

Geometric Langlands for Physicists / Kapustin–Witten for Mathematicians

Lecture 1

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Plan for Today

- Introduce the [Hitchin system](#), and talk about some of its structure.
- Discuss what it means to [quantize](#) the Hitchin system.
- Talk about how to describe this quantization; this will lead us to a version of the [geometric Langlands conjecture](#).

Next time we'll talk about how all the objects in today's lecture arise in gauge theory.

Higgs Bundles

Let C be a Riemann surface, and let G be a complex Lie group (e.g. $G = \mathrm{GL}(n; \mathbb{C})$).

Definition

A **Higgs bundle** on C for the group G is a pair (P, ϕ) where:

- P is a holomorphic G -bundle on C (e.g. for $\mathrm{GL}(n; \mathbb{C})$ think of a holomorphic vector bundle E).
- ϕ is a holomorphic 1-form taking values in \mathfrak{g}_P (e.g. for $\mathrm{GL}(n; \mathbb{C})$ think of an endomorphism of E i.e. a matrix-valued section).

There is a **moduli space** of Higgs bundles on C that we denote as

$$\mathrm{Higgs}_G(C).$$

The name **Higgs bundle** comes by analogy to Yang–Mills–Higgs theory.

- Holomorphic structure on $P \approx$ certain connections A on P solving the Yang–Mills equations (in 4d: instantons).
- Holomorphic 1-form $\phi \approx$ d_A -closed 1-form ϕ .

The space of Higgs bundles is a holomorphic version of the space of minimum energy solutions to the Yang–Mills–Higgs equations (in 2d, but a similar definition can be made in other dimensions).

The space $\text{Higgs}_G(C)$ has a nice description as follows. There's a forgetful map down to the moduli space of holomorphic G -bundles (forget ϕ)

$$\pi: \text{Higgs}_G(C) \rightarrow \text{Bun}_G(C).$$

It turns out that this map is identified with the projection from the **cotangent bundle** of $\text{Bun}_G(C)$

$$\text{Higgs}_G(C) \cong T^* \text{Bun}_G(C).$$

Using this fact, we can identify a few things straight away:

- $\dim_{\mathbb{C}}(\text{Higgs}_G(C)) = 2 \dim_{\mathbb{C}}(\text{Bun}_G(C)) = (2g - 2) \dim(G)$ where g is the genus of C (if $g > 1$).
- $\text{Higgs}_G(C)$ has a **holomorphic symplectic** structure.

Structures on $\text{Higgs}_G(C)$

Actually the moduli space has even richer structure! Two different closely related stories.

1. Because of the symplectic structure, it makes sense to talk about the **Poisson bracket** of two holomorphic functions on $\text{Higgs}_G(C)$. It turns out that we can find a whole collection

$$h_1, \dots, h_d$$

of holomorphic functions where $d = (g - 1) \dim(G)$ is half the dimension of $\text{Higgs}_G(C)$ with the property that $\{h_i, h_j\} = 0$ for any i and j . This is the structure of an **integrable system**. Geometric point of view: this is the same as a holomorphic map called the **Hitchin fibration**

$$p: \text{Higgs}_G(C) \rightarrow \mathbb{C}^d$$

with some nice properties. You can describe p concretely in terms of the characteristic polynomial of the Higgs field.

2. There is more than one complex structure on $\text{Higgs}_G(C)$, in fact there is a whole \mathbb{CP}^1 family of such structures!
 $\text{Higgs}_G(C)$ is an example of a **hyperkähler** space.

Quantization

We will study the **canonical quantization** of $\text{Higgs}_G(C)$. Using the identification with a cotangent bundle $T^* \text{Bun}_G(C)$, we can quantize the ring of functions on $\text{Higgs}_G(C)$:

$$\mathcal{O}(\text{Higgs}_G(C)) \rightsquigarrow \text{DiffOp}(\text{Bun}_G(C)).$$

Question (Quantization of the Hitchin system)

Can we find commuting differential operators H_1, \dots, H_d that quantize the classical Hamiltonians h_1, \dots, h_d ?

Answer ([Beilinson–Drinfeld 1991](#)): Yes, and they can be described in terms of the classical Hitchin system of a different “dual” group!

Question (Spectral problem)

Describe the simultaneous solutions to the equation

$$H_j(\phi) = a_j(\phi),$$

for $a \in \mathbb{C}^d$.

Fancy terminology: think of the space of solutions as the “D-module”

$$M_a = \text{DiffOp}(\text{Bun}_G(C)) / (H_j - a_j).$$

The Langlands dual group

To the group G we can cook up its **Langlands dual** group G^\vee combinatorially. “Reverse the root data”: roots/characters of G are coroots/cocharacters of G^\vee and vice versa. In some examples:

$$\begin{aligned} \mathrm{GL}(n; \mathbb{C}) &\longleftrightarrow \mathrm{GL}(n; \mathbb{C}) \\ \mathrm{SL}(n; \mathbb{C}) &\longleftrightarrow \mathrm{PGL}(n; \mathbb{C}) \\ \mathrm{SO}(2n+1; \mathbb{C}) &\longleftrightarrow \mathrm{Sp}(2n; \mathbb{C}). \end{aligned}$$

Suggestively, this duality operation also shows up in electric-magnetic duality.

Geometric Langlands duality: first picture

The solutions to the two quantization problems take the following flavour.

Quantization of $\text{Higgs}_G(C) \leftrightarrow$ Deformation of $\text{Higgs}_{G^\vee}(C)$.

Here **deformation** means deformation of the complex structure: going from $\text{Higgs}_{G^\vee}(C)$ to $\text{Flat}_{G^\vee}(C)$.

For example, the D-modules generated by $H_j - a_j$ can be associated to **points** in $\text{Flat}_{G^\vee}(C)$. I'll say more about this in a second, but first let me state a strong (conjectural) version of this duality.

Conjecture (Beilinson–Drinfeld “best hope”)

There is an equivalence of categories

$$\{D\text{-modules on } \mathrm{Bun}_G(C)\} \leftrightarrow \{Coherent \text{ sheaves on } \mathrm{Flat}_{G^\vee}(C)\}.$$

Examples of coherent sheaves are given by choosing a subspace $Y \subseteq \mathrm{Flat}_{G^\vee}(C)$ and specifying a vector bundle on Y . The simplest case of this is when Y is just a point!

Hecke Eigensheaves

Rough idea: there are nice operations one can perform on D-modules on $\text{Bun}_G(C)$ called **Hecke transformations**. They take place locally at a point in C . The operations commute with one another, so one can try to decompose D-modules into simultaneous eigenvectors: such things are called **Hecke eigensheaves**.

By eigenvector in this context, we mean $A_x(M) = V_x \otimes M$, where x is a point in C , and V_x is a vector space.

Theorem

The D-module M_a solving the spectral problem is a Hecke eigensheaf with eigenvalue $V(a)$ in $\text{Flat}_{G^\vee}(C)$.

The geometric Langlands conjecture would say that **all** points in $\text{Flat}_{G^\vee}(C)$ arise as eigenvalues for some Hecke eigensheaf.