Chris Elliott – Research Statement

1 Introduction

My work uses tools from derived algebraic geometry and homotopical algebra to rigorously investigate classical and quantum field theories, with a particular focus on examples that can be applied to deepen our understanding of important problems in geometric representation theory, such as the geometric Langlands conjecture.

One of the most central problems for 21st century mathematical physics is the question: "how can we mathematically formalize the notion of a non-perturbative quantum field theory?" Formulations of aspects of quantum field theory developed by mathematicians (e.g. [Seg88], [Ati88], [FLM89], [Lur09], [BD04]) have generated conceptual frameworks for problems in mathematics far removed from physics. What's more, in recent decades the number of potential purely mathematical applications of quantum field theory has grown dramatically, with many striking claims made by physicists that dualities between quantum field theories have, as consequences, deep equivalences between algebro-geometric, representation theoretic and topological objects (see e.g. [Kon94], [SYZ96], [KW06], [AGT10], [MO12], [BFN14], [MNOP06a], [MNOP06b], [GV98]). As such, a deeper formal mathematical understanding of physically interesting quantum field theories and the dualities between them has the potential to lead to progress on fundamental problems in these fields. My work addresses aspects of this formalization problem in several critical examples, principally the example of $N = 4$ supersymmetric gauge theory.

In this research statement I'll describe ongoing and future work in two main directions.

1. I'm exploring mathematical formulations of $N = 4$ super Yang-Mills theory, with the goal of making rigorous and extending the connection discovered by Kapustin and Witten between S-duality for topological twists of $N = 4$ theories and the geometric Langlands correspondence. In a paper joint with Philsang Yoo [EY15] I developed a notion of twisting in derived algebraic geometry reflecting the physical intuition, and computed the Kapustin-Witten twists of $N = 4$ theory at a classical level, recovering the representation-theoretic moduli spaces relevant for geometric Langlands. In our joint paper [EY15] we prove the following theorem (the terminology is explained in the subsequent section):

   Theorem 1.1. The A- and B- topological twists of $N = 4$ super Yang-Mills theory assign to a 3-manifold of form $S^1 \times \Sigma$, where Σ has the structure of a smooth proper complex algebraic curve, the moduli spaces $T^*(\mathcal{L}\,\mathrm{Bun}_G(\Sigma)_{\mathrm{dR}})$ and $T^*(\mathcal{L}\,\mathrm{Loc}_G(\Sigma))$ respectively.

   Our calculation in particular reflects the algebraic structure on these moduli spaces, which was not visible in the work of Kapustin-Witten. In ongoing work we are describing a geometric quantization of these classical field theories on spaces to obtain 2d topological field theories related to the Hochschild homology of the geometric Langlands categories, and giving a physical explanation of the corrections to the statement of the geometric Langlands conjecture introduced by Arinkin and Gaitsgory [AG12]. I also intend to prove from this point of view that standard objects that appear in the current state of the art results in geometric Langlands [Gai13] (spaces of opers for a Levi subgroup of the gauge group) arise as classical boundary conditions in the B-twisted theory, providing a mathematical perspective on work of Gaiotto and Witten [GW09] [GW12].

   Gaiotto [Gai12] argued that S-duality for $N = 4$ gauge theories was only one example of a family of generalised S-dualities between $N = 2$ supersymmetric gauge theories with matter. I plan on extending the calculations of my joint work with Yoo to describe classical field theories and 2d topological quantum field theories associated to general $N = 2$ gauge theories, thus suggesting a generalization of the geometric Langlands conjecture for gauge groups of type A. While these "theories of class S" typically don't have classical descriptions, we will begin our analysis with the rank 1 case where this issue doesn't arise.

2. I will use the understanding of twists of classical field theories developed in my work on supersymmetric gauge theories to investigate twists of supergravity theories, especially $N = 1$ supergravity in 11 dimensions and type IIB supergravity in 10 dimensions. The prototypical example for these high-dimensional twists is $N = 1$ supergravity in dimension 4; this theory admits a holomorphic twist classically describing moduli spaces of Kähler-Einstein metrics on a 4-manifold. Twists of 10- and 11-dimensional supergravity theories will have conjectural descriptions in terms of algebro-geometric and topological objects in mathematics; for example, a description of a twist in 10 dimensions related to Calabi-Yau geometry was recently proposed by Costello and Li [CL15].

2 Twists of $N = 4$ Gauge Theories and Geometric Langlands

2.1 Background

$N = 4$ supersymmetric Yang-Mills theories in dimension 4 form a rich field of study with a range of applications both for physicists and for mathematicians. A very striking application to the field of geometric representation theory was introduced by Kapustin and Witten [KW06]. They argued that the categorical geometric Langlands conjecture, whose statement we will shortly recall, arises as a consequence of S-duality for $N = 4$ gauge theories by the following procedure: $N = 4$ gauge theories admit a $\mathbb{CP}^1$ family of "topological twists" – topological quantum field theories obtained by modifying the classical field theory using the action of a certain supersymmetry – such that duality provides an equivalence between antipodal points. In these theories, the geometric Langlands categories arise as categories of branes along a fixed surface Σ, and therefore the duality provides an equivalence of these categories.

Conjecture 2.1 (Arinkin-Gaitsgory [AG12])**.** For a complex reductive group G with Langlands dual $G^\vee$, and a smooth projective curve Σ, there is an equivalence of ∞-categories

$$\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Loc}_{G^\vee}(\Sigma)) \cong D(\mathrm{Bun}_G(\Sigma)),$$

where $\mathrm{Loc}_{G^\vee}(\Sigma)$ and $\mathrm{Bun}_G(\Sigma)$ are the derived moduli stacks of $G^\vee$ algebraic flat connections and of algebraic G-bundles on Σ respectively, and where $\mathrm{IndCoh}_{\mathcal{N}}$ denotes ind-coherent sheaves with nilpotent singular support.

From this point of view, Kapustin and Witten's analysis of the two sides of this duality was incomplete: they didn't give an explanation of the choice of algebraic structure on the moduli stacks, and they didn't give an explanation of the Arinkin-Gaitsgory singular support condition (without this condition the conjecture is false even for $\Sigma = \mathbb{P}^1$ [Laf09]).

Even addressing these subtleties would not provide a proof of the geometric Langlands conjecture: there is still no complete mathematical model for the quantum field theories in question, or for the duality between them. However, the quantum field theoretic story has provided mathematicians with profitable analogies – most prominently those coming from the theory of two-dimensional topological field theories – informing current work on the geometric Langlands program. The aim of my work is to improve these analogies, and suggest deeper and more precise connections with physical ideas, by connecting the Kapustin-Witten theories to existing mathematical constructions.

2.2 Classical Descriptions of the Kapustin-Witten Twists

In work joint with Philsang Yoo [EY15], I computed the moduli spaces of equations of motion for the classical Kapustin-Witten twists of $N = 4$ supersymmetric gauge theories as -1-shifted symplectic derived stacks. This calculation proves that $N = 4$ gauge theory naturally encodes, even at a classical level, the geometry of derived moduli stacks crucial in modern geometric representation theory, including the moduli stacks of Higgs bundles, algebraic flat connections, and the de Rham prestack of the moduli stack algebraic G-bundles on a smooth projective curve over $\mathbb{C}$. In order to obtain the quantum theory compactified on a surface – the theory most directly related to the geometric Langlands correspondence – the key calculations are the following.

Theorem 2.2 ([EY15, 4.16])**.** The moduli space of germs of solutions to the equations of motion in the B-twist of $N = 4$ super Yang-Mills near $\Sigma \times S^1$, where Σ is a compact curve, is equivalent to

$$\mathrm{EOM}_B(\Sigma \times S^1) \cong T^*(\mathcal{L}\,\mathrm{Loc}_G(\Sigma))$$

as a 0-shifted symplectic derived stack, where $\mathcal{L}\,\mathrm{Loc}_G(\Sigma)$ is the derived loop space of $\mathrm{Loc}_G(\Sigma)$.

Theorem 2.3 ([EY15, 4.25])**.** The moduli space of germs of solutions to the equations of motion in the A-twist of $N = 4$ super Yang-Mills near $\Sigma \times S^1$, where Σ is a compact curve, is equivalent to

$$\mathrm{EOM}_A(\Sigma \times S^1) \cong T^*((\mathcal{L}\,\mathrm{Bun}_G(\Sigma))_{\mathrm{dR}})$$

as a 0-shifted symplectic derived stack, where X_{dR} is the de Rham prestack of X.

2.3 Quantization of the Phase Spaces of Kapustin-Witten Twisted Theories

In the sequel to my paper with Yoo, we will partially quantize this classical data to obtain 2d topological quantum field theories encoding the Hochschild homologies of the categories in the geometric Langlands conjecture, including the data of tensoring and Hecke operators. The theorems above describing the phase spaces of the A- and B-twisted theories are amenable to canonical quantization, which yield two-dimensional topological field theories which assign to the circle the spaces of distributions (meaning sections of the dualizing sheaf) on the bases of these cotangent spaces. These rings conjecturally agree with the Hochschild homologies of the (naïve) geometric Langlands categories, i.e.

$$\mathrm{HH}_\bullet(\mathrm{IndCoh}(\mathrm{Loc}_{G^\vee}(\Sigma))) \cong \omega(\mathcal{L}\,\mathrm{Loc}_{G^\vee}(\Sigma))$$
$$\text{and } \mathrm{HH}_\bullet(D(\mathrm{Bun}_G(\Sigma))) \cong \omega_{\mathrm{dR}}(\mathcal{L}\,\mathrm{Bun}_G(\Sigma)).$$

where the notation $\omega(X)$ and ω_{dR} refer to the complexes $p_* p^! \mathbb{C}$, where p is the map $X \to \mathrm{pt}$, in the context of ind-coherent sheaves and D-modules respectively. Both the above claims are closely related to a theorem of Ben-Zvi and Nadler [BZN13, 4.2]: their methods only apply to D-modules or ind-coherent sheaves on quasi-compact stacks, but results of Drinfeld and Gaitsgory [DG13] demonstrate that $\mathrm{Bun}_G(\Sigma)$ has well-behaved sheaf theory even though it isn't quasi-compact. This is the physically expected result, since the extended 2d topological field theory that assigns one of the geometric Langlands categories to the point assigns its Hochschild homology to the circle. We can go one step further and introduce a nilpotent singular support condition on the ind-coherent side as discussed below; applying the theorem of Ben-Zvi and Nadler here will require an additional argument.

We will also describe the action of line operators in four dimensions on these complexes. We'll verify that these operators can be described as Wilson and 't Hooft operators from the point of view of gauge theory, and as the image under Hochschild homology of the Hecke and tensoring operators on the geometric Langlands categories, verifying the claim of Kapustin and Witten that this local structure is visible from the point of view of supersymmetric gauge theory.

By studying moduli spaces of vacua in these quantum field theories we have a proposal (following a suggestion of David Ben-Zvi) for obtaining a physical explanation for the nilpotent singular support condition in the work of Arinkin and Gaitsgory on geometric Langlands [AG12]. In the 4 dimensional A- and B-twisted quantum field theories, there is an action of the algebra $Z(S^3)$ of local operators on the category $Z(\Sigma)$ associated to a curve Σ. This action yields a morphism of E_2 algebras from $Z(S^3)$ to the Hochschild cohomology $\mathrm{HH}^\bullet(Z(\Sigma))$.

One can define the moduli space of vacua in a topological field theory as the spectrum of the algebra of local operators. In the B-twisted theory, we can check that the algebra of local operators is equivalent to $\mathcal{O}(\mathfrak{g}[2]/G)$, and therefore to $\mathcal{O}(\mathfrak{h}[2]/W)$. The moduli space of vacua in this theory is therefore given by the affine dg-scheme $\mathfrak{h}[2]/W$. A naïve geometric quantization procedure suggests assigning to the curve Σ the category $Z_{\text{naïve}}(\Sigma) = \mathrm{IndCoh}(\mathrm{Loc}_G(\Sigma))$. This is indeed acted upon by $Z(S^3)$, and therefore for any object $\mathcal{F} \in \mathrm{IndCoh}(\mathrm{Loc}_G(\Sigma))$ there is a morphism

$$v_{\mathcal{F}} \colon \mathcal{O}(\mathfrak{h}[2]/W) \to \mathrm{HH}^\bullet(\mathrm{IndCoh}(\mathrm{Loc}_G(\Sigma)) \to \mathrm{End}^\bullet(\mathcal{F}).$$

Choosing a point $x \in \mathfrak{h}/W$ yields a maximal ideal $\mathfrak{m}_x$ in $\mathcal{O}(\mathfrak{h}[2]/W)$ (although this is not a dg-ideal unless $x = 0$). We think of this as a *choice of vacuum state*. We expect to recover Arinkin-Gaitsgory's nilpotent singular support condition by making a choice of vacuum, in the following way.

Conjecture 2.4. For the vacuum state $x = 0$, the full subcategory generated by those objects $\mathcal{F}$ such that $v_{\mathcal{F}}(\mathfrak{m}_0) = 0$ is the category $\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Loc}_G(\Sigma))$ of objects with nilpotent singular support.

This construction makes sense for general choices of vacuum states in topological field theories. We intend to verify that for the analogous story on the A-side, the condition is vacuous – that every D-module already has nilpotent singular support in this sense. What's more, we intend to investigate the categories appearing for more general choices of vacuum. For instance, choosing a regular element of $\mathfrak{h}/W$ should yield – on the B-side – the category $\mathrm{IndCoh}_{\mathrm{reg}}(\mathrm{Loc}_H(\Sigma))$ of ind-coherent sheaves whose singular support is regular, on the moduli stack of local systems for a maximal torus; this choice of vacuum can be thought of as degenerating the boundary conditions to an abelian subgroup of the gauge group. Further work is required to identify the dual theory on the A-side, but it should exhibit a similar degeneration to an abelian theory. There will be a rich family of intermediate correspondences between the two extreme possibilities of $0 \in \mathfrak{h}$ and a regular element in $\mathfrak{h}$, corresponding to intermediate strata in $\mathfrak{h}/W$.

2.4 Boundary Conditions and Opers

A full understanding of the Kapustin-Witten analogy, and indeed a full understanding of the nature of any classical field theory, requires an understanding of boundary conditions for classical equations of motion, and quantizations of these objects. I'd like to begin my investigation of this structure with a fundamental family of examples, introduced by Gaiotto and Witten [GW09] [GW12]. These will be classical boundary conditions for an $N = 4$ gauge theory which satisfy a maximal amount of supersymmetry: invariance under exactly half of the supersymmetries (this is maximal, because with any more invariance than this, one can derive invariance under all infinitesimal translations by taking Lie brackets of invariant supercharges, which is impossible, because boundary conditions can never be invariant under translations perpendicular to the boundary). The subalgebra consisting of these invariant symmetries is referred to as the 1/2-BPS subalgebra of the supersymmetry algebra.

Gaiotto and Witten describe such supersymmetric boundary conditions on $\Sigma \times \mathbb{R} \times \mathbb{R}_{>0}$ for a Riemann surface Σ associated to an $\mathfrak{sl}_2$ triple in the Lie algebra of the gauge group. These boundary conditions are described as being determined by the residues of three scalar fields at the boundary, which generate a subalgebra of the gauge Lie algebra isomorphic to $\mathfrak{sl}_2$.

A priori, these boundary conditions describe subspaces of the moduli space of supersymmetry invariant solutions to the equations of motion near the boundary. These subspaces allow us to obtain natural subspaces of the space of solutions near the boundary in the *B-twisted* theory. Indeed, the fermionic piece of the 1/2-BPS subalgebra is generated by a supercharge intermediate between the holomorphic and B supercharges, so interpreting the boundary conditions considered by Gaiotto and Witten as derived stacks mapping to the stack of derived 1/2-BPS invariant solutions to the equations of motion near the boundary, we can pull the boundary conditions back to the space of derived invariants for the B-twist.

Conjecture 2.5.
1. The canonical inclusion of the derived invariants with respect to the B-supercharge into the classical B-twisted theory on Σ admits a natural splitting $\pi\colon \mathrm{EOM}_B(\Sigma) = T^*[1]\,\mathrm{Loc}_G(\Sigma) \to \mathrm{EOM}(\Sigma)^{Q_B}$.
2. For the principal $\mathfrak{sl}_2$ associated to a Levi subgroup $L \subseteq G$, the pullback of the Gaiotto-Witten boundary condition along the splitting map π coincides with the shifted conormal space
$$N^*[1]\mathrm{Op}_L(\Sigma) \to T^*[1]\,\mathrm{Loc}_G(\Sigma)$$
where $\mathrm{Op}_L(\Sigma) \to \mathrm{Loc}_G(\Sigma)$ is the moduli space of opers for the Levi subgroup.

Gaiotto and Witten [GW09] gave a related description of these boundary conditions in terms of opers at a physical level of rigour, but only for the group $G = \mathrm{SL}_n$, and without explicitly investigating the relationship with the B-twisted theory.

Given this description, these Gaiotto-Witten boundary conditions have a natural origin from the point of view of geometric Langlands. Sheaves of the form $\mathcal{O}_{\mathrm{Op}_L(\Sigma)} \in \mathrm{QC}(\mathrm{Loc}_G(\Sigma))$ play an important role in the work of Arinkin, Drinfeld and Gaitsgory [Gai13] towards a proof of the geometric Langlands conjecture. These sheaves arise from the structure sheaf of the moduli space of G-opers under the geometric Eisenstein series functor $\mathrm{QC}(\mathrm{Loc}_L(\Sigma)) \to \mathrm{QC}(\mathrm{Loc}_G(\Sigma))$. The Langlands dual of $\mathcal{O}_{\mathrm{Op}_G(\Sigma)}$ is known by work of Beilinson and Drinfeld, and the geometric Eisenstein series functor and its analogue on the A-side intertwines Langlands duality. The objects $\mathcal{O}_{\mathrm{Op}_L(\Sigma)}$ therefore can be used as "building blocks" in a proof of the Langlands correspondence. The shifted conormal $N^*[1]\mathrm{Op}_L(\Sigma)$ can be thought of as a classical limit of this sheaf: up to a shift it coincides with the sheaf's singular support.

2.5 $N = 2$ Theories and Gaiotto Duality

The projects discussed so far build a bridge between supersymmetric gauge theory and geometric representation theory; in work joint with Thel Seraphim and Philsang Yoo I plan to describe new correspondences in geometric representation theory coming from dualities between supersymmetric gauge theories systematically described by Gaiotto [Gai12], simultaneously generalising S-duality and a duality for SU(3) $N = 2$ gauge theories of Argyres and Seiberg [AS07].

We will begin by describing the moduli spaces of solutions to the equations of motion in twists of $N = 2$ gauge theories with matter, guided by the example of adjoint matter, which recovers the $N = 4$ theory discussed above. It's a straightforward algebraic problem to classify the possible twists, which in particular include A- and B- type topological twists – as in $N = 4$ – and a holomorphic-topological twist first described by Kapustin [Kap06] which is fixed by S-duality. With a description of the phase spaces in these twisted theories associated to the product of a Riemann surface with a circle, we can describe the geometric quantization and the moduli spaces of vacua to obtain conjectural descriptions of the categories of branes in these $N = 2$ theories.

Having done this, Gaiotto duality provides conjectural equivalences between these categories, generalising the geometric Langlands correspondence. As an exploratory example, we will begin with Gaiotto's initial family of examples: SU(2) quiver gauge theories. These are $N = 2$ supersymmetric theories that admit classical Lagrangian descriptions, so our classical techniques always apply. Here Gaiotto's duality acts as the mapping class group on a Teichmüller space viewed as a moduli space of coupling constants, so appropriately dualising the twist we obtain a collection of dual theories, for instance, one associated to each cusp in the moduli space. This will serve as an access point in to the far more complicated family of examples associated to higher rank type A gauge groups.

2.6 Abelian Duality and Geometric Class Field Theory

It is worthwhile to consider the example of $N = 4$ gauge theory with an abelian gauge group. This theory is free, therefore much simpler to analyse. In my 2014 paper [Ell14], I described dualities between generalized gauge theories with dual abelian gauge group in all dimensions, as a version of the Fourier transform. In particular, one obtains an abelian S-duality by considering ordinary abelian gauge theories in dimension 4. I intend to extend this work to also include supersymmetric gauge theories in dimension 4.

It would be very interesting to relate this abelian duality for free $N = 4$ theories to the Kapustin-Witten story, obtaining a new perspective on geometric class field theory. There are several steps necessary to understand this relationship. Firstly, I will need to investigate the supersymmetry action on the two sides of the correspondence; I expect the duality to descend to a duality between appropriate twisted theories, which would follow from a supersymmetry equivariance for the correspondence. Secondly, I will need to understand the relationship between the duality I've constructed – a correspondence of dual factorization algebras – and the categories of boundary conditions considered by Kapustin and Witten. A natural first step towards this understanding would be to apply theorems of Scheimbauer [Sch14], which associate to a translation-invariant factorization algebra an extended topological field theory (in the sense of Lurie) whose target is an E_n Morita category. In particular, to a surface Σ, the theorem will associate an E_2 algebra (with additional structure) whose module category we can compare to the expected categories from geometric class field theory.

3 Twists of Supergravity Theories

3.1 Background and Initial Computations

The idea of twisting has been very profitable in the study of supersymmetric gauge theories. In particular it allows mathematicians to study physically interesting dualities between gauge theories at the level of topological field theories where the link to important objects in mathematics is comparatively well understood. It is natural to ask whether the same idea can be used for supersymmetric field theories describing gravity, which play a role in dualities like the famous AdS/CFT correspondence: by understanding twists of gravitational theories this duality may lead to interesting mathematical discoveries.

There is a well-developed theory of classical supergravity theories in a wide range of dimensions, with the largest natural example being provided by $N = 1$ supergravity in 11-dimensions [CJS78], with many other theories obtainable from this theory by dimensional reduction. This theory, and its descendents, play an important role in string theory; 11-dimensional supergravity is supposed to arise as the low-energy limit of M-theory. The construction of these theories, however, still has a rather ad-hoc flavour; the action functional in particular is determined from the unique Lagrangian density which is polynomial in the fields and their field strengths, which is locally invariant under the action of $\mathrm{SO}(10,1) \times \mathbb{C}^\times$ by brute force computation (see for example [CDF91]). I intend to apply my understanding of the theory of twisting for classical field theories to define classical field theories which admit more natural geometric descriptions.

Example 3.1. For a representative example of such a geometric description, in work in progress I am computing the holomorphic twist of $N = 1$ supergravity in dimension 4. Based on perturbative calculations using BV complex techniques introduced by Costello [Cos11], I believe the resulting twisted theory has as its classical solutions the moduli space of *Kähler-Einstein metrics* on a 4-manifold. Proving this at a global level will require adapting the techniques from my preprint with Yoo [EY15] to supergravity theories.

3.2 10 and 11d Supergravity

I'd like to extend this calculation to describe twists of ten and eleven dimensional supergravity theories, in particular addressing some recent conjectures of Costello and Li. Eleven dimensional $N = 1$ supergravity dimensionally reduces to a ten dimensional theory with local $(2, 0)$ supersymmetry called type IIB supergravity. In recent work [CL15], Costello and Li investigated the existence of a perturbative quantization for BCOV theory on $\mathbb{C}^d$. Motivated by string theory, they give a conjectural relationship between (classical) BCOV theory and type IIB supergravity.

Conjecture 3.2. There is a holomorphic twist of 10-dimensional type IIB supergravity equivalent to BCOV theory on $\mathbb{C}^5$.

The Kählerian condition in four dimensions appears because the stabiliser of a Weyl spinor in dimension 4 is isomorphic to $\mathrm{SU}(2)$; this has the result – perturbatively – of restricting the holonomy of a gauge field to this subgroup. This calculation suggests that interesting geometric conditions can arise in dimension 11, because the classification of possible stabilisers of spinors in 11-dimensions [Igu70] is rich, with – for instance – the groups $\mathrm{Sp}(4;\mathbb{C})$, $\mathrm{Spin}(7)$ and G_2 appearing as factors of possible stabilizer groups. For example, Costello recently gave a conjectural perturbative description of the twist by a Dirac spinor Q_7 such that the image of the map $[Q_9, -]\colon S \to \mathbb{C}^{11}$ is 9-dimensional. According to Igusa's classification, the stabiliser of such a spinor is equivalent to $(G_2 \times \mathrm{SL}(2;\mathbb{C})) \otimes U_{15}$, where U_{15} is a unipotent group of dimension 15.

Conjecture 3.3. Perturbatively, the twist of 11-dimensional $N = 1$ supergravity with respect to the spinor Q_9 is equivalent to the classical theory with fields $\Omega^\bullet(\mathbb{R}^7) \otimes \Omega^{0,\bullet}(\mathbb{C}^2)$ and action functional

$$S(\phi) = \int \phi(d_{\mathrm{dR}} + \overline{\partial})\phi + \phi\{\phi, \phi\} dz_1 dz_2$$

where ϕ is a general inhomogeneous differential form in the space of fields.

The twisting language I am developing will lead to a non-perturbative description of this twisted theory, which should extend to a well-defined theory on any manifold of form $M_1 \times M_2$, where M_1 is a G_2 manifold and M_2 is a complex surface.

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