Lecture on Group Cosets – Context in a Course

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My provided ten minute teaching sample on group cosets would take place as part of the course Math0006: Algebra 2, within the segment of the course covering introductory group theory. I have modelled the plan for this part of the course using a similar course that I taught at the University of Massachusetts, Amherst in 2021, on an introduction to group theory (see the syllabus here: `https://people.math.umass.edu/~celliott/411_Syllabus_2021.pdf`).

Prerequisite Material

My teaching sample builds on the following material, that I would have covered earlier in the course.

- Important examples of groups, including the integers $\mathbb{Z}$, finite cyclic groups, and dihedral groups D_n.
- Subgroups, and the notion of the subgroup generated by an element of a group.
- Equivalence relations and equivalence classes, including the definition, examples, and the idea that the equivalence classes for an equivalence relation R on a set X provides a partition of X into disjoint subsets.

Subsequent Material

Having introduced group cosets, I would be able to use them to study some other foundational ideas in group theory.

- Lagrange's theorem. After the material in this lecture sample, I could point out that all (left) cosets for a subgroup H of a finite group G must have the same size, by observing that multiplication by g provides a bijection $H \to gH$. Combined with the observation from the lecture – that left cosets form a partition of G, I would define the index $[G : H]$ and prove Lagrange's theorem.
- Quotient groups. I would start by attempting to define a group operation on the set G/H of left cosets of H in G for a general group, and ask whether it makes sense (of course, the answer will turn out to be if and only if H is normal). We would work our way to the answer by trying the examples where $G = D_3$, and H is generated either by an element of order two or of order three. This would lead me to emphasise the idea of an operation being well-defined, and eventually to the definition of a normal subgroup.

Lesson Plan

To give a bit more detail, here is an outline for a 50 minute lesson plan that includes the 10 minute sample. The teaching sample covers points 2 and 3 in this lesson plan.

1. Introduction: summarize the goals of today's lecture.
2. Motivate the definition by talking about the decomposition of $\mathbb{Z}$ into three residue classes modulo 3 (cosets for the subgroup $3\mathbb{Z} \subseteq \mathbb{Z}$).
3. Give the general definition of a (left) coset, and give another example, namely the cosets for the cyclic subgroup of D_3 generated by an element of order 3.
4. Prove that left cosets for $H \subseteq G$ are equivalence classes for the equivalence relation gRg' if $g^{-1}g' \in H$.
5. Remark on the fact that there is an analogous story for right cosets, defined very similarly.
6. Extended example: for the group D_3 again, consider a subgroup H generated by an element of order 2 now. Describe the left and the right cosets, and notice that they are different to one another.

This would lead into the following lecture, in which I would introduce Lagrange's theorem, with some of its examples and applications.