Spinors in Four-Dimensions

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This is a quick note on spinors to establish notation and try to clear some things up.

1 Spaces of Spinors

Definition 1.1. In **Lorentzian signature**, one has the following spinorial representations. The *Dirac spinors* S are the four-complex-dimensional spin representation of $\mathrm{Spin}(1,3) \cong \mathrm{SL}(2;\mathbb{C})$. They split into two two-complex-dimensional irreducible subrepresentations $S_+ \oplus S_-$, the *Weyl spinors*, which are isomorphic to the fundamental representation of $\mathrm{SL}(2;\mathbb{C})$ and its dual respectively. The *Majorana spinors* $S_\mathbb{R}$ are the four-real-dimensional spin representation of $\mathrm{Spin}(1,3)$, which is isomorphic to the fundamental real representation of $\mathrm{SO}(1,3)$.

Definition 1.2. In **Riemannian signature**, one has the following spinorial representations. The *Dirac spinors* S are the four-complex-dimensional spin representation of $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$. They split into two two-complex-dimensional irreducible subrepresentations $S_+ \oplus S_-$, the *Weyl spinors*, which are isomorphic to the fundamental representations of the two $\mathrm{SU}(2)$ factors. The *Majorana spinors* $S_\mathbb{R}$ are the four-real-dimensional spin representation of $\mathrm{Spin}(4)$, which is isomorphic to the fundamental real representation of $\mathrm{SO}(4)$.

Definition 1.3. (For completeness) in **Anti de Sitter signature**, one has the following spinorial representations. The *Dirac spinors* S are the four-complex-dimensional spin representation of $\mathrm{Spin}(2,2) \cong \mathrm{SL}(2;\mathbb{R}) \times \mathrm{SL}(2;\mathbb{R})$. They split into two two-complex-dimensional irreducible subrepresentations $S_+ \oplus S_-$, the *Weyl spinors*, which are isomorphic to the complexified fundamental representations of the two $\mathrm{SL}(2;\mathbb{R})$ factors. The *Majorana spinors* $S_\mathbb{R}$ are the four-real-dimensional spin representation of $\mathrm{Spin}(2,2)$. They split into two two-real-dimensional irreducible subrepresentations $S_{\mathbb{R}+} \oplus S_{\mathbb{R}-}$, which are isomorphic to the fundamental representations of the two $\mathrm{SL}(2;\mathbb{R})$ factors.

The complex spinorial representations S admit actions of the *complexified* spin group $\mathrm{SL}(2;\mathbb{C}) \times \mathrm{SL}(2;\mathbb{C})$, from complexifying the representation $S_\pm$ in either signature, or the representation $S_\mathbb{R}$ when it's defined. The result is the representation $S = S_+ \oplus S_-$ where $S_\pm$ are the fundamental representations of the two $\mathrm{SL}(2;\mathbb{C})$ factors.

2 Scalar and Vector Pairings

We'll write $\mathbb{R}^{i,j}$ for the $i+j$-dimensional vector space with metric and natural action of $\mathrm{SO}(i,j)$, and $\mathbb{C}^{i+j}$ for its complexification, with natural action of the complexification of the group. In this section I'm referring to chapter 4 of [Del99].

- In Lorentzian signature there is a natural evaluation pairing between the fundamental representation of $\mathrm{SL}(2;\mathbb{C})$ and its dual. This yields a $\mathrm{Spin}(1,3)$-invariant bilinear pairing on the Dirac spinors

  $$(,)' \colon S \otimes S \to \mathbb{C}$$

  which is zero on the subspaces $S_\pm \otimes S_\pm$.

- In Riemannian signature this pairing fails to be Spin(4)-invariant. Instead, we define a pairing by $(s,t)'' = s^\dagger t$ on each Weyl spinor factor, which is manifestly SU(2)-invariant. This yields a nondegenerate equivariant bilinear pairing
$$(,)'' \colon S \otimes S \to \mathbb{C}$$
which is zero on the subspaces $S_\pm \otimes S_\mp$. This is not invariant for the action of $\mathrm{Spin}(1,3) \cong \mathrm{SL}(2;\mathbb{C})$, or for the action of the complexified group $\mathrm{SL}(2;\mathbb{C}) \times \mathrm{SL}(2;\mathbb{C})$ (take any $A \in \mathrm{SL}(2;\mathbb{C})$ such that $A \neq A^\dagger$, and let s and t vary over a basis for S_+).

- However, there is a pairing which is equivariant even for the complexified group. We can see this from the Clebsch-Gordan decomposition $S_+ \otimes S_+ \cong \wedge^2 S_+ \oplus \mathbb{C}$, by projection onto the second factor. Concretely, this is given by the trace pairing $(s,t) = \mathrm{Tr}(s \otimes t)$ by identifying the tensor product with the endomorphism algebra. This extends to a nondegenerate $\mathrm{SL}(2;\mathbb{C}) \times \mathrm{SL}(2;\mathbb{C})$-equivariant bilinear pairing
$$(,) \colon S \otimes S \to \mathbb{C}$$
which is zero on the subspaces $S_\pm \otimes S_\mp$, and which is a fortiori also equivariant for $\mathrm{Spin}(i,j)$ where $i+j=4$.

The fact that our representations are spinorial means that they extend to the Clifford algebra, and so define a Clifford multiplication map $\rho \colon S \otimes \mathbb{C}^4 \to S$, which is Spin-equivariant. The vector representation is self-dual as a representation, and we can use a non-degenerate invariant pairing to identify S with its dual, yielding an equivariant map
$$\Gamma \colon S \otimes S \to \mathbb{C}^4.$$
The three pairings $(,)$, $(,)'$ and $(,)''$ yield three Γ pairings of the opposite parity, since Clifford multiplication always reverses parity, and the Γ pairing satisfies the identity
$$\langle \Gamma(s \otimes t), v \rangle = (\rho(v \otimes s), t)^{(')}.$$
Thus $(,)'$ yields a pairing which is $\mathrm{Spin}(1,3)$-equivariant and odd, and the others yields pairings which are even.

References

[Del99] Pierre Deligne. *Quantum Fields and Strings; A Course for Mathematicians: Notes on Spinors*, volume 1. AMS, 1999.

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