pi R\mathbb{Z} \leq \mathbb{R}$.

The construction of the previous chapter yields a classical factorization algebra of observables from the above data. I can describe the local sections fairly concretely. We start from the algebra of polyvector fields

$$PV(\Phi_p(U)) = \text{Sym}(T_0\Phi_p(U)) \otimes \mathcal{O}(\Phi_p(U)).$$

We take each term individually. Functions on the Deligne complex $\Phi_p(U)$ were given as an abstract homotopy colimit so aren't easy to describe directly. However, the tangent fibre $T_0\Phi_p(U)$ is much more accessible in that it comes from an actual complex of vector spaces. We find

$$T_0\Phi_p(U) = \Omega^0(U; \mathbb{C})[p-1] \rightarrow \Omega^1(U; \mathbb{C})[p-2] \rightarrow \cdots \rightarrow \Omega^{p-1}(U; \mathbb{C}),$$

i.e. the shifted truncated de Rham complex, or more precisely the associated simplicial algebraic group, as in example 2.4. The Poincaré lemma tell us that this is quasi-isomorphic to the sheaf of closed p -forms via the exterior derivative, as a sheaf of cochain complexes.

Now, we describe the classical differential coming from the action functional (or its first variation dS). This is an operator $T_0\Phi_p(U)_c \rightarrow \mathcal{O}(\Phi_p(U))$, or using the above description, an operator $\Omega_c^{p-1}(U; \mathbb{C}) \rightarrow \mathcal{O}(\Phi_p(U))$ that vanishes on exact forms (i.e. a cochain map out of the shifted truncated de Rham complex). We define such an operator via the curvature map $F: \Phi_p(U) \rightarrow \Omega_{cl}^p(U)$. This induces a pullback map $F^*: \mathcal{O}(\Omega_{cl}^p(U)) \rightarrow \mathcal{O}(\Phi_p(U))$. By composing with the curvature map it suffices to define the classical differential as the map $\Omega_c^{p-1}(U; \mathbb{C}) \rightarrow \mathcal{O}(\Omega_{cl}^p(U; \mathbb{C})) \xrightarrow{F^*} \mathcal{O}(\Phi_p(U))$

$$\alpha \mapsto \left(\beta \mapsto \int_U \beta \wedge *d\alpha \right) \mapsto \left(A \mapsto \int_U F_A \wedge *d\alpha \right) = \iota_{dS}(\alpha).$$

This functional, sending a field A to $\int_U F_A \wedge *d\alpha$, clearly recovers the first variation of the required action functional.

3.1.2 The Quantum Factorisation Algebra

Now, we know abstractly how to quantise this factorization algebra, but we should see what it actually means in this context. There's an evaluation map $\Omega_c^{p-1}(U; \mathbb{C}) \otimes \mathcal{O}(\Phi_p(U)) \rightarrow \mathcal{O}(\Phi_p(U))$ which, again, is defined via the curvature map. We'll also identify the 1-forms on the fields $\Omega^1(\Phi_p(U))$ with the quasi-isomorphic complex $\mathcal{O}(\Phi_p(U)) \otimes \Omega_{cl}^p(U; \mathbb{C})^\vee$. The evaluation map is given by pairing a vector field on $\Phi_p(U)$ with a 1-form. We can spell this out in two steps:

1. Start with $\chi \otimes f \in \Omega_c^{p-1}(U; \mathbb{C}) \otimes \mathcal{O}(\Phi)$. We first take the exterior derivative of both χ and f to yield

$$d\chi \otimes df \in \Omega_{c,cl}^p(U; \mathbb{C}) \otimes \Omega^1(\Phi(U)) \cong \Omega_{c,cl}^p(U; \mathbb{C}) \otimes \mathcal{O}(\Phi_p(U)) \otimes \Omega_{cl}^p(U; \mathbb{C})^\vee.$$

2. Use the evaluation pairing between $\Omega_{cl}^p(U; \mathbb{C})^\vee$ and $\Omega_{c,cl}^p(U; \mathbb{C})$ (that is, between linear vector fields and linear 1-forms on the fields) to produce a contracted element $df(d\chi) \in \mathcal{O}(\Phi_p(U))$ as required.

This evaluation map then extends to a differential on the whole complex of observables by the BD operator formula: this is the quantum differential in the generalised Maxwell theory.

From now on, when I write $\text{Obs}^q(U)$ I'm referring specifically to the quantum observables in a generalised Maxwell theory (which one will generally be clear from context). I'll write $\text{Obs}^q(U)_0$ to refer specifically to the *gauge invariant degree zero* observables: the part of the cochain complex that refers to actual observables in the usual sense of the word, as opposed to encoding relationships between observables. The notation doesn't refer to the entire degree zero part of the cochain complex, but rather to the subcomplex $\mathcal{O}(H^0(\Phi(U))) \leq \text{Obs}_\Phi^q(U)$, where the projection $\Phi(U) \rightarrow H^0(\Phi(U))$ induces a pullback map $\mathcal{O}(H^0(\Phi(U))) \rightarrow \mathcal{O}(\Phi(U))$.

For generalised Maxwell theories specifically, the local degree zero observables are given by functions on the 0th hypercohomology of the Deligne complex (which, with our degree conventions, is the degree p differential cohomology group $\hat{H}^p(U)$). We compute this using the long exact sequence on hypercohomology associated to the short exact sequence of sheaves

$$0 \rightarrow \tau_{<p} \Omega_{\mathbb{C}}^*[p-1] \rightarrow \mathbb{Z}(p)_{\mathcal{D}, \mathbb{C}} \rightarrow 2\pi R\mathbb{Z}[p] \rightarrow 0$$

yielding $\mathbb{H}^0(U; \mathbb{Z}(p)_{\mathcal{D}})$ isomorphic to the product of a torus (on which there are no non-constant global functions, so we can safely ignore it) and the group $\Omega_{cl, \mathbb{Z}}^p(U; \mathbb{C})$ of closed p -forms whose cohomology class lies in the subgroup $H^p(U; 2\pi R\mathbb{Z}) \leq H^p(U; \mathbb{C})$. The calculation is described in [Bry93], theorem 1.5.3.

3.2 Free Theories from p -forms

In order to do calculations with Maxwell theories we will relate them to much easier free field theories where the fields are sheaves of p -forms. This will correspond, intuitively, to considering observables that factor through the curvature map $\hat{H}^p(U) \rightarrow \Omega_{cl}^p(U)$. These theories will be especially easy in that the action functional will involve no derivatives at all, so the classical BV operator is just a scalar.

Definition 3.3. Fix $0 < p < n$ as before. The *free p -form theory* on X with coupling constant R is the Lagrangian field theory with sheaf of fields given by the sheaf of vector spaces Ω^p and action functional

$$S_R(\alpha) = R^2 \|\alpha\|_2^2 = R^2 \int_X \alpha \wedge * \alpha$$

given by the L^2 -norm. The *free closed p -form theory* with coupling constant R is the subtheory with sheaf of fields Ω_{cl}^p – the sheaf of closed p -forms with the same action functional.

We can build classical and quantum factorization algebras directly from this data as a very easy application of the BV formalism described above: I'll denote them by $\text{Obs}_{\Omega^p}^{cl}$, $\text{Obs}_{\Omega^p}^q$ etc, with the choice of R suppressed. For the general p -form theory we compute

$$\text{Obs}_{\Omega^p}^{cl}(U) = \text{Sym}(\Omega_c^p(U)[1] \xrightarrow{R^2 \iota} \Omega^p(U)^\vee)$$

where ι is the inclusion of $\Omega_c^p(U)$ into the dual space given by the L^2 -pairing. This follows directly from the construction of the action functional given in section 2.3: this operator describes the first variation of the action functional.

The quantum BV operator D is induced from the evaluation pairing $\Omega_c^p(U) \otimes \Omega^p(U)^\vee \rightarrow \mathbb{R}$. We can produce a smeared complex of quantum observables using the standard L^2 pairing on p -forms coming from the Riemannian metric on X . That is, we have local smeared quantum observables given by

$$(\text{Sym}(\Omega_c^p(U)[1] \oplus \Omega_c^p(U)), \text{Sym}(\cdot R^2) + D)$$

where $\cdot R^2$ is now just a scalar multiplication operator, and D is the operator extended from the L^2 pairing as a map $\text{Sym}^2 \rightarrow \text{Sym}^0$ according to the usual BD formula. This complex is quasi-isomorphic to \mathbb{R} for every U . Indeed, by a spectral sequence argument it suffices to check this for the classical observables, where we're computing Sym of a contractible complex.

The closed p -form theory is similar until we smear. That is, the quantum factorization algebra is given as the complex

$$\text{Obs}_{\Omega_{cl}^p}^q(U) = (\text{Sym}(\Omega_{c,cl}^p(U)[1] \oplus \Omega_{cl}^p(U)^\vee), \text{Sym}(R^2 \iota) + D)$$

where D is induced from the evaluation pairing as above. The theory is not quite the same after smearing, in particular it is no longer locally contractible. The smeared version of the factorization algebra is now

$$\text{Obs}_{\Omega_{cl}^p}^{\text{sm}}(U) = (\text{Sym}(\Omega_{c,cl}^p(U)[1] \oplus \Omega_c^p(U)/d^* \Omega_c^{p+1}(U)), \text{Sym}(R^2 \pi) + D)$$

where now π is the projection $\Omega_c^p(U) \rightarrow \Omega_c^p(U)/d^* \Omega_c^{p+1}(U)$, and where D is induced by the L^2 -pairing. The embedding $\Omega_c^p(U)/d^* \Omega_c^{p+1}(U) \rightarrow \Omega_{cl}^p(U)^\vee$ is given, as usual, by the metric, and we use the fact that local closed and local coexact forms are orthogonal with respect to the L^2 -pairing. This complex is only quasi-isomorphic to \mathbb{R} *globally*, i.e. on compact U where we can use Hodge theory to identify closed forms as an orthogonal complement to coexact forms

This story all proceeds identically for complex-valued forms, which we'll use from now on. The complexification of the closed p -form theory is – by design – closely related to the generalised Maxwell theory of order p : the complexes of observables are isomorphic on certain open sets, and one has a natural map on *degree zero* observables $\text{Obs}_{\Omega_{cl}^p}^q(U; \mathbb{C})_0 \rightarrow \text{Obs}_p^q(U)_0$ for any U . This map should be thought of as the inclusion of those observables that factor through the curvature map. Concretely, the map is induced from the map of sheaves of cochain complexes

$$F: \mathbb{Z}(p)_{\mathcal{D}}(A) \rightarrow \Omega_{cl}^p \otimes_{\mathbb{C}} A$$

(the *curvature* map) for a commutative \mathbb{C} -algebra A induced by the derivative. This gives a map of sheaves of simplicial algebraic groups upon taking homotopy colimits. If $U \subseteq X$ is contractible then the map of complexes F on the open set U is a quasi-isomorphism, and so the induced map of simplicial algebraic groups is a homotopy equivalence.

The map F can be promoted to a morphism of factorisation algebras. Indeed, we first extend the degree zero map to a map on all classical observables by pulling back polyvector fields. Concretely this is the map

$$F: \text{Sym}(\Omega_{c,cl}^p(U; \mathbb{C})[1]) \otimes \text{Sym}(\Omega_{cl}^p(U; \mathbb{C})^\vee) \rightarrow \text{Sym}(T_0 \Phi_c(U)[1]) \otimes \mathcal{O}(\Phi(U))$$

given by F^* in the second factor and a quasi-inverse d^{-1} to the quasi-isomorphism of cosheaves of cochain complexes $d: T_0 \Phi_c \rightarrow \Omega_{c,cl}^p$ in the second factor. Specifically, on global sections we can define $d^{-1}: \Omega_{cl}^p(X; \mathbb{C}) \rightarrow T_0 \Phi(X)$ using the Hodge decomposition. This preserves the support of a form, so restricts to the compactly supported local sections on an open set U , and defines a quasi-inverse to the quasi-isomorphism of complexes on contractible open sets.

One needs to check that this is compatible with the classical BV operator Q , which requires observing that the square

$$\begin{array}{ccc} \Omega_{c,cl}^p(U; \mathbb{C}) & \longrightarrow & \mathcal{O}(\Omega_{cl}^p(U; \mathbb{C})) \\ d^{-1} \downarrow & & \downarrow F^* \\ T_0 \Phi_c(U) & \longrightarrow & \mathcal{O}(\Phi(U)) \end{array}$$

commutes, where the horizontal arrows are those maps defining the Poisson brackets. If, further, the open set U is contractible then this map defines a *quasi-isomorphism* of classical observables.

We then need to check that our map commutes with the quantum BV operator; that is, we check that the triangle

$$\begin{array}{ccc} \Omega_{c,cl}^p(U; \mathbb{C}) \otimes \Omega_{cl}^p(U; \mathbb{C}) & \xrightarrow{\langle -, - \rangle} & \mathbb{C} \\ d^{-1} \otimes F^* \downarrow & D \nearrow & \\ T_0 \Phi_p(U) \otimes \mathcal{O}(\Phi(U)) & & \end{array}$$

commutes, for D the quantum BV operator in the generalised Yang-Mills theory. This is clear from the definition of the map D , which first applies the exterior derivative to the linear vector field and the linear functional, then pairs the result via the L^2 -pairing. Again, if U is in fact contractible then the resulting map on local observables is actually a quasi-isomorphism.

4 Expectation Values

In this section we'll explain how to compute vacuum expectation values of observables in free theories where the fields are given by a cochain complex of vector spaces, such as the closed p -form theories introduced in the previous section. While the method does not apply directly to free theories with a more complicated space of fields, we show that the expectation value can be computed by functional integrals, which *will* generalise to more complicated settings like that of generalised Maxwell theories.

4.1 Expectation Values from Free Quantum Factorisation Algebras

For an elliptic theory (Φ, \mathcal{L}) , consider the complex of global classical observables $\text{Obs}(X)$ with underlying graded vector space $\mathcal{O}(T^*[-1]\Phi)$. This cdga is freely generated by a cochain complex, and its smeared version is assumed to be freely generated by an elliptic complex \mathcal{E} as described in section 2.5. Now, Hodge theory gives us a Laplacian operator $\Delta : \mathcal{E} \rightarrow \mathcal{E}$ and a splitting in each degree: $\mathcal{E}_i = \mathcal{H}_i \oplus \mathcal{H}_i^\perp$, where \mathcal{H}_i denotes the finite-dimensional vector space of *harmonic* elements in degree i . In particular, we can apply this to the degree zero elements \mathcal{E}_0 , which represent a linearisation of the space of the (degree zero) *fields* in our theory.

Definition 4.1. The Hodge decomposition defines a splitting

$$\mathcal{E}_0 = \mathcal{M} \oplus \mathcal{M}^\perp$$

where \mathcal{M} denotes the finite-dimensional space of harmonic fields. We call \mathcal{M} the space of *massless modes* of the theory, and \mathcal{M}^\perp the space of *massive modes*.

Remark 4.2. In the terminology of Costello's work on perturbation theory, massless modes provide an example of non-propagating degrees of freedom, as in [Cos11b, Definition 13.1.1].

The terminology comes from the example of the free scalar field.

Example 4.3. Let $\Phi(U) = C^\infty(U)$ with the action for a free scalar field of mass $m \geq 0$, i.e.

$$S(\phi) = \int_X (\phi \Delta \phi - m^2 |\phi|^2) \, \text{dvol}.$$

The complex of smeared quantum global observables here, as remarked upon in 2.21 has underlying graded vector space $\text{Sym}(C^\infty(X)[1] \oplus C^\infty(X))$, classical differential $\text{Sym}(\Delta - m^2)$ and quantum differential induced from the L^2 -pairing on functions. The relevant elliptic complex then is the two-step complex

$$C^\infty(X) \xrightarrow{\Delta - m^2} C^\infty(X)$$

in degrees -1 and 0 . The cohomology of this complex is finite-dimensional since X is assumed to be compact, so the m^2 -eigenspace of the Laplacian is finite-dimensional (or just by Hodge theory, since the complex is elliptic).

Focusing on the case $m = 0$, the Hodge decomposition splits $C^\infty(X)$ in each degree as a sum of eigenspaces for Δ^2 , or equivalently as a sum of eigenspaces for Δ . The eigenspaces are the spaces of solutions to $\Delta\phi = \lambda\phi$, which we think of as the energy $\sqrt{\lambda}$ pieces of the space of fields (by analogy with the Klein-Gordon equation in Lorentzian signature). The cohomology is then represented by the harmonic / massless piece.

From now on we'll always suppose that the complex \mathcal{E} has no cohomology outside of degree zero. We don't expect this hypothesis to be necessary, but including it makes the argument below simpler, and is satisfied by the theories we study in this paper. With this condition relaxed, the expectation value would involve projecting to the subspace $\text{Sym}(H^0(\mathcal{E})) \leq H^0(\text{Sym } \mathcal{E})$ corresponding to global degree zero observables.

Suppose now that we're considering a theory with no massless modes; for instance, we might restrict to the subcomplex with \mathcal{M}^\perp in degree zero. Then the elliptic complex \mathcal{E} is contractible, therefore the cohomology of $\text{Obs}^{\text{sm}} = \text{Sym}(\mathcal{E})$ is isomorphic to $\text{Sym}(0) = \mathbb{C}$ in degree zero. To see this is also true for the smeared *quantum* observables we use a simple spectral sequence argument, using the filtration of the complex by Sym degree. The BV operator is extended from the map from Sym^2 to Sym^0 by the L^2 pairing, so in general lowers Sym-degree by two. The E_1 page of the spectral sequence computes the cohomology of the classical complex of smeared observables (i.e. the cohomology with respect to only the Sym degree 0 part of the differential), and the spectral sequence converges to the cohomology of the complex of smooth quantum observables (i.e. the cohomology with respect to the entire differential). Since the E_1 page is quasi-isomorphic to \mathbb{C} in degree 0, so must be the E_∞ page. Finally we observe that there is a *unique* quasi-isomorphism from this complex of smooth observables on U to \mathbb{C} characterised by the property that 1 in Sym^0 maps to 1. We call this the *expectation value* map. It takes a degree zero smeared quantum observable and returns a number.

Remark 4.4. The reader might find this assumption surprising, since generalised Maxwell theories are far from being of this type. However, there is a map onto the generalised Maxwell theory from a closed p -form theory, which is an elliptic theory with no massless modes, and the image of this map is dense on contractible open sets. In the last part of this paper we'll define expectation values and duals for the observables in the image of this map, using the above assumptions for the closed p -form theory.

Remark 4.5. With the setup we've been using, having no massless modes was essential. The massless modes correspond to the locus in \mathcal{E}_0 where the action functional vanishes, so the locus where the exponentiated action is *degenerate*. Since we'll be computing expectation values as a limit of finite dimensional Gaussian integrals it will be important to ensure that there are no massless modes so that the Gaussian is non-degenerate, and so the finite dimensional Gaussian integrals give finite answers.

It might be possible, in a somewhat different formalism, to work with a non-linear space of fields splitting into a *linear* space of massive modes and a *compact finite-dimensional* moduli space of massless modes. One could then describe an expectation value by integrating out the space of massive modes over each point in the moduli space of massless modes (using the formalism we will describe below) to produce a section of a rank 1 local system. If this local system was actually trivialisable then such a section could be integrated to give a number. Failure of trivialisability would be an example of an *anomaly* for a free field theory.

4.2 Computing Expectation Values

The ideas of this section are not original, but are merely a recollection of familiar physical techniques, whose analysis is well understood, in the present context. A modern mathematical account of the relationship between functional integrals, Feynman diagrams and homological algebra can be found in [GJF12].

4.2.1 Feynman Diagrams for Free Theories

So, let's fix a free elliptic theory with fields Φ and no massless modes: for instance a p -form or closed p -form theory on a compact manifold X . The idea of the Feynman diagram expansion is to compute expectation values of observables in our theory combinatorially. The crucial idea that we'll use in order to check that we can do this is that –for smeared observables– the expectation value map is *uniquely characterised*. That is, for smeared observables there is a unique quasi-isomorphism from global smeared observables to \mathbb{C} that sends 1 to 1. Therefore to check that a procedure for computing expectation values is valid it suffices to check that it is a non-trivial quasi-isomorphism, then rescale so the map is appropriately normalised.

Take a global degree zero smeared observable $\mathcal{O} \in \text{Sym}(\Phi(X)_0)$ which is *gauge invariant*. That is, we consider observables that can be written as a product of linear observables

$$\mathcal{O} = \mathcal{O}_1^{n_1} \mathcal{O}_2^{n_2} \dots \mathcal{O}_k^{n_k}$$

where $\mathcal{O}_1, \dots, \mathcal{O}_k$ are linearly independent linear smeared observables in $\Phi(X)_0$ (not necessarily gauge invariant themselves), and where \mathcal{O} is *closed* in the classical (or equivalently quantum) complex of smeared observables. This corresponds to gauge invariance because the only differential *from* the degree zero observables in either the classical or quantum complexes of smeared observables comes from Sym of the underlying differential on Φ . Closed elements are observables in the kernel of $\text{Sym}(d^\vee)$ where $d: \Phi_{-1} \rightarrow \Phi_0$ is the underlying differential, and where the adjoint d^\vee is defined on Φ_0 using the invariant pairing on fields. The kernel of d^\vee corresponds exactly to the cokernel of d , which is the space of degree zero observables we think of as invariant under gauge transformations.

Generally gauge invariant polynomial observables are sums of monomial observables of this form, and we extend the procedure of computing duals linearly, so it suffices to consider \mathcal{O} of this form.

We compute the expectation value of \mathcal{O} combinatorially as follows. Depict \mathcal{O} as a graph with k vertices, and with n_i half edges attached to vertex i . The expectation value $\langle \mathcal{O} \rangle$ of \mathcal{O} is computed as a sum of terms constructed by gluing edges onto this frame in a prescribed way. Specifically, we attach *propagator edges* – which connect together two of these half-edges – in order to leave no free half-edges remaining. A propagator between linear observables \mathcal{O}_i and \mathcal{O}_j receives weight via the pairing

$$\frac{1}{2} \int_X \langle \mathcal{O}_i, Q^{-1} \mathcal{O}_j \rangle$$

where Q is the classical BV operator, Q^{-1} is defined by inverting Q on each eigenspace for the corresponding Laplacian (using the non-existence of massless modes), and $\langle -, - \rangle$ is the invariant pairing on smeared observables. A diagram is weighted by the product of all these edge weights. The expectation value is the sum of these weights over all such diagrams.

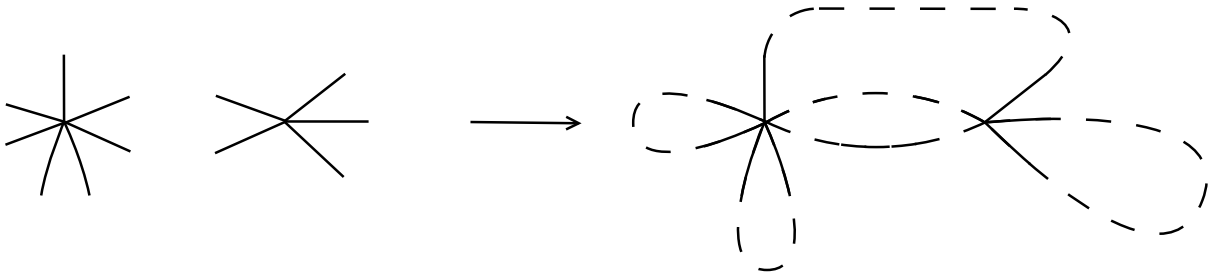


Figure 1: One of the terms in the Feynman diagram expansion computing the expectation value of an observable of form $\mathcal{O}_1^7 \mathcal{O}_2^5$. On the left we see the starting point, with half-edges, and on the right we see one way of connecting these half-edges with propagator edges (indicated by dashed lines).

To check that this computes the expectation value, we must show that it is non-zero, and that it vanishes on the image of the differential in the complex of quantum observables. The former is easy: the observable 1 has expectation value 1 (so we're also already appropriately normalised). For the latter, we'll show that the path

integral computation for degree zero global observables in $\text{Obs}^q(U)_0$ arise as a limit of finite-dimensional Gaussian integrals, and that the images of the quantum BV differential are all divergences, so vanish by Stokes' theorem.

4.2.2 Regularisation and the Path Integral

The classical complex of linear observables in our theory is elliptic, so induces a Laplacian operator Δ acting on \mathcal{E}_0 with discrete spectrum $0 < \lambda_1 < \lambda_2 < \dots$ and finite-dimensional eigenspaces. Let $F^k H^0(\Phi(X))$ denote the sum of the first k eigenspaces: this defines a filtration of the global degree zero (linearised) fields by finite-dimensional subspaces. We recall a standard result about infinite dimensional Gaussian integrals.

Proposition 4.6. Let \mathcal{O} be a smeared global observable. The finite-dimensional Gaussian integrals

$$\frac{1}{Z_k} \int_{F^k H^0(\Phi(X))} \mathcal{O}(a) e^{-S(a)} da,$$

where Z_k is the volume $\int_{F^k H^0(\Phi(X))} e^{-S(a)} da$, converge to a real number $I(\mathcal{O})$ as $k \rightarrow \infty$, and this number agrees with the expectation value computed by the Feynman diagrammatic method.

Proof. We check that for each k the Gaussian integral admits a diagrammatic description, and observe that the expressions computed by these diagrams converge to the expression we want. We may assume as usual that \mathcal{O} splits as a product of linear smeared observables $\mathcal{O} = \mathcal{O}_1^{n_1} \mathcal{O}_2^{n_2} \dots \mathcal{O}_\ell^{n_\ell}$. The \mathcal{O}_i describe linear operators on the filtered pieces. We can write the Gaussian integral using a generating function as

$$\int_{F^k H^0(\Phi(X))} \mathcal{O}(a) e^{-S(a)} da = \frac{\partial^{n_1+\dots+n_\ell}}{\partial t_1^{n_1} \dots \partial t_\ell^{n_\ell}} \Big|_{t_1=\dots=t_\ell=0} \int_{F^k H^0(\Phi(X))} e^{-\int_X \langle a, Qa \rangle + t_1 \mathcal{O}_1(a) \dots + t_\ell \mathcal{O}_\ell(a)} da,$$

provided that k is large enough that upon projecting to $F^k H^0(\Phi(X))$ the \mathcal{O}_i are linearly independent. Call this projection $\mathcal{O}_i^{(k)}$. This expression is further simplified by completing the square, yielding

$$Z_k \frac{\partial^{n_1+\dots+n_\ell}}{\partial t_1^{n_1} \dots \partial t_\ell^{n_\ell}} \Big|_{t_1=\dots=t_\ell=0} e^{\frac{1}{2} \int_X \langle (t_1 \mathcal{O}_1^{(k)} + \dots + t_\ell \mathcal{O}_\ell^{(k)}), Q^{-1}(t_1 \mathcal{O}_1^{(k)} + \dots + t_\ell \mathcal{O}_\ell^{(k)}) \rangle}$$

where we've identified the linear smeared observables with differential forms. We can now compute the Gaussian integral diagrammatically. The $t_1^{n_1} \dots t_\ell^{n_\ell}$ -term of the generating function is the sum over Feynman diagrams as described above, where a diagram is weighted by a product of matrix elements $\frac{1}{2} \int_X \langle \mathcal{O}_i^{(k)}, Q^{-1} \mathcal{O}_j^{(k)} \rangle$ corresponding to the edges. We see that as $k \rightarrow \infty$ this agrees with the weight we expect. \square

Now we can justify why the expectation value vanishes on the image of the quantum differential. Let $\mathcal{O} \in \text{Obs}_\Phi^{\text{sm}}(X)_0$ be a smeared degree 0 global observable, and suppose $\mathcal{O} = d_\Phi V + (D - \iota_{dS})W$ is in the image of the quantum differential. The exact term $d_\Phi V$ is zero in H^0 of the fields, so it suffices to consider the W piece. The restriction of W to a filtered piece is a vector field on the vector space $F^k H^0(\Phi(X))$, and we can compute the divergence

$$\text{div}(e^{-S(a)} W) = \mathcal{O} e^{-S(a)},$$

where the restriction to the filtered piece is suppressed in the notation. So the expectation value of \mathcal{O} is a limit of integrals of divergences, which vanish by Stokes' theorem, and the expectation value is zero. This implies that the procedure described above does indeed compute the cohomology class of a global smeared observable in the canonically trivialised cohomology.

5 Fourier Duality for Polynomial Observables

In its simplest form, Fourier duality is an isomorphism on degree 0 observables in the free p -form theories: $\text{Obs}_{\Omega^p, R}^q(U)_0 \cong \text{Obs}_{\Omega^{n-p}, 1/2R}^q(U)_0$. It will not extend to any kind of cochain maps in these theories, and in particular will not be compatible with the expectation value maps, but we'll show that it *is* compatible with the expectation values after the restriction $\text{Obs}_{\Omega^p, R}^q \rightarrow \text{Obs}_{\Omega_{cl}^p, R}^q$ to the closed p -form theories.

5.1 Feynman Diagrams for Fourier Duality

We'll construct the Fourier transform in an explicit combinatorial way using Feynman diagrams extending the Feynman diagram expression computing expectation values. Take a smeared monomial observable $\mathcal{O} \in \text{Obs}_{\Omega^p, R}^{\text{sm}}(U)_0$. As above, we write \mathcal{O} as

$$\mathcal{O}_1^{n_1} \mathcal{O}_2^{n_2} \dots \mathcal{O}_k^{n_k}$$

where $\mathcal{O}_1, \dots, \mathcal{O}_k$ are linearly independent linear smeared observables in $\Omega_c^p(U)$.

We compute the Fourier dual of \mathcal{O} in a similar diagrammatic way to the method we used to compute expectation values. Depict \mathcal{O} as a graph with k vertices, and with n_i half edges attached to vertex i . Now, we can attach any number of *propagator edges* as before, and also any number of *source terms* – which attach to an initial half-edge and leave a half-edge free – in such a way as to leave none of the original half-edges unused. The source terms have the effect of replacing a linear term \mathcal{O}_i with its Hodge dual $*\mathcal{O}_i$, i.e. the observable obtained by precomposition with the Hodge star operator. The result is a new observable

$$(*\mathcal{O}_1)^{m_1} (*\mathcal{O}_2)^{m_2} \dots (*\mathcal{O}_k)^{m_k}$$

where m_i is the number of source edges connected to vertex i , now thought of as a degree zero observable in $\text{Obs}_{\Omega^{n-p}, 1/2R}^{\text{sm}}(U)_0$.

Definition 5.1. The total Fourier dual observable $\tilde{\mathcal{O}}$ is the sum of these observables over all such graphs, where each observable is weighted by the product over all edges of the corresponding graph of the following weights.

- A propagator between linear observables \mathcal{O}_i and \mathcal{O}_j receives weight

$$\frac{1}{2R^2} \langle \mathcal{O}_i, \mathcal{O}_j \rangle = \frac{1}{2R^2} \int_X \mathcal{O}_i \wedge *\mathcal{O}_j.$$

- A source term attached to a linear observable \mathcal{O} receives weight $i/2R^2$.

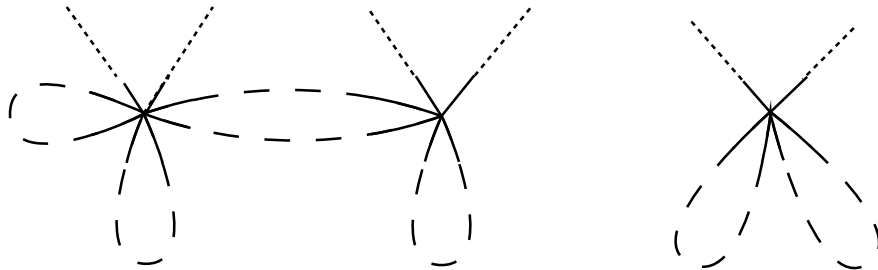


Figure 2: The Feynman diagram corresponding to a degree 6 term in the Fourier dual of an observable of form $\mathcal{O}_1^8 \mathcal{O}_2^6 \mathcal{O}_3^6$. Propagators are indicated by dashed lines and sources by dotted lines.

From a path integral perspective, these terms have natural interpretations. The Fourier transform of \mathcal{O} can be thought of as the expectation value of an observable of form $\mathcal{O}e^{i\langle a, \tilde{a} \rangle}$ where a is a field and \tilde{a} is its Fourier dual variable. Alternatively, this can be thought of as a functional derivative of an exponential of form $e^{-S(a) + i\langle a, \tilde{a} \rangle}$. The propagator terms arise from applying a functional derivative to the action term, while the source terms arise from applying it to the second term, implementing the Fourier dual. The Hodge star in the source term arises from the specific pairing in the p -form theory, namely the L^2 pairing $\int a \wedge *\tilde{a}$.

Example 5.2. To demonstrate the idea, we compute the Fourier dual observable to \mathcal{O}^4 for \mathcal{O} a linear smeared observable. There is one term with no propagator edges and four sources, six with one propagator edge and two sources, and three with two propagator edges and no sources. The dual is therefore

$$\tilde{\mathcal{O}}^4 = \frac{1}{16R^8} (*\mathcal{O})^4 - \frac{6}{8R^6} \|\mathcal{O}\|^2 (*\mathcal{O})^2 + \frac{3}{4R^4} \|\mathcal{O}\|^4.$$

If $R^2 = 1/2$ and $\|\mathcal{O}\| = 1$ this recovers the fourth Hermite polynomial $\text{He}_4(*\mathcal{O})$.

We can compute the dual of a general global observable by smearing first, then dualising: the result is that an observable has a uniquely determined smeared dual for each choice of smearing. In order to compare expectation values of an observable and its dual, the crucial tool that we'll use is Plancherel's formula, which we can rederive in terms of Feynman diagrams. The first step is to prove a Fourier inversion formula in this language. In doing so we'll need to remember that after dualising once, the new observable lives in the *dual* theory, with a different action: therefore the weights assigned to edges will be different, corresponding to a different value of the parameter R .

We'll also use the convention that the second application of the Fourier transform is the inverse Fourier transform, which assigns weight $-i/2R^2$ to a source edge, but is otherwise identical.

Proposition 5.3. A smeared observable \mathcal{O} is equal to its Fourier double dual $\tilde{\tilde{\mathcal{O}}}$.

Proof. Let $\mathcal{O} = \mathcal{O}_1^{n_1} \mathcal{O}_2^{n_2} \dots \mathcal{O}_k^{n_k}$ as above. The Fourier double dual of \mathcal{O} is computed as a sum over diagrams with two kinds of edges: those coming from the first dual and those coming from the second. We'll show that these diagrams all naturally cancel in pairs apart from the diagram with no propagator edges. We depict such diagrams with solid edges coming from the first dual, and dotted edges coming from the second dual.

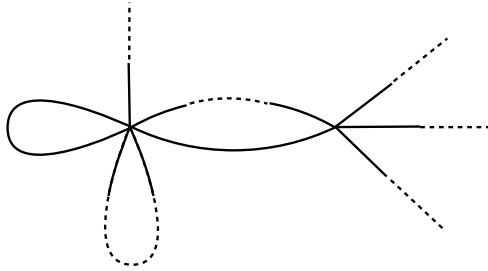


Figure 3: A diagram depicting a summand of the Fourier double dual of an observable of form $\mathcal{O}_1^7 \mathcal{O}_2^5$.

So choose any diagram D with at least one propagator, and choose a propagator edge in the diagram. We produce a new diagram D' by changing this propagator edge from solid to dotted or from dotted to solid.

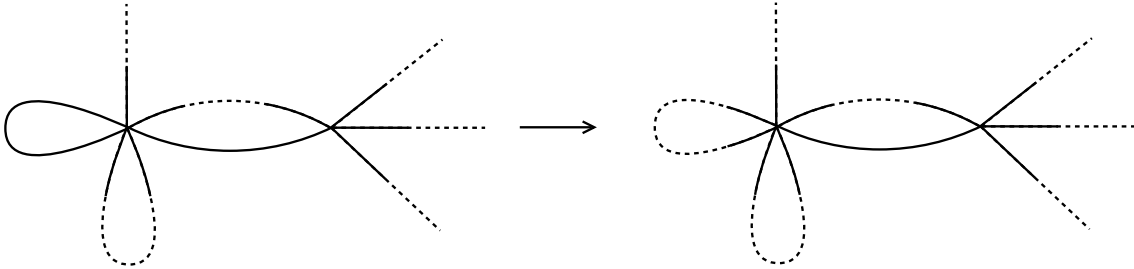


Figure 4: In this diagram we chose the solid leftmost propagator loop (coming from the first dual), and replaced it by two source terms (solid lines) connected with a dotted propagator loop coming from the second dual.

It suffices to show that the weight attached to this new diagram is -1 times the weight attached to the original diagram, so that the two cancel. This is easy to see: the propagator from the first term contributes a weight $\frac{1}{2R^2} \int_X \mathcal{O}_i \wedge * \mathcal{O}_j$. In the second dual, the weights come from the source terms in the original theory, but the propagator in the *dual* theory, which contributes a weight using the dual theory. So the total weight is

$$2R^2 \left(\frac{i}{2R^2} \right)^2 \int_X \mathcal{O}_i \wedge * \mathcal{O}_j$$

Which is -1 times the weight of the other diagram, as required.

Finally, we note that the weight assigned to the diagram with no propagator edges in the Fourier double dual is 1. Indeed, at each free edge, we have a composite of two source terms, contributing a factor of $\left(\frac{i}{2R^2}\right)\left(\frac{-i}{2(1/2R)^2}\right) = 1$. \square

For further justification for these choices of weights, we should compare this combinatorial Fourier dual to one calculated using functional integrals (for global smeared observables). We'll perform such a check by defining a sequence of Gaussian integrals on filtered pieces, and checking that they converge to a Fourier dual that agrees with the one combinatorially described above. As before, $F^k\Omega^p(X)$ refers to the filtration by eigenspaces of the Laplacian, this time on the space of all p -forms, not just closed p -forms.

Proposition 5.4. Let \mathcal{O} be a smeared global observable. The finite-dimensional Gaussian integrals

$$\tilde{\mathcal{O}}(\tilde{a}) = \left(\frac{1}{Z_k} \int_{F^k\Omega^p(X)} \mathcal{O}(a) e^{-S_R(a) + i \int_X \tilde{a} \wedge a} da \right) e^{S_{1/2R}(\tilde{a})}$$

where \tilde{a} is an $(n-p)$ -form, converge to a smeared global observable which agrees with the Fourier dual observable computed by the Feynman diagrammatic method.

Proof. We use the same method of proof as for 4.6, writing the integral as a derivative of a generating function. Specifically, for $\mathcal{O} = \mathcal{O}_1^{n_1} \mathcal{O}_2^{n_2} \cdots \mathcal{O}_\ell^{n_\ell}$ we expand

$$\begin{aligned} \frac{1}{Z_k} \int_{F^k\Omega^p(X)} \mathcal{O}(a) e^{-S_R(a) + i \int_X \tilde{a} \wedge a} da &= \frac{\partial^{n_1+\cdots+n_\ell}}{\partial t_1^{n_1} \cdots \partial t_\ell^{n_\ell}} \Big|_{t_1=\cdots=t_\ell=0} \frac{1}{Z_k} \int_{F^k\Omega(X)} e^{S_R(a) + \sum t_i \int_X \mathcal{O}_i \wedge *a + i \int_X \tilde{a} \wedge a} da \\ &= \frac{\partial^{n_1+\cdots+n_\ell}}{\partial t_1^{n_1} \cdots \partial t_\ell^{n_\ell}} \Big|_{t_1=\cdots=t_\ell=0} \\ &\quad e^{-S_{1/2R}(\tilde{a})} e^{\frac{1}{4R^2} \int_X (t_1 \mathcal{O}_1^{(k)} + \cdots + t_\ell \mathcal{O}_\ell^{(k)}) \wedge * (t_1 \mathcal{O}_1^{(k)} + \cdots + t_\ell \mathcal{O}_\ell^{(k)}) + \frac{i}{2R^2} \int_X (t_1 \mathcal{O}_1^{(k)} + \cdots + t_\ell \mathcal{O}_\ell^{(k)}) \wedge \tilde{a}} \end{aligned}$$

(by completing the square) and extract the $t_1^{n_1} \cdots t_\ell^{n_\ell}$ -term. Once again we're denoting by $\mathcal{O}_i^{(k)}$ the projection of \mathcal{O}_i onto the k^{th} filtered piece $F^k\Omega^p(X)$. We choose the level in the filtration large enough so that the upon projecting to the filtered piece the forms \mathcal{O}_i are linearly independent. One then observes that in the limit as $k \rightarrow \infty$ the relevant term is given by a sum over diagrams as described with the correct weights. \square

Now, for any open set $U \subseteq X$ we have a restriction map of degree zero local observables $r(U): \text{Obs}_{\Omega^p(U)_0}^{\text{sm}} \rightarrow \text{Obs}_{\Omega_{cl}^p(U)_0}^{\text{sm}}$ induced by the projection $\Omega_c^p(U) \rightarrow \Omega_c^p(U)/d^*\Omega_c^{p+1}(U)$. This gives us a candidate notion of duality in the closed p -form theory. So, we might take a degree 0 observable in the image of $r(U)$, choose a preimage, compute the dual then restrict once more. Of course, this is not quite canonical, because the map $r(U)$ is not injective: the resulting dual observable might depend on the choice of preimage we made. However, in certain circumstances we might be able to choose a consistent scheme for choosing such a preimage, therefore a canonical duality map. We'll give such an example in section 5.3, but first we'll prove that for *any* choice of lift, the resulting dual observable in the Ω_{cl}^p theory has the same expectation value as the original theory.

Remark 5.5. We can also consider duality for closed p -form theories with coefficients in a vector space V , and – as we'll observe shortly – generalised Maxwell theories with gauge group a higher rank torus $T = V/L$, as mentioned in section 3. The theory generalises in a natural way, with a p -form theory with gauge group T dual to an $(n-p)$ -form theory with gauge group \hat{T} , the dual torus. Indeed, there is an identical relationship between the generalised Maxwell theory with gauge group a higher rank torus and a closed p -form theory where the forms have coefficients in a vector bundle, and where the classical BV operator is given by the matrix describing the lattice L . By diagonalising this matrix the system separates into a sum of rank one theories, with monomial smeared observables likewise splitting into products of monomial observables in rank one theories which one can dualise individually.

5.2 Fourier Duality and Expectation Values

At this point we have two equivalent ways of thinking about both the Fourier transform and the expectation value map for smeared observables: by Feynman diagrams (which allowed us to describe the dual locally) and by functional integration (which allow us to perform calculations, but only globally). We'll compare the expectation values of dual observables using a functional integral calculation, in which the restriction to *closed* p -forms will be crucial.

For an actual equality of expectation values as described above we'll have to restrict to observables on a contractible open set U . Recall this is the setting where the local observables in the closed p -form theory agree with observables in the original generalised Maxwell theory. On more general open sets connections on higher circle bundles are related to only those closed p -forms with *integral periods*. As such there is a map on degree zero observables $\text{Obs}_{\Omega_{cl}^p, R}^{\text{sm}}(U)_0 \rightarrow \text{Obs}_R^{\text{q}}(U)_0$ for *any* U , which sends a compactly supported closed p -form a to the local observable

$$A \mapsto \int_U F_A \wedge *a.$$

However, this is generally not an isomorphism. Still, from a functional integral point-of-view we can define the expectation value of such an observable in the generalised Maxwell theory, even if X has non-vanishing degree p cohomology. Given $\mathcal{O} \in \text{Obs}_{\Omega_{cl}^p, R}^{\text{sm}}(U)_0$ we extend \mathcal{O} to a global degree zero observable, and define its *expectation value* to be

$$\langle \mathcal{O} \rangle_R = \lim_{k \rightarrow \infty} \int_{F^k \Omega_{cl, \mathbb{Z}}^p(X)} \mathcal{O}(a) e^{-S_R(a)} da$$

Recall here that $\Omega_{cl, \mathbb{Z}}^p(X)$ is our notation for the closed p -forms with integral periods. This is the product of a finite-rank lattice with a vector space, and our filtration is the intersection of this subgroup with the filtration $F^k \Omega_{cl}^p(X)$ defined previously; this intersection is well-behaved since the lattice part is contained in the harmonic forms, thus in the intersection of all the filtered pieces. We notice that if $H^p(U) = 0$ then this definition agrees with the one we used in section 4.6, so in particular the limit converges. In general, the integrand is dominated in absolute value by the integrand over all closed p -forms, which we already know converges (since the proof of 4.6 still applies with \mathcal{O}_i replaced by $|\mathcal{O}_i|$).

Using this definition (and bearing in mind its relationship to the notion of expectation value considered above on certain open sets), we'll prove the main compatibility with duality.

Theorem 5.6. Let \mathcal{O} be a local observable in $\text{Obs}_{\Omega_{cl}^p, R}^{\text{sm}}(U)_0$, and let $\tilde{\mathcal{O}} \in \text{Obs}_{\Omega^{n-p}, 1/2R}^{\text{sm}}(U)_0$ be its Fourier dual observable. Let $r(\mathcal{O})$ and $r(\tilde{\mathcal{O}})$ be the restrictions to local observables in $\text{Obs}_{\Omega_{cl}^p}^{\text{sm}}(U)_0$ and $\text{Obs}_{\Omega_{cl}^{n-p}}^{\text{sm}}(U)_0$ respectively. Then, computing the expectation values of $r(\mathcal{O})$ and $r(\tilde{\mathcal{O}})$, we find

$$\langle r(\mathcal{O}) \rangle_R = \langle r(\tilde{\mathcal{O}}) \rangle_{\frac{1}{2R}}.$$

Remark 5.7. Using the map discussed above we can just as well consider the observables $r(\mathcal{O})$ and $r(\tilde{\mathcal{O}})$ as observables in the appropriate generalised Maxwell theories.

Proof. We know by 5.3 that $\mathcal{O} = \tilde{\tilde{\mathcal{O}}}$, so in particular $\langle r(\mathcal{O}) \rangle_R = \langle r(\tilde{\tilde{\mathcal{O}}}) \rangle_R$. By the calculation in Proposition 5.4 we can write this expectation value as the limit as $k \rightarrow \infty$ of the Gaussian integrals

$$\begin{aligned} \frac{1}{Z_k} \int_{F^k \Omega_{cl, \mathbb{Z}}^p(X)} \mathcal{O}(a) e^{-S_R(a)} da &= \frac{1}{Z_k} \int_{F^k \Omega_{cl, \mathbb{Z}}^p(X)} \tilde{\tilde{\mathcal{O}}}(a) e^{-S_R(a)} da \\ &= \frac{1}{Z_k} \int_{F^k \Omega_{cl, \mathbb{Z}}^p(X)} \int_{F^k \Omega^{n-p}(X)} \tilde{\mathcal{O}}(\tilde{a}) e^{-S_{1/2R}(\tilde{a}) - i \int_X \tilde{a} \wedge a} d\tilde{a} da \\ &= \frac{1}{Z_k} \int_{F^k \Omega^p(X)} \int_{F^k \Omega^{n-p}(X)} \tilde{\mathcal{O}}(\tilde{a}) e^{-S_{1/2R}(\tilde{a}) - i \int_X \tilde{a} \wedge a} \delta_{\Omega_{cl, \mathbb{Z}}^p(X)}(a) d\tilde{a} da \end{aligned}$$

The last line needs a little explanation. The distribution $\delta_{\Omega_{cl,\mathbb{Z}}^p(X)}$ is the delta-function on the closed and integral p -forms sitting inside all p -forms (restricted to the filtered piece): pairing with this distribution and integrating over all p -forms in the filtered piece is the same as integrating only over the relevant subgroup.

Now, for a fixed value of k , we can reinterpret the final integral above by changing the order of integration. This computes the Fourier dual of the delta function $\delta_{\Omega_{cl,\mathbb{Z}}^p(X)}$ and then pushes forward along the Hodge star. The Fourier dual of the delta function is $\delta_{\Omega_{cocl,\mathbb{Z}}^p(X)}$, the delta function on the group of *coclosed* p -forms with integral d^* cohomology class. That is, the external product $\delta_{d^*\Omega^{p+1}(X)} \boxtimes \delta_{\mathcal{H}_{\mathbb{Z}}^p}$ where $\mathcal{H}_{\mathbb{Z}}^p$ is the lattice in the space of harmonic p -forms corresponding to the integral cohomology via Hodge theory. Pushing this distribution forward along the Hodge star yields the delta function $\delta_{\Omega_{cl,\mathbb{Z}}^{n-p}(X)}$ on the closed $(n-p)$ -forms with integral periods. Therefore

$$\begin{aligned} \langle r(\mathcal{O}) \rangle_R &= \lim_{k \rightarrow \infty} \frac{1}{Z_k} \int_{F^k \Omega_{cl,\mathbb{Z}}^{n-p}(X)} \tilde{\mathcal{O}}(\tilde{a}) e^{-S_{1/2R}(\tilde{a})} \delta_{\Omega_{cl,\mathbb{Z}}^{n-p}(X)} d\tilde{a} \\ &= \lim_{k \rightarrow \infty} \frac{1}{Z_k} \int_{F^k \Omega_{cl,\mathbb{Z}}^{n-p}(X)} \tilde{\mathcal{O}}(\tilde{a}) e^{-S_{1/2R}(\tilde{a})} d\tilde{a} \\ &= \langle r(\tilde{\mathcal{O}}) \rangle_{\frac{1}{2R}} \end{aligned}$$

as required. \square

So to summarise, duality gives the following structure to the factorization algebra of quantum observables in our theories.

- For each open set U , we have a subalgebra $\text{Obs}_{\Omega_{cl,R}^p(U)_0}^{\text{sm}} \leq \text{Obs}_R^q(U)_0$ of the space of degree 0 local observables. If U is contractible (for instance for local observables in a small neighbourhood of a point) this subalgebra is dense.
- For a local observable \mathcal{O} living in this subalgebra we can define a *Fourier dual* observable in $\text{Obs}_{\Omega_{cl}^{n-p}, 1/2R}(U)_0$. This depends on a choice of extension of \mathcal{O} to a functional on all p -forms, rather than just closed p -forms.
- For any choice of dual observable, we can compute their expectation values in the original theory and its dual, and they agree. If $H^p(U) = 0$ then this expectation value map agrees with a natural construction from the point of view of the factorization algebra.

We can rephrase the theorem in the language of factorization algebras. Note that $\text{Obs}_{\Omega^p(U)_0}^{\text{sm}}$ and $\text{Obs}_{\Omega_{cl}^p(U)_0}^{\text{sm}}$ form factorization algebras themselves as U varies, concentrated in degree zero. The inclusion maps $\text{Obs}_{\Omega^p(U)_0}^{\text{sm}} \rightarrow \text{Obs}_{\Omega^p(U)}^{\text{sm}}$ and $\text{Obs}_{\Omega_{cl}^p(U)_0}^{\text{sm}} \rightarrow \text{Obs}_{\Omega_{cl}^p(U)}^{\text{sm}}$ are cochain maps since the target complexes are concentrated in non-positive degrees, and factorization algebra maps because the factorization algebra structure maps preserve the degree zero piece. Likewise, the restriction maps $\text{Obs}_{\Omega^p(U)_0}^{\text{sm}} \rightarrow \text{Obs}_{\Omega_{cl}^p(U)_0}^{\text{sm}}$ clearly commute with the factorization algebra structure maps, so define a factorization algebra map.

It's also easy to observe that the Fourier duality map $\text{Obs}_{\Omega^p, R}(U)_0 \rightarrow \text{Obs}_{\Omega^{n-p}, \frac{1}{2R}}(U)_0$ defines a factorization algebra map: the degree zero observables are just the free cdga on the local sections of a cosheaf of vector spaces, so the structure maps are just the maps induced on the free algebra from the cosheaf structure maps. The structure maps are therefore given by the ordinary product

$$\begin{aligned} \text{Sym}(\Omega_c^p(U_1)/d^*\Omega_c^{p+1}(U_1)) \otimes \text{Sym}(\Omega_c^p(U_2)/d^*\Omega_c^{p+1}(U_2)) &\rightarrow \text{Sym}(\Omega_c^p(V)/d^*\Omega_c^{p+1}(V)) \\ (\mathcal{O}_1 \cdots \mathcal{O}_n) \otimes (\mathcal{O}'_1 \cdots \mathcal{O}'_m) &\mapsto \mathcal{O}_1 \cdots \mathcal{O}_n \cdot \mathcal{O}'_1 \cdots \mathcal{O}'_m \end{aligned}$$

where \mathcal{O}_i and \mathcal{O}'_j are local linear smeared observables: sections of $\Omega_c^p/d^*\Omega_c^{p+1}$ with compact support in disjoint open sets U_1 and U_2 respectively. The Fourier transform of a product of observables with disjoint support is the product of the Fourier transforms, since if \mathcal{O} and \mathcal{O}' have disjoint support then their L^2 inner product is zero, so all Feynman diagrams with propagator edges between their vertices contribute zero weight. Therefore duality gives a factorization algebra map on the factorization algebra consisting of degree zero observables only.

Combining all of these statements we have a correspondence of factorization algebras of form

$$\begin{array}{ccccc}
 (\text{Obs}_{\Omega^p, R}^{\text{sm}})_0 & \xleftarrow{\sim} & (\text{Obs}_{\Omega^{n-p}, \frac{1}{2R}}^{\text{sm}})_0 & & \\
 \swarrow r_R & & \searrow r_{\frac{1}{2R}} & & \\
 \text{Obs}_{p, R}^{\text{q}} \xleftarrow{F_R} \text{Obs}_{\Omega_{\text{cl}}^p, R}^{\text{sm}} & & \text{Obs}_{\Omega_{\text{cl}}^{n-p}, \frac{1}{2R}}^{\text{sm}} \xrightarrow{F_{\frac{1}{2R}}} \text{Obs}_{n-p, \frac{1}{2R}}^{\text{q}} & &
 \end{array}$$

where the top arrow is the isomorphism given by Fourier duality, and the diagonal arrows are given by restriction, then inclusion of degree zero observables into all observables. We say a pair of local observables $\mathcal{O}, \mathcal{O}'$ in $\text{Obs}_{p, R}^{\text{q}}(U)$ and $\text{Obs}_{n-p, \frac{1}{2R}}^{\text{q}}(U)$ respectively are *incident* if they are the images under the restriction maps of Fourier dual degree zero observables. In this language, theorem 5.6 can be rephrased in the following way.

Corollary 5.8. If \mathcal{O} and \mathcal{O}' are incident local observables in dual generalised Maxwell theories then $\langle \mathcal{O} \rangle_R = \langle \mathcal{O}' \rangle_{\frac{1}{2R}}$.

At the end of the next section we'll observe that it is not possible to improve this statement from a correspondence to a *map* of factorization algebras (even of the factorization algebras in the closed p -form theories) in any natural way.

5.3 Wilson and 't Hooft Operators

In this section we'll give a concrete example of observables that admit canonical duals in generalised Maxwell theories, corresponding to familiar observables in the usual Maxwell theory, i.e. the case $p = 2$. For Wilson and 't Hooft operators in dimension 4 specifically, the behaviour under abelian duality is discussed in a paper of Kapustin and Tikhonov [KT09].

Wilson and 't Hooft operators can be defined classically in Yang-Mills theory with any compact gauge group G , just as functionals on the space of fields (from which we will – for an abelian gauge group – construct classical and quantum observables). So let X , for the moment, be a Riemannian 4-manifold.

Definition 5.9. Let ρ be an irreducible representation of G . The *Wilson operator* $W_{\gamma, \rho}$ around an oriented loop γ in X is the functional on the space of connections on principal G -bundles sending a connection A to

$$W_{\gamma, \rho}(A) = \text{Tr}(\rho(\text{Hol}_{\gamma}(A))),$$

where $\text{Hol}_{\gamma}(A)$ denotes the holonomy of the connection around the loop γ . Equivalently we can compute the Wilson operator as a path-ordered exponential

$$W_{\gamma, \rho}(A) = \text{Tr}(\rho(\mathcal{P}e^{i \oint_{\gamma} A})).$$

Suppose γ bounds a disc D . In this case there is a candidate dual observable to the Wilson operator.

Definition 5.10. Let $\mu: U(1) \rightarrow G$ be a cocharacter for the group G . The *'t Hooft operator* $T_{\gamma, \mu}$ around the loop γ in X is the functional on the space of connections on principal G -bundles sending a connection A to

$$T_{\gamma, \mu}(A) = e^{i \int_D \mu^*(*F_A)}$$

where F_A is the curvature of A , $*F_A$ is its Hodge star, and where $\mu^*: \Omega^2(X; \mathfrak{g}^*) \rightarrow \Omega^2(X)$ is the pullback along the cocharacter.

The relationship between these two kinds of operator is clearest in the abelian case, so let $G = U(1)$ (or, with minor modifications, any torus). The irreducible representations of $U(1)$ are given by the n -power maps $z \mapsto z^n$ for $n \in \mathbb{Z}$, so we can write our Wilson operators as

$$W_{\gamma, n}(A) = e^{in \oint_{\gamma} A} = e^{in \int_D F_A}$$

assuming $\partial D = \gamma$ as above. Cocharacters are also indexed by integers, so similarly we can describe the 't Hooft operators as

$$T_{\gamma,m}(A) = e^{im \int_D *FA}.$$

As described above we compute the dual of an observable in abelian Yang-Mills theory by taking its Fourier dual as a functional on all 2-forms, then precomposing with the Hodge star. The Fourier dual of a plane wave is a plane wave, so we should expect Wilson and 't Hooft observables to be dual to one another. In the rest of this section we'll prove this, and generalise it to higher degree theories.

Consider the degree p generalised Maxwell theory with coupling constant R on an n -manifold X . I'll first describe degree zero gauge invariant observables associated to a complex number r and a singular chain $C \in C_p(U)$, for $U \subseteq X$ an open set. Recall that the local degree zero observables are given by $\mathcal{O}(H^0(\Phi(U))) \leq \mathcal{O}(T^*[-1]\Phi(U))$, and that in this case the degree zero cohomology $H^0(\Phi)$ is given by the group $\Omega_{cl,\mathbb{Z}}^p(U;\mathbb{C})$ of closed p -forms with integral periods (periods in the lattice $H^p(U;2\pi R\mathbb{Z})$). So, analogously to the above we define a *Wilson*-type operator by

$$W_{C,r}(\alpha) = e^{ir \int_C \alpha}.$$

Similarly, if C is instead a chain in $C_{n-p}(U)$, we can define an 't Hooft-type operator by first applying the Hodge star:

$$T_{C,r}(\alpha) = e^{ir \int_C *\alpha}.$$

Remark 5.11. These operators don't quite arise from our definitions: they aren't polynomial functions in linear observables. However, they can be arbitrarily well approximated by polynomials by taking a finite number of terms in the Taylor series. We should either note that our constructions, in which the observables are described by a symmetric algebra, extend to completed symmetric algebras, or equivalently just interpret claims about duality for these observables as claims about these polynomial approximations at every degree.

Now, let's investigate duality for these observables. Firstly, suppose U is an open set with $H^p(U) = 0$, so that the condition of having integral periods is trivial. Then the observables defined above immediately lift to observables in the closed p -form theory, and admit canonical extensions to observables in the theory where fields are *all* p -forms (given by precisely the same formula). We can also investigate approximations for these observables by *smeared* observables. Integration over a p -chain C can be written as the L^2 -pairing with a particular current: the delta function δ_C . This current can, in turn, be approximated in L^2 by p -forms supported on small neighbourhoods of C . So let's investigate the Fourier dual of the smeared observable

$$\mathcal{O}_\beta(a) = e^{ir \int_X a \wedge * \beta}$$

where β is a p -form.

We'll compute this dual using the functional integral at a regularised level:

$$\begin{aligned} \widetilde{\mathcal{O}_{\beta,r}}(\tilde{a}) &= \lim_{k \rightarrow \infty} \frac{1}{Z_k} \left(\int_{F^k \Omega^p(X)} e^{-S_R(a) + i \int_X \tilde{a} \wedge a + ir \int_X a \wedge * \beta} da \right) e^{S_{1/2R}(\tilde{a})} \\ &= e^{-\frac{r^2}{4R^2} \|\beta\|^2} e^{\frac{-r}{2R^2} \int_X \tilde{a} \wedge \beta} \\ &= e^{-\frac{r^2}{4R^2} \|\beta\|^2} \cdot \mathcal{O}_{*\beta, ir/2R^2}(\tilde{a}). \end{aligned}$$

This calculation allows us to produce the dual of the original Wilson operator by dualising increasingly good smooth approximations. We find

$$\widetilde{W_{C,r}} = e^{-\frac{r^2}{4R^2} \|C\|^2} T_{C, ir/2R^2}$$

where $\|C\|$ is the L^2 -norm of the chain C : the usual L^2 norm with respect to the metric of its image under Poincaré duality.

To summarise, duality for (generalised) Wilson and 't Hooft operators tells us the following.

Corollary 5.12. There is an equality of expectation values in generalised Maxwell theories

$$\langle W_{C,r} \rangle_R = e^{-\frac{r^2}{4R^2} \|C\|^2} \langle T_{C, ir/2R^2} \rangle_{\frac{1}{2R}}.$$

Remark 5.13. We described a canonical dual for Wilson and 't Hooft operators, using a natural choice of lift from operators acting on closed p -forms to operators acting on all p -forms. It's natural to ask whether it's possible to do this for all observables, thus promoting abelian duality from a correspondence to a genuine map of factorization algebras. It turns out however that this is impossible. We'll demonstrate this in a specific example.

Let X be a $2p$ -manifold satisfying $H^p(X) = 0$, so $\Omega^p(X)$ splits as $d\Omega^{p-1}(X) \oplus d^*\Omega^{p+1}(X)$. We'll discuss global linear smeared observables in the closed p -form theory on X . Such an observable is an element of $d\Omega^{p-1}(X)^\vee$ given by L^2 -pairing with an exact p -form. Prescribing an extension of an observable in the closed p -form theory to an observable in the *full* p -form theory is equivalent to prescribing the action of the lifted observable on coexact p -forms, so a choice of such an extension for all linear smeared observables is a map

$$f: d\Omega^{p-1}(X) \rightarrow d^*\Omega^{p+1}(X).$$

Having specified such a map we obtain a canonical dual observable for every linear observable in the closed p -form theory. Applying this duality procedure twice should bring us back to the observable we started with, i.e. $*f*f = \text{id}$, which means in particular that f must be an isomorphism.

However, we also need compatibility with the canonical duals constructed above for Wilson and 't Hooft operators. Let I be a linear observable of form “integrate over a p -cycle”, and let $\alpha_i \rightarrow I$ be a sequence of smeared observables approximating I . The duals of Wilson observables of form e^I are determined by the duals of linear observables since e^x is Fourier self-dual, and to obtain the required dual for our Wilson operator we need to choose the trivial lift for I , i.e. we need $f(\alpha_i) \rightarrow 0$. But then $*f*f(\alpha_i) \rightarrow 0$, so $\alpha_i \rightarrow 0$ which is false. So there is no possible canonical lift f compatible with the natural duals for Wilson and 't Hooft operators, and in particular no way of improving abelian duality to a genuine map rather than just a correspondence.

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