A Brief Introduction to Twistors

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1 Compactified Minkowski Space

In this section we'll describe the geometry of the conformal compactification of Minkowski space in a few different ways. The aim of the game here is to describe a compact 4-manifold M^c with a Lorentzian metric such that a Lorentzian manifold M conformally equivalent to Minkowski space $\mathbb{R}^{1,3}$ embeds isometrically into M^c. The complement of the image of Minkowski space is called *conformal infinity*. We'll give a construction of such an M^c, and describe lots of ways of thinking about it.

Remark 1.1. As a manifold, M^c has to be more complicated than S^4, and thus conformal infinity has to be more complicated than a point. This is because S^4 doesn't admit a Lorentzian metric. To prove this, note that if h was a Lorentzian metric on S^4 and g was any Riemannian metric, then we could cook up an endomorphism A of the tangent bundle by the formula

$$h(-,-) = g(A-,-)$$

with a single negative eigenvalue. An eigenvector of A with this eigenvalue yields a non-vanishing section of TS^4, but no such section exists.

1.1 Several Constructions of M^c

1. [1] One basic idea is the following: we view $M = \mathbb{R}^{1,3}$ as a closed subspace of $\mathbb{R}^{2,4}$. Let (t, x_1, x_2, x_3) be orthonormal coordinates on M, and let (s, u) be orthonormal coordinates orthogonal to M. The null cone in $\mathbb{R}^{2,4}$ is the quadric of points of zero norm; intersecting this cone with a slice $\{s = u + 1\}$ (diagonal in the su-plane) say gives a space isometric to M.

   Remark 1.2. We get some neat spaces by intersecting the null cone with other slices. The intersection with a vertical slice $\{s = 1\}$ is a hyperboloid with constant positive scalar curvature usually called *de Sitter space*, and the intersection with a horizontal slice $\{u = 1\}$ is a hyperboloid with constant negative scalar curvature usually called *anti-de Sitter space*.

   We get a *compact* submanifold by, rather than intersecting the cone with a slice, intersecting the cone with a *sphere*, namely the unit 5-sphere

   $$\{s^2 + t^2 + u^2 + |x|^2 = 1\} \subseteq \mathbb{R}^{2,4}.$$

   The resulting intersection is a Lorentzian 4-manifold diffeomorphic to $S^1 \times S^3$. That's all well and good, but is it a compactification of Minkowski space? Sadly no, but it is a compactification of *two disjoint copies* of M. Indeed, intersecting with the hemisphere as above but with $\{s > 0\}$ gives an isometric copy of M, as does the intersection with the hemisphere $\{s < 0\}$. Thus we obtain a *true* compactification M^c of Minkowski space by quotienting this 4-manifold by the antipodal involution of the 5-sphere.

   So, let's take stock of what we've done. The 5-sphere modulo involution is precisely the real projective space $\mathbb{P}(\mathbb{R}^{2,4}) = \mathbb{RP}^5$. Intersecting with this locus corresponds to considering a *projective quadric* in $\mathbb{P}(\mathbb{R}^{2,4})$, corresponding to the projectivised null cone.

[1] In this section I referred to a nice description given in a blog post by Willie Wong, available online at http://williewong.wordpress.com/2009/10/26/conformal-compactification-of-space-time/.

2. There's a more constructive way of understanding this compactification, by considering *Lie spheres* in a spacelike slice $\mathbb{R}^3$ [Jad11a]. We can identify a point (t, x) in $\mathbb{R}^{1,3}$ as an *oriented sphere* in $\mathbb{R}^3$. That is, we take a sphere centred at x of radius $|t|$, and with orientation corresponding to the sign of t. We view points with $t = 0$ as degenerate spheres, aka points. With this in mind, consider the embedding constructed above into the null cone of $\mathbb{RP}^5$. In coordinates it can be written explicitly as

$$(t, x) \mapsto \left(t : \frac{1}{2}(|x|^2 - t^2 + 1) : \frac{1}{2}(|x|^2 - t^2 - 1) : x \right).$$

   Now, what points in the projectivised null cone are we missing? If $t = 0$ then the above is the only point on the cone unless $x = 0$ also, in which case there is an additional point: $(0 : 1 : 1 : 0)$. We identify this with a *point at spacial infinity* in $\mathbb{R}^3$, since it is obtained from the above formula by taking $|x|$ to infinity.

   If $t \neq 0$ then the above is the only point on the cone unless $|x|^2 = t^2$ also, in which case there is a whole $\mathbb{R}$ family of additional points of form $(t : 0 : 0 : x)$. We identify this with a *sphere at temporal infinity* centred at x with radius $1/t$, or as a *plane* in $\mathbb{R}^3$ with normal vector x, distance $|x|$ from the origin. Indeed, this occurs from the above formula by setting $|x| = t$ and taking the x_i and t to infinity simultaneously.

3. Here's another way of producing the projectivised null cone in $\mathbb{RP}^5$, which gives another description of the compactified Minkowski space. Recall the *Plücker embedding* of Grassmannians: this is the map $\mathrm{Gr}_k(V) \to \mathbb{P}(V^{\wedge k})$ sending the k-plane spanned by $v_1, \ldots, v_k$ to the line spanned by $v_1 \wedge \cdots \wedge v_k$. Setting $k = 2, n = \dim V = 4$ yields a map

$$P\colon \mathrm{Gr}_2(\mathbb{R}^4) \to \mathbb{P}(\mathbb{R}^6).$$

   If one chooses a basis $v_1, \ldots, v_4$ for $\mathbb{R}^4$, let $v_{ij} = v_i \wedge v_j$. These coordinates for $i < j$ form a basis for $(\mathbb{R}^4)^{\wedge 2} \cong \mathbb{R}^6$. The image of the Plücker map is then the projective quadric defined by the homogeneous polynomial

$$-v_{13}v_{24} + v_{12}v_{34} + v_{14}v_{23} = 0.$$

   Performing a linear change of variables identifies this quadric with the null cone for a pseudo-Riemannian metric on $\mathbb{R}^6$ of signature $(3, 3)$. This is therefore not quite the compactified Minkowski space we've already studied: it's out by a "Wick rotation". We fix this by doing a slightly more subtle construction.

   Consider now the *complex* Plücker embedding

$$P_{\mathbb{C}}\colon \mathrm{Gr}_2(\mathbb{C}^4) \to \mathbb{P}(\mathbb{C}^6).$$

   We'll pick out a suitable real form of this map that gives a quadric of the correct signature. Choose a pseudo-Hermitian form on $\mathbb{C}^4$ of signature $(2, 2)$ [2]. This naturally induces a Hodge star map $*\colon (\mathbb{C}^4)^{\wedge k} \to (\mathbb{C}^4)^{\wedge(4-k)}$, an $\mathbb{R}$-linear map which squares to 1 as an endomorphism of $(\mathbb{C}^4)^{\wedge 2}$. This splits $(\mathbb{C}^4)^{\wedge 2}$ as a sum of positive and negative eigenspaces: $(\mathbb{C}^4)^{\wedge 2} \cong V_+ \oplus V_-$ where the $V_\pm$ are *real* 6-dimensional subspaces, interchanged by multiplication by i. Now, consider the intersection of the image of $P_{\mathbb{C}}$ with the real submanifold $\mathbb{P}(V_+) \subseteq \mathbb{P}(\mathbb{C}^6)$. That means we must find a form of the non-degenerate quadric using self-dual coordinates. Suitable such coordinates are given by $w_{ij} = v_{ij} + iv_{kl}$ for i, j, k, l all distinct, and in these coordinates the quadric becomes the projectivisation of

$$-w_{13}^2 - w_{24}^2 + w_{12}^2 + w_{34}^2 + w_{14}^2 + w_{23}^2 = 0,$$

   which defines the projectivised null cone as before, and therefore realises the preimage inside $\mathrm{Gr}_2(\mathbb{C}^4)$ as a compactification of Minkowski space. The full space $\mathrm{Gr}_2(\mathbb{C}^4)$ is sometimes given the cumbersome name *complexified compactified Minkowski space.*

   We naturally would like a description of this preimage. To furnish one, we consider the locus $\mathrm{Gr}_2^+(\mathbb{C}^4)$ of *isotropic* subspaces in $\mathbb{C}^4$ with respect to the pseudo-Hermitian metric introduced above. That is, subspaces contained in the zero locus of the metric (so another null cone condition). One calculates that such planes map to lines in V_+.

[2] The reason this is a good idea is that there's an exceptional isomorphism $\mathrm{Spin}(2, 4) \cong SU(2, 2)$, so if we're looking for a space with an action of $SO(2, 4)$ we might look instead at things a double-cover of which are acted on by $SU(2, 2)$.

4. Finally, we can describe compactified Minkowski space using the group $U(2)$ [Jad11b]. We can naturally view Minkowski space as the space of 2x2 Hermitian matrices, via the bijection

$$(t, x_1, x_2, x_3) \mapsto \begin{pmatrix} t - x_3 & x_1 + ix_2 \\ x_1 - ix_2 & t + x_3 \end{pmatrix}$$

where the quadratic form of a Hermitian matrix is given by the determinant. There's a natural map from Hermitian matrices to unitary matrices given by the *Cayley transform*. That is, the map

$$A \mapsto \frac{A + iI}{A - iI}.$$

This map is injective, and its image is the set of unitary matrices U such that $U - I$ is invertible. Indeed, on this locus the Cayley transform is invertible with inverse

$$U \mapsto i\frac{I + U}{I - U}.$$

We'd like to identify $U(2)$ with the compactified Minkowski space as described above. A unitary matrix $U \in U(2)$ induces an isotropic plane in $\mathrm{Gr}_2(\mathbb{R}^4)$, hence a point in the compactified Minkowski space. We define this plane by taking

$$\left\{ \begin{pmatrix} Uv \\ v \end{pmatrix} : v \in \mathbb{C}^2 \right\} \subseteq \mathbb{C}^4.$$

This is clearly isotropic since U is unitary, so describes an injective map $U(2) \to \mathrm{Gr}_2^+(\mathbb{C}^4)$, which must therefore be an isomorphism.

2 Twistor Space

Twistor space is a complex manifold whose geometry is closely related to that of (compactified) Minkowski space. At its root, twistor space $\mathbb{PT}$ is just the complex manifold $\mathbb{CP}^3$, but we can describe it in a way that explains why it might be related to the geometry of $\mathbb{R}^{1,3}$. Consider the space $\mathbb{T} = S \oplus S$ of pairs of Weyl spinors in signature $(1, 3)$. Concretely, there is an exceptional isomorphism $\mathrm{Spin}(1, 3) \cong SL(2; \mathbb{C})$, and the copies of S are two copies of the fundamental representation of this group. Thus $\mathbb{T}$ is a 4-complex-dimensional vector space. The *twistor space* $\mathbb{PT}$ is then the space of complex lines in $\mathbb{T}$.

Recall that the space S of spinors admits a Hermitian inner product. The space $\mathbb{T} = S \oplus S$ therefore admits a pseudo-Hermitian structure by

$$((\alpha_1, \beta_1), (\alpha_2, \beta_2)) \mapsto \langle \alpha_1, \beta_2 \rangle + \langle \beta_1, \alpha_2 \rangle$$

which we observe has signature $(2, 2)$. This is called the *twistor norm* [3]. The space of twistors with vanishing twistor norm is denoted $\mathbb{N} \subseteq \mathbb{T}$ and forms a seven-real-dimensional submanifold. Looking at complex lines contained in $\mathbb{N}$ defines $\mathbb{PN} \subseteq \mathbb{PT}$, a five-real-dimensional compact submanifold.

We can relate points in $\mathbb{PN}$ with points in compactified Minkowski space (or points in $\mathbb{PT}$ with points in complexified compactified Minkowski space) by means of the *Penrose correspondence*, a correspondence referred to by Penrose as an "incidence relation". The complexified story is slightly easier: there's a natural correspondence

$$\begin{tikzcd} & \mathrm{Fl}_{1,2}(\mathbb{C}^4) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{PT} \cong \mathbb{P}(\mathbb{C}^4) & & \mathrm{Gr}_2(\mathbb{C}^4) \cong M^c_{\mathbb{C}} \end{tikzcd}$$

where $\mathrm{Fl}_{1,2}(\mathbb{C}^4)$ is the space of $(1, 2)$-flags in $\mathbb{C}^4$, and the maps π_i are the two forgetful maps. We say that a twistor in $\mathbb{PT}$ and a point in $M^c_{\mathbb{C}}$ are *incident* if the line described by the twistor is contained in the plane described by

[3] So one of our descriptions of compactified Minkowski space was of planes in $\mathbb{T}$ isotropic with respect to the twistor norm.

the spacetime point. Choosing a point $x \in M^c_{\mathbb{C}}$, the set of points in $\mathbb{PT}$ incident to x consists of all those lines contained in the plane corresponding to x, thus a $\mathbb{CP}^1 \subseteq \mathbb{PT}$. These lines are called *twistor lines*. There's a similar correspondence for genuine spacetime points:

$$\begin{tikzcd} & \mathrm{Fl}^+_{1,2}(\mathbb{C}^4) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{PN} \cong \mathbb{P}^+(\mathbb{C}^4) & & \mathrm{Gr}_2^+(\mathbb{C}^4) \cong M^c \end{tikzcd}$$

where now we've restricted to isotropic subspaces with respect to the twistor norm throughout. Given a point $x \in M^c$, the set of points in $\mathbb{PN}$ incident to x are those isotropic lines contained in the isotropic plane corresponding to x. By this procedure, each point x defines a $\mathbb{CP}^1 \subseteq \mathbb{PN}$.

Now, following [WW91] we can identify $\mathbb{T}$ with $\mathbb{H}^2$. Under this identification we obtain projective lines in $\mathbb{PT}$, or loci of lines in $\mathbb{T}$, by taking all lines in $\mathbb{H}^2$ reachable by simultaneous (left) quaternion multiplication when starting from a particular vector. Those lines in $\mathbb{PN}$ occur when starting from a null vector. This defines a map $\pi\colon \mathbb{PT} \to \mathbb{HP}^1 \cong S^4$ [4] whose fibres are twistor lines. In terms of the description of twistor lines we gave above, we've obtained those corresponding to points in $\mathrm{Gr}_2(\mathbb{T})$ which are invariant under the involution "multiply by j" after identifying $\mathbb{T}$ with $\mathbb{H}^2$. This cuts out a copy of S^4 in $\mathrm{Gr}_2(\mathbb{C}^4) \cong M^c_{\mathbb{C}}$. Intersecting with the real part M^c yields an S^3 compactifying a hyperplane in Minkowski space.

Remark 2.1. It's worth describing subspaces of twistor space corresponding to loci in S^4. Firstly, the preimage of $\mathbb{R}^4 \subseteq S^4$ is just the complement $\mathbb{CP}^3 \setminus \mathbb{CP}^1$ of a single twistor line. This can be thought of as the locus of twistor lines corresponding to Minkowski space inside its compactification. In twistor coordinates $(Z_0\colon Z_1\colon Z_2\colon Z_3)$ for $\mathbb{PT}$, the image in $\mathbb{HP}^1$ is the point $(Z_0 + \boldsymbol{j} Z_1\colon Z_2 + \boldsymbol{j} Z_3)$, so the pre-image under π of the point $(0:1)$ at infinity is the locus of points $(0:0:Z_2:Z_3)$.

When studying 4d field theories with boundary conditions, we are also interested in the half space $\mathbb{R}^3 \times \mathbb{R}_{\geq 0} \subseteq S^4$. The preimage of this half-space in twistor space is the locus where Z_0 and Z_1 are not both 0 and the quaternion $(Z_0 + \boldsymbol{j} Z_1)^{-1}(Z_2 + \boldsymbol{j} Z_3)$ has non-negative real part. This is precisely the subspace of $\mathbb{PT}$ of points with *non-negative twistor norm*, excluding the $\mathbb{CP}^1$ at infinity. The boundary (real) hypersurface is the space $\mathbb{PN}$ of null twistors, and the open interior is usually denoted $\mathbb{PT}^+$.

3 Super-Twistor Spaces

We can enrich the twistor space $\mathbb{PT}$ to a superspace admitting the action of a complexified supersymmetry algebra by infinitesimal symmetries.

Definition 3.1. The $N = k$ *super-twistor space* $\mathbb{PT}^{N=k}$ is the total space of the odd vector bundle $\Pi(\mathcal{O}(1) \otimes \mathbb{C}^k) \to \mathbb{PT}$. As a supermanifold this is just $\mathbb{CP}^{3|k}$.

Costello discusses super-twistor spaces from this perspective in [Cos11], although it's worth noting that in his notation, $\mathbb{PT}$ refers to the complement of the preimage of the point at infinity in S^4 under the twistor line map π. Describing an action of a supersymmetry algebra by infinitesimal symmetries means producing a subsheaf of the sheaf of vector fields on $\mathbb{PT}^{N=k}$ locally isomorphic to the $N = k$ supersymmetry algebra as a super Lie algebra. Recall the $N = k$ complexified supersymmetry algebra can be described as

$$\big(\mathfrak{sl}(2;\mathbb{C})_+ \oplus \mathfrak{sl}(2;\mathbb{C})_- \oplus \mathbb{C}^4 \oplus \mathfrak{g}_R\big) \oplus \Pi(S_+ \otimes W \oplus S_- \otimes W^*)$$

where $S_\pm$ is the fundamental representation of $\mathfrak{sl}(2;\mathbb{C})$, the $\mathfrak{sl}(2;\mathbb{C})_+ \oplus \mathfrak{sl}(2;\mathbb{C})$ act on the translations $\mathbb{C}^4$ by block diagonal matrices, $\mathfrak{g}_R$ is the R-symmetry algebra, and W is a k-complex-dimensional representation of $\mathfrak{g}_R$.

[4] It's a familiar quaternionic Hopf map.

The super group $PGL(4|k;\mathbb{C})$ acts on super-twistor space $\mathbb{PT}^{N=k}$, inherited from the matrix multiplication action on $\mathbb{C}^{4|k}$. The corresponding infinitesimal symmetries are given by its Lie algebra $\mathfrak{sl}(4|4;\mathbb{C})$. Concretely, this Lie algebra has form $\mathfrak{sl}(4;\mathbb{C}) \oplus \Pi(V)$, where V is a $4k$-complex-dimensional representation of the bosonic piece comprising k copies of the fundamental representation. By embedding $\mathfrak{sl}(2;\mathbb{C})_+ \oplus \mathfrak{sl}(2;\mathbb{C})_-$ in $\mathfrak{sl}(4;\mathbb{C})$ block diagonally we produce an action of $(\mathfrak{sl}(2;\mathbb{C})_+ \oplus \mathfrak{sl}(2;\mathbb{C})_-) \oplus \Pi(S_+ \otimes W \oplus S_- \otimes W^*)$ on supertwistor space. It remains to produce an action of the R-symmetries and an action of translation compatible with the brackets.

For the translations we have a natural guess: the translation algebra $\mathbb{R}^4$ acts infinitesimally on S^4 by rotations. Concretely we embed $\mathbb{R}^4 \hookrightarrow \mathfrak{so}(5)$ diagonally: $(x_1, x_2, x_3, x_4) \mapsto \mathrm{diag}(x_1, x_2, x_3, x_4, -x_1 - x_2 - x_3 - x_4)$. These Lie commuting local vector fields then pull back under the projection map π to vector fields on $\mathbb{PT}^{N=k}$, where the pullback is defined using the twistor norm to lower and raise indices. To verify this guess we need to check that it interacts correctly with the Lorentz transformations, and modify the fermionic symmetries so that they bracket to translations. (Note: This is still in progress. In particular I'm not sure whether we really expect a $\mathfrak{g}_R$-action, either in general or in the example of $N = 4$.)

References

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