An Example of Abelian Duality

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In this note I will describe the simplest non-trivial example of abelian duality: relationships between a family of free field theories where the fields are maps $S^2 \to S^1$. I'll describe the factorisation algebras associated to these theories, and prove interesting relationships between these, which I expect to arise as part of an isomorphism of factorisation algebras.

By a *dg-vector space* V we mean a cochain complex of nuclear Fréchet spaces over a field k of characteristic zero (always $\mathbb{R}$ or $\mathbb{C}$ in this note). By the dual $V^\vee$ of such a complex, we will mean the continuous dual graded vector space, with the dual differential, equipped with the strong topology. A dg-space is a space X equipped with a sheaf $\mathcal{O}_X$ of dg-vector spaces such that $(X, H^0(\mathcal{O}_X))$ is a smooth scheme.

1 Description of the Factorisation Algebras

Let Φ be the sheaf of $\mathbb{R}/2\pi R\mathbb{Z}$-valued functions on S^2, thought of as a sheaf of spaces. The exterior derivative map $d\colon \Phi \to d\Omega^0 \subseteq \Omega^1$ is surjective, and has kernel the constant sheaf $\mathbb{R}/2\pi R\mathbb{Z}$. After choosing a Riemannian metric g on S^2 we can define a map of sheaves $S_R\colon \Phi \to \mathbb{R}$ by

$$S_R(\phi) = -\|d\phi\|^2 = -\int d\phi \wedge *d\phi$$

This implicitly depends on the radius R of the target circle: the dependence is in the exterior derivative map, which involves locally lifting functions from $\mathbb{R}/2\pi R\mathbb{Z}$ fo $\mathbb{R}$. The kernel of S_R also consists of precisely the constant functions.

1.1 Classical Observables

The *classical observables* associated to this classical field theory are described as a sheaf of dg-spaces on S^2 built from Φ and the action map S_R. We define

$$\mathrm{Obs}^{\mathrm{cl}}(U) = (\mathrm{PV}(\Phi(U)), \iota_{dS}),$$

that is, the dg-space with the same underlying *space* as $\Phi(U)$, but with structure sheaf the sheaf of polyvector fields on $\Phi(U)$, equipped with a differential by contracting with the 1-form ι_{dS}. For a more explicit description, on any open set U, by choosing a basepoint $u \in U$ we can split the derivative map, that is, produce an isomorphism

$$\Phi(U) \leftrightarrow \mathbb{R}/2\pi R\mathbb{Z} \times d\Omega^0(U)$$

by sending ϕ to $(\phi(u), d\phi)$, and in the other direction (θ, α) to $\theta + d^{-1}\alpha$ where the preimage is chosen so that $d^{-1}\alpha(u) = 0$. So we need only understand $\mathcal{O}(\mathbb{R}/2\pi R\mathbb{Z}) \otimes \mathrm{PV}(d\Omega^0(U))$, and the differential, which acts only on the second factor. This factor has a nice description as

$$\mathrm{Sym}(d\Omega^0(U)^\vee \oplus d\Omega^0(U)[1]),$$

and the differential ι_{dS} is extended as a derivation from the embedding $d\Omega^0(U)[1] \to d\Omega^0(U)^\vee$ given by the metric. This complex has no cohomology, so in fact

$$\mathrm{Obs}^{\mathrm{cl}}(U) \cong \mathcal{O}(\mathbb{R}/2\pi R\mathbb{Z}).$$

The sheaf of polyvector fields with the differential ι_{dS} gives a model for functions on the moduli space of classical observables: in this case the solutions to the equations of motion are exactly the constant functions.

1.2 Smearing and Quantisation

To quantise these classical observables we introduce the BV operator on polyvector fields. This operators is defined by identifying $PV(\Phi(U))$ as functions on the shifted cotangent bundle $\mathcal{O}(T^*[-1]\Phi(U))$. As such, there is a natural Poisson bracket coming from the symplectic pairing on this cotangent bundle which we can use to deform the differential on the classical observables.

Using the local description above away from constant functions, the BV operator D is easy to describe: one uses the natural degree 1 pairing $d\Omega^0(U)^\vee \otimes d\Omega^0(U)[1] \to \mathbb{R}$ from $\mathrm{Sym}^2 \to \mathrm{Sym}^0$, and extend to all of $\mathrm{PV}(d\Omega^0(U))$ by the formula

$$D(\mathcal{O}_1\mathcal{O}_2) = D(\mathcal{O}_1)\mathcal{O}_2 + (-1)^{|\mathcal{O}_1|}\mathcal{O}_1 D(\mathcal{O}_2) + \{\mathcal{O}_1, \mathcal{O}_2\}$$

where $\{-,-\}$ denotes the natural pairing on $\mathrm{Sym}(d\Omega^0(U)^\vee) \otimes \mathrm{Sym}(d\Omega^0(U)[1]) \leq \mathrm{Sym}(d\Omega^0(U)^\vee \oplus d\Omega^0(U)[1])$, and is zero on other elements of the algebra.

We can compute this operator explicitly: the dual space $d\Omega^0(U)^\vee$ can be identified as the space orthogonal to the constant functions in a space of distributions on U. Focusing on the global observables – i.e. $U = S^2$ – we find tempered distributions on S^2. The natural Poisson bracket pairing $d\Omega^0(S^2)^\vee \otimes d\Omega^0(S^2)$ then corresponds to the pairing

$$(\psi, \phi) \mapsto R^2 \int_{S^2} \psi(\Delta\phi)\,\mathrm{dvol}$$

for $\psi \in \mathbb{R}^\perp \leq \mathrm{Dist}(S^2)$ and $\phi \in \mathbb{R}^\perp \leq \Omega^0(S^2)$, where Δ is the Laplacian for the given metric on S^2. One can imagine a similar description on $U \subseteq S^2$, but where one restricts to an appropriate collection of distributions such that the integrals are guaranteed to be finite.

We'd like to define the factorisation algebra of *quantum observables* to be

$$\mathrm{Obs}^{\mathrm{q}}(U) = (\mathrm{PV}(\Phi(U)), \iota_{dS} + D),$$

but this is not quite the right definition: this complex has infinitely generated cohomology. We resolve this by *smearing*, which is a two-step procedure most easily described on the quotient sheaf of vector spaces $\mathrm{PV}(d\Omega^0)$ where we have a more concrete description of the differentials. First, the metric on S^2 gives us an inclusion map on sections over U of form

$$\iota_1\colon \mathrm{Sym}(d\Omega^0(U)^\vee \oplus d\Omega^0(U)[1]) \hookrightarrow \mathrm{Sym}(d\Omega^0(U)^\vee \oplus d\Omega^0(U)^\vee[1]).$$

That is, we include the shifted tangent bundle into the shifted cotangent bundle. At this level the Laplacian has finite-dimensional kernel and cokernel, but we no longer have a natural symplectic pairing. We have both a BV operator with finite-dimensional cohomology *and* a natural symplectic pairing on the subalgebra

$$\iota_2\colon \mathrm{Sym}(d\Omega^0(U) \oplus d\Omega^0(U)[1]) \hookrightarrow \mathrm{Sym}(d\Omega^0(U)^\vee \oplus d\Omega^0(U)^\vee[1]).$$

generated only by functions rather than distributions. We choose a map P in the other direction split by ι_2, and call P a *parametrix*. This complex inherits a differential from the differential $\iota_{dS} + D$ by translating along $P \circ \iota_1$: the resulting cosheaf is called the factorisation algebra of *smeared quantum observables*. The smearing described extends to the whole dg-space including the constant maps since the map $P \circ \iota_2$ doesn't depend on the choice of basepoint $u \in U$, so we can choose a basepoint, smear, then forget the basepoint.

1.3 Expectation Values

We're particularly interested in those observables in degree 0 of the complex (or in terms of dg-spaces the degree 0 sections of the structure sheaves). These are the "functions on fields" one thinks of as observables for the theory. We'll denote this sheaf of spaces $\mathrm{Obs}^{\mathrm{q}}_0$. By understanding the cohomology of $\mathrm{Obs}^{\mathrm{q}}$ we'll be able to construct an *expectation value* map from $\mathrm{Obs}^{\mathrm{q}}_0(S^2) \to \mathbb{R}$.

We saw that the classical observables $\mathrm{Obs}^{\mathrm{cl}}(U)$ were quasi-isomorphic to functions on the classical moduli space – the circle – by construction. In fact *globally* (but not locally) this is also true of the smeared quantum observables. We can check this using a spectral sequence argument, using the filtration of the complex of global observables by Sym degree. We restrict as above to the quotient complex $\mathrm{PV}(d\Omega^0(S^2))$, or rather its smeared version

$$\mathrm{Sym}(\Omega^0_\perp(S^2) \oplus \Omega^0_\perp(S^2)[1])$$

as a dg vector space (where the $\perp$ symbol indicates that we're working with the orthogonal complement to the constant functions). The classical differential in this language is just *the identity*, so the classical complex is isomorphic to $\mathbb{R}$ in degree 0 (as only Sym^0 survives).

The BV operator in this language is extended from the Laplacian pairing map on functions

$$(\phi_1, \phi_2) \to R^2 \int_{S^2} \phi_1 \Delta \phi_2 \,\mathrm{dvol}$$

from $\mathrm{Sym}^2 \to \mathrm{Sym}^0$, so in general it lowers Sym degree by two. Consider the spectral sequence of a filtered complex using the Sym degree filtration. The E_1 page of this spectral sequence computes the cohomology of the classical complex of smeared observables (i.e. the cohomology with respect to only the Sym degree 0 part of the differential), and the spectral sequence converges to the cohomology of the complex of smeared quantum observables (i.e. the cohomology with respect to the entire differential). Since the E_1 page is quasi-isomorphic to $\mathbb{R}$ in degree 0, so must be the E_∞ page. There is a unique quasi-isomorphism from this complex of global smeared observables to $\mathbb{R}$ characterised by the property that 1 in Sym^0 maps to 1.

So we define the *expectation value* $\langle \mathcal{O} \rangle$ of an observable $\mathcal{O} \in \mathrm{Obs}^{\mathrm{q}}_0(U)$ as follows:

- Use the parametrix to map $\mathcal{O}$ to a smeared observable.
- Extend this smeared observable to a global smeared observable via the factorisation algebra structure.
- This is now an element of $\mathcal{O}(\mathbb{R}/2\pi R\mathbb{Z}) \otimes \mathrm{Sym}(\Omega^0_\perp(S^2))$, so we compute its image using the distinguished map $\mathcal{O}(\mathbb{R}/2\pi R\mathbb{Z}) \otimes \mathrm{Sym}(\Omega^0_\perp(S^2)) \to \mathcal{O}(\mathbb{R}/2\pi R\mathbb{Z}) \otimes \mathbb{R}$ characterised above.
- Compute the image of this element under the integration map $\mathcal{O}(\mathbb{R}/2\pi R\mathbb{Z}) \to \mathbb{R}$.

This procedure depends on the parametrix P we chose above, but on no other data.

1.4 Computing Expectation Values with Feynman Diagrams

The idea of Feynman diagrams is to compute expectation values of observables combinatorially. The crucial idea that we'll use to check that we an do this is that –for smeared observables – the expectation value map is *uniquely characterised*. Let's ignore for now the final step in computing expectation values: integrating out the constant maps. Then for smeared observables there is a unique quasi-isomorphism from global smeared observables to $\mathcal{O}(\mathbb{R}/2\pi R\mathbb{Z})$ that sends 1 to 1. Therefore to check that a procedure for computing expectation values is valid it suffices to check that it is a non-trivial quasi-isomorphism, then rescale so the map is appropriately normalised.

Take a smeared observable $\mathcal{O} \in \mathrm{Obs}^{\mathrm{q}}_0(U)$. Suppose can write $\mathcal{O}$ as a product of linear observables

$$\mathcal{O} = \mathcal{O}_1^{n_1} \mathcal{O}_2^{n_2} \cdots \mathcal{O}_k^{n_k}$$

where $\mathcal{O}_1, \ldots \mathcal{O}_k$ are linearly independent linear smeared observables in $\Omega^0_\perp(U)$ (general observables are sums of observables of this form, and we extend the procedure of computing duals linearly). More generally, after choosing a local splitting of $\mathrm{Obs}^q_0(U)$ we can take a local section of the dg-space $\mathrm{Obs}^q_0(U)$ over some open set $V \times d\Omega^0(U)$, which will have the form above times a local section of the structure sheaf of the circle over V. We'll ignore this aspect for now – we compute the expectation value by integrating it out at the last step, and it won't play an interesting role in the duality – and just concentrate on the quotient factorisation algebra away from the constant functions.

We compute the expectation value of $\mathcal{O}$ combinatorially as follows. Depict $\mathcal{O}$ as a graph with k vertices, and with n_i half edges attached to vertex i. The Fourier dual $\widetilde{\mathcal{O}}$ of $\mathcal{O}$ consists of a sum of observables constructed by gluing edges onto this frame in a prescribed way. Specifically, we attach *propagator edges* – which connect together two of these half-edges – in order to leave no free half-edges remaining. A propagator between linear observables $\mathcal{O}_i$ and $\mathcal{O}_j$ receives weight

$$\frac{1}{2R^2}\int_{S^2} \mathcal{O}_i \Delta^{-1} \mathcal{O}_j \,\mathrm{dvol}$$

and a diagram is weighted by the product of all these edge weights. The expectation value is the sum of these weights over all such diagrams.

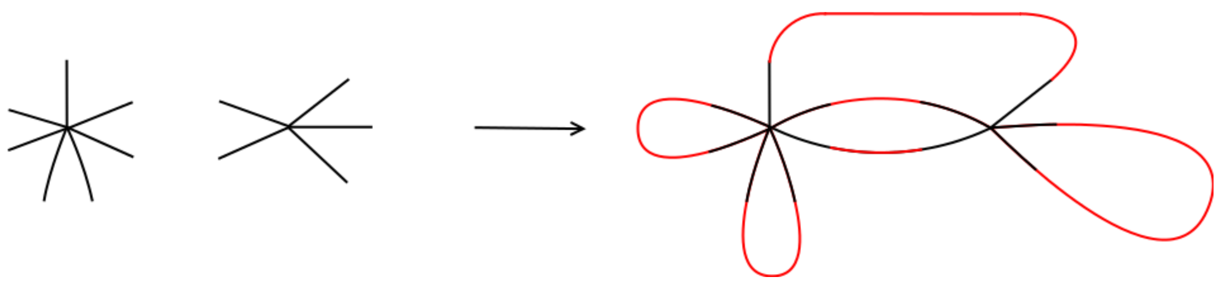

Figure 1: One of the terms in the Feynman diagram expansion computing the expectation value of an observable of form $\mathcal{O}_1^7\mathcal{O}_2^5$. On the left we see the starting point, with half-edges (in black), and on the right we see one way of connecting these half-edges with propagator edges (in red).

To check that this computes the expectation value, we must show that it is non-zero, and that it vanishes on the image of the differential in the complex of quantum observables. The former is easy: the observable 1 has expectation value 1 (so we're also already appropriately normalised). For the latter, we'll show that the path integral computation for global observables in $\mathrm{Obs}^q_0(S^2)$ arises as a limit of finite-dimensional Gaussian integrals, and that the images of the quantum BV differential are all divergences, so vanish by Stokes' theorem.

To set up this limit, recall that the Laplacian Δ acting on $\Omega^0_\perp(S^2)$ has a discrete spectrum $0 < \lambda_1 < \lambda_2 < \cdots$, with finite-dimensional eigenspaces (spanned by the spherical harmonics Y^m_ℓ). Let $F^k\Omega^0(S^2)$ denote the sum of the first k eigenspaces: this defines an exhaustive increasing filtration of $\Omega^0_\perp(S^2)$ by finite-dimensional vector spaces.

Proposition 1.1. Let $\mathcal{O}$ be a smeared global observable. The finite-dimensional Gaussian integrals

$$\frac{1}{Z_k}\int_{F^k\Omega^0(S^2)} \mathcal{O}(\phi)e^{S(\phi)}d\phi,$$

where Z_k is the volume $\int_{F^k\Omega^0(S^2)} e^{S(\phi)}d\phi$, converge to a real number $I(\mathcal{O})$ as $k \to \infty$, and this number agrees with the expectation value computed by the Feynman diagrammatic method.

Proof. We check that for each k the Gaussian integral admits a diagrammatic description, and observe that the expressions computed by these diagrams converge to the expression we want. We may assume as usual that $\mathcal{O}$ splits as a product of linear smeared observables $\mathcal{O} = \mathcal{O}_1^{n_1}\mathcal{O}_2^{n_2}\cdots\mathcal{O}_k^{n_k}$. The $\mathcal{O}_i$ describe linear operators on the filtered pieces. We can write the Gaussian integral using a generating function as

$$\int_{F^k\Omega^0(S^2)} \mathcal{O}(\phi)e^{S(\phi)}d\phi = \left.\frac{\partial^{n_1+\cdots n_k}}{\partial t_1^{n_1}\cdots\partial t_k^{n_k}}\right|_{t_1=\cdots=t_k=0}\int_{F^k\Omega^0(S^2)} e^{S(\phi)+t_1\mathcal{O}_1(\phi)\cdots+t_k\mathcal{O}_k(\phi)}d\phi$$

(here S denotes the action, thought of as a negative definite quadratic form). We can simplify the integral on the right-hand side by completing the square to get

$$Z_k \int_{F^k\Omega^0(S^2)} e^{\frac{1}{4}(t_1\mathcal{O}_1(\phi)\cdots+t_k\mathcal{O}_k(\phi))A^{-1}(t_1\mathcal{O}_1(\phi)\cdots+t_k\mathcal{O}_k(\phi))}d\phi$$

where A is the restriction of the non-degenerate pairing

$$(\phi_1,\phi_2)\mapsto R^2\int_{S^2}\phi_1\Delta\phi_2\,\mathrm{dvol}$$

to the k^{th} filtered piece of the space of fields $F^k\Omega^0(S^2)$. The linear observables $\mathcal{O}_i$ are given by pairing with fields ψ_i. These fields project onto the filtered pieces, so from now on choose we restrict to k large enough that the projections of the ψ_i are linearly independent. Choose a basis for $F^k\Omega^0(S^2)$ that includes the projections of all the ψ_i, and let A^{-1}_{ij} denote the matrix elements of A^{-1} corresponding to these basis elements: that is

$$A^{-1}_{ij}=\frac{1}{R^2}\int_{S^2}\psi_i\Delta^{-1}\psi_j.$$

We can now compute the Gaussian integral diagrammatically. The $t_1^{n_1}\cdots t_k^{n_k}$-term of the integral is the sum over Feynman diagrams as described above, where a diagram is weighted by a product of matrix elements $\frac{1}{2}A^{-1}_{ij}$ corresponding to the edges. We see that this is constant for sufficiently large level in the filtration, and agrees with the expression we gave above. □

So we can now compute the expectation values of all degree 0 observables, by smearing, and applying the Feynman diagram method. The next task is to investigate duality, and how this procedure interacts with it.

2 Fourier Duality for Quantum Observables

In this section I'll describe a Fourier duality map $\mathcal{F}\colon \mathrm{Obs}^{\mathrm{q}}_0(U)\to\mathrm{Obs}^{\mathrm{q}}_0(U)^\vee$, then explain how this map leads to an interesting isomorphism of prefactorisation algebras of smeared observables.

2.1 Feynman Diagrams for Fourier Duality

We'll construct $\mathcal{F}$ in an explicit combinatorial way using Feynman diagrams extending the Feynman diagram expression computing expectation values. Take a smeared monomial observable $\mathcal{O}\in\mathrm{Obs}^{\mathrm{q}}_0(U)$. As above, we write $\mathcal{O}$ as

$$\mathcal{O}_1^{n_1}\mathcal{O}_2^{n_2}\cdots\mathcal{O}_k^{n_k}$$

where $\mathcal{O}_1,\ldots\mathcal{O}_k$ are linearly independent linear smeared observables in $\Omega^0_\perp(U)$.

We compute the Fourier dual of $\mathcal{O}$ much as before. Depict $\mathcal{O}$ as a graph with k vertices, and with n_i half edges attached to vertex i. Now, we can attach any number of *propagator edges* as before, and also any number of *source terms* – which attach to an initial half-edge and leave a half-edge free – in such a way as to leave none of the original half-edges unused. The result is a new observable

$$\mathcal{O}_1^{m_1}\mathcal{O}_2^{m_2}\cdots\mathcal{O}_k^{m_k}$$

where m_i is the number of source edges connected to vertex i. The total Fourier dual observable $\widetilde{\mathcal{O}}$ is then the sum of all these observables with appropriate weightings.

Again, we weight such a diagram by taking a product of weights attached to each edge. Edges are weighted in the following way:

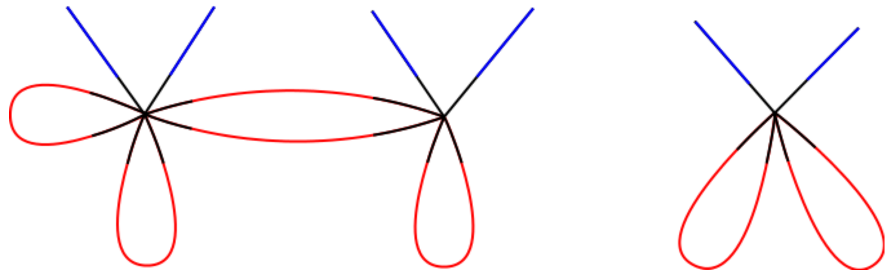

Figure 2: The Feynman diagram corresponding to a degree 6 term in the Fourier dual of an observable of form $\mathcal{O}_1^8\,\mathcal{O}_2^6\,\mathcal{O}_3^6$. Propagators are drawn in red and sources in blue.

- A propagator between linear observables $\mathcal{O}_i$ and $\mathcal{O}_j$ receives weight

$$\frac{1}{2R^2}\int_{S^2}\mathcal{O}_i\Delta^{-1}\mathcal{O}_j\,\mathrm{dvol}\,.$$

- A source term attached to a linear observable $\mathcal{O}$ receives weight

$$\frac{i}{R^2}\int_{S^2}\Delta^{-1}\mathcal{O}\,\mathrm{dvol}\,.$$

We can compute the dual of a general observable by smearing first, then dualising: the result is that an observable has a uniquely determined smeared dual for each choice of smearing. We should be a little careful with exactly what theory these dual observables naturally live in. As degree zero observables they live in the the same vector space $\mathrm{Obs}_0^{\mathrm{q}}(U)$ as the original observable, but the BV-differential in the dual theory will be different. Specifically, the BV operator will be replaced by its inverse: coming from the pairing

$$\langle\phi_1,\phi_2\rangle R^2\int_{S^2}\phi_1\Delta^{-1}\phi_2\,\mathrm{dvol}\,.$$

We'll bear this in mind when we compute Fourier double-duals in a minute.

In order to compare expectation values of an observable and its dual, the crucial tool that we'll use is Plancherel's formula, which we can rederive in terms of Feynman diagrams. The first step is to prove a Fourier inversion formula in this language.

Proposition 2.1. A smeared observable $\mathcal{O}$ is equal to its Fourier double dual $\widetilde{\widetilde{\mathcal{O}}}$.

Proof. Let $\mathcal{O}=\mathcal{O}_1^{n_1}\mathcal{O}_2^{n_2}\cdots\mathcal{O}_k^{n_k}$ as above. The Fourier double dual of $\mathcal{O}$ is computed as a sum over diagrams with two kinds of edges: those coming from the first dual and those coming from the second. We'll show that these diagrams all naturally cancel in pairs apart from the diagram with no propagator edges. We depict such diagrams with blue edges coming from the first dual, and red edges coming from the second dual.

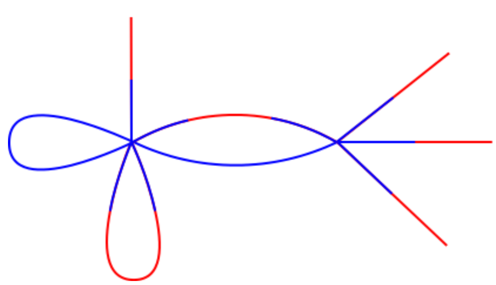

Figure 3: A diagram depicting a summand of the Fourier double dual of an observable of form $\mathcal{O}_1^7\,\mathcal{O}_2^5$.

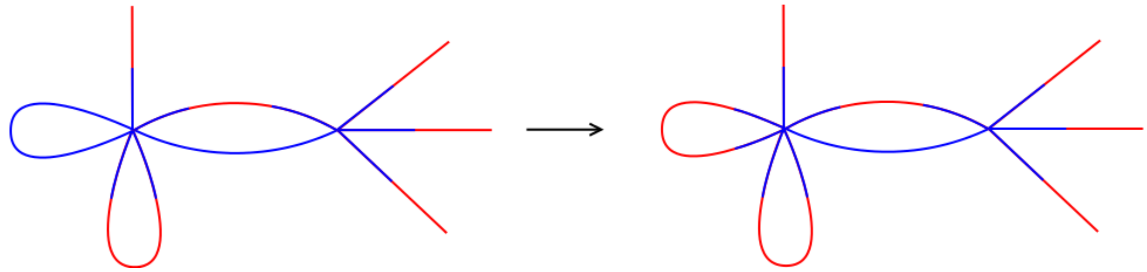

Figure 4: In this diagram we chose the blue leftmost propagator loop (coming from the first dual), and replaced it by two blue source terms connected with a red propagator loop (coming from the second dual).

So choose any diagram D with at least one propagator, and choose a propagator edge in the diagram. We produce a new diagram D' by swapping the colour of this propagator edge.

It suffices to show that the weight attached to this new diagram is -1 times the weight attached to the original diagram, so that the two cancel. The propagator from the first term contributes a weight $\frac{1}{2R^2}\int_{S^2}\mathcal{O}_i\Delta^{-1}\mathcal{O}_j\,\mathrm{dvol}$. In the second dual, the weights come from the source terms in the original theory, but the propagator in the *dual* theory, which contributes a weight using the dual theory. So the total weight is

$$\begin{aligned}&\left(\frac{i}{R^2}\int_{S^2}\Delta^{-1}\mathcal{O}_i\,\mathrm{dvol}\right)\left(\frac{i}{R^2}\int_{S^2}\Delta^{-1}\mathcal{O}_j\,\mathrm{dvol}\right)\left(\frac{R^2}{2}\int_{S^2}\mathcal{O}_i\Delta\mathcal{O}_j\,\mathrm{dvol}\right)\\ &=\frac{-1}{2R^2}\left(\int_{S^2}\Delta^{-1}\mathcal{O}_i\,\mathrm{dvol}\right)\left(\int_{S^2}\Delta^{-1}\mathcal{O}_j\,\mathrm{dvol}\right)\left(\int_{S^2}\mathcal{O}_i\Delta\mathcal{O}_j\,\mathrm{dvol}\right).\end{aligned}$$

To see that this is equal to -1 times the weight from the other diagram, we expand $\mathcal{O}_i$ and $\mathcal{O}_j$ as infinite sums of Laplace eigenfunctions, and use orthogonality. □

2.2 Fourier Duality as a Map of Factorisation Algebras

We go further and discuss the relationship between a factorisation algebra and its dual. Duality as described provided a map between the degree 0 smeared observables on any open set in two theories, which depended on a choice of parametrix. This map preserved expectation values, which means it provided an isomorphism between the whole *complexes* of global smeared observables. These maps are compatible with the extension maps in the factorisation algebras. We have to investigate whether duality can be extended to an isomorphism of the whole factorisation algebras.

In order to do this, we have to extend duality to a map between *all* observables: not just degree 0 observables. If we can extend duality in this way to a cochain map compatibly with extensions then we'll have produced an isomorphism of factorisation algebras. That is, we should extend Fourier Duality to a map between the whole cochain complexes $\mathrm{Obs}^{\mathrm{q}}_{\mathrm{sm}}(U)$ of smeared quantum observables, using Feynman diagrammatic methods (which will manifestly commute with extensions). In particular we'll recover the equality of expectation values proved above.

The complex $\mathrm{Obs}^{\mathrm{q}}_{\mathrm{sm}}(U)$ has underlying graded vector space $\mathrm{Sym}(\Omega^0_\perp(U)\oplus\Omega^0_\perp(U))$. We can write a general element as a sum of monomials

$$\mathcal{O}=\mathcal{O}_1^{n_1}\cdots\mathcal{O}_k^{n_k}\cdot\mathcal{P}_1^{m_1}\cdots\mathcal{P}_\ell^{m_\ell}$$

where $\mathcal{O}_i$ are linear observables in degree 0 and $\mathcal{P}_j$ are linear observables in degree -1 (linear vector fields in the complex of polyvector fields). We depict this diagrammatically much as before: with $k+\ell$ vertices (labelled by elements of $\Omega^0_\perp(U)$) each with a number of incident half-edges, except now the vertices fall into two categories: those corresponding to the $\mathcal{O}_i$ and those corresponding to the $\mathcal{P}_j$. We'll depict these respectively with circles and squares in our diagrams.

The quantum differential $\iota_{dS}+D$ can be understood in terms of these diagrams. The classical part ι_{dS} is the sum of all ways of turning exactly one square into a circle (lowering the degree of a linear term), and the quantum part

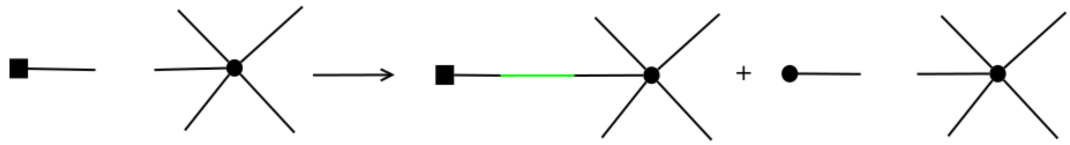

Figure 5: The Fourier dual for a degree -1 observable of form $\mathcal{P}\mathcal{O}^5$. There are terms of two forms: the term including a propagator edge will be weighted, and there will be five such terms corresponding to the five half-edges on the degree 0 vertex.

D is the sum of all ways of connecting a square to a circle with a propagator edge, weighted by the pairing

$$R^2 \int_{S^2} \mathcal{O}_i \Delta \mathcal{P}_j \,\mathrm{dvol}\,.$$

As we expect, the classical differential preserves Sym degree while the quantum part lowers Sym degree by two.

Now we can extend our description of the Fourier dual to this setting in a way that will manifestly commute with extensions. As before, the Fourier dual is a sum over diagrams where we glue on arbitrarily many propagator and source edges, leaving no free half-edges in the diagram. The edges can be glued onto the degree 0 vertices exactly as before. We are only allowed to attach *source* edges to the degree -1 vertices, where they receive a weight

$$\frac{-2i}{R^2} \int_{S^2} \Delta^{-1} \mathcal{O} \,\mathrm{dvol}\,.$$

Proposition 2.2. The Fourier transform commutes with the quantum differential $\iota_{dS} + D$.

Proof. We'll use a similar technique to the one we used to prove 2.1. It suffices to check that the differential from degree -1 to degree 0 commutes with the Fourier transform since the diagrams for $\mathcal{F} \circ (\iota_{dS} + D)$ and $(\iota_{dS} + D) \circ \mathcal{F}$ (where $\mathcal{F}$ is the Fourier transform map) both involve only one degree -1 vertex in an interesting way. So let $\mathcal{O} = \mathcal{P}\mathcal{O}_1^{n_1} \cdots \mathcal{O}_k^{n_k}$ be a degree -1 monomial observable. We'll try to compute $(\mathcal{F} \circ (\iota_{dS} + D))(\mathcal{O})$ and $((\iota_{dS} + D) \circ \mathcal{F})(\mathcal{O})$, match up the diagrams bijectively and check that the weights assigned to them agree.

To be precise, we match up the diagrams in the following way. There are two cases: either the $\mathcal{P}$ vertex remains a source, or it ends up connected to another vertex by a propagator. In the former case there is a clear bijective correspondence of diagrams in $(\mathcal{F} \circ (\iota_{dS} + D))(\mathcal{O})$ and $((\iota_{dS} + D) \circ \mathcal{F})(\mathcal{O})$. In the latter case this doesn't quite hold, as we can see in figure 6: one diagram in $((\iota_{dS} + D) \circ \mathcal{F})(\mathcal{O})$ where the $\mathcal{P}$ vertex is connected to an $\mathcal{O}$ vertex by a propagator is matched to *two* diagrams on the other side: one where the vertices are connected by a propagator and one where they are connected to source edges then contracted by the quantum differential. We'll need to compare the weights assigned to all of these diagrams.

Now, let's compute the weights. The diagrams where the $\mathcal{P}$ vertex is attached to a source are identical on the two sides, so we must compute the weights of the diagrams where this vertex is attached to a propagator. Both the propagator edge and the contraction edge coming from D are assigned the same weight, giving a total weight of

$$\frac{1}{R^2} \int_{S^2} \mathcal{O} \Delta^{-1} \mathcal{P} \,\mathrm{dvol}$$

for the diagrams in $(\mathcal{F} \circ (\iota_{dS} + D))(\mathcal{O})$. On the other side we have a source edge attached to a square, a source edge attached to a circle and a contraction edge in the *dual* theory, for a total weight of

$$\left(\frac{-2i}{R^2} \int_{S^2} \Delta^{-1} \mathcal{P} \,\mathrm{dvol} \right) \left(\frac{i}{R^2} \int_{S^2} \Delta^{-1} \mathcal{O} \,\mathrm{dvol} \right) \left(\frac{R^2}{2} \int_{S^2} \mathcal{O} \Delta \mathcal{P} \,\mathrm{dvol} \right).$$

As in 2.1 we observe that these weights are equal by expanding $\mathcal{O}$ and $\mathcal{P}$ in terms of Laplace eigenfunctions. $\square$

We notice that Fourier inversion as in 2.1 still applies in this setting, with exactly the same argument. Therefore the Fourier transform actually defines an *isomorphism* of cochain complexes. It commutes with extensions, so further defines an isomorphism of prefactorisation algebras.

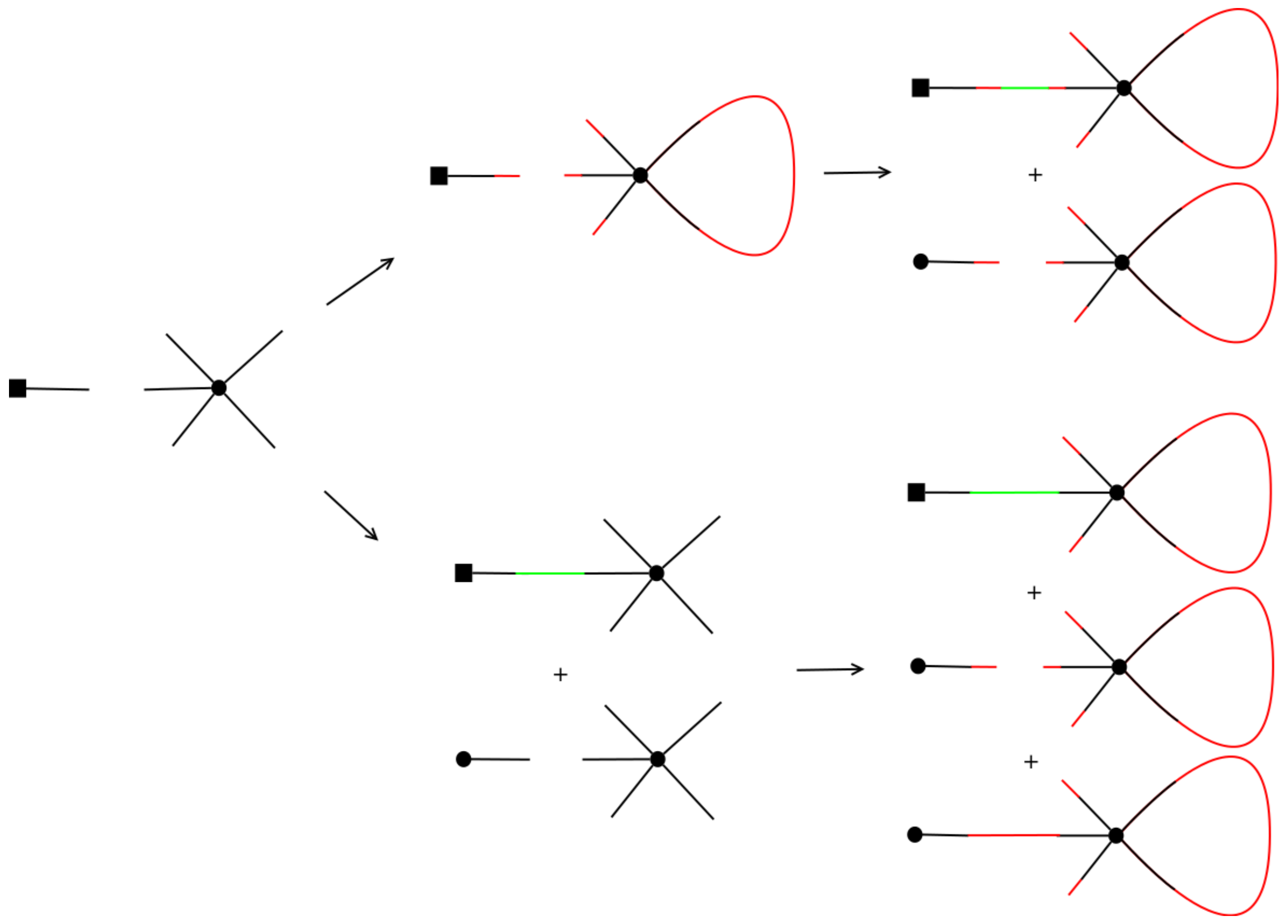

Figure 6: Diagrams in the Feynman diagram expansion of $((\iota_{dS}+D)\circ\mathcal{F})(\mathcal{P}\mathcal{O}^5)$ (top row) and $(\mathcal{F}\circ(\iota_{dS}+D))(\mathcal{P}\mathcal{O}^5)$ (bottom row). The edges added by the Fourier dual are in red, and the edge added by the quantum differential is in green.

Remark 2.3. Let $\mathcal{O}\in \mathrm{Obs}_0^{\mathrm{q}}(U)$ be a quantum observable, not necessarily smeared. We can compute its Fourier dual using a smearing. Let's change notation to make the choice of parametrix specific, so the Fourier dual smeared observable coming from parametrix P is written $\mathcal{F}_P(\mathcal{O})$. We want to reconstruct a local observable $\widetilde{\mathcal{O}}$ on U from these smearings. In order to do so it will help to understand the space of parametrices. We can produce parametrices by smooth approximations of the Green's function for the Laplacian on U: specifically, for a linear observable $\mathcal{O}$ given by a local distribution ψ supported on U we have

$$\mathcal{O}(\phi)=\int_{S_2}\phi\Delta(G*\psi)\,\mathrm{dvol}$$

where G is the fundamental solution to Laplace's equation on the sphere. Let G_n be smooth functions such that $G_n\to G$ (in L^2, say). This gives a sequence of parametrices P_n defined on linear observables by

$$(P_n(\mathcal{O}))(\phi)=\int_{S^2}\phi\Delta(G_n*\psi)\,\mathrm{dvol}$$

since the convolutions are now smooth functions. Thus, we can define the Fourier dual $\widetilde{\mathcal{O}}$ of an observable $\mathcal{O}$ by

$$\widetilde{\mathcal{O}}(\phi)=\lim_{n\to\infty}(\mathcal{F}_{P_n}(\mathcal{O}))(\phi),$$

which is independent of the approximations we chose if it exists.

Of course, this limit might not exist: the weights assigned to loops will generally diverge for non-smeared observables. For certain observables (e.g. n-point functions with all points distinct) however, this defines a dual without smearing.

2.3 Duality and Expectation Values

In this section we'll show by explicit calculation that dual observables have equal expectation values, using the method for computing duals and expectation values we've already described. This follows from the isomorphism we constructed above, but we'll see it in a more explicit way, but calculating both sides. We'll also rewrite the dual theory in a way that is more manifestly similar to the original theory. Our method is to use a version of Plancherel's theorem. Again we'll compare our combinatorial calculations with calculations done by functional integrals on a filtered piece. We'll need to check the following version of 1.1 for the Fourier transform.

Proposition 2.4. Let $\mathcal{O}$ be a smeared global observable. The finite-dimensional Gaussian integrals

$$\widetilde{\mathcal{O}}(\widetilde{\phi}) = \frac{1}{Z_k}\int_{F^k\Omega^0(S^2)} \mathcal{O}(\phi)e^{S(\phi)+iR^2\int_{S^2}\widetilde{\phi}\Delta\phi\,\mathrm{dvol}}d\phi$$

converge to a smeared global observable, which agrees with the Fourier dual observable computed by the Feynman diagrammatic method.

Proof. We use the same method of proof as for 1.1, writing the integral as a derivative of a generating function. Specifically, for $\mathcal{O} = \mathcal{O}_1^{n_1}\mathcal{O}_2^{n_2}\cdots\mathcal{O}_k^{n_k}$ we expand

$$\begin{aligned}\frac{1}{Z_k}\int_{F^k\Omega^0(S^2)} \mathcal{O}(\phi)e^{S(\phi)+iR^2\int_{S^2}\widetilde{\phi}\Delta\phi\,\mathrm{dvol}}d\phi &= \left.\frac{\partial^{n_1+\cdots n_k}}{\partial t_1^{n_1}\cdots\partial t_k^{n_k}}\right|_{t_1=\cdots=t_k=0}\frac{1}{Z_k}\int_{F^k\Omega^0(S^2)} e^{S(\phi)+\sum t_i\mathcal{O}_i(\phi)+i\int_{S^2}\widetilde{\phi}\Delta\phi\,\mathrm{dvol}}d\phi \\ &= \left.\frac{\partial^{n_1+\cdots n_k}}{\partial t_1^{n_1}\cdots\partial t_k^{n_k}}\right|_{t_1=\cdots=t_k=0}\int_{F^k\Omega^0(S^2)} e^{\frac{1}{4}(i\widetilde{\phi}+\sum t_i\mathcal{O}_i(\phi))A^{-1}(i\widetilde{\phi}+\sum t_i\mathcal{O}_i(\phi)))}d\phi\end{aligned}$$

and extract the $t_1^{n_1}\cdots t_k^{n_k}$-term. We choose the level in the filtration large enough so that we can choose a basis that includes functions ψ_i, where the smeared linear observable $\mathcal{O}_i$ is given by pairing with the function ψ_i. One then observes that the relevant term is given by a sum over diagrams as described with the correct weights. □

Remark 2.5. We could've also used this characterisation to prove the Fourier inversion theorem 2.1 above. I gave an alternative proof purely in terms of Feynman diagrams just to show that we could understand Fourier inversion without extending to global observables.

Now, we can compare expectation values of $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ using a Plancherel's theorem calculation.

Theorem 2.6. Let $\mathcal{O}$ be a local observable in $\mathrm{Obs}_0^{\mathrm{q}}(U)$ for the sigma model with target $\mathbb{R}/2\pi R\mathbb{Z}$, and let $\widetilde{\mathcal{O}}$ be its Fourier dual observable. We can compute the expectation values of $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ in theories where the target circle has any radius; denote this expectation value by $\langle - \rangle_R$. Then

$$\langle\mathcal{O}\rangle_R = \langle *\widetilde{\mathcal{O}}\rangle_{\frac{1}{R}}$$

where $*\widetilde{\mathcal{O}}(d\phi) = \widetilde{\mathcal{O}}(*d\phi)$ for the smeared observable $\widetilde{\mathcal{O}}$ extended naturally to act on all 1-forms

Proof. We know by 2.1 that $\mathcal{O} = \widetilde{\widetilde{\mathcal{O}}}$, so in particular $\langle\mathcal{O}\rangle_R = \langle\widetilde{\widetilde{\mathcal{O}}}\rangle_R$. By the calculation in 2.4 we can write this expectation value as the limit as $k \to \infty$ of the Gaussian integrals

$$\begin{aligned}\frac{1}{Z_k}\int_{F^k\Omega^0(S^2)} \mathcal{O}(\phi)e^{S_R(\phi)}d\phi &= \frac{1}{Z_k}\int_{F^k\Omega^0(S^2)} \widetilde{\widetilde{\mathcal{O}}}(\phi)e^{S_R(\phi)}d\phi \\ &= \lim_{\ell\to\infty}\frac{1}{Z_k}\int_{F^\ell\Omega^0(S^2)}\int_{F^k\Omega^0(S^2)} \widetilde{\mathcal{O}}(\widetilde{\phi})e^{S_R(\phi)+\frac{i}{R^2}\int_{S^2}\widetilde{\phi}\Delta^{-1}\phi\,\mathrm{dvol}}d\widetilde{\phi}\,d\phi \\ &= \lim_{\ell\to\infty}\frac{1}{Z_k}\int_{F^\ell d\Omega^0(S^2)}\int_{F^k\Omega^1(S^2)} \widetilde{\mathcal{O}}(\widetilde{\alpha})e^{S_{1/R}(\widetilde{\alpha})+i\int_{S^2}\alpha\wedge*\widetilde{\alpha}}\,\delta_{d\Omega^0(S^2)}(\alpha)\,d\widetilde{\alpha}\,d\alpha\end{aligned}$$

The last line needs a little explanation. We've translated the functional integral from an expression involving $\phi \in \Omega^0_\perp(S^2)$ to an expression involving $\alpha = d\phi \in \Omega^1(S^2)$. The space of 1-forms also admits a filtration by finite-dimensional vector spaces using the eigenvalues of the Laplacian, which I've also written as $F^\ell\Omega^1(S^2)$. The distribution $\delta_{d\Omega^0(S^2)}$ is the delta-function on the exact 1-forms sitting inside all 1-forms (restricted to the filtered piece). The smeared observable $\widetilde{\mathcal{O}}$ is given by pairing an exact 1-form with a coexact 1-form, so is well-defined on all 1-forms. Finally, we've written $S_{1/R}(\widetilde{\alpha})$ for the Fourier dual action functional, which is also well-defined on all 1-forms, given by

$$S_{1/R}(\widetilde{\alpha}) = \frac{1}{R^2}\|\widetilde{\alpha}\|^2$$

in terms of the usual L^2-norm on 1-forms.

Now, for a fixed value of ℓ, we can reinterpret the final integral above by changing the order of integration. This computes the Fourier dual of the delta function $\delta_{d\Omega^0(S^2)}$, (which is just $\delta_{d^*\Omega^2(S^2)} = *\delta_{d\Omega^0(S^2)}$: the pushforward along $*$ of the original delta function). Therefore

$$\begin{aligned}
\langle \mathcal{O}\rangle_R &= \lim_{k\to\infty}\frac{1}{Z_k}\int_{F^\ell d\Omega^0(S^2)} \widetilde{\mathcal{O}}(\widetilde{\alpha})e^{S_{1/R}(\widetilde{\alpha})}\delta_{d^*\Omega^2(S^2)}d\widetilde{\alpha}\\
&= \lim_{k\to\infty}\frac{1}{Z_k}\int_{F^\ell \Omega^0(S^2)} \widetilde{\mathcal{O}}(*d\widetilde{\phi})e^{S_{1/R}(*d\widetilde{\phi})}d\widetilde{\phi}\\
&= \langle *\widetilde{\mathcal{O}}\rangle_{\frac{1}{R}}
\end{aligned}$$

as required. □

Remark 2.7. We wrote this in a "self-dual" form, where the dual theory came with the same space of fields as the original theory. More naturally we might think of the dual theory as having fields $d^*\Omega^2(U) \cong \Omega^2_\perp(U)$ – identified as a subspace of the dual space $\Omega^0_\perp(U)^\vee$ – with the natural dual action $\widetilde{S}(v) = \frac{1}{R^2}\|v\|^2$.