Notes on $N = 4$ Supersymmetry

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These are some personal notes on real $N = 4$ supersymmetry algebras in four dimensions. My goal is to understand the 1/2 BPS subalgebras of the (Lorentzian) $N = 4$ supersymmetry algebra studied by Gaiotto and Witten [GW09]. They claim that there is an $\mathbb{RP}^1$ worth of maximal subalgebras of the $N = 4$ supersymmetry algebra that each preserve a spacelike hypersurface $\mathbb{R}^{1,2} \subseteq \mathbb{R}^{1,3}$.

1 The $N = 4$ Supersymmetry Algebra

We'll begin by reviewing the available real and complex spinor representations in four dimensions. Throughout we'll work in Lorentzian signature; the real supersymmetry algebra would be quite different in other signatures. Recall the exceptional isomorphism $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2;\mathbb{C})$ as real Lie algebras. The complexification is therefore given by $\mathfrak{so}(4;\mathbb{C}) \cong \mathfrak{sl}(2;\mathbb{C}) \oplus \mathfrak{sl}(2;\mathbb{C})$. We write V for the four real-dimensional fundamental representation of $\mathfrak{so}(1,3)$, and $V_{\mathbb{C}}$ for its complexification.

Proposition 1.1. A complete list of irreducible complex representations of $\mathfrak{so}(4;\mathbb{C}) \cong \mathfrak{sl}(2;\mathbb{C}) \oplus \mathfrak{sl}(2;\mathbb{C})$ is provided by

$$V_{k_+,k_-} \cong \mathrm{Sym}^{k_+} S_+ \otimes_{\mathbb{C}} \mathrm{Sym}^{k_-} S_-$$

where $S_\pm$ are the fundamental representations of the two summands. When we restrict these to real representations of $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2;\mathbb{C})$, S_+ becomes the fundamental representation, and S_- becomes its complex conjugate. The irreducible spinorial representations are precisely S_+ and S_-. We call these the positive and negative *Weyl spinor* modules.

Proposition 1.2. An irreducible *real* spinorial representation of $\mathfrak{so}(1,3)$ is obtained by taking the $\mathbb{Z}/2$-fixed points

$$S_{\mathbb{R}} = (S_+ \oplus S_-)^c$$

where c is the $\mathbb{Z}/2$-action given by taking the complex conjugate and swapping the two factors. We call this the *Majorana spinor* module.

Remark 1.3. There are real linear (but not complex linear) isomorphisms $S_{\mathbb{R}} \cong S_+ \cong S_-$, but we'll find the description above the most useful.

Definition 1.4. The *complex* Γ*-pairing* is the symmetric $\mathfrak{so}(4;\mathbb{C})$-equivariant map

$$\Gamma\colon S_+ \otimes_{\mathbb{C}} S_- \to V_{\mathbb{C}}$$

given by the classification of representations in 1.1. It is projectively unique.

The *real* Γ*-pairing* is the projectively unique symmetric $\mathfrak{so}(1,3)$-equivariant map

$$\Gamma\colon S_{\mathbb{R}} \otimes_{\mathbb{R}} S_{\mathbb{R}} \to V.$$

This is obtained from the $\mathfrak{so}(4;\mathbb{C})$-equivariant map

$$\begin{aligned}(S_+ \oplus S_-) \otimes_{\mathbb{C}} (S_+ \oplus S_-) &\cong (S_+ \otimes_{\mathbb{C}} S_-) \oplus (S_+ \otimes_{\mathbb{C}} S_-) \oplus (S_+ \otimes_{\mathbb{C}} S_+) \oplus (S_- \otimes_{\mathbb{C}} S_-)\\ &\to S_+ \otimes_{\mathbb{C}} S_- \cong V_{\mathbb{C}}\end{aligned}$$

given by summing the first two factors. Each side admits a $\mathbb{Z}/2$ action, by $c \otimes c$ on the left and by complex conjugation on the right, so by rescaling by a suitable complex number we can ensure that $\Gamma = \overline{\Gamma \circ (c \otimes c)}$, so that the map Γ descends to the $\mathbb{Z}/2$ fixed points.

We can also define Γ-pairings on reducible spinorial representations.

Definition 1.5. Let W be a complex vector space. The Γ pairing on $S_+ \otimes W \oplus S_- \otimes W^*$ is given by applying the Γ pairing above to the S_+ and S_- factors, and pairing the W and W^* factors using the evaluation pairing.

Let W be a complex unitary vector space. The Γ pairing on $(S_+ \otimes_{\mathbb{C}} W \oplus S_- \otimes_{\mathbb{C}} W^*)^c$, where c is a $\mathbb{Z}/2$ action given by conjugation and swapping the factors, is given by applying the pairing as in definition 1.4, pairing the W and W^* factors using the evaluation pairing, and noting that there exists a complex rescaling so that this pairing commutes with the $\mathbb{Z}/2$ actions on the two sides.

We can now define the real and complex supersymmetry algebras. Write $\mathcal{P} = \mathfrak{so}(1,3) \ltimes \mathbb{R}^4$ for the Poincaré algebra, and $\mathcal{P}_{\mathbb{C}} = \mathfrak{so}(4;\mathbb{C}) \ltimes \mathbb{C}^4$ for its complexification.

Definition 1.6. Let W be a k-dimensional complex vector space. The $N = k$ complex *super Poincaré algebra* is the super Lie algebra

$$\mathcal{A}_{\mathbb{C}}^{N=k} = \mathcal{P}_{\mathbb{C}} \ltimes \Pi(S_+ \otimes_{\mathbb{C}} W \oplus S_- \otimes_{\mathbb{C}} W^*)$$

with fermionic bracket given by the Γ pairing.

Let W be a k-dimensional unitary vector space. The $N = k$ real *super Poincaré algebra* is the super Lie algebra

$$\mathcal{A}_{\mathbb{R}}^{N=k} = \mathcal{P} \ltimes \Pi(S_+ \otimes_{\mathbb{C}} W \oplus S_- \otimes_{\mathbb{C}} W^*)^c$$

with fermionic bracket given by the Γ pairing.

To define the full supersymmetry algebra, fix a subalgebra $\mathfrak{g}_R$ of $\mathfrak{gl}(W)$ (in the complex case) or of $\mathfrak{u}(W)$ (in the real case). We call this the algebra of *R-symmetries*.

Definition 1.7. The $N = k$ real or complex *supersymmetry algebra* is the super Lie algebra $\mathfrak{g}_R \ltimes \mathcal{A}^{N=k}$, where $\mathfrak{g}_R$ acts on W by the restriction of the fundamental representation of $\mathfrak{gl}(W)$ or $\mathfrak{u}(W)$.

Remark 1.8. We'll be most interested in the case $N = 4$, in which case we consider the supersymmetry algebra associated to – in the real case – $\mathfrak{g}_R = \mathfrak{su}(4)$.

2 The 1/2 BPS Subalgebra

Choose a spacelike hypersurface $\mathbb{R}^{1,2} \subseteq \mathbb{R}^{1,3}$. A 1/2 BPS subalgebra of the supersymmetry algebra is a maximal subalgebra preserving this subspace. We'll classify such subalgebras in the case of $N = 4$.

First investigate the bosonic piece. The subalgebra of the Poincaré algebra fixing this subspace is isomorphic to $\mathfrak{so}(1,2) \ltimes \mathbb{R}^3$, so the bosonic piece of the subalgebra is the sum of this with some subalgebra $\mathfrak{h}_R \subseteq \mathfrak{su}(4)$ of the R-symmetry algebra. We choose (following Gaiotto and Witten [GW09]) the subalgebra $\mathfrak{h}_R \cong \mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Using the exceptional isomorphism $\mathfrak{so}(1,2) \cong \mathfrak{sl}(2;\mathbb{R})$, the space of supersymmetries (i.e. fermions in the supersymmetry algebra) becomes

$$\begin{aligned}(S_+ \otimes_{\mathbb{C}} W \oplus S_- \otimes_{\mathbb{C}} W^*)^c &\cong ((S' \oplus iS') \otimes_{\mathbb{C}} (W_+ \oplus W_-) \oplus (S' \oplus iS') \otimes_{\mathbb{C}} (W_+^* \oplus W_-^*))^c\\ &\cong (S' \oplus iS') \otimes_{\mathbb{C}} (W_+ \oplus W_-)\end{aligned}$$

where S' is the fundamental representation of $\mathfrak{sl}(2;\mathbb{R})$, and $W_\pm$ are the fundamental representations of the two copies of $\mathfrak{su}(2)$ in the R-symmetry algebra. On the second line we used the fact that the fundamental representation of $\mathfrak{su}(2)$ is isomorphic to its dual to identify the c-fixed points with one of the summands. We might write this as

$$(S' \otimes_{\mathbb{C}} (W_+ \oplus W_-))_{\mathbb{R}} \oplus (S' \otimes_{\mathbb{C}} (W_+ \oplus W_-))_{\mathbb{R}}$$

where S' is thought of as a one-dimensional complex vector space, and where we write the subscript $\mathbb{R}$ to indicate that we forget down to a real vector space. This is embedded diagonally in the real summands and anti-diagonally in the imaginary summands of the complexified representation.

Let's describe the Γ pairing in these terms. Choose $\mathbb{C}$-bases $\langle a_1^j + ib_1^j, a_2^j + ib_2^j \rangle$ and $\langle c_1^j + id_1^j, c_2^j + id_2^j \rangle$ for W_+ and W_- respectively, where $j = 1, 2$. The algebra $\mathfrak{sl}(2;\mathbb{R})$ acts by treating each (a, b) or (c, d) pair as an element of the fundamental representation. In this basis, the non-trivial brackets are those of form

$$\begin{aligned}
\Gamma(a_k^j, a_m^l) = \Gamma(c_k^j, c_m^l) &= \delta_{jl}\delta_{km}x_0 \\
\Gamma(a_k^j, b_m^l) = \Gamma(c_k^j, d_m^l) &= \delta_{jl}\delta_{km}x_1 \\
\Gamma(b_k^j, a_m^l) = \Gamma(d_k^j, c_m^l) &= \delta_{jl}\delta_{km}x_2 \\
\Gamma(b_k^j, b_m^l) = \Gamma(d_k^j, d_m^l) &= \delta_{jl}\delta_{km}x_3
\end{aligned}$$

where $\{x_0, x_1, x_2, x_3\}$ is a basis for $\mathbb{R}^{1,3}$.

Definition 2.1. A 1/2 *BPS subalgebra* of the superalgebra is given by choosing an irreducible submodule of $(S' \otimes_{\mathbb{C}} (W_+ \oplus W_-))_{\mathbb{R}} \oplus (S' \otimes_{\mathbb{C}} (W_+ \oplus W_-))_{\mathbb{R}}$, which corresponds to a choice of $(\alpha : \beta) \in \mathbb{CP}^1$.

Question. Now I'm confused. We've followed Gaiotto and Witten exactly here, as far as I can tell. However, per the description of the brackets above, if we restrict to one of these submodules, the Γ matrix is *still* surjective onto $\mathbb{R}^4$, so the algebra is not well-defined.

References

[GW09] Davide Gaiotto and Edward Witten. Supersymmetric boundary conditions in $\mathcal{N} = 4$ super Yang-Mills theory. *Journal of Statistical Physics*, 135(5-6):789–855, 2009.

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