Notes on Representations of the Poincaré Group

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1 Background

1.1 Special Relativity

Let's start with a little background on the geometry of spacetime.

Definition 1.1. *Minkowski space* M is $\mathbb{R}^4$ with a pseudo-Riemannian metric of signature $(3,1)$. We often choose a set of orthonormal coordinates on Minkowski space and denote them (t,x,y,z), so the metric has form $-dt^2 + dx^2 + dy^2 + dz^2$.

The idea that this might be a model for "spacetime" in the universe goes back to Poincaré in the early 20th century, as a mathematical framework for Einstein's theory, built on the postulate that the speed of light should be the same in every reference frame. Einstein's principle of (special) relativity – a consequence of this postulate – says that there is no preferred set of coordinates on M. In other words, any experimental result must be invariant under the action of the isometry group of Minkowski space.

Remark 1.2. This isn't quite true: in real life we *can* distinguish a "future" and a "past" time direction. So really we'd only expect experimental results to be invariant under the group of isometries that preserve a positive time direction. Such isometries are called *orthochronous*.

Definition 1.3. The *Poincaré group* $\mathcal{P}$ is the group of orthochronous isometries of M. It can be expressed as a semidirect product $\mathrm{SO}^+(3,1) \ltimes \mathbb{R}^4$, where $\mathrm{SO}(3,1)$ is the group of automorphisms of $\mathbb{R}^4$ which leave the metric invariant, i.e. the group of four-by-four matrices A such that

$$A \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} A^T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and where $\mathrm{SO}^+(3,1)$ is the connected component of the identity.

1.2 Relativistic Quantum Mechanics

Let's start to mesh this principle with a mathematical model that's supposed to describe physical systems: *quantum mechanics*. I'll tell you everything you need to know to make sense of our arguments today.

In quantum mechanics, the *state* of a physical system is described by a line in some fixed Hilbert space $\mathcal{H}$, i.e. a point in the projectivization $\mathbb{P}\mathcal{H}$. Experiments one can perform correspond to *self-adjoint operators* on $\mathcal{H}$, and the result of performing an experiment $\mathcal{O}$ if the universe is in state v is given by the inner product

$$\frac{\langle v, \mathcal{O}v\rangle}{\langle v, v\rangle}.$$

Thus, if we want to make sense of Einstein's principle of relativity – the results of experiments are fixed by isometries – we need our Hilbert space $\mathcal{H}$ to carry a *unitary projective representation* of $\mathcal{P}$, i.e a continuous homomorphism

$$\rho\colon \mathcal{P} \to \mathrm{PU}(\mathcal{H}).$$

Let's simplify things by getting rid of that projectivization.

Proposition 1.4. A projective representation of a Lie group G corresponds to an ordinary representation of its universal cover.

So instead, we should classify representations of the universal cover of the Poincaré group. It'll suffice to describe the universal cover of $\mathrm{SO}^+(3,1)$ and its action on $\mathbb{R}^4$.

Proposition 1.5. The universal cover of $\mathrm{SO}^+(3,1)$ is isomorphic to $\mathrm{SL}(2;\mathbb{C})$.

Proof. Consider the linear action of $\mathrm{SL}(2;\mathbb{C})$ on the 4-dimensional real vector space of *two-by-two Hermitian matrices*, by conjugation. The determinant of such a matrix defines a Lorentzian metric, which we can see by writing

$$\det \begin{pmatrix} z-t & x+iy \\ -x+iy & z+t \end{pmatrix} = -t^2 + x^2 + y^2 + z^2.$$

Conjugation by matrices in $\mathrm{SL}(2;\mathbb{C})$ preserves the determinant, and $\mathrm{SL}(2;\mathbb{C})$ is connected, so we've defined a homomorphism $\mathrm{SL}(2;\mathbb{C}) \to \mathrm{SO}^+(3,1)$. What's more, $\mathrm{SL}(2;\mathbb{C})$ is simply connected, and by looking at the induced map on Lie algebras one can check that the map is surjective, which means it's the universal covering map. $\square$

Thus we've motivated why, in physics, we'd like to understand the unitary representations of the semidirect product $\widetilde{\mathcal{P}} = \mathrm{SL}(2;\mathbb{C}) \ltimes \mathbb{R}^4$. We'll classify its irreducible unitary representations.

2 Representations of the Poincaré Group

Let $G = H \ltimes N$ be a semidirect product, and for simplicity suppose N is abelian. There's a general algorithm for finding all the irreducible representations of G if one knows about the irreducible representations of H and its subgroups. We'll describe the algorithm in general, then apply it for the example we're interested in, the Poincaré group.

Remark 2.1. When we state general theorems, all groups will be locally compact topological groups, and all representations will be weakly continuous.

2.1 Action of the Translations

First of all, let's discuss representations where H acts trivially.

Definition 2.2. A one-dimensional unitary representation of a topological group G, i.e. a continuous map $\chi\colon G \to U(1)$, is called a *characters*. The space of characters is denoted by $\hat{G}$. Actually, $\hat{G}$ can itself be made into a group under convolution, and if one gives $\hat{G}$ the compact-open topology it becomes a topological group such that $\hat{\hat{G}} \cong G$ canonically, as topological groups. This is *Pontryagin duality.*

Theorem 2.3. All irreducible unitary representations of an abelian group are given by characters.

What about general unitary representations? This will be a collection of commuting unitary matrices in $U(\mathcal{H})$, and we should be able to "simultaneously diagonalise" the operators in such a representation to split our representation as something like a direct sum. In fact, with a little analytic subtlety, this is possible!

Theorem 2.4. All unitary representations of an abelian group N on a Hilbert space $\mathcal{H}$ are unitarily equivalent to representations

$$\rho\colon N \to U(L^2(X))$$

where $X \subseteq \hat{N}$ is a measurable subset, and $\rho(n)$ is the operator

$$\rho(n)(f)(\chi) = \int_X f(\chi)\chi(n)d\mu$$

where $d\mu$ is the Haar measure on $\hat{N}$.

This construction is called the *direct integral* of the characters over the subset $X \subseteq \hat{N}$, because it recovers the direct sum in the case where X is a finite set. We might denote this representation by $\int_X \chi$.

We'll now focus on the example we're interested in, where $N = M$ is Minkowski space. The space of irreducible representations is just the dual space M^*. We'll call points in this dual space *4-momenta*, and denote the dual basis to an orthonormal basis (t, x, y, z) by $(E, \rho_x, \rho_y, \rho_z)$. This is not just arbitrary, it has a real interpretation: states that have a definite position in spacetime are the *eigenvectors* for the action of the translation group M, so states that have a definite 4-momentum are the eigenvectors for the Fourier dual action of M^*.

To understand this interpretation, let's do the one-dimensional version, so $N = \mathbb{R} \cdot x$ with dual group $\hat{N} = \mathbb{R}^* \cdot p$. So characters of N are representations of form

$$\rho(ax)\colon |p\rangle \mapsto e^{-ibp(ax)}|p\rangle,$$

where $\{|p\rangle\}$ is just a basis for a one-dimensional vector space. Per our description above, more general representations "want to" have eigenvectors of form $\delta_{|bp\rangle}$ – simultaneous eigenvectors for all ax with eigenvalue $e^{-ibp(ax)}$. These are our states with definition position in spacetime. Of course these distributions don't literally live in the space of L^2 functions, but I'm just saying something heuristic here. In the space of distributions on $\mathbb{R}$, delta functions are eigenvalues for the "position" operator – multiplication by the coordinate function. So N is acting by position operators, and its Fourier dual $\hat{N}$ most naturally acts by "momentum" operators: differentiation in the coordinate direction.

With this interpretation in mind, we interpret the norm-squared in momentum space M^*

$$m^2 = E^2 - \rho_x^2 - \rho_y^2 - \rho_z^2$$

as the *mass squared*. This formula relating mass and 4-momentum should look familiar if you've studied special relativity, and even if you haven't, if the 3-momentum vanishes we're left with $m^2 = E^2$, or in differently normalised units, $m^2 = E^2c^4$. If you haven't seen this before, all you need to know is that classically, the norm of the 4-momentum of a particle travelling through space is always the same as its mass.

2.2 The Algorithm, and Little Groups

Now I'll tell you how to compute the representations of any semi-direct product with N abelian, assuming you understand the representations of the factors.

Theorem 2.5. Irreducible unitary representations of $G = H \ltimes N$ are precisely those representations arising from the following algorithm

1. Choose a character $\chi \in \hat{N}$.
2. Compute the H-orbit $\mathcal{O}_\chi$ of χ in $\hat{N}$, and the stabiliser $H_\chi \leq H$ of χ. Here H_χ is called the *little group*.
3. Choose an irreducible unitary representation ρ_χ of H_χ, and form the representation of $H_\chi \ltimes N$ where H_χ acts by $\rho(\chi)$ and N acts by the direct integral of characters over $\mathcal{O}_\chi$.

4. Induce this to a representation of all of G.

Therefore irreducible unitary representations of G are classified by orbits $\mathcal{O}_\chi$ and representations of the corresponding little group H_χ.

Remark 2.6. You're probably familiar with induction for finite groups, but what does it mean for topological groups? There's actually a nice geometric description. If H is a subgroup of G, and V is a representation of H then we can form a vector bundle over the homogeneous space G/H with fibre V from the tautological H-bundle as the fibre product

$$E = G \times_H V \to G.$$

The induced representation is just the space $L^2(E)$ of L^2-sections of E, where G acts by left multiplication on the fibre product.

So let's describe the orbits in M^* and their corresponding little groups.

1. First observe that $\mathrm{SL}(2;\mathbb{C})$ preserves the norm $m^2 = E^2 - \boldsymbol{\rho}^2$ on M^*. Recall that the action of $\mathrm{SL}(2;\mathbb{C})$ on M was identified as the conjugation action on the space of two-by-two Hermitian matrices, and the norm was the determinant. Conjugation will preserve the determinant, thus identically in the dual representation.
2. If $m^2 > 0$ then the space of points of norm m^2 is a hyperboloid of two sheets. Since $\mathrm{SL}(2;\mathbb{C})$ is connected it can't swap the sheets, and indeed each sheet is an orbit, which we denote by $\mathcal{O}_m^\pm$, $m \in \mathbb{R}_{>0}$. The stabiliser of a point in such an orbit (for instance the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$) is $\mathrm{SU}(2)$.
3. If $m^2 < 0$ then the space of points of norm m^2 is a hyperboloid of one sheet. The group $\mathrm{SL}(2;\mathbb{C})$ acts transitively on these hyperboloids, so they are orbits, which we denote by $\mathcal{O}_{im}$, $m \in \mathbb{R}_{>0}$. The stabiliser of a point in such an orbit (for instance the matrix $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$) is $\mathrm{SL}(2;\mathbb{R})$.
4. If $m^2 = 0$ then the space of points of norm m^2 is a cone. Clearly $\{0\}$ comprises an orbit with stabiliser all of $\mathrm{SL}(2;\mathbb{C})$, and the remaining two open cones are each orbits, which we denote by $\mathcal{O}_0^\pm$. The stabiliser of a point in such an orbit (for instance the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$) is $\mathrm{SO}(2) \ltimes \mathbb{R}^2$ where $\mathrm{SO}(2)$ acts on $\mathbb{R}^2$ with weight two.

This is now enough information to classify all the irreducible representations of $\widetilde{\mathcal{P}}$ using our algorithm. We just need to describe the irreducible unitary representations of the little groups.

1. For $\mathcal{O}_m^\pm$, the irreducible representations of $\mathrm{SU}(2)$ are given by symmetric powers of the two-dimensional fundamental representations. These are traditionally denoted by half integers $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ called *spins* (they're the dominant weights in the dual Cartan). They're named so essentially for historical reasons; the Lie algebra $\mathfrak{su}(2)$ is isomorphic to $\mathfrak{so}(3)$, so the operators corresponding to a representation of $\mathrm{SU}(2)$ are like *angular momenta*, just like the operators corresponding to the action of $\mathbb{R}^4$ are like 4-momenta. Here however we think of the $\mathrm{SU}(2)$ action as describing internal degrees of freedom, rather than actual angular momenta. So we interpret the representations of $\widetilde{\mathcal{P}}$ that arise as describing the space of states of a particle of mass m and spin s. Only $\mathcal{O}_m^+$ (the *positive energy* representations) are thought of as physical: representations corresponding to $\mathcal{O}_m^-$ look like particles with negative energy, or travelling backwards in time.
2. For $\mathcal{O}_{im}$ we get unphysical states, which would describe particles of imaginary mass called *tachyons* (because they would travel faster than light). Still, there are perfectly good representations of $\widetilde{\mathcal{P}}$ coming from irreducible unitary representations of $\mathrm{SL}(2;\mathbb{R})$. There's a well-known classification consisting of various infinite series of representations (principal, discrete and supplementary series) which I won't describe. It's worth noting though that it's done using a similar kind of induction trick, but from a Borel subgroup.
3. For $\mathcal{O}_0^\pm$ we can use the same trick again to describe representations of $\mathrm{SO}(2) \ltimes \mathbb{R}^2$. The $\mathrm{SO}(2)$ orbits in $\mathbb{R}^2$ are given by the origin, and the circles of radius $r > 0$, with little groups $\mathrm{SO}(2)$ and the trivial group respectively.

We thus obtain representations of $\mathrm{SO}(2) \ltimes \mathbb{R}^2$ as $\int_{S^1_r} \chi$, and from the irreducible representations of $\mathrm{SO}(2)$, which are given by weights $w \in \mathbb{Z}$. The induced representations are then given by radii r, or by half-integers $\frac{w}{2} \in \frac{1}{2}\mathbb{Z}$ which we call *helicities*. The former case doesn't have any physical applications that I know of, but the latter case describes particles travelling at the speed of light. Again only those corresponding to $\mathcal{O}_0^+$ are physical.

4. Finally, there's the trivial orbit $\{0\}$, yielding representations of $\widetilde{\mathcal{P}}$ coming from irreducible unitary representations of $\mathrm{SL}(2;\mathbb{C})$. These correspond to states that are spread out uniformly over all of space, so aren't so physically interesting (except for the *vacuum state*: the one corresponding to the trivial representation). However, again, there are lots of valid representations of $\widetilde{\mathcal{P}}$ where the translations all act trivially, and there's a well-studied classification (principal and supplementary series).

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