Notes on the $N = 4$ Supersymmetry Algebra

Chris Elliott

These are some notes on supersymmetry algebras written when trying to understand some aspects of the papers [GW09] and [CY13].

1 The Supersymmetry Algebra in Dimension 10

We begin with a discussion of the ($N = 1$) supersymmetry algebra in dimension 10. Throughout we'll work in Lorentzian signature. The usual – bosonic – symmetry algebra of $\mathbb{R}^{1,9}$ is a semidirect product $\mathfrak{so}(1,9) \ltimes \mathbb{R}^{1,9}$, where the rotations act on the translations by the fundamental representation. Extending this to a super-algebra means choosing a spinorial representation of $\mathfrak{so}(1,9)$, so let's classify the possible such representations.

Abstractly, Clifford theory tells us to expect a pair of mutually dual irreducible spinorial representations of $\mathfrak{so}(1,9)$ over the real numbers, each of dimension 16. In physicists' terminology, the Majorana spin representation is 32-dimensional, and splits into two Majorana-Weyl spin representations. We can actually describe this representation very concretely; the details are described by Deligne in [Del99] chapter 6.

It suffices to construct a non-trivial 32-dimensional module for the algebra $Cl(V,Q)$, where V is 10-dimensional, and Q is a quadratic form of signature (1,9). Concretely, we'll set $V = \mathbb{O} \oplus H$ with $\mathbb{O}$ 8-dimensional and $H = \langle e, f \rangle$ 2-dimensional, and we set

$$Q(\omega + ae + bf) = \omega \cdot \overline{\omega} - ab$$

where $\omega\overline{\omega}$ is the octonion norm-squared. Let $S = (\mathbb{O}^2) \oplus (\mathbb{O}^2)$ be a 32-dimensional real vector space. We must describe a Clifford multiplication $\rho\colon V \otimes S \to S$ making S into a module for $Cl(V,Q)$. This is concretely given by

$$\rho\colon \mathbb{O} \oplus H \to \mathrm{End}(S)$$

where

$$\rho(\omega) = \begin{pmatrix} 0 & \begin{pmatrix} m_\omega & 0 \\ 0 & m_\omega \end{pmatrix} \\ \begin{pmatrix} m_\omega & 0 \\ 0 & m_\omega \end{pmatrix} & 0 \end{pmatrix} \text{ for } \omega \in \mathcal{O},\ m_\omega(\alpha) = \overline{\omega} \cdot \overline{\alpha}$$

$$\rho(e) = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \quad \text{and} \quad \rho(f) = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \end{pmatrix}.$$

One can check that this gives a well-defined Clifford multiplication, and thus defines a 32-dimensional real spin representation which splits as a sum of two 16-dimensional representations of the even part of the Clifford algebra: call them S_+, spanned by the first and third components of $\mathbb{O}^4$, and S_- spanned by the second and forth. There is also the induced pairing $\Gamma\colon S_\pm \otimes S_\pm \to V$, which one checks is given by (on S_+ say)

$$\Gamma((\alpha_1, \alpha_2), (\beta_1, \beta_2)) = \overline{\alpha}_1 \cdot \overline{\beta}_2 + \overline{\alpha}_2 \cdot \overline{\beta}_1 + \mathrm{Tr}(\alpha_2 \cdot \overline{\beta}_1)e + \mathrm{Tr}(\alpha_1 \cdot \overline{\beta}_2)f$$

where $\mathrm{Tr}(\alpha) = \alpha + \overline{\alpha}$ is the octonionic reduced trace, and where the calculation is done using the identity $\langle \Gamma(s,t), v\rangle = (\rho(v)s, t)$ for spinors s, t and vectors v. This now gives us a complete description of the supersymmetry algebra in 10-dimensions: it is given by

$$(\mathfrak{so}(1,9) \oplus \mathbb{R}^{1,9}) \oplus \Pi(S_+)$$

with brackets given by the internal bracket on $\mathfrak{so}(1,9)$, the action of $\mathfrak{so}(1,9)$ on the translations, the action of $\mathfrak{so}(1,9)$ on the supersymmetries, and the pairing $\Gamma\colon S_+ \otimes S_+ \to \mathbb{R}^{1,9}$.

Remark 1.1. We could also extend this algebra to include R-symmetries, which would be important if we were trying to twist the theories. The R-symmetries here are given by automorphisms of S_+ which preserve the pairing Γ. The only such automorphisms are given by $U(1) \subseteq \mathbb{C} \subseteq \mathbb{O}$ acting on α_1 and β_2 by multiplication and on α_2 and β_1 by conjugate-multiplication. The R-symmetry algebra is then given by $\mathfrak{u}(1)$ with brackets given by the derivative of this action. (Note: certainly these are R-symmetries, but I can't quite see how to prove that they're the only ones.)

Remark 1.2. Finally, we can complexify the supersymmetry algebra to obtain a superalgebra of form

$$(\mathfrak{so}(10;\mathbb{C}) \ltimes \mathbb{C}^{10}) \oplus \Pi(S_+ \otimes \mathbb{C}).$$

The complexification $S_+ \otimes \mathbb{C}$ is a 16-complex dimensional Weyl spinor representation of $\mathfrak{so}(10;\mathbb{C})$. Clifford theory says that $\mathfrak{so}(10;\mathbb{C})$ embeds in the (even part of the) Clifford algebra $Cl_{10}^+ \cong \mathrm{Mat}_{16}(\mathbb{C}) \oplus \mathrm{Mat}_{16}(\mathbb{C})$ as the elements of spinor norm one. The Weyl spinors are the fundamental representation of the first matrix algebra factor.

More concretely, we write $S_+ \otimes \mathbb{C}$ as $\mathbb{O}^2 \oplus i\mathbb{O}^2$ where $\mathbb{O}$ is a 4-complex dimensional vector space. We write $\mathbb{C}^{10}$ as $\mathbb{O} \oplus i\mathbb{O} \oplus \mathbb{C}\langle e, f\rangle$. The Clifford multiplication is then given by

$$\rho(\omega) = \begin{pmatrix} \begin{pmatrix} 0 & m_\omega \\ m_\omega & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & m_\omega \\ m_\omega & 0 \end{pmatrix} \end{pmatrix}, \quad \rho(i\omega) = \begin{pmatrix} 0 & \begin{pmatrix} 0 & m_\omega \\ m_\omega & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & m_\omega \\ m_\omega & 0 \end{pmatrix} & 0 \end{pmatrix} \quad \text{for } \omega \in \mathbb{O}$$

$$\rho(e) = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \quad \text{and} \quad \rho(f) = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \end{pmatrix}.$$

2 The $N = 4$ Supersymmetry Algebra in Dimension 4

To understand the dimensional reduction to four dimensions, choose an embedding $\mathbb{R}^{1,3} \hookrightarrow \mathbb{R}^{1,9}$ and consider the subalgebra of the symmetry algebra fixing this subspace. This has the form $\mathfrak{so}(1,3) \ltimes \mathbb{R}^{1,3} \oplus \mathfrak{so}(6)$, where the $\mathfrak{so}(6)$ fixes the subspace pointwise. We must understand the structure of the space S_+ of spinors as a representation of the subalgebra $\mathfrak{so}(1,3) \oplus \mathfrak{so}(6) \leq \mathfrak{so}(1,9)$.

It's easy to understand the action of the two factors separately. There are exceptional isomorphisms $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2;\mathbb{C})$ and $\mathfrak{so}(6) \cong \mathfrak{su}(4)$. The restricted representations for each factor are still spinorial, so we can argue using the classification of spin representations. Firstly, $\mathfrak{so}(6) \cong \mathfrak{su}(4)$ has an 8-dimensional space of Majorana spinors, corresponding to the 4-complex-dimensional fundamental representation of $\mathfrak{su}(4)$. The module S_+ has no invariant elements under the $\mathfrak{so}(6)$ action, so splits as two copies of this 8-dimensional representation. Similarly, $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2;\mathbb{C})$ has a 4-dimensional space of Majorana spinors, corresponding to the 2-complex dimensional fundamental representation of $\mathfrak{sl}(2;\mathbb{C})$. Again, S_+ has no invariant elements for the restricted action, so splits as four copies of this representation.

To understand the interaction of these representations, we have to specify a Lie algebra homomorphism

$$\mathfrak{su}(4) \to \mathrm{End}_{\mathfrak{sl}(2;\mathbb{C})}(S_+) \cong \mathbb{C} \otimes \mathfrak{gl}(4;\mathbb{R}) \cong \mathfrak{gl}(4;\mathbb{C})$$

where the $\mathfrak{sl}(2;\mathbb{C})$-endomorphisms must act as multiplication by a complex number within each of the four irreducible summands, but may freely move them around. The splitting here corresponds to the standard embedding of $\mathfrak{su}(4)$ into $\mathfrak{gl}(4;\mathbb{C})$. This doesn't split as a sum of tensor products of representations for dimension reasons.

The dimensionally reduction of the 10-dimensional supersymmetry algebra that we have described is the $N = 4$ *supersymmetry algebra* in four dimensions. It has form

$$(\mathfrak{so}(1,3) \oplus \mathfrak{so}(6) \oplus \mathbb{R}^{1,3}) \oplus \Pi(S_{16})$$

where S_{16} is a 16-real-dimensional space of spinors. The brackets are given by the internal brackets on $\mathfrak{so}(1,3)$ and $\mathfrak{so}(6)$, the action of $\mathfrak{so}(1,3)$ on $\mathbb{R}^{1,3}$, the actions of the $\mathfrak{so}(1,3)$ and $\mathfrak{so}(6)$ on the spinors described above, and four sets of brackets from the pairing

$$\Gamma\colon S_4 \otimes S_4 \to \mathbb{R}^{1,3}$$

where S_4 is the four-dimensional Majorana spinor representation of $\mathfrak{so}(1,3)$ (so $S_{16} = S_4^{\oplus 4}$ as an $\mathfrak{so}(1,3)$-module), and where $\Gamma = (i\gamma^0, i\gamma^1, i\gamma^2, i\gamma^3)$ is the map given by the four-dimensional gamma matrices (in Majorana form). We use the fact that S_4 is isomorphic to its dual as an $\mathfrak{so}(1,3)$-module. (Note: I think this is right: the gamma matrices normally define the Clifford multiplication, but I think they also give the pairing, via the defining property of the pairing: $\langle \Gamma(s,t), v\rangle = (\rho(v)s, t)$, where $\langle , \rangle$ is the inner product on spacetime, and $(,)$ is the inner product on spinors from the identification with the dual.)

(Note: It couldn't hurt to remind you what the Majorana basis for the gamma matrices is. They have the form

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix}$$

and give a natural description of the real Clifford multiplication in signature 1,3. Note the additional factors of i: this is the physicists' standard convention.)

Remark 2.1. The bosonic subalgebra $\mathfrak{so}(6)$ consists of R-symmetries, i.e. outer automorphisms fixing the bosonic piece of the supersymmetry algebra. (Note: I think they should be all the R-symmetries. Can you see whether this is true, and if so why?)

Remark 2.2. After complexification the supersymmetry algebra is easier to describe. The complexified spinors form a 16-complex-dimensional representation of $\mathfrak{sl}(2;\mathbb{C}) \oplus \mathfrak{sl}(2;\mathbb{C}) \oplus \mathfrak{sl}(4;\mathbb{C})$ which *does* split nicely. It can be written in the form $V_+ \otimes W \oplus V_- \otimes W^*$, where $V_\pm$ are the 2-complex-dimensional fundamental representations of the two copies of $\mathfrak{sl}(2;\mathbb{C})$, and where W is the fundamental representation of the $\mathfrak{sl}(4;\mathbb{C})$. The supersymmetry algebra is then

$$(\mathfrak{sl}(2;\mathbb{C}) \oplus \mathfrak{sl}(2;\mathbb{C}) \oplus \mathfrak{sl}(4;\mathbb{C}) \oplus \mathbb{C}^4) \oplus \Pi(V_+ \otimes W \oplus V_- \otimes W^*).$$

3 The ½-BPS Subalgebra

We pick out a certain subalgebra of the $N = 4$ supersymmetry algebra: a maximal subalgebra fixing a spacelike hyperplane in $\mathbb{R}^{1,3}$. In the bosonic piece, the subalgebra of the Poincaré algebra fixing a hyperplane is just $\mathfrak{so}(1,2) \ltimes \mathbb{R}^{1,2} \oplus \mathfrak{so}(6)$. However, we can't keep the whole $\mathfrak{so}(6)$ of R-symmetries if we want any supersymmetries to remain: the supersymmetries are irreducible as an $\mathfrak{so}(1,2) \oplus \mathfrak{so}(6)$-module, and by bracketing supersymmetries we can obtain any translation in $\mathbb{R}^{1,3}$, including those not lying on our hyperplane.

We'll take instead the diagonal subalgebra $\mathfrak{so}(3) \oplus \mathfrak{so}(3) \leq \mathfrak{so}(6)$, or equivalently the diagonal $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \leq \mathfrak{su}(4)$. For this subalgebra the supersymmetries will split into four irreducible summands. Indeed, we investigate the action of this subalgebra on the space of supersymmetries. We first look at the actions of the residual Lorentz transformations and R-symmetries separately, then observe that the action splits naturally. (Note: I'm not really sure in what sense this is "maximal"...)

The R-symmetries $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ act by the restriction of two copies of the fundamental representation of $\mathfrak{su}(4)$, so as an $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$-module the supersymmetries comprise two copies of the fundamental representation of each $\mathfrak{su}(2)$. The Lorentz transformations $\mathfrak{sl}(2;\mathbb{R})$ act by the restriction of four copies of the fundamental representation of $\mathfrak{sl}(2;\mathbb{C})$, so as eight copies of the fundamental representation of $\mathfrak{sl}(2;\mathbb{R})$. The interaction is described by the homomorphism

$$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \to \mathrm{End}_{\mathfrak{sl}(2;\mathbb{R})}(S) \cong \mathfrak{gl}(8;\mathbb{R})$$

sending a pair of matrices (U_1, U_2) to the block diagonal matrix $\mathrm{diag}(\mathrm{Re}(U_1), \mathrm{Im}(U_1), \mathrm{Re}(U_2), \mathrm{Im}(U_2))$. (Note: Again, this is worth checking more carefully, but it's just precomposing the interaction I describe above with $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \to \mathfrak{su}(4)$, and post-composing with the forgetful map $\mathrm{End}_{\mathfrak{sl}(2;\mathbb{C})}(S) \to \mathrm{End}_{\mathfrak{sl}(2;\mathbb{R})}(S)$.) This means the representation splits as

$$S \cong (V_{R+} \oplus V_{R_-}) \otimes V_\Lambda$$

where $V_{R\pm}$ are the four-real-dimensional representations of the two copies of the fundamental representations of $\mathfrak{su}(2)$, and V_Λ is the two-real-dimensional fundamental representation of the restricted Lorentz symmetries $\mathfrak{sl}(2;\mathbb{R})$. (Note: In the physics notation used by for instance [GW09], this is $(\mathbf{4} \oplus \overline{\mathbf{4}}) \otimes \mathbf{2}$.)

Now, this has two reducible components, so for the ½ BPS subalgebra we choose only one, say $V_{R+} \otimes V_\Lambda$. To make sure that such spinors can't bracket to form translations perpendicular to our chosen hyperplane we investigate the pairing to $\mathbb{R}^{1,3}$. This acts diagonally on four copies of the fundamental representation of $\mathfrak{sl}(2;\mathbb{C})$, each of which occurs in the form $\langle s_+, s_- \rangle \otimes V_\Lambda$ with $s_\pm \in V_{R\pm}$. If we restrict to $V_{R+} \otimes V_\Lambda$ then the restricted image of the pairing is a two-dimensional spacelike surface inside the given $\mathbb{R}^{1,2}$. (Note: Here's my reasoning. The given restriction corresponds to choosing a two-dimensional subspace of S_4. In the Majorana basis this means the image of the pairing is the span of the four 4-vectors γ^μ_{ij} where $i, j = 1, 2$, which is computed to be two-dimensional. Still, this could be checked more carefully.)

So, to summarise, we've described a maximal subalgebra (the ½ BPS subalgebra) of the $N = 4$ supersymmetry algebra fixing a spacelike hypersurface. It has form

$$(\mathfrak{sl}(2;\mathbb{R}) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathbb{R}^{1,2}) \oplus \Pi(V_{R+} \otimes V_\Lambda)$$

with brackets coming from the internal brackets on the matrix algebra, the adjoint action of $\mathfrak{sl}(2;\mathbb{R})$ on $\mathbb{R}^{1,2}$, the fundamental action of $\mathfrak{sl}(2;\mathbb{R})$ on V_Λ, the fundamental action of the first $\mathfrak{su}(2)$ on V_{R+}, and the restricted gamma matrix pairing of the supersymmetries to $\mathbb{R}^{1,2}$.

Remark 3.1. Again, we can complexify the ½ BPS subalgebra. We find

$$(\mathfrak{sl}(2;\mathbb{C})_\Lambda \oplus \mathfrak{sl}(2;\mathbb{C})_{R+} \oplus \mathfrak{sl}(2;\mathbb{C})_{R-} \oplus \mathbb{C}^3) \oplus \Pi(V_\Lambda \otimes V_{R+} \oplus V_\Lambda^* \otimes V_{R+}^*)$$

where now the V refer to the two-complex-dimensional fundamental representations of the various copies of $\mathfrak{sl}(2;\mathbb{C})$. (Note: no $R-$ action. Does this make sense?)

4 Some Holomorphic Twists

We begin with the complexified supersymmetry algebra in 10-dimensions, which – including the R-symmetries – we might write as

$$(\mathfrak{so}(10;\mathbb{C}) \ltimes \mathbb{C}^{10} \oplus \mathbb{C}^\times) \oplus \Pi(S)$$

where S is a Weyl spinor representation of $\mathfrak{so}(10;\mathbb{C})$: what we called $S_+ \otimes \mathbb{C}$ in section 1. The R-symmetries act as scalar multiplication on V_F and conjugate scalar multiplication on V_F^*. To twist we must choose a supersymmetry Q so that $[Q, Q] = 0$. The twist is *holomorphic* if, upon identifying $\mathbb{C}^{10}$ with $\mathbb{C}^5 \otimes_\mathbb{R} \mathbb{C}$, the image of $[Q, -]$ is contained in the subspace $\mathbb{R}^5 \otimes_\mathbb{R} \mathbb{C}$.

For any Q, the image of $[Q, -]$ consists of translations v in $\mathbb{C}^{10}$ such that

$$\langle \Gamma(Q, Q'), v \rangle = (\rho(v)Q, Q') \neq 0$$

for some Q'. Here $\rho(v)$ is the image of v in the even part of the 10-dimensional complex Clifford algebra, $Cl_{10}^{+}(\mathbb{C}) \cong \mathrm{Mat}_{16}(\mathbb{C}) \oplus \mathrm{Mat}_{16}(\mathbb{C})$, where the first factor acts on the Weyl spinor Q. We've interested therefore in the situation where $\rho(v)Q = 0$ for a half-dimensional space of $\mathbb{C}^{10}$, which we view as the anti-holomorphic translations. Spinors Q satisfying such a constraint are usually called *pure spinors* [Cha97].

To a spinor we can associate an *isotropic subspaces* of $\mathbb{C}^{10}$, i.e. vector subspaces where the complex-valued metric obtained by extended from the standard metric on $\mathbb{R}^{10}$ by complex linearity vanishes identically. The association is the map sending a spinor Q to its null-space: $N(Q) = \{v \in \mathbb{C}^{10} \colon \rho(v)Q = 0\}$, which is isotropic because

$$\langle v, w\rangle Q = \frac{1}{2}(\rho(v)\rho(w)Q + \rho(w)\rho(v)Q) = 0$$

for $v, w \in N(Q)$. The pure spinors are therefore those spinors whose associated isotropic subspace is *maximal*, i.e. half-dimensional, and in fact this map gives a bijection between pure spinors modulo rescaling and maximal isotropic subspaces. In keeping with our discussion above, we can think of this null space as identifying a choice of *real structure*, i.e. the image of a map $\mathbb{R}^5 \otimes_{\mathbb{R}} \mathbb{C} \hookrightarrow \mathbb{C}^5 \otimes_{\mathbb{R}} \mathbb{C}$. In 10-dimensions, a spinor Q is pure if and only if $[Q, Q] = \Gamma(Q, Q) = 0$, so the space of pure spinors is given by a quadratic subvariety in $\mathbb{C}^{10}$ cut out by a system of 10 quadratic equations.

Each pure spinor therefore defines a holomorphic twist of the 10-dimensional supersymmetry algebra: let's try to describe the twisted supersymmetry algebra, i.e. the cohomology with respect to the differential $[Q, -]$. The bosonic piece of the cohomology is easy: it is the direct sum of the kernel of the differential $\mathfrak{so}(10; \mathbb{C}) \to S$ and the cokernel of the differential $S \to \mathbb{C}^{10}$. For Q a pure spinor the latter is isomorphic to $\mathbb{C}^5$ by definition, and the former is the stabiliser of Q under the $\mathfrak{so}(10; \mathbb{C})$ action. This action corresponds to the rotation action on isotropic subspaces of $\mathbb{C}^{10}$, so the stabiliser of a point is an $\mathfrak{so}(5; \mathbb{C}) \oplus \mathfrak{so}(5; \mathbb{C}) \cong \mathfrak{sp}(4; \mathbb{C}) \oplus \mathfrak{sp}(4; \mathbb{C})$. (Note: This is different from (smaller than) the complexification of the algebra preserving a given complex structure, which looks like $\mathfrak{u}(5) \otimes \mathbb{C} \cong \mathfrak{gl}(5; \mathbb{C})$. I'm a little concerned as to which one is right. I'm also concerned by the fact that isotropic subspaces only correspond to *projective* pure spinors: might this make a difference?)

Now, for the fermionic piece we must compute the quotient of the set of spinors commuting with Q by the orbit of Q under $\mathfrak{so}(10; \mathbb{C}) \oplus \mathfrak{u}(1)$. The orbit of Q is precisely the set of pure spinors: quotienting by $\mathfrak{u}(1)$ correponds to quotienting by rescaling, and the action of $\mathfrak{so}(10; \mathbb{C})$ on rescaling equivalence classes is precisely the action on isotropic subspaces, on which the orbit is all such subspaces of the same dimension.

Proposition 4.1. The piece of the Q-cohomology in fermionic degree vanishes.

Proof. We must show that the only spinors in the kernel of the differential $[Q, -]$ are pure. Firstly

$$\begin{aligned}\ker(Q) &= \{Q' \colon \langle \Gamma(Q, Q'), v\rangle = 0 \ \forall v\} \\ &= \{Q' \colon (\rho(v)Q, Q') = 0 \ \forall v\} \\ &= \{Q' \colon N(\rho(v)Q) \cap N(Q') \neq \{0\} \ \forall v\}.\end{aligned}$$

where the final equality is [Cha97] Lemma 2.3.35 part 3. Then, note that impure spinors have $\dim N(Q') = 1$, by [Cha97] Theorem 3.4.1, so it suffices to show that $N(\rho(v)Q)$ can be made to miss any vector by choosing a suitable v. Applying the lemma again, this means finding v, w so that $(\rho(v)Q, \rho(w)Q) \neq 0$, or finding w so that $\Gamma(Q, \rho(w)Q) \neq 0$ which is possible for any non-zero Q. (Note: Is this right? It's a little sketchy.) □

References

[Cha97] Philip Charlton. *The geometry of pure spinors, with applications*. PhD thesis, University of Newcastle, 1997.

[CY13] Chi-Ming Chang and Xi Yin. 1/16 BPS states in $\mathcal{N} = 4$ SYM. *arXiv preprint arXiv:1305.6314*, 2013.

[Del99] Pierre Deligne. *Quantum Fields and Strings; A Course for Mathematicians: Notes on Spinors*, volume 1. AMS, 1999.

[GW09] Davide Gaiotto and Edward Witten. Supersymmetric boundary conditions in $\mathcal{N} = 4$ super Yang-Mills theory. *Journal of Statistical Physics*, 135(5-6):789–855, 2009.