An Example of Abelian Duality

Chris Elliott

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In this note I will describe the simplest non-trivial example of abelian duality: relationships between a family of free field theories where the fields are maps $\Sigma \to S^1$ where Σ is a compact Riemannian 2-manifold. I'll describe the factorisation algebras associated to these theories, and prove interesting relationships between these: a procedure for assigning a "dual" to an observable preserving an expectation value map.

By a *dg-vector space* V we mean a cochain complex of nuclear Fréchet spaces over a field k of characteristic zero (always $\mathbb{R}$ or $\mathbb{C}$ in this note). By the dual $V^\vee$ of such a complex, we will mean the continuous dual graded vector space, with the dual differential, equipped with the strong topology. A dg-space is a space X equipped with a sheaf $\mathcal{O}_X$ of dg-vector spaces such that $(X, H^0(\mathcal{O}_X))$ is a smooth scheme.

1 Description of the Factorisation Algebras

Let Φ be the sheaf of $\mathbb{R}/2\pi R\mathbb{Z}$-valued functions on Σ, thought of as a sheaf of spaces, and let Φ_c be the cosheaf of compactly supported such functions. We can define an exterior derivative map

$$d \colon \Phi \to \Omega^1_{cl}$$

as a map of sheaves of spaces, by lifting a function $\phi \in \Phi(U)$ to a real-valued function on each set in a cover of U by contractible sets and applying the usual exterior derivative. The resulting 1-forms are independent of the choice of lift and are compatible on overlaps, so glue to a closed 1-form. We can similarly define a map $\Phi_c \to \Omega^1_{c,cl}$ of cosheaves of spaces. We scale the map d so that its image on $\Phi(U)$ is the subgroup of closed 1-forms with integral periods $\Omega^1_{cl,\mathbb{Z}}(U)$ (i.e. 1-forms whose cohomology class lies in the integral lattice $H^1(U;\mathbb{Z}) \leq H^1_{dR}(U)$), and the kernel is the subsheaf of locally constant $\mathbb{R}/2\pi R\mathbb{Z}$-valued functions. Using a Riemannian metric on Σ we can define a map of cosheaves $S_R \colon \Phi_c \to \mathbb{R}$ by

$$S_R(\phi) = -R^2\|d\phi\|^2 = -R^2 \int d\phi \wedge *d\phi$$

The dependence on the radius R of the target circle has been made explicit here: this action is the image of the sigma-model action with target $\mathbb{R}/2\pi R\mathbb{Z}$ under the scaling of the map d defined above. The kernel of S_R also consists of precisely the constant functions. It was necessary to use the compactly supported functions, otherwise the integral may not have been well-defined.

1.1 Classical Observables

The *classical field theory* associated to this data is described as a cosheaf of dg-spaces on Σ built from Φ_c and the action map S_R. We define

$$\mathcal{T}^{\mathrm{cl}}(U) = (\mathrm{PV}(\Phi_c(U)), -\iota_{dS}),$$

that is, the dg-space with the same underlying *space* as $\Phi_c(U)$, but with structure sheaf the sheaf of polyvector fields on $\Phi_c(U)$, equipped with a differential by contracting with the 1-form dS. We can give a more concrete description on a fixed open set U in a non-canonical way by choosing a point $u \in U$, giving us a splitting of the space of fields

$$\Phi_c(U) \cong \mathbb{R}/2\pi R\mathbb{Z} \times \Omega^1_{c,cl,\mathbb{Z}}(U)$$

by mapping ϕ to $(\phi(u), d\phi)$, and in the other direction (θ, α) to $\theta + d^{-1}\alpha$ where the preimage is chosen so that $d^{-1}\alpha(u) = 0$. So we need only understand $\mathcal{O}(\mathbb{R}/2\pi R\mathbb{Z}) \otimes \mathrm{PV}(\Omega^1_{c,cl,\mathbb{Z}}(U))$, and the differential, which acts only on the second factor. As a space, this factor has a collection of connected components indexed by the discrete group $H^1(U;\mathbb{Z})$, each of which is a torsor for $\mathrm{PV}(d\Omega^0_c(U))$. The sections of the structure sheaf over such a component ($\Omega^1_h(U)$ for $h \in H^1(U;\mathbb{Z})$ say) have a nice description as

$$\mathrm{Sym}(\Omega^1_h(U)^\vee \oplus \Omega^1_h(U)[1]),$$

and the differential $-\iota_{dS}$ is extended as a derivation from the inclusion $\Omega^1_h(U)[1] \to \Omega^1_h(U)^\vee$ given by the metric:

$$\alpha \mapsto \left(\beta \mapsto R^2 \int_U \alpha \wedge *\beta\right)$$

where the factor of R^2 is introduced for explicit compatibility with the $\mathbb{R}/2\pi R\mathbb{Z}$-valued function theory, as above.

1.2 Quantum Observables

To quantise these classical observables we introduce the BV operator on polyvector fields. This operator is defined by identifying $PV(\Phi_c(U))$ as functions on the shifted cotangent bundle $\mathcal{O}(T^*[-1]\Phi_c(U))$. As such, there is a natural Poisson bracket coming from the symplectic pairing on this cotangent bundle which we can use to deform the differential on the classical observables.

Using the local description above away from constant functions, the BV operator D is easy to describe for a single component: one uses the natural degree 1 pairing $\Omega^1_h(U)^\vee \otimes \Omega^1_h(U)[1] \to \mathbb{R}$ from $\mathrm{Sym}^2 \to \mathrm{Sym}^0$, and extend to all of $\mathrm{PV}(\Omega^1_h(U))$ by the formula

$$D(\mathcal{O}_1\mathcal{O}_2) = D(\mathcal{O}_1)\mathcal{O}_2 + (-1)^{|\mathcal{O}_1|}\mathcal{O}_1 D(\mathcal{O}_2) + \{\mathcal{O}_1, \mathcal{O}_2\}$$

where $\{-,-\}$ denotes the natural pairing on $\mathrm{Sym}(\Omega^1_h(U)^\vee) \otimes \mathrm{Sym}(\Omega^1_h(U)[1]) \leq \mathrm{Sym}(\Omega^1_h(U)^\vee \oplus \Omega^1_h(U)[1])$, and is zero on other elements of the algebra.

We define the *quantum field theory* associated to this Lagrangian theory in question to be the precosheaf of dg-spaces whose sections over U are given by

$$\mathcal{T}^{\mathrm{q}}(U) = (\mathrm{PV}(\Phi_c(U)), D - \iota_{dS}),$$

as described above. However, this definition will need some modification in order to define expectation values of observables, and to compute the Fourier duals of observables. We'll describe a factorisation algebra where duality can be defined in a well-behaved way, then explain how it's related to the global sections of this factorisation space

We define a factorisation space $\mathcal{T}^{\mathrm{q}}_{\Omega^1_{cl}}$ on Σ closely related to the above: specifically, we form the factorisation space of quantum observables for the classical field theory whose fields are closed 1-forms and whose action is the squared L^2 norm $\|\alpha\|_2^2$. Specifically, this is the presheaf whose sections over U are the vector space of closed 1-forms $\Omega^1_{cl}(U)$, with structure sheaf, again, the sheaf of polyvector fields with the quantum differential. In particular, the global sections define a factorisation algebra modelled by the cochain complex

$$\mathrm{Sym}\left(\Omega^1_{c,cl}(U)[1] \oplus (\Omega^1_{cl}(U))^\vee\right)$$

with differential $\mathrm{Sym}(R^2\iota) + D$, where ι is the inclusion $\Omega^1_{c,cl}(U)[1] \to (\Omega^1_{cl}(U))^\vee$ given by the metric as above, and where D is the BV operator coming from the evaluation pairing on this two-step complex.

Writing $\mathrm{Obs}(U)$ for the factorisation algebras obtained by taking (derived) global sections of these factorisation spaces, if U is simply-connected then we have a natural identification

$$\mathrm{Obs}^{\mathrm{q}}(U) = \mathrm{Obs}^{\mathrm{q}}_{\Omega^1_{cl}}(U).$$

1.3 Expectation Values

We can understand expectation values of observables in a very natural way in this simplified theory of closed 1-forms. We'll define a dense contractible factorisation subalgebra of $\mathrm{Obs}^{\mathrm{q}}_{\Omega^1_{cl}}$, corresponding to the smooth forms sitting naturally inside of distributional forms. There will be a canonical quasi-isomorphism from the local sections of this trivial subalgebra to $\mathbb{R}$, and a (non-canonical) map to this subalgebra for *global* observables.

Firstly, recall that as a graded vector space we defined

$$\mathrm{Obs}^{\mathrm{q}}_{\Omega^1_{cl}}(U) = \mathrm{Sym}\left(\Omega^1_{c,cl}(U)[1] \oplus (\Omega^1_{cl}(U))^\vee\right).$$

Sitting inside this graded vector space we have a dense subspace, namely

$$\mathrm{Obs}^{\mathrm{sm}}_{\Omega^1_{cl}}(U) = \mathrm{Sym}\left(\Omega^1_{c,cl}(U)[1] \oplus \Omega^1_c(U)/d^*\Omega^2_c(U)\right)$$

where the embedding $\Omega^1_c(U)/d^*\Omega^2_c(U) \to (\Omega^1_{cl}(U))^\vee$ in degree zero is given by the metric as usual, noting that the local coexact forms $d^*\Omega^2_c(U)$ are orthogonal to the local closed 1-forms under the L^2-pairing. Here *sm* stands for either "smooth" or "smeared" observables. This graded vector subspace inherits a differential from the quantum differential on the whole theory; we'll show that its global cohomology with respect to this differential is canonically isomorphic to $\mathbb{R}$ in degree zero. Indeed, this is clear in the case of the complex equipped with the *classical differential.* Notice first that globally, Hodge theory implies that $\Omega^1_{cl}(\Sigma) \cong \Omega^1(\Sigma)/d^*\Omega^2(\Sigma)$. Restricted to the smooth observables the complex with this differential becomes

$$\begin{aligned} \mathrm{Sym}\left(\Omega^1_{cl}(\Sigma) \overset{\times R^2}{\to} \Omega^1_{cl}(\Sigma)\right) &\sim \mathrm{Sym}(0) \\ &= \mathbb{R}. \end{aligned}$$

To see this is also true with respect to the *quantum* observables we use a simple spectral sequence argument, using the filtration of the complex by Sym degree. The BV operator is extended from the map from Sym^2 to Sym^0 by the L^2 pairing, so in general lowers Sym-degree by two. Consider the spectral sequence of a filtered complex using the Sym degree filtration. The E_1 page of this spectral sequence computes the cohomology of the classical complex of smeared observables (i.e. the cohomology with respect to only the Sym degree 0 part of the differential), and the spectral sequence converges to the cohomology of the complex of smooth quantum observables (i.e. the cohomology with respect to the entire differential). Since the E_1 page is quasi-isomorphic to $\mathbb{R}$ in degree 0, so must be the E_∞ page. Finally we observe that there is a *unique* quasi-isomorphism from this complex of smooth observables on U to $\mathbb{R}$ characterised by the property that 1 in sym^0 maps to 1.

To produce an expectation value map for the factorisation algebra $\mathrm{Obs}^{\mathrm{q}}_{\Omega^1_{cl}}(U)$ we'll define a *smearing* map on global observables:

$$\mathrm{Obs}^{\mathrm{q}}_{\Omega^1_{cl}}(\Sigma) \to \mathrm{Obs}^{\mathrm{sm}}_{\Omega^1_{cl}}(\Sigma).$$

This allows us to associate a number $\langle \mathcal{O} \rangle$ to a local observable $\mathcal{O}$ (in the degree zero piece of the cochain complex of observables) by the following procedure:

1. Take a degree zero observable $\mathcal{O} \in \mathrm{Obs}^{\mathrm{q}}_{\Omega^1_{cl}}(U)$ and extend it to a degree zero global observable using the factorisation algebra structure.
2. Use the smearing map to associate to this global observable a *smeared* global observable in $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^1_{cl}}(\Sigma)$.
3. Use the canonical quasi-isomorphism $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^1_{cl}}(\Sigma) \to \mathbb{R}$ to produce a number: the *expectation value* of $\mathcal{O}$.

A smearing map is defined as follows: first we must *choose* a *parametrix for the Laplacian.* That is to say, a map $P\colon (\Omega^1_{cl}(\Sigma))^\vee \to \Omega^1_{cl}(\Sigma)$ such that $\Delta \circ P \circ \iota$ is the identity on $\Omega^1_{cl}(\Sigma)$. One can construct parametrices by convolving with smooth approximations to the Green's kernel for Δ, but there is no canonical choice. By definition the square

$$\begin{tikzcd} \Omega^1_{cl}(\Sigma) \arrow[r, "\iota"] \arrow[d, "1"'] & (\Omega^1_{cl}(\Sigma))^\vee \arrow[d, "\Delta\circ P"] \\ \Omega^1_{cl}(\Sigma) \arrow[r, "1"] & \Omega^1_{cl}(\Sigma) \end{tikzcd}$$

commutes, so defines a map of cochain complexes on the level of classical observables. The map is not a cochain map for the quantum differential, but it is the identity on $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^1_{cl}}(\Sigma)$ thought of as a subcomplex.

1.4 Computing Expectation Values

The idea of Feynman diagrams is to compute expectation values of observables combinatorially. The crucial idea that we'll use to check that we an do this is that –for smeared observables – the expectation value map is *uniquely characterised*. Let's ignore for now the final step in computing expectation values: integrating out the constant maps. Then for smeared observables there is a unique quasi-isomorphism from global smeared observables to $\mathbb{R}$ that sends 1 to 1. Therefore to check that a procedure for computing expectation values is valid it suffices to check that it is a non-trivial quasi-isomorphism, then rescale so the map is appropriately normalised.

Again, we'll focus on expectation values in the closed 1-form theory. Take a global degree 0 smeared observable $\mathcal{O} \in \mathrm{Obs}^{\mathrm{sm}}_{\Omega^1_{cl}}(\Sigma)_0$. Suppose can write $\mathcal{O}$ as a product of linear observables

$$\mathcal{O} = \mathcal{O}_1^{n_1}\mathcal{O}_2^{n_2}\cdots\mathcal{O}_k^{n_k}$$

where $\mathcal{O}_1, \ldots \mathcal{O}_k$ are linearly independent linear smeared observables in $\Omega^1_{cl}(\Sigma)$ (general observables are sums of observables of this form, and we extend the procedure of computing duals linearly).

We compute the expectation value of $\mathcal{O}$ combinatorially as follows. Depict $\mathcal{O}$ as a graph with k vertices, and with n_i half edges attached to vertex i. The expectation value $\langle\mathcal{O}\rangle$ of $\mathcal{O}$is computed as a sum of terms constructed by gluing edges onto this frame in a prescribed way. Specifically, we attach *propagator edges* – which connect together two of these half-edges – in order to leave no free half-edges remaining. A propagator between linear observables $\mathcal{O}_i$ and $\mathcal{O}_j$ receives weight via the L^2-pairing

$$\frac{1}{2R^2}\langle\mathcal{O}_i, \mathcal{O}_j\rangle = \frac{1}{2R^2}\int_\Sigma \mathcal{O}_i \wedge *\mathcal{O}_j$$

and a diagram is weighted by the product of all these edge weights. The expectation value is the sum of these weights over all such diagrams.

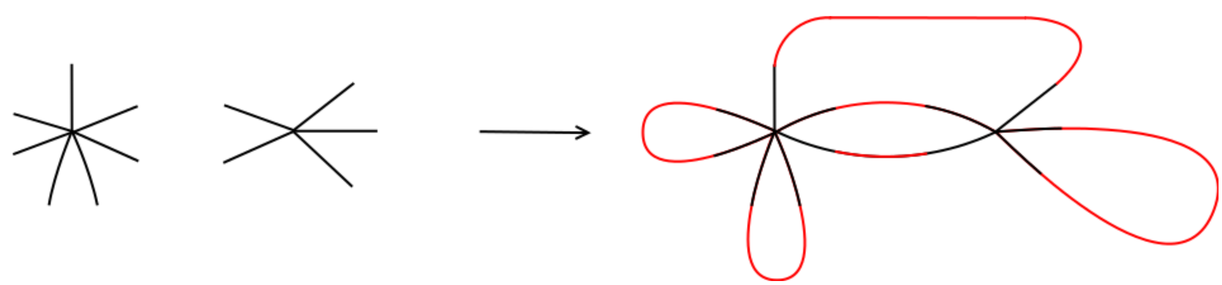

Figure 1: One of the terms in the Feynman diagram expansion computing the expectation value of an observable of form $\mathcal{O}_1^7\,\mathcal{O}_2^5$. On the left we see the starting point, with half-edges (in black), and on the right we see one way of connecting these half-edges with propagator edges (in red).

To check that this computes the expectation value, we must show that it is non-zero, and that it vanishes on the image of the differential in the complex of quantum observables. The former is easy: the observable 1 has expectation value 1 (so we're also already appropriately normalised). For the latter, we'll show that the path integral computation for degree zero global observables in $\mathrm{Obs}^{\mathrm{q}}_{\Omega^1_{cl}}(U)_0$ arise as a limit of finite-dimensional Gaussian integrals, and that the images of the quantum BV differential are all divergences, so vanish by Stokes' theorem.

To set up this limit, recall that the Laplacian Δ acting on $\Omega^1_{cl}(\Sigma)$ has a discrete spectrum $0 < \lambda_1 < \lambda_2 < \cdots$, with finite-dimensional eigenspaces. Let $F^k\Omega^1_{cl}(\Sigma)$ denote the sum of the first k eigenspaces: this defines an exhaustive increasing filtration of $\Omega^1_{cl}(\Sigma)$ by finite-dimensional vector spaces.

Proposition 1.1. Let $\mathcal{O}$ be a smeared global observable. The finite-dimensional Gaussian integrals

$$\frac{1}{Z_k}\int_{F^k\Omega^1_{cl}(\Sigma)} \mathcal{O}(a)e^{-S(a)}da,$$

where Z_k is the volume $\int_{F^k\Omega^1_{cl}(\Sigma)} e^{S(a)}da$, converge to a real number $I(\mathcal{O})$ as $k \to \infty$, and this number agrees with the expectation value computed by the Feynman diagrammatic method.

Proof. We check that for each k the Gaussian integral admits a diagrammatic description, and observe that the expressions computed by these diagrams converge to the expression we want. We may assume as usual that $\mathcal{O}$ splits as a product of linear smeared observables $\mathcal{O} = \mathcal{O}_1^{n_1}\mathcal{O}_2^{n_2}\cdots\mathcal{O}_\ell^{n_\ell}$. The $\mathcal{O}_i$ describe linear operators on the filtered pieces. We can write the Gaussian integral using a generating function as

$$\int_{F^k\Omega^1_{cl}(\Sigma)} \mathcal{O}(a)e^{-S(a)}da = \frac{\partial^{n_1+\cdots n_\ell}}{\partial t_1^{n_1}\cdots\partial t_\ell^{n_\ell}}\bigg|_{t_1=\cdots=t_\ell=0}\int_{F^k\Omega^1_{cl}(\Sigma)} e^{-R^2\|a\|+t_1\mathcal{O}_1(a)\cdots+t_\ell\mathcal{O}_\ell(a)}da,$$

provided that k is large enough that upon projecting to $F^k\Omega^1_{cl}$ the $\mathcal{O}_i$ are linearly independent. Call this projection $\mathcal{O}_i^{(k)}$. This expression is further simplified by completing the square, yielding

$$Z_k\,\frac{\partial^{n_1+\cdots n_\ell}}{\partial t_1^{n_1}\cdots\partial t_\ell^{n_\ell}}\bigg|_{t_1=\cdots=t_\ell=0} e^{\frac{1}{4R^2}\int_\Sigma(t_1\mathcal{O}_1^{(k)}+\cdots t_\ell\mathcal{O}_\ell^{(k)})\wedge*(t_1\mathcal{O}_1^{(k)}+\cdots t_\ell\mathcal{O}_\ell^{(k)})}.$$

We can now compute the Gaussian integral diagrammatically. The $t_{n_1}\cdots t_{n_\ell}$-term of the generating function is the sum over Feynman diagrams as described above, where a diagram is weighted by a product of matrix elements $\frac{1}{2R^2}\langle\mathcal{O}_i^{(k)},\mathcal{O}_j^{(k)}\rangle$ corresponding to the edges. We see that as $k\to\infty$ this agrees with the weight we expect.

□

Now we can justify why the expectation value vanishes on the image of the quantum differential. Let $\mathcal{O} \in \mathrm{Obs}^{\mathrm{sm}}_{\Omega^1_{cl}}(\Sigma)_0$ be a smeared global observable, and suppose $\mathcal{O} = (D - \iota_{dS})\Phi$ is in the image of the quantum differential. So the restriction of Φ to a filtered piece is a vector field on the vector space $F^k\Omega^1_{cl}(\Sigma)$, and we can compute the divergence

$$\mathrm{div}(\Phi e^{-S(a)}) = \mathcal{O}e^{-S(a)},$$

where the restriction to the filtered piece is suppressed in the notation. So the expectation value of $\mathcal{O}$ is a limit of integrals of divergences, which vanish by Stokes' theorem, and the expectation value is zero. This implies that the procedure described above does indeed compute the cohomology class of a global smeared observable in the canonically trivialised cohomology.

Remark 1.2. This also allows us to compute expectation values in the circle-valued function theory, at least for observables supported on simply connected sets. On a simply connected open set, the observables in the closed 1-form theory and in the circle-valued function theory agree, so this procedure gives a definition of the expectation value in the circle-valued function setting. We'll describe a way of extending this definition to expectation values of certain observables in the circle-valued function theory even when the support is not simply-connected later on, in section 2.2.

2 Fourier Duality for Quantum Observables

To define the Fourier dual of an observable, we'll introduce yet another factorisation algebra on Σ. Instead of the factorisation algebra $\mathrm{Obs}^{\mathrm{q}}_{\Omega^1_{cl}}$ built from closed 1-forms and the L^2-norm, we can work with *all* 1-forms and the L^2-norm. We'll call this factorisation algebra $\mathrm{Obs}^{\mathrm{q}}_{\Omega^1}$. At this point we'll also stop supressing the parameter R in our notation: we'll write $\mathrm{Obs}^{\mathrm{q}}_{\Omega^1,R}$ for the factorisation algebra with fields all 1-forms and action

$$S_R(a) = R^2\|a\|^2.$$

In its simplest form, Fourier duality is an isomorphism on degree 0 observables in these theories: $\mathrm{Obs}^{\mathrm{q}}_{\Omega^1,R}(U)_0 \cong \mathrm{Obs}^{\mathrm{q}}_{\Omega^1,1/2R}(U)_0$. It will not be extend to any kind of cochain maps in these theories, and in particular will not be compatible with the expectation value maps, but we'll show that it *is* compatible with the expectation values after restriction $\mathrm{Obs}^{\mathrm{q}}_{\Omega^1,R} \to \mathrm{Obs}^{\mathrm{q}}_{\Omega^1_{cl},R}$.

2.1 Feynman Diagrams for Fourier Duality

We'll construct $\mathcal{F}$ in an explicit combinatorial way using Feynman diagrams extending the Feynman diagram expression computing expectation values. Take a smeared monomial observable $\mathcal{O} \in \mathrm{Obs}^{\mathrm{sm}}_{\Omega^1,R}(U)_0$. As above, we write $\mathcal{O}$ as

$$\mathcal{O}_1^{n_1}\mathcal{O}_2^{n_2}\cdots\mathcal{O}_k^{n_k}$$

where $\mathcal{O}_1,\ldots\mathcal{O}_k$ are linearly independent linear smeared observables in $\Omega^1_c(U)$.

We compute the Fourier dual of $\mathcal{O}$ much as before. Depict $\mathcal{O}$ as a graph with k vertices, and with n_i half edges attached to vertex i. Now, we can attach any number of *propagator edges* as before, and also any number of *source terms* – which attach to an initial half-edge and leave a half-edge free – in such a way as to leave none of the original half-edges unused. The source terms have the effect of replacing a linear term $\mathcal{O}_i$ with its Hodge dual $*\mathcal{O}_i$. The result is a new observable

$$(*\mathcal{O}_1)^{m_1}(*\mathcal{O}_2)^{m_2}\cdots(*\mathcal{O}_k)^{m_k}$$

where m_i is the number of source edges connected to vertex i, now thought of as a degree zero observable in $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^1,1/2R}(U)_0$. The total Fourier dual observable $\widetilde{\mathcal{O}}$ is then the sum of all these observables with appropriate weightings.

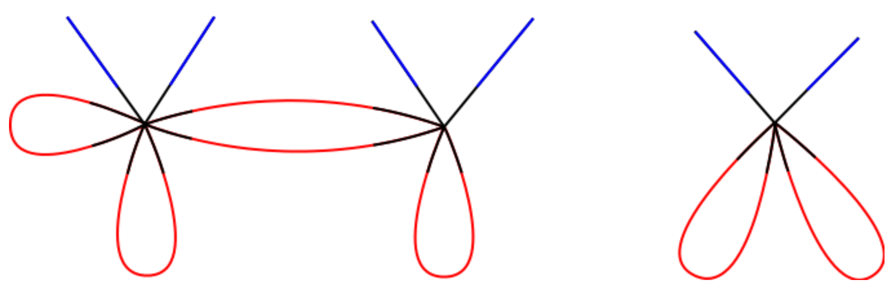

Figure 2: The Feynman diagram corresponding to a degree 6 term in the Fourier dual of an observable of form $\mathcal{O}_1^8\mathcal{O}_2^6\mathcal{O}_3^6$. Propagators are drawn in red and sources in blue.

Again, we weight such a diagram by taking a product of weights attached to each edge. Edges are weighted in the following way:

- A propagator between linear observables $\mathcal{O}_i$ and $\mathcal{O}_j$ receives weight

$$\frac{1}{2R^2}\langle\mathcal{O}_i,\mathcal{O}_j\rangle = \frac{1}{2R^2}\int_{S^2}\mathcal{O}_i\wedge *\mathcal{O}_j.$$

- A source term attached to a linear observable $\mathcal{O}$ receives weight $i/2R^2$.

We can compute the dual of a general global observable by smearing first, then dualising: the result is that an observable has a uniquely determined smeared dual for each choice of smearing. In order to compare expectation values of an observable and its dual, the crucial tool that we'll use is Plancherel's formula, which we can rederive in terms of Feynman diagrams. The first step is to prove a Fourier inversion formula in this language. In doing so we'll need to remember that after dualising once, the new observable lives in the *dual* theory, with a different action: therefore the weights assigned to edges will be different.

Proposition 2.1. A smeared observable $\mathcal{O}$ is equal to its Fourier double dual $\tilde{\tilde{\mathcal{O}}}$.

Proof. Let $\mathcal{O} = \mathcal{O}_1^{n_1}\mathcal{O}_2^{n_2}\cdots\mathcal{O}_k^{n_k}$ as above. The Fourier double dual of $\mathcal{O}$ is computed as a sum over diagrams with two kinds of edges: those coming from the first dual and those coming from the second. We'll show that these diagrams all naturally cancel in pairs apart from the diagram with no propagator edges. We depict such diagrams with blue edges coming from the first dual, and red edges coming from the second dual.

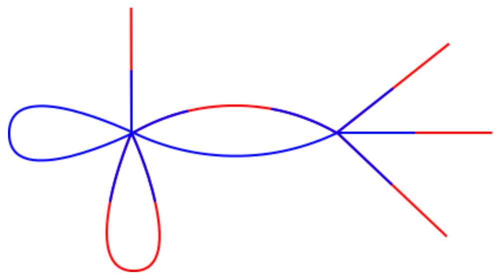

Figure 3: A diagram depicting a summand of the Fourier double dual of an observable of form $\mathcal{O}_1^7\,\mathcal{O}_2^5$.

So choose any diagram D with at least one propagator, and choose a propagator edge in the diagram. We produce a new diagram D' by swapping the colour of this propagator edge.

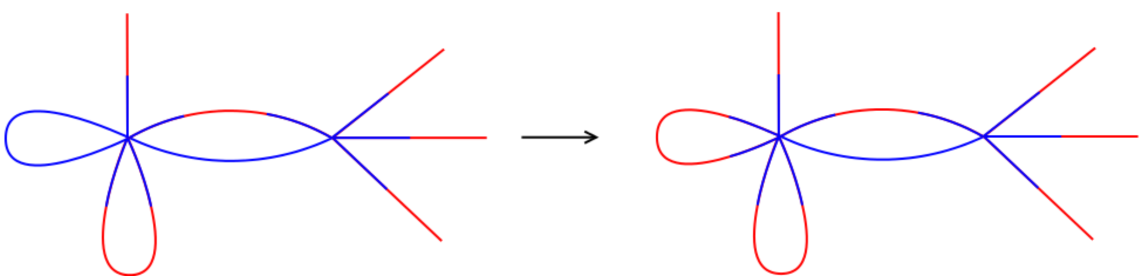

Figure 4: In this diagram we chose the blue leftmost propagator loop (coming from the first dual), and replaced it by two blue source terms connected with a red propagator loop (coming from the second dual).

It suffices to show that the weight attached to this new diagram is -1 times the weight attached to the original diagram, so that the two cancel. This is easy to see: the propagator from the first term contributes a weight $\frac{1}{2R^2}\int_\Sigma \mathcal{O}_i \wedge *\mathcal{O}_j$. In the second dual, the weights come from the source terms in the original theory, but the propagator in the *dual* theory, which contributes a weight using the dual theory. So the total weight is

$$2R^2\left(\frac{i}{2R^2}\right)^2\int_\Sigma \mathcal{O}_i \wedge *\mathcal{O}_j$$

Which is -1 times the weight of the other diagram, as required. $\square$

For further justification for these choices of weights, we should compare this combinatorial Fourier dual to one calculated using functional integrals (for global smeared observables). We'll perform such a check by defining a sequence of Gaussian integrals on filtered pieces, and checking that they converge to a Fourier dual that agrees with the one combinatorially described above. As before, $F^k\Omega^1(\Sigma)$ refers to the filtration by eigenspaces of the Laplacian, this time on the space of all 1-forms, not just closed 1-forms.

Proposition 2.2. Let $\mathcal{O}$ be a smeared global observable. The finite-dimensional Gaussian integrals

$$\widetilde{\mathcal{O}}(\widetilde{a}) = \left(\frac{1}{Z_k}\int_{F^k\Omega^1(\Sigma)} \mathcal{O}(a)e^{-S_R(a)+i\int_\Sigma \widetilde{a}\wedge a}da\right)e^{S_{1/2R}(\widetilde{a})}$$

converge to a smeared global observable, which agrees with the Fourier dual observable computed by the Feynman diagrammatic method.

Proof. We use the same method of proof as for 1.1, writing the integral as a derivative of a generating function. Specifically, for $\mathcal{O} = \mathcal{O}_1^{n_1}\mathcal{O}_2^{n_2}\cdots\mathcal{O}_\ell^{n_\ell}$ we expand

$$\begin{aligned}\frac{1}{Z_k}\int_{F^k\Omega^1(\Sigma)} \mathcal{O}(a)e^{-S_R(a)+i\int_\Sigma \widetilde{a}\wedge a}da &= \left.\frac{\partial^{n_1+\cdots n_\ell}}{\partial t_1^{n_1}\cdots\partial t_\ell^{n_\ell}}\right|_{t_1=\cdots=t_\ell=0}\frac{1}{Z_k}\int_{F^k\Omega^1(\Sigma)} e^{S_R(a)+\sum t_i\int_\Sigma \mathcal{O}_i\wedge * a+i\int_\Sigma \widetilde{a}\wedge a}da \\ &= \left.\frac{\partial^{n_1+\cdots n_\ell}}{\partial t_1^{n_1}\cdots\partial t_\ell^{n_\ell}}\right|_{t_1=\cdots=t_\ell=0} \\ &\quad e^{-S_{1/2R}(\widetilde{a})}e^{\frac{1}{4R^2}\int_\Sigma(t_1\mathcal{O}_1^{(k)}+\cdots+t_\ell\mathcal{O}_\ell^{(k)})\wedge*(t_1\mathcal{O}_1^{(k)}+\cdots+t_\ell\mathcal{O}_\ell^{(k)})+\frac{i}{2R^2}\int_\Sigma(t_1\mathcal{O}_1^{(k)}+\cdots+t_\ell\mathcal{O}_\ell^{(k)})\wedge\widetilde{a}}\end{aligned}$$

(by completing the square) and extract the $t_1^{n_1}\cdots t_\ell^{n_\ell}$-term. once again we're denoting by $\mathcal{O}_i^{(k)}$ the projection of $\mathcal{O}_i$ onto the k^{th} filtered piece $F^k\Omega^1(\Sigma)$. We choose the level in the filtration large enough so that the upon projecting to the filtered piece the forms $\mathcal{O}_i$ are linearly independent. One then observes that in the limit as $k\to\infty$ the relevant term is given by a sum over diagrams as described with the correct weights. $\square$

Now, for any open set $U\subseteq\Sigma$ we have a restriction map of degree zero local observables $r(U)\colon \mathrm{Obs}^{\mathrm{sm}}_{\Omega^1}(U)_0 \to \mathrm{Obs}^{\mathrm{sm}}_{\Omega^1_{cl}}(U)_0$ induced by the projection $\Omega^1_c(U)\to\Omega^1_c(U)/d^*\Omega^2_c(U)$. This gives us a candidate notion of duality in the closed 1-form theory. So, we might take a degree 0 observable in the image of $r(U)$, choose a preimage, compute the dual then restrict once more. Of course, this is not quite canonical, because the map $r(U)$ is not injective: the resulting dual observable might depend on the choice of preimage we made. However, in certain circumstances we might be able to choose a consistent scheme for choosing such a preimage, therefore a canonical duality map. We'll return to this later in section 2.3, but first we'll prove that for *any* choice of lift, the resulting dual observable in the Ω^1_{cl} theory has the same expectation value as the original theory.

2.2 Fourier Duality and Expectation Values

At this point we have two equivalent ways of thinking about both the Fourier transform and the expectation value map for smeared observables: by Feynman diagrams (which allowed us to describe the dual locally) and by functional integration (which allow us to peform calculations, but only globally). We'll compare the expectation values of dual observables using a functional integral calculation, in which the restriction to *closed* 1-forms will be crucial.

For an actual equality of expectation values as described above we'll have to restrict to observables on a *simply connected* open set. Recall this is the setting where the local observables agree with observables in the circle-valued function theory. On more general open sets circle-valued functions are related to only those closed 1-forms with *integral periods*. As such there is a map on degree zero observables $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^1_{cl},R}(U)_0 \to \mathrm{Obs}^{\mathrm{q}}_{S^1,R}(U)_0$ for *any* U, which sends a compactly supported closed 1-form a to the local observable

$$\phi \mapsto \int_U d\phi\wedge *a.$$

However, this is generally not an isomorphism. Still, from a functional integral point-of-view we can define the expectation value of such an observable in the circle-valued function theory, even if Σ has genus > 0 so is not simply-connected. Given $\mathcal{O} \in \mathrm{Obs}^{\mathrm{sm}}_{\Omega^1_{cl},R}(U)_0$ we extend $\mathcal{O}$ to a global degree zero obervable, and define its *expectation value* to be

$$\langle \mathcal{O} \rangle_R = \lim_{k\to\infty} \int_{F^k\Omega^1_{cl,\mathbb{Z}}(\Sigma)} \mathcal{O}(a) e^{-S_R(a)} da$$

Recall here that $\Omega^1_{cl,\mathbb{Z}}(\Sigma)$ is our notation for the closed 1-forms with integral periods. This is the product of a finite-rank lattice with a vector space, and our filtration is the intersection of this subgroup with the filtration $F^k\Omega^1_{cl}(\Sigma)$ defined previously. We notice that if U is simply connected then this definition agrees with the one we used in 1.1, so in particular the limit converges. In general, the integrand is dominated in absolute value by the integrand over all closed 1-forms, which we already know converges (since the proof of 1.1 still applies with $\mathcal{O}_i$ replaced by $|\mathcal{O}_i|$).

Using this definition (and bearing in mind its relationship to the notion of expectation value considered above on simply-connected sets), we'll prove the main compatibility with duality.

Theorem 2.3. Let $\mathcal{O}$ be a local observable in $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^1,R}(U)_0$, and let $\widetilde{\mathcal{O}} \in \mathrm{Obs}^{\mathrm{sm}}_{\Omega^1,1/2R}(U)_0$ be its Fourier dual observable. Let $r(\mathcal{O})$ and $r(\widetilde{\mathcal{O}})$ be the restrictions to local observables in $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^1_{cl}}(U)_0$. Then, computing the expectation values of $r(\mathcal{O})$ and $r(\widetilde{\mathcal{O}})$, we find

$$\langle r(\mathcal{O}) \rangle_R = \langle r(\widetilde{\mathcal{O}}) \rangle_{\frac{1}{2R}}.$$

Proof. We know by 2.1 that $\mathcal{O} = \widetilde{\widetilde{\mathcal{O}}}$, so in particular $\langle r(\mathcal{O})\rangle_R = \langle r(\widetilde{\widetilde{\mathcal{O}}})\rangle_R$. By the calculation in Proposition 2.2 we can write this expectation value as the limit as $k \to \infty$ of the Gaussian integrals

$$\begin{aligned}
\frac{1}{Z_k}\int_{F^k\Omega^1_{cl,\mathbb{Z}}(\Sigma)} \mathcal{O}(a) e^{-S_R(a)} da &= \frac{1}{Z_k}\int_{F^k\Omega^1_{cl,\mathbb{Z}}(\Sigma)} \widetilde{\widetilde{\mathcal{O}}}(a) e^{-S_R(a)} da \\
&= \lim_{\ell\to\infty} \frac{1}{Z_k}\int_{F^k\Omega^1_{cl,\mathbb{Z}}(\Sigma)} \int_{F^\ell\Omega^1(\Sigma)} \widetilde{\mathcal{O}}(\widetilde{a}) e^{-S_R(a) + i\int_\Sigma \widetilde{a}\wedge a} d\widetilde{a}\, da \\
&= \lim_{\ell\to\infty} \frac{1}{Z_k}\int_{F^k\Omega^1(\Sigma)} \int_{F^\ell\Omega^1(\Sigma)} \widetilde{\mathcal{O}}(\widetilde{a}) e^{-S_{1/2R}(\widetilde{a}) + i\int_\Sigma \widetilde{a}\wedge a} \delta_{\Omega^1_{cl,\mathbb{Z}}(\Sigma)}(a)\, d\widetilde{a}\, da
\end{aligned}$$

The last line needs a little explanation. The distribution $\delta_{\Omega^1_{cl,\mathbb{Z}}(\Sigma)}$ is the delta-function on the closed and integral 1-forms sitting inside all 1-forms (restricted to the filtered piece): pairing with this distribution and integrating over all 1-forms in the filtered piece is the same as integrating only over the relevant subgroup.

Now, for a fixed value of ℓ, we can reinterpret the final integral above by changing the order of integration. This computes the Fourier dual of the delta function $\delta_{\Omega^1_{cl,\mathbb{Z}}(\Sigma)}$ and then pushes forward along the Hodge star. THe Fourier dual of the delta function is $\delta_{\Omega^1_{cocl,\mathbb{Z}}(\Sigma)}$, the delta function on the group of *coclosed* 1-forms with integral periods. That is, the external product $\delta_{d^*\Omega^2(\Sigma)} \boxtimes \delta_{\mathcal{H}^1_{\mathbb{Z}}}$ where $\mathcal{H}^1_{\mathbb{Z}}$ is the lattice in the space of harmonic 1-forms corresponding to the integral cohomology via Hodge theory. Pushing this distribution forward along the Hodge star recovers the original delta function. Therefore

$$\begin{aligned}
\langle r(\mathcal{O})\rangle_R &= \lim_{k\to\infty} \frac{1}{Z_k}\int_{F^k\Omega^1(\Sigma)} \widetilde{\mathcal{O}}(\widetilde{a}) e^{-S_{1/2R}(\widetilde{a})} \delta_{\Omega^1_{cl,\mathbb{Z}}(\Sigma)} d\widetilde{a} \\
&= \lim_{k\to\infty} \frac{1}{Z_k}\int_{F^k\Omega^1_{cl,\mathbb{Z}}(\Sigma)} \widetilde{\mathcal{O}}(\widetilde{a}) e^{-S_{1/2R}(\widetilde{a})} d\widetilde{a} \\
&= \langle r(\widetilde{\mathcal{O}})\rangle_{\frac{1}{2R}}
\end{aligned}$$

as required. $\square$

So to summarise, duality gives the following structure to the factorisation algebra of quantum observables in our sigma models.

- For each open set U, we have a subalgebra $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^1_{cl},R}(U)_0 \leq \mathrm{Obs}^{\mathrm{q}}_{S^1,R}(U)_0$ of the space of degree 0 local observables. If U is simply connected (for instance for local observables in a small neighbourhood of a point) this subalgebra is dense.
- For a local observable $\mathcal{O}$ living in this subalgebra we can define a *Fourier dual* observable in $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^1_{cl},1/2R}(U)_0$. This depends on a choice of extension of $\mathcal{O}$ to a functional on all 1-forms, rather than just closed 1-forms.
- For any choice of dual observable, we can compute their expectation values in the original theory and its dual, and they agree. If U is simply connected this expectation value map agrees with a natural construction from the point of view of the factorisation algebra.

2.3 A Canonical Dual

Although we do not have a canonical method for producing a smeared observable in $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^1}(U)_0$ from a general local smeared observable in the closed 1-form theory $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^1_{cl}}(U)_0$, we can do so for *certain* observables. Specifically, there is a subalgebra of the smeared observables:

$$\mathrm{Sym}(\Omega^1_{c,cl}(U)) \leq \mathrm{Sym}(\Omega^1_c(U)/d^*\Omega^2_c(U)).$$

That is, observables arising by pairing with a closed 1-form. For $U = \Sigma$ all smeared observables arise this way, by Hodge theory.

Now, there is a naïve way of extending such an observable to a functional on all 1-forms: that is, the map on smeared observables induced by the inclusion of closed 1-forms into all 1-forms. However, this extension is undesirable for purposes of duality. Indeed, let U be a simply connected open set, and let $\mathcal{O} \in \mathrm{Obs}^{\mathrm{sm}}_{\Omega^1_{cl}}(U)_0$ be a polynomial in local closed (hence exact) 1-forms $\mathcal{O}_i \in d\Omega^0_c(U)$. Dualising using the naïve extension given above gives an observable $\widetilde{\mathcal{O}}$ which is now a polynomial in $*\mathcal{O}_i \in d^*\Omega^2_c(U)$. Restricting to a local observable in the closed 1-form theory once more, all the $*\mathcal{O}_i$ act by zero, since the exact and coexact forms are orthogonal, so the dual observable is just a constant. Now, if we try to dualise once more the canonical extension of a constant to a functional on all 1-forms remains a constant, so we do not recover $\mathcal{O}$ by dualising again. So this naïve extension does not give an isomorphism on the local smeared observables $\mathrm{Sym}(\Omega^1_{c,cl}(U))$.

Another extension on this subalgebra performs better, and *does* give an isomorphism. Given such a local observable $\mathcal{O}$ on a simply-connected set U, we can extend to a smeared observable in $\mathrm{Obs}^{\mathrm{sm}}_{\Omega^1}(U)_0$ of the form

$$\mathcal{O} - *\mathcal{O} + c_{*\mathcal{O}},$$

where $*\mathcal{O}$ is the functional on 1-forms given by $a \mapsto \mathcal{O}(*a)$, and where $c_{*\mathcal{O}}$ is the constant term of the polynomial observable $*\mathcal{O}$. Dualising this extension and projecting onto $\mathrm{Sym}(\Omega^1_{c,cl}(U))$ again yields the dual observable

$$-\widetilde{*\mathcal{O}} + c_{*\mathcal{O}} + c_{\widetilde{\mathcal{O}}},$$

and dualising again recovers the original observable, $\mathcal{O}$.