Notes on Character Sheaves

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1 Introduction

The theory of character sheaves was introduced by Lusztig in the series of papers beginning with [13] in part to solve the following problem:

Problem. Let G be a reductive algebraic group defined over $\mathbb{F}_q$, so $G(\mathbb{F}_q)$ is a finite group. Compute the character tables of such groups in a unified way.

The theory of characters of finite groups gives us effective methods of computing the character tables of any individual finite group, but not methods that are applicable for such a wide class of groups simultaneously. For a unified calculation, we'd really like to be able to compute a version of character theory for $G(\overline{\mathbb{F}_p})$, and obtain the characters of each $G(\mathbb{F}_{p^n})$ by a process of restriction. Lusztig's character theory fulfills such a goal. He defined the notion of a *character sheaf* for an algebraic group – a categorification of a character, thought of as a class function – and by a decategorification procedure recovered an orthonormal basis for the class functions of the finite groups of Lie type.

More precisely, the irreducible character sheaves are certain adjoint equivariant ℓ-adic sheaves on an algebraic field G defined over an algebraically closed field k of characteristic p. They correspond to the irreducible representations of the group. If $k = \overline{\mathbb{F}_q}$, we can take a character sheaf which is isomorphic to its preimage under the Frobenius map for $\mathbb{F}_q$, and recover a class function for $G(\mathbb{F}_q)$ by taking an alternating sum of traces of Frobenius on the cohomology groups of the sheaf. Under a mild assumption on the characteristic (we forbid characteristics 2, 3, and 5 for groups containing certain exceptional Lie groups as closed normal subgroups), these functions give an orthonormal basis for the class functions of the group, which is (at least conjecturally) the same as the basis of "almost characters". From this basis, one can recover the character table of the group (see [12] 13.7.)

In more recent work of Ben-Zvi and Nadler [3] and Bezrukavnikov, Finkelberg and Ostrik [5] the theory of character sheaves is related to a certain (unoriented) extended 2d topological quantum field theory, assigning to a point the Hecke category of a reductive algebraic group and assigning the character sheaves to the circle. This provides and organises a large amount of structure to the character theory of the group, for instance it provides operations labelled by 2-manifolds with boundary to the character sheaves.

In these notes, we will describe the construction of the character sheaves of a group in a way as natural as possible. We will stick closely to the motivating analogy with the character theory of a finite group. We will also describe the related story of character sheaves on a Lie algebra, with some hints as to how the two stories are related.

1.1 Motivation – Character Theory of a Finite Group

In order to motivate some of the constructions we'll be describing, I'll give a construction of the irreducible characters of a finite group G analogous to the constructions of character sheaves of an algebraic group. In the most simple-minded terms, we produce a character χ of G by taking a finite dimensional representation $\rho\colon G \to GL(V)$ of G, and taking the trace

$$\chi_V(g) = \operatorname{Tr} \rho(g).$$

Viewing V as a (left) kG-module, we interpret the character as the image of the identity under a *universal trace* map:

Definition 1.1. The *universal trace* for a kG-module is the map

$$\mathrm{Tr}\colon \mathrm{End}_{kG\text{-mod}}(V) \to k(G)^G$$

given by sending an endomorphism θ to the function in kG given by $g \mapsto \mathrm{Tr}(\alpha_g \theta) = \mathrm{Tr}(\theta\alpha_g)$, where α_g is the action by g map. This function is clearly conjugation invariant. Furthermore, the identity map 1_V maps to the character of V.

This map makes sense for any group G and field k, but we can interpret it in another way for a finite group and algebraically closed field of characteristic zero. In this case, the group algebra kG breaks up into a finite sum indexed by irreducible representations

$$kG \cong \bigoplus_{\text{irreps}} (\dim V) \cdot V$$

as a kG module (the decomposition of the regular representation). Alternatively, we can write this decomposition

$$kG \cong \bigoplus_{\text{irreps}} \mathrm{End}_k(V)$$

where $\mathrm{End}_k(V)$ has the structure of a left kG-module by precomposition by the action endomorphisms. This is a simple special case of the Peter-Weyl theorem where G is a finite group. Thus we have inclusion and projection maps

$$\mathrm{End}_k(V) \rightleftarrows kG \ ,$$

or, after taking G-invariants on both sides

$$\mathrm{End}_{kG}(V) \rightleftarrows k(G)^G \ .$$

The resulting map from left to right is exactly the universal trace described above for the irreducible representation V. What's more, in this situation we have *Schur's Lemma* available to exactly describe endomorphisms of a kG-module, since representation theory of finite groups tells us that kG is a semisimple left Artinian ring:

Lemma 1.2 (Schur)**.** Let V_1, V_2 be irreducible kG-modules. Then $\mathrm{Hom}_{kG}(V_1, V_2) = 0$ if $V_1 \ncong V_2$, and $\mathrm{Hom}_{kG}(V_1, V_2) = k$ if $V_1 \cong V_2$.

So in particular, $\mathrm{End}_{kG}(V) \cong k^n$, where n is the number of irreducible summands of V. If V is irreducible, we only need to identify the image of 1 under the map. This is the image of the identity matrix under the inclusion

$$\mathrm{End}_k(V) \hookrightarrow kG$$

which is exactly the character of V, so this construction agrees with the universal trace.

Example 1.3. Let H be a subgroup of a finite group G. Then we can consider the kG-module $k(G/H)$: the H-invariants on kG under the right multiplication action. This is the same as the induced representation $\mathrm{Ind}_H^G(k)$, as we will see in 3.2 below. The universal trace gives us a map

$$\mathrm{End}_{kG}(k(G/H)) \to k(G)^G$$

sending the identity to the character of this induced representation. Given a kG-submodule $V \subseteq k(G/H)$, we can consider the projector π_V as an endomorphism. Our above discussion implies that the image of this endomorphism under the universal trace is precisely the character of V. We can describe another construction of the universal trace map in this case.

The algebra $\mathrm{End}_{kG}(k(G/H))$ is usually called the *Hecke algebra* for the pair (G, H), and denoted $\mathcal{H}_{G,H}$. We'll just write $\mathcal{H}$ when it's clear which groups we are referring to. It is isomorphic to the algebra of double cosets $k(H\backslash G/H)$ with a certain convolution product (more on this later, in 3.2).

For (G, H) as above, consider the correspondence of action groupoids

$$\begin{tikzcd} & G/_{ad}\, H \arrow[dl, "p"'] \arrow[dr, "q"] & \\ G/_{ad}\, G & & H\backslash G/H. \end{tikzcd}$$

where the maps are the natural forgetful (inclusion) maps. Given a function $f \in k(H\backslash G/H)$, we can first pull it back to $G/_{ad}\, H$ along q (corresponding simply to forgetting some of the action), then pushforward to $G/_{ad}\, G$ along p, (corresponding to averaging over the set of H-orbits contained in a G-orbit). This gives an algebra homomorphism

$$\mathcal{H} \to k(G)^G.$$

Does it agree with the universal trace map? Well, $\mathrm{End}_{kG}(k(G/H))$ has a natural basis given by the projectors π_V onto the irreducible submodules. We'll work in the most general case, that is, $H = \{e\}$. So for every irreducible representation V we have an element $\pi_V \in \mathrm{End}_{kG}(kG)$, corresponding to the function $\pi_V(|G|\delta_e) \in kG$ where δ_e is the constant function supported at the identity (and where we've normalised by the size of the group: $|G|\delta_e$ is the character of the regular representation). Consider the commutative square

$$\begin{tikzcd} kG \arrow[r, "\pi_V"] \arrow[d] & kG \arrow[d] \\ k(G)^G \arrow[r, "\pi_{\chi_V}"] & k(G)^G \end{tikzcd}$$

where the vertical arrows are given by averaging a function over conjugacy classes. The algebra $k(G)^G$ has a basis given by character of irreducible representations, and the map π_{χ_V} is the projection onto the one-dimensional factor corresponding to V. That this square commutes says that $\pi_V(|G|\delta_e)$ averages to $\pi_{\chi_V}(|G|\delta_e) = \chi_V \in k(G)^G$, the character of V. But this averaging is precisely the push forward along the projection $G \to G/_{ad}\, G$ in our correspondence above, so the image of π_V under this correspondence is the character of V. An analogous story holds if we instead worked with H right invariant functions for H a non-trivial subgroup of G.

1.2 Outline of the Construction

In what way will we try to generalise this? We'd like to start with an *algebraic* group, defined over some field k (so for instance the k-points give us a possibly infinite group), and we'd like to define a functor from a category of representations to a category of class functions analogous to the character of a representation of a finite group. In particular, if the group G is defined over $\mathbb{F}_p$, we'd like this functor to be suitable natural that we can produce the character tables of every group $G(\mathbb{F}_{p^n})$ in a universal way.

In order to achieve this, we'll work in a *categorified* setting: instead if adjoint invariant functions, we'll produce adjoint equivariant *sheaves* on the algebraic group G, from which we might recover functions via some process of decategorification. The starting category of representations itself will have to be replaced with some categorified analogue. The language of Hecke algebras lends itself particularly nicely to this task: just as the representations contained in a given kG-module $k(G/H)$ were controlled by a finite Hecke algebra $\mathcal{H}$, a categorical analogue will be controlled by a *finite Hecke category*: a symmetric monoidal category $\boldsymbol{\mathcal{H}}$.

More precisely, instead of trying to work with the group algebra kG of functions on G, we work with a derived category of sheaves on G, $D(G)$. In the case where G has topology, just as we would need to specify exactly which functions we'd like to work with (smooth, L^2, some form of distributions, ...), we must specify which kind of sheaves. There are three key possibilities:

- We could consider D-modules on G. Recall, these are modules for the sheaf of differential operators on G, roughly systems of differential equations on G.

- We could consider constructible sheaves on G, over a field of characteristic zero. A sheaf $\mathcal{S}$ is constructible if there exists a stratification of G by subvarieties such that on each stratum $\mathcal{S}$ is a locally constant sheaf of finite dimensional vector spaces. Constructible sheaves are equivalent to a certain category of maximally determined D-modules (so-called *holonomic* D-modules). From such a D-module we recover a constructible sheaf by taking the "solutions" of the system of differential equations.

- We could consider ℓ-adic sheaves on G, where ℓ is a prime different from the characteristic of the base field k. In the case where k has positive characteristic (the relevant case when computing the character tables of finite groups of Lie type), we need to work with this version of constructible sheaves.

these categories have similar properties, and there are functors relating them, but they are not the same. There's a story in each case, so whenever possible I'll describe constructions in a way that will work in all three examples. I'll be clearer when certain results are only true in certain settings.

We can produce categorical analogues of the Hecke algebras in an analogous way to the finite Hecke algebras described above. In the simplest case, choose a Borel subgroup $B \leq G$. Then the category $D(G/B)$ (where $D(-)$ refers to a category of sheaves as described above) is acted on by the category $D(G)$ by convolution, coming from the action of G on G/B on the left. This is analogous to the kG-module $k(G/B)$ in the classical theory. Then by analogy, we define the *finite Hecke category* to be

$$\boldsymbol{\mathcal{H}} = D(B\backslash G/B),$$

just as the endomorphism ring of $k(G/B)$ was given by the Hecke algebra $k(B\backslash G/B)$ of bi-invariant functions. This should be thought of as describing a part of the representation theory of G: that part contained within the induction of the trivial representation of B to G. The module category $D(G/B)$ can be thought of as the induction of the category of local systems on T with unipotent monodromy to a $D(G)$-module. Similarly, there are categories playing the role of the induction of the other "representations of B", namely the so-called λ-*twisted* D-*modules* on G/B, where λ is a character of the torus T, and twisted Hecke categories playing the roles of their endomorphisms.

Now, we need to produce a version of the universal trace map, assigning a character to an element of this Hecke category. We can naturally do this by the correspondence description we gave before. In the "unipotent" case, i.e. for the usual finite Hecke category, we can consider the correspondence of stacks

$$\begin{tikzcd} & G/_{ad}\, B \arrow[dl, "p"'] \arrow[dr, "q"] & \\ G/_{ad}\, G & & B\backslash G/B. \end{tikzcd}$$

where the maps are the obvious projection maps. Then, just like we did for finite groups, we can push and pull along this adjunction to define a functor

$$H = p_! q^* \colon \boldsymbol{\mathcal{H}} \to D(G/_{ad}\, G)$$

with a right adjoint $q_* p^!$. This recovers our universal trace map when G is a finite group, and B is any subgroup. Just like the image of the trace map was contained in the *characters* of representations of the group G, the image of the functor H should be contained in the *character sheaves* of the group G: some special adjoint equivariant sheaves on G.

Now, what is a character sheaf in general? We can construct some examples by this so-called "horocycle" correspondence, and its twisted analogues, and take the category they generate. But is this the right analogue of the whole set of characters of the group? To justify why the answer should be 'yes', we appeal to the *Beilinson-Bernstein correspondence*. This says –approximately – that the category of λ-twisted D-modules on the flag variety G/B is equivalent to a certain subcategory of representations of the Lie algebra $\mathfrak{g}$: those with fixed "central character" λ. So the categories of twisted D-modules for various twists describe the whole category of representations of the group G. Thus, our candidate module categories for $D(G)$ actually include the data of the whole category of $\mathfrak{g}$-modules.

1.3 Conventions on Stacks and Equivariant Sheaves

By a *stack*, we will mean the following:

Definition 1.4. A stack $\mathcal{X}$ is a sheaf of groupoids over the étale site for a field k admitting a smooth surjective representable morphism from a scheme X, such that the diagonal map $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is representable and quasicompact.

These are usually called *Artin stacks*, and the map $\underline{X} \to \mathcal{X}$ is called an *atlas*. We can define various sorts of sheaves on stacks by descent from an atlas, i.e. if our stack has atlas $\pi \colon \underline{X} \to \mathcal{X}$ then we define – say – quasicoherent sheaves on $\mathcal{X}$ by a Barr-Beck argument (more on this kind of thing later). Less concretely, sheaves on $\mathcal{X}$ can be defined to be a compatible system of sheaves on schemes U admitting smooth surjective maps to $\mathcal{X}$.

The only stacks we will need to work with are quotient stacks of the form X/G, where X is an algebraic variety acted on by an algebraic group G. The notion of *equivariant sheaves* on X will allow us to avoid subtle definitional issues involved with sheaves on stacks. Categories of such sheaves are studied in detail in the book [4] of Bernstein and Lunts. The derived category of equivariant sheaves is *not* the same as the derived category of the category of equivariant sheaves. It is instead defined in the following way. Say a map of G-spaces $P \to X$ is a *free resolution* if P is a free G-spaces.

Definition 1.5. For a free resolution $f \colon P \to X$, define the category $D^b_G(X, P)$ to be the category of triples $(\mathcal{F}_X, \overline{\mathcal{F}}, \beta)$, where $\mathcal{F}_X \in D^b(X)$, $\overline{\mathcal{F}} \in D^b(P/G)$, and β is an isomorphism $f^*\mathcal{F}_X \xrightarrow{\sim} p^*\overline{\mathcal{F}}$ where p is the projection $P \to P/G$.

If P and Q are n-acyclic free resolutions of X, then there is a natural isomorphism between $D^{[-n,n]}_G(X, P)$ and $D^{[-n,n]}_G(X, Q)$. Thus we can uniquely define $D^{[-n,n]}_G(X)$ to be this common category, and hence define

$$D^b_G(X) = \lim_n D^{[-n,n]}_G(X).$$

This definition of the derived category of equivariant sheaves is equivalent to a derived category of simplicial sheaves on the simplicial space

$$\cdots G \times G \times X \substack{\to \\ \to \\ \to} G \times X \rightrightarrows X$$

where all the structure morphisms are isomorphisms. It is by such a simplicial resolution that we might define the derived category of sheaves on the stack X/G. See [4] Appendix B.

With this in mind, we will use the following two notations equivalently. We could work in various categories of sheaves, but most often we'll be concerned with the bounded derived (dg) category of constructible sheaves over a field k of characteristic zero. Denote the category of such sheaves on a scheme or stack X by $D(X)$. If G is an algebraic group acting on a variety X, then we have the category $D_G(X)$ of G-equivariant sheaves on X in the sense of [4], or equivalently we have the category of sheaves on the quotient stack $D(X/G)$. These two notations will be used interchangeably. We might also occasionally refer to the derived category of D-modules on a variety or stack. This is strictly larger than the category referred to above, which is equivalent to *holonomic* D-modules with *regular singularities*. We might abuse notation by also referring to this category as $D(X)$, but if so we will take care to point out the change in notation.

Let $f \colon X \to Y$ be a map of varieties or stacks, on which we have a notion of constructible sheaves or D-modules. We'll often use the *six functors* formalism. We have a collection of functors between derived categories: $f_*, f^\dagger, f_!, f^!, \boxtimes, \mathcal{H}om$. These come with certain adjunctions between them: $f^\dagger \dashv f_*$, and $f_! \dashv f^!$. We use the notation $f^\dagger$ to indicate that the correct left-adjoint of the derived pushforward functor isn't just the pullback, but the pullback with a shift by the difference in dimensions:

$$f^\dagger = f^*[\dim X - \dim Y]$$

so that the natural t-structure on the derived category of constructible sheaves is preserved.

2 Character Sheaves on a Torus

Just like in the representation theory of finite groups, an easy initial example of character sheaves will come from abelian groups. In the classical story representations of abelian groups are all one-dimensional, so agree with their characters. We'll see how to make sense of this in a categorified setting.

So, throughout this section, let k be a field, and let T be an algebraic torus defined over k. That is, T is locally isomorphic to $\mathbb{G}_m^r$ as a k-scheme. For simplicity I'll mostly assume k is algebraically closed, in which case $T(k)$ is just $(k^\times)^r$ (i.e. we'll only be concerned with *split* tori). I might also write simply T when I mean the k-points $T(k)$. The motivating example will be the case $k = \mathbb{C}$.

2.1 Characters of Abelian Groups

Let's review the most classical story. Let A be a finite abelian group, and let V be a finite-dimensional representation of A over k, i.e. a kA-module. The k-algebra kA splits up as a product $k(\mathbb{Z}/q_1\mathbb{Z})\times\cdots\times k(\mathbb{Z}/q_n\mathbb{Z})$ by the classification of finitely generated abelian groups, so to describe kA-modules we need only describe $k(\mathbb{Z}/q\mathbb{Z}) \cong k[t]/(t^q)$-modules.

A $k[t]$-module is a k-vector space equipped with an endomorphism. A $k[t]/(t^q)$-module is a k-vector space equipped with an endomorphism whose q^{th} power is the identity. As k is assumed to be algebraically closed, represent this endomorphism by a matrix in Jordan Normal form. It is easy to check that this matrix must be diagonal, with eigenvalues all q^{th} roots of unity. Thus there are : q one-dimensional irreducible representations of $\mathbb{Z}/q\mathbb{Z}$.

One could go further and classify all representations of *finitely generated* abelian groups, by understanding the representation theory of $\mathbb{Z}$. m-dimensional representations of $\mathbb{Z}$ are classified by conjugacy classes in $GL(m,k)$, i.e. by Jordan types. So there is an irreducible representation of each dimension with each possible eigenvalue $\lambda \in k$. We might call λ the *central character* of this representation: a piece of terminology which will re-occur later on.

2.2 Characters and Representations of Tori

Definition 2.1. A *character* of an algebraic torus T is a homomorphism of algebraic groups

$$T \to \mathbb{G}_m.$$

The characters of T form an abelian group under pointwise multiplication, denoted $X^\bullet(T)$.

Remark 2.2. The group of characters of T is also called the lattice of *weights* of T. We can define the weight lattice $X^\bullet(G)$ for any algebraic group G in an analogous way: as the group of homomorphisms into the multiplicative group.

In the split case – where k is algebraically closed – the group of characters is easy to define. Indeed, we have

$$\begin{aligned} X^\bullet(T) &= \{\text{homomorphisms } \mathbb{G}_m^r \to \mathbb{G}_m\} \\ &= \{\text{homomorphisms } \mathbb{G}_m \to \mathbb{G}_m\}^r \\ &= \mathbb{Z}^r \end{aligned}$$

since the endomorphisms of $\mathbb{G}_m$ are given by the n-power maps $t \mapsto t^n$. This justifies calling the group of characters a *lattice*. At least when $k = \mathbb{C}$, it actually canonically embeds in the dual Lie algebra $\mathfrak{t}^*$, where $\mathfrak{t}$ is the Lie algebra of T.

Now, given a representation of T, we can describe its *weight decomposition*. As usual, a *representation* of T is a homomorphism

$$\rho\colon T \to GL(V)$$

for some k-vector space V. Analogously to the representation theory of finite groups, we can split representations of tori into sums of characters.

Proposition 2.3. Any finite dimensional representation V of T splits up as a finite sum of characters:

$$V \cong \bigoplus_{\chi \in X^\bullet(T)} V^\chi$$

where V^χ is the summand where the representation acts by the character χ.

The χ appearing in this sum with $V^\chi \neq 0$ are called the *weights* of the representation ρ.

Proof. This is really the fact that homomorphisms of algebraic groups preserve the Jordan decomposition of elements. Indeed, let $\rho\colon T \to GL(V)$ be a finite dimensional representation. It suffices to show that each $\rho(t)$ is diagonal, since a homomorphism into the subgroup of diagonal matrices clearly splits into a direct sum of characters as required. Preservation of Jordan decomposition implies precisely this, since every element in T is purely semisimple, and this fact is proved in [6] Theorem 4.4.

To sketch, we split ρ as the composite of a surjective map and a closed embedding. That closed embeddings into $GL(V)$ preserve Jordan type is easy from the usual Jordan decomposition. If $f\colon G \to G'$ is surjective, then the induced map $f^*\colon kG' \hookrightarrow kG$ is an injective homomorphism of k-algebras. So if r_g denotes right translation by g, we have $r_{fg} = (r_g)|_{kG'}$, which means the Jordan decomposition of g agrees with the decomposition of fg. (TODO: rewrite more clearly). □

2.3 Local Systems on a Torus

In this section, we'll explain why the category of *local systems* on T gives a good categorification of the characters, or representations, of the torus. Throughout, we'll work in the setting of derived (or dg) categories, as the setting with the most natural notion of functoriality.

Definition 2.4. A K*-local system* on an algebraic variety X is a *locally constant sheaf*, i.e. a sheaf $\mathcal{L}$ of K-vector spaces with an étale open cover (U_i), such that $\mathcal{L}|_{U_i}$ is a constant sheaf for all i. Equivalently, if X is a variety over $\mathbb{C}$, we can instead use the analytic topology, and ask for the sheaf to be analytically locally constant.

Denote by $\mathrm{Loc}(X)$ the derived (dg) category of local systems on X. So objects are complexes of local systems, and quasi-isomorphic such complexes are isomorphic. The field K will usually be omitted in our notation. It will generally be an algebraically closed field of characteristic zero.

Local systems are equivalent to various other mathematical objects, most notably representations of the fundamental group of X. We can see this most geometrically in the case where everything is defined over $k = \mathbb{C}$. Then associated to a rank r local system $\mathcal{L}$, we produce a representation of $\pi_1(X, *)$ by choosing a collection γ_i of paths generating π_1, covering each path with a finite collection of open sets on which $\mathcal{L}$ is trivial, and producing an element A_i of $GL(m, \mathbb{C})$ by composing the transition functions as we pass between trivialisations around the loop γ_i. This gives a homomorphism

$$\pi_1(X, *) \to GL(m, \mathbb{C}),$$

once we do the simple check that the matrix A_i does not depend on the representative of γ_i within its homotopy class, or the choice of trivialisation around the loop. This representation is called the *monodromy* of the local system.

One can go back the other way, producing a local system from a representation of π_1. In fact, the two categories are equivalent.

$$\mathrm{Loc}(T) \sim \mathrm{Rep}(\pi_1(T, *))$$

where our notation always refers to the bounded derived categories.

In our example, $X = T$, an algebraic torus. What is $\pi_1(T)$? Suppose $k = \mathbb{C}$, so we can work with the usual fundamental group. Then of course $\pi_1(T) = \mathbb{Z}^r$, where r is the rank of the torus. Crucially, this is essentially the *dual* lattice to the lattice of characters we defined in 2.1. Why? Well, an element of $\pi_1(T, *)$ is a based map from the circle to T. In every homotopy class there is a map that extends to a map $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times \to T$, which we can see by noting that it is true for a generating set. Thus

$$\pi_1(T, *) = X_\bullet(T) = \{\text{homomorphisms } \mathbb{G}_m \to T\},$$

over the complex numbers at least. The lattice $X_\bullet(T)$ is called the lattice of *cocharacters* of T, and is dual to the lattice of characters.

In particular, rank one local systems on T, i.e. one-dimensional representations of $\pi_1(T)$, are homomorphisms $X_\bullet(T) \to \mathbb{G}_m$, i.e. elements of $X^\bullet(T)$, i.e. characters of T. Thus the derived category of local systems gives a suitable derived geometric analogue to the theory of representations of the algebraic group T. We should note however, that not all local systems on T split as a direct sum of those given by characters. It is possible to have monodromy around a generator of $\pi_1(T)$ given by a matrix with non-trivial Jordan blocks. For instance, if T is rank 1, the category of local systems generated by the character λ consists of those representations whose monodromy is given by a matrix all of whose eigenvalues are λ. However, some of the Jordan blocks could be bigger than one-by-one.

2.4 Characters of the Lie algebra, and Descent

This simple example lends itself to a natural description via descent, which we might hope to generalise to the non-abelian setting. Lets work over the complex numbers for simplicity. Then the complex algebraic group T has Lie algebra $\mathfrak{t} = T_e T$. The category of $\mathbb{C}$-local systems on $\mathfrak{t}$ is simply the derived category of chain complexes over $\mathbb{C}$, because $\mathfrak{t}$ has trivial fundamental group. We'll denote this category by Vect.

We'll define an adjunction

$$\mathrm{Loc}(T) \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} \mathrm{Vect}\ ,$$

and prove that the functor π^* is *comonadic*, so $\mathrm{Loc}(T)$ is equivalent to a category of comodules for a comonad on Vect which we can describe explicitly. In this setting, we'll just recover the theorem that we described above: that local systems are the same as representations of the fundamental group

The reason we can do this so easily in this example is because $\mathfrak{t}$ is isomorphic to the *universal cover* of T. Our maps come from the projection map $\pi \colon \mathfrak{t} \to T$ from the universal cover, or equivalently the *exponential map*. We should be careful here: this map is a map of complex manifolds, but not a map of complex algebraic varieties. Still, by GAGA we have well-defined maps π^* and π_* on the derived categories of local systems (the derived pullback and pushforward). The inverse image functor π^* is left adjoint to π_*. To prove the result we want, we'll need to introduce some category theory: namely the notion of the category of coalgebras for a comonad, and Beck's (co)monadicity theorem.

Definition 2.5. Associated to a pair of adjoint functors $\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$ we can associate categories of *Eilenberg-Moore modules* and *comodules* as follows:

A *module* for the endofunctor $T = RL \colon \mathcal{D} \to \mathcal{D}$ (T carries the structure of a *monad*) is an object $A \in \mathcal{D}$ with a morphism $\alpha \colon TA \to A$ satisfying unit and associativity axioms (see [1], though they refer to *triples* instead of monads).

A *comodule* for the endofunctor $U = LR \colon \mathcal{C} \to \mathcal{C}$ (U carries the structure of a *comonad*) is an object $A \in \mathcal{C}$ with a morphism $c \colon A \to UA$ satisfying natural counit and coassociativity axioms (dual to the above).

Definition 2.6. A functor $R \colon \mathcal{D} \to \mathcal{C}$ is *monadic* if it has a left adjoint L, and the category $\mathcal{D}^{RL}$ of modules for the monad is equivalent to $\mathcal{C}$ by the natural comparison functor. A functor $L \colon \mathcal{C} \to \mathcal{D}$ is *comonadic* if it has a right adjoint R, and the category $\mathcal{C}^{LR}$ of comodules for the comonad is equivalent to $\mathcal{D}$ by the natural comparison functor.

Theorem 2.7 (Beck's (co)Monadicity Theorem)**.** A functor $R\colon \mathcal{D} \to \mathcal{C}$ with left adjoint L is monadic if and only if

1. It reflects isomorphisms.
2. $\mathcal{D}$ has, and R preserves, coequalizers of R split pairs.

Alternatively, R is monadic if

1. It reflects isomorphisms.
2. $\mathcal{D}$ has, and R preserves, reflexive coequalizers.

A functor L is comonadic if / if and only if the obvious dual conditions hold.

This is theorem 3.14 in [1]. We can use this result to construct the equivalence we want. The functor π^* has a right adjoint, so if we can prove that it satisfies necessary conditions for comonadicity then we'll have established an equivalence with a category of comodules

$$\mathrm{Loc}(T) \sim \mathrm{Vect}^{\pi^*\pi_*}.$$

This is fairly easy. Indeed the category $\mathrm{Loc}(T)$ is triangulated, and the functor π^* is a morphism of triangulated categories. This gives it very strong continuity properties. All we need to do is check that π^* reflects isomorphisms, which in this setting can be reduced to checking that if $\pi^*\mathcal{L} = 0$, then $\mathcal{L} = 0$, which is obvious. (TODO: Mention a similar argument for more general G?)

Now, we need only understand what it means to be a comodule for the comonad $U = \pi^*\pi_*$. Firstly, what is this functor U? An element V of Vect is a constant sheaf on the flat space $\mathfrak{t} = \mathrm{Lie}\, T = \widetilde{T}$. We first compute $\pi_* V$.

$$\begin{aligned} \pi_* V(B) &= V(\pi^{-1}B) \\ &= \prod_{X_\bullet(T)} V \end{aligned}$$

where B is a small open ball in T. The universal covering map π has fibres indexed by the fundamental group of T, i.e. by the cocharacter lattice. Thus

$$\begin{aligned} UV(B') &= \pi^*\pi_* V(B') \\ &= \pi_* V(\pi(B')) \\ &= \prod_{X_\bullet(T)} V \end{aligned}$$

where now B' is a small open ball in $\mathfrak{t}$. The functor U takes a chain complex V to a cocharacter indexed product $\prod_{X_\bullet(T)} V$.

What is a comodule for this comonad? It is an object $V \in \mathrm{Vect}$ equipped with a morphism $c\colon V \to UV$ (the "coaction") satisfying certain conditions, namely:

- Counitarity: The triangle

$$\begin{tikzcd} V \arrow[r, "c"] \arrow[dr, "1_V"'] & UV \arrow[d, "\varepsilon_V"] \\ & V \end{tikzcd}$$

  commutes.

- Coassociativity: the square

$$\begin{tikzcd} V \arrow[r, "c"] \arrow[d, "c"'] & UV \arrow[d, "Uc"] \\ UV \arrow[r, "\Delta_V"'] & U^2V \end{tikzcd}$$

  commutes.

In our situation, the first condition says that c is a left inverse for the counit, which is projection onto the $\lambda = 0$ factor. Thus c must act as the identity from $V \to V_0 \subseteq \Pi_{\lambda \in X_\bullet(T)} V_\lambda$. The second condition tells us how the different λ components of c relate to one another. Precisely, it says that map to the components $V_\lambda \subseteq \Pi_{\lambda \in X_\bullet(T)} V_\lambda$ satisfy the identity

$$c_{\lambda_1} \cdot c_{\lambda_2} = c_{\lambda_1 + \lambda_2}$$

as elements of $GL(V)$. This is because the comultiplication map sends an element (λ_i) to (λ'_{ij}) where $\lambda'_{ij} = \lambda_{i+j}$.

Thus a comodule for the comonad is an object $V \in \mathrm{Vect}$ equipped with an action of the fundamental group $\pi_1(T, *) \cong X_\bullet(T)$, i.e. precisely a representation of the fundamental group. So we have indeed recovered the theorem that a local system on T is equivalent to a representation of $\pi_1(T)$.

3 Hecke Categories

3.1 Bruhat Decomposition

The Bruhat Decomposition of an algebraic group is a generalisation of the classical *Gauss-Jordan algorithm*. Let's recall how this works. We aim to transform an invertible matrix into the identity using row and column operations, but we restrict ourselves to only *scale* rows and columns by a constant, and add a multiple of the i^{th} row or column to the j^{th} row or column *where i is less than j*. Generically, this is possible, but in general the best result one can hope for is a permutation matrix. So we conclude that the Gauss-Jordan algorithm gives a method of writing an invertible matrix A as

$$A = U_1 P U_2$$

where U_1 and U_2 are upper triangular, and P is a permutation matrix.

The Bruhat decomposition generalises this. To be precise, let G be a connected reductive algebraic group defined over k, and let B be a borel subgroup. Let $T \subseteq B$ be a maximal torus, and express the Weyl group as usual as $W = N_G(T)/T$. Let $\dot{w}_1, \ldots, \dot{w}_m \in N_G(T)$ be a set of choices of lifts of the m elements of W to elements of G. Here I'm often conflating algebraic groups with their sets of k-points for brevity.

Theorem 3.1 (Bruhat Decomposition)**.** The double cosets $B\dot{w}_iB \subseteq G$ are disjoint and independent of the choice of lift, and they exhaust G, i.e.

$$G = \coprod_{i=1}^{m} B\dot{w}_iB.$$

In light of this notation, we'll usually refer to a coset as simply BwB, omitting in our notation the choice of lift of the Weyl group element.

We'll describe a proof in the special case where $k = \mathbb{C}$ (following [8] 3.1.9), using the following decomposition theorem

Theorem 3.2 (Białynicki-Birula, [8] 2.4.3)**.** Let X be a smooth complex projective variety equipped with an action of $\mathbb{C}^\times$. Let W denote the set of fixed points under the action. Then X decomposes as

$$X \coprod_{w \in W} X_w$$

where $X_w = \{x \in X \colon \lim_{\lambda\to 0} \lambda \cdot x = w\}$ is the *attracting set* associated to w. Furthermore, the attracting sets can be computed directly as

$$X_w = T_w^+ X$$

where $T_w^+ X$ is the sum of the positive eigenspaces for the $\mathbb{C}^\times$ action on the tangent space $T_w X$.

Proof of 3.1. Since $T \subseteq B$, the independence of the choice of $\dot{w}$ is obvious. We must show that under the left B-action, the flag variety G/B contains a unique point of the form wB for $w \in W$. In order to do this, we'll use the Białynicki-Birula decomposition. We cook up a $\mathbb{C}^\times$ action by choosing a one parameter subgroup $\mathbb{C}^\times \subseteq T$ in *general position*, i.e. on the level of Lie algebras we ask for $\operatorname{Lie}\mathbb{C}^\times$ to be generated by a regular semisimple element $t \in \mathfrak{t}$. We must study the fixed point set for this action: fixed points will turn out to correspond exactly to elements of the Weyl group.

The generator $t \in \mathfrak{t}$ is the infinitesimal generator of a vector field on G/B, and so the fixed points of the action are precisely the zeroes of this vector field. Viewing the flag variety as the variety of Borel subalgebras $\mathfrak{b} \subseteq \mathfrak{g}$, these zeroes are precisely the $\mathfrak{b}$ such that $t \in \mathfrak{b}$, since in this language, the action is given by conjugating the subalgebra. The general type assumption implies that this is equivalent to $\mathfrak{t} \subseteq \mathfrak{b}$, so we must study the collection of Borel subalgebras containing a fixed $\mathfrak{t}$. One can show that a Borel subalgebra contains $\mathfrak{t}$ if and only if it is of the form

$$w\mathfrak{b}_0 w^{-1}$$

for $w \in W = N_G(T)/T$, where $\mathfrak{b}_0$ is the Lie algebra of B. Thus the fixed point set of our $\mathbb{C}^\times$ action is parameterised exactly by the Weyl group, as required.

Thus, the Białynicki-Birula decomposition gives

$$G/B = \coprod_{w\in W} X_w \cong \coprod_{w \in W} T_w^+(G/B)$$

and we must analyse the attracting sets. To be precise, we want to show that they agree with the left B-orbits. Since the maximal torus T fixes the fized points, hence the attracting sets, we can equivalently show that they agree with the N-orbits, where N is the unipotent radical of B (as $B = N \cdot T$). Now, the tangent space to G/B at the fixed point w is the quotient of the tangent space $\mathfrak{g}$ to G at w by the Lie algebra of the stabiliser of w under the B-action. This Lie algebra certainly contains $\mathfrak{t}$, so $T_w(G/B)$ is a quotient of

$$\mathfrak{g}/\mathfrak{t} \cong \mathfrak{n}^+ \oplus \mathfrak{n}^-$$

where the decomposition of $\mathfrak{g}$ as $\mathfrak{t} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-$ is the decomposition into zero, positive and negative eigenspaces under the $\mathbb{C}^\times$ action. Clearly $T_w^+(G/B) = \mathfrak{n}^+ \cdot w$.

Now, the orbit $N \cdot w$ is contained in the attracting set X_w. This is an easy computation: choose $u \in N$, and look at $t \in T$.

$$\begin{aligned} tu(w) &= (tut^{-1})t(w) \\ &= (tut^{-1})(w) \\ &\to 1(w) = w \text{ as } t \to 0 \end{aligned}$$

Furthermore we can check that this is actually a closed immersion. However $\dim N \cdot 0 = \dim \mathfrak{n}^+ \cdot 0 = \dim X_w$. Therefore they must agree, as required. □

Remark 3.3. The Bruhat decomposition immediately gives us a set of $|W|$ disjoint embedded maximal tori in G, one for each Bruhat orbit, namely $T \hookrightarrow BwB$ by $t \mapsto BwtB$. Apart from the torus corresponding to $w = 1$ these are not subgroups: they are more like cosets wT. In fact, the Bruhat orbit BwB has the structure of an affine space $\times$ the torus T. One checks this by noting

$$BwB = NwB = (NwN)TN$$

and observing that $NwNtN$ is an affine space for each t.

Remark 3.4. Along these lines, we can analyse the Bruhat cells in a little more detail. Then cell $BwB = BwN$ can be written as

$$BwN = N^w TN$$

where N^w is the product of the root subgroups $N_\alpha \leq N$ with $w^{-1}N_\alpha w$ not contained in N. (TODO: what does this mean? Use it in setting up Lusztig sheaves)

3.2 The Finite Hecke Category

So the Bruhat decomposition has allowed us to describe the double quotient $B\backslash G/B$. In this section, we'll see that this double quotient, and functions on it, is an important representation theoretic object. It describes endomorphisms of the induced representation G/B.

As before, fix a Borel subgroup $B \subseteq G$. We can produce a representation of G by inducing the trivial representation of B up to a representation of G (we'll review induction in detail in 4.1). That is, we're producing the kG-module

$$\mathrm{Hom}_{kB}(k, kG) \cong k\left(G/B\right)$$

by noting $k(G/B)$ is just the ring of B-invariants in kG.

We can compute the endomorphisms of this representation by Frobenius reciprocity. The following computation is precise for B a finite subgroup of a finite group G, and merely heuristic in our setting:

$$\begin{aligned}
\mathrm{End}(k\left(G/B\right)) &\cong \mathrm{Hom}_{kG}(k\left(G/B\right), k\left(G/B\right)) \\
&\cong \mathrm{Hom}_{kB}(k, k\left(G/B\right)) \\
&\cong \mathrm{Hom}_{k}(k, k\left(G/B\right))^B \\
&\cong k(B\backslash G/B)
\end{aligned}$$

where all isomorphisms are as k-vector spaces. Transferring the algebra structure from the endomorphism ring to $k(B\backslash G/B)$ gives the usual description of the *finite Hecke algebra* $\mathcal{H}(G,B)$: the algebra of bi-invariant functions on G. To be truly accurate in this setting, one would have to be clear about what one *means* by the group algebra of a Lie group. In general we don't want to consider all functions, but rather some nice subset, for instance smooth functions with compact support.

It is natural at this point to want to categorify this story. As ever, there are multiple choices as to what kind of "function" (sheaf) we take on our group. Throughout this section I'll assume char $k = 0$, and work with *constructible* sheaves on $G(k)$ (again, I'll usually omit the k in my notation). In characteristic p we could instead work with ℓ-adic sheaves for ℓ some prime different from the characteristic of k.

Definition 3.5. The *finite Hecke category* associated to the pair $B \subseteq G$ is the bounded derived category of constructible sheaves on the stack $B\backslash G/B$, i.e.

$$\boldsymbol{\mathcal{H}} = D^b(B\backslash G/B).$$

Equivalently, this is the derived category of B equivariant constructible sheaves on the flag variety G/B.

This category has a monoidal structure under *convolution*. Just like in the case of the Hecke algebra, we can identify this Hecke category with a category of endofunctors leading to a natural description of the monoidal structure. We would expect to find

$$\begin{aligned}
D^b(B\backslash G/B) &\sim \mathrm{Fun}_{D(G)}(D(G/B), D(G/B)) \\
&\sim \mathrm{Fun}_{\bullet/G}(D(\bullet/B), D(\bullet/B))
\end{aligned}$$

by analogy. This is a reflection of the fact that the double quotient stack can be built as a fibre product:

$$B\backslash G/B \cong (\bullet/B) \times_{\bullet/G} (\bullet/B)$$

which one checks by computing functions on each side. (TODO: Rewrite this section more carefully.) We can also compute the convolution product directly using this representation. Given two elements $\mathcal{S}_1, \mathcal{S}_2 \in \boldsymbol{\mathcal{H}}$, we compute the product $\mathcal{S}_1 * \mathcal{S}_2$ via the diagram

$$\begin{tikzcd} & B\backslash G/B & \\ & B\backslash G \times G/B \arrow[u, "\mu"] \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ B\backslash G/B & & B\backslash G/B \end{tikzcd}$$

where the map π_1 is given by projection first onto the first factor $B\backslash G$, then onto the double quotient, and similarly for π_2. The map μ is the multiplication map. The convolution product is given by the push-pull construction

$$\mathcal{S}_1 * \mathcal{S}_2 = \mu_*(\pi_1^* \mathcal{S}_1 \otimes \pi_2^* \mathcal{S}_2).$$

This is the general pattern for a convolution product in a Hecke algebra, modulo the choice of what one means by pushforward and pullback along the canonical maps. It acts on the category $D(G/B)$ in a similar way, via pushing and pulling along the analogous diagram

$$\begin{tikzcd} & G/B & \\ & G \times G/B \arrow[u, "\mu"] \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ B\backslash G/B & & G/B \end{tikzcd} .$$

By the Bruhat decomposition, there are $|W|$ special skyscraper-like elements in the Hecke category. For each element of the Weyl group, there is an inclusion

$$j_w \colon B\backslash BwB/B \hookrightarrow B\backslash G/B$$

corresponding to choosing a Bruhat orbit. Thus, one gets elements $k_w \in \boldsymbol{\mathcal{H}}$ for each $w \in W$ by pushing forward the constant sheaf under these inclusions. Of course, we need to specify what kind of pushforward we mean by this: we might mean the $*$, !, or intersection cohomology extension, i.e. the minimal extension $j_{!*}k = \mathrm{Image}(j_! k \to j_* k)$. In the end it won't matter which choice we make: the category of character sheaves we generate will be the same in any event, but in a sense it is nicest to choose the middle – or "IC" – extension, which has the property of preserving perversity. These elements *generate* the Hecke category as a monoidal category: they form a "basis". (TODO: Elaborate on this.)

3.3 Harish-Chandra Bimodules

Yet another way to view the finite Hecke category is as the category of *Harish-Chandra bimodules* with trivial central character. This perspective will be particularly useful in figuring out how to generalise the category to other central characters, acting on other pieces of the representation theory of G. The utility of this will become clear with we discuss the Beilinson-Bernstein correspondence later on.

Definition 3.6. Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a Cartan subalgebra of a Lie algebra. The *Harish-Chandra homomorphism* is an algebra homomorphism $\gamma\colon \mathfrak{z}(\mathfrak{g}) \to U(\mathfrak{t})$ defined as follows. The *Weyl vector* $\rho \in \mathfrak{t}^*$ is defined to be the sum of half positive roots

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$$

for the Lie algebra $\mathfrak{g}$. Let π be the projection $U(\mathfrak{g}) \to U(\mathfrak{t})$ (using Poincaré-Birkhoff-Witt to extend the projection $\mathfrak{g} \to \mathfrak{t}$ to enveloping algebras). The Harish-Chandra homomorphism is defined to be

$$\gamma(z) = \pi(z) - \rho(\pi(z)) \cdot 1.$$

The Harish-Chandra homomorphism is injective, and defines an isomorphism onto its image: the Weyl group invariant elements $U(\mathfrak{t})^W$. Given a weight λ of $\mathfrak{t}$, there is a corresponding ideal I_λ of $U(\mathfrak{t})$, generated by elements of the form $(t - \lambda(t))$. Under the Harish-Chandra homomorphism, this corresponds to a maximal ideal Z_λ of $\mathfrak{z}(\mathfrak{g})$. Let $U(\mathfrak{g})_\lambda$ be the quotient of $U(\mathfrak{g})$ by the ideal generated by Z_λ.

Definition 3.7. The category $\mathcal{HC}^\lambda$ with central character $\lambda \in \mathfrak{t}$ is the category of finitely generated bimodules for the ring $U(\mathfrak{g})_\lambda$ such that the adjoint action by $\mathfrak{g}$ is locally finite. That is (TODO: interpretation...)

Unpacking this definition in the case where $\lambda = 0$, we find the following. The ideal I_0 is the maximal ideal of the polynomial algebra $U(\mathfrak{t}) \cong \mathrm{Sym}(\mathfrak{t})$ generated by $\mathfrak{t}$ itself. It corresponds to the ideal $Z_0 =$ of $\mathfrak{z}(\mathfrak{g})$, with quotient (TODO: ...)

3.4 The Hecke category and Representations of G

One way of thinking about the finite Hecke algebra and its categorification is as an object that controls part of the representation theory of the group G, either as an analogue of the group algebra, or at the categorified level as part of the category of representations. By "part", I mean the collection of representations that appear as subrepresentations of the induced representation G/B. First we'll see how such representations appear as elements of the Hecke category, and then we'll argue that this gives a complete description.

Firstly, as we argued above, the Hecke category describes the endomorphisms of the category $D(G/B)$. The monoidal unit $\mathbf{1}$ of the Hecke category corresponds to the identity endomorphism of this category: tracing through the correspondences we can see that it corresponds to the skyscraper sheaf at $B\backslash 1/B$: the identity Bruhat orbit. The category $D(G/B)$ is a categorified analogue of the kG-module $k(G/B)$, and does indeed carry an action of the sheaf $D(G)$ by convolution.

What about the elements corresponding to the other Bruhat orbits? These elements are fundamental – they generate the category as a monoidal category in the sense that (TODO: ...) The element of the Hecke category corresponding to the skyscraper sheaf on BwB corresponds under our equivalence to the endomorphism

On a simpler level, given a splitting of an algebra $A \cong B \oplus C$ say, an endomorphism of A is supplied by the projection onto the subalgebra B. In the case of the group algebra of a finite group over a field of characteristic zero, or any subalgebra thereof, we have Maschke's theorem, implying that any subalgebra is complementable. Furthermore, Schur's lemma tells us that the only endomorphisms of an irreducible representation are given by constants, and there are no morphisms between distinct irreducible representations. Thus, the endomorphism algebra of kG is generated as a k-algebra by the projections onto simple submodules. In other words, the endomorphism algebra of a kG-module precisely encodes the data of its irreducible sub-representations: it is a free algebra on the irreducible submodules.

In what sense is this still true in the categorified setting? For the category $D(G/H)$ where G is a finite group, an analogous categorified story is still true. Indeed (TODO: ...)

3.5 Unipotent Subgroups and Twisted Hecke Categories

We produced the finite Hecke category $\boldsymbol{\mathcal{H}} = D(B\backslash G/B)$, heuristically at least, as the endomorphism category of $D(G/B)$. There is a meaningful sense in which this category arises from the category generated by the trivial local system on T by induction, just like the algebra $k(G/H)$ for a finite group G arose by inducing the trivial representation on H to G. Let's describe this idea.

First of all, the inclusion $T \subseteq B$ is a homotopy equivalence, or equivalently N is contractible. This means the pushforward map $\mathrm{Loc}(T) \to \mathrm{Loc}(B)$ is essentially surjective, thus gives an equivalence of categories. The trivial rank one local system on T is the local system whose monodromy around any loop is 1, corresponding to the character $1 \in X^\bullet(T)$. This object generates a full subcategory of $\mathrm{Loc}(T) \sim \mathrm{Loc}(B)$ by shifts, summands and extensions, so this is the subcategory whose eigenvalue around any loop is a matrix all of whose eigenvalues are 1: a *unipotent matrix*. Of course, this category includes monodromy given by non-trivial Jordan blocks:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now, let's consider the category generated by the induction of these elements from B to G. We'll discuss induction of characters in the categorified setting in section 4.1, but this is a subtler notion, namely induction of representations.

Let's consider the unipotent case first. We can consider the category of ∞-categories with a $D(G)$- or $D(B)$-module structure. There is a forgetful functor from the former to the latter, and by purely category theoretic considerations it has a left adjoint functor, which we might call *induction*. Indeed, this functor clearly preserves limits (TODO: probably the adjoint functor theorem). We can identify this left adjoint with a tensor product functor

$$\mathcal{M} \mapsto \mathcal{M} \otimes_{D(B)} D(G).$$

What do we mean by this? We're using the symmetric monoidal structure on the ∞-category of $D(B)$-module categories given by (TODO: ? Then we need to actually compute it)

(TODO: We want to get the category of D-modules on G/N which are strongly equivariant for the T action.)

Definition 3.8. Let $Y \to X$ be a principal T-bundle. The exponential map $\exp\colon \mathfrak{t} \to T$ combined with the action of T on Y defines a map

$$U(\mathfrak{t}) \to \Gamma(Y, D_Y)$$

We say an element $\mathcal{S} \in D(Y)$ is *weakly T-equivariant* if it is equivariant with respect to the action induced by this inclusion.

There is another inclusion $U(\mathfrak{t}) \to \Gamma(Y, D_Y)$ by differentiating the action map. Thus taking the difference of these two maps defines an action of $\mathfrak{t}$ on $\mathcal{S}$. We say $\mathcal{S}$ is *strongly T-equivariant* if this action is identically zero. We say $\mathcal{S}$ is *T-monodromic* with monodromy λ if this action is given by a character λ, via the exponential map. We'll denote the category of such monodromic sheaves by $D_\lambda(Y)$.

The example that we have in mind is the projection map $G/N \to G/B$, which is a T-torsor. So we can consider the category generated by all monodromic D-modules on G/N, for various λ. Call it $D_{\mathrm{mon}}(G/N)$. The category decomposes as

$$D_{\mathrm{mon}}(G/N) \cong \bigoplus_{\lambda \in X^\bullet(T)} D_{\widetilde{\lambda}}(G/N)$$

where $D_{\widetilde{\lambda}}(G/N)$ is the subcategory of monodromic D-modules where T acts with generalised eigenvalue λ. In some sense $D_{\widetilde{\lambda}}(G/N)$ is generated by $D_\lambda(G/N)$ under extensions.

Both here, and for a more general principal bundle $Y \to X$, there is an equivalence between monodromic D-modules on Y and *twisted* D-modules on X for some character λ. This is important, from the point of view of the Beilinson-Bernstein correspondence, which we'll discuss in 4.2 below.

We can enrich the heuristic equivalence

$$D(B\backslash G/B) \sim \mathcal{E}nd_{D(G)}(D(G/B)$$

to give a "twisted" version of the Hecke category. We should produce a category acting like the endomorphism category of the monodromic sheaves on G/N. At least, we should describe a category acting on the category of monodromic sheaves.

Definition 3.9. The *horocycle space* for the pair $B \subseteq G$ is the space

$$Y = ((G/N) \times (G/N))\,/T$$

where the torus T acts diagonally on the two factors. The space Y lives as a T-bundle over the product $G/B \times G/B$, by the natural projection.

There are natural equivalences of categories $D(B\backslash G/B) \sim D_G(G/B \times G/B)$ and

$$D((N\backslash G/N)/T) \sim D_G(((G/N) \times (G/N))\,/T)$$

. We'll discuss such equivalences further in 4.4.

Definition 3.10. The *twisted Hecke category* is the category of G-equivariant sheaves on the horocycle space, or equivalently

$$D_G(Y) \sim D((N\backslash G/N)\,/T).$$

This category acts on the category $D_{\mathrm{mon}}(G/N)$ on the left in a similar way to the Hecke category acting on $D(G/B)$: by a natural push-pull diagram. The diagonal T-quotient is necessary to ensure this action preserves the property of being monodromic.

What does this category look like? The space $(N\backslash G/N)\,/T$ is a T-bundle over $B\backslash G/B$, which we know is discrete as a space, with points indexed by the Weyl group W. So we can think of this version of the horocycle space as a collection of tori indexed by W, the tori being given by the Bruhat orbits

$$\iota_w \colon (N\backslash BwB/N)/T \hookrightarrow (N\backslash G/N)\,/T.$$

We can then produce elements of the twisted Hecke category by taking a rank one local system $\mathcal{L}$ on T corresponding to the character λ, and taking the pushforward $(\iota_w)_!(\mathcal{L})$ of this local system under the inclusion.

Proposition 3.11. This element $(\iota_w)_!(\mathcal{L})$ of the Hecke category preserves the category of λ-monodromic D-modules on G/N.

Proof. (TODO: ?) □

We should view these elements as generators for the whole twisted Hecke category, with the elements for a particular character λ viewed as the endomorphisms of the monodromic D-modules on G/N with monodromy λ (generating a subcategory: the twisted Hecke category for a fixed character $\boldsymbol{\mathcal{H}}_\lambda$).

4 Character Sheaves on a Lie Group

4.1 Induction and Restriction Functors

Before we continue with a definition of character sheaves, it will be useful to include a proper discussion of induction and restriction functors on this categorified level. First, recall the classical story for representations of finite groups, which we have been implicitly assuming in previous sections. Let G be a finite group, and let H be a subgroup of G. We can define a *restriction* functor from (left) kG-modules down to kH-modules. It has a two-sided adjoint functor called *induction*.

The group algebra kG is naturally a bimodule over itself. We'll produce functors between left kG-modules and kH-modules by viewing it either as a kG-kH bimodule, or as a kH-kG bimodule. A lot of the below discussion would make sense not only for infinite groups, but for a more general algebra and subalgebra: we just use the standard Hom tensor adjunction for an A-B bimodule.

Definition 4.1. The *restriction* functor of representations $\mathrm{Res}_H^G\colon \mathrm{Rep}(G) \to \mathrm{Rep}(H)$ is just the forgetful functor. There is a corresponding functor on characters. Alternatively, in terms of the group algebra, the restriction functor is the functor

$$M \mapsto kG \otimes_{kG} M \cong \mathrm{Hom}_{kG}(kG, M)$$

where in the first case the target inherits a left kH-module structure from the right kH-module structure of kG, and in the second case from the left kH-module structure of kG.

Definition 4.2. Define *induction* and coinduction functors $\mathrm{Rep}(H) \to \mathrm{Rep}(G)$ by

$$\mathrm{Ind}_H^G(M) = kG \otimes_{kH} M \qquad \mathrm{coInd}_H^G(M) = \mathrm{Hom}_{kH}(kG, M)$$

using the right and left kG-module structures on kG respectively. By construction, Ind_H^G and coInd_H^G are left and right adjoints respectively to the restriction functor Res_H^G, viewed in its two guises above.

For a *finite* group G the induction and coinduction functors actually coincide, giving a two-sided adjoint to the restriction functor. To see this isomorphism, we should apply a duality argument. Using finiteness of G, the group algebra is non-canonically isomorphic to its dual, so we have

$$\begin{aligned}\mathrm{Hom}_{kH}(kG, M) &\cong \mathrm{Hom}_{kH}(kG^*, M)\\ &\cong \mathrm{Hom}_{kH}(M, kG^*)^*\\ &\cong (M^* \otimes_{kH} kG^*)^* \cong M \otimes_{kH} kG\end{aligned}$$

as required. Thus, the restriction functor has both a left and a right adjoint, which we might call induction and coinduction functors.

We can also define induction for *characters* directly by means of a formula:

$$\mathrm{Ind}_H^G(\chi)(g) = \frac{1}{|H|} \sum_{k \in G, kgk^{-1} \in H} \chi(kgk^{-1}). \tag{1}$$

This formula makes sense for any class function on the group H.

There are various ways of extending these ideas to topological groups, often requiring some subtle analysis. However, if we work with categorical representations of algebraic groups then there is a very nice geometric picture in terms of derived functors on categories of sheaves. Let $H \leq G$ be a subgroup of an algebraic group. Suppose G acts on an algebraic variety X: so to generalise restriction and induction of class functions we should consider the category $D(X/G) \sim D_G(X)$ categorifying class functions.

Definition 4.3. The *restriction* functor $\mathrm{Res}_H^G\colon D_G(X) \to D_H(X)$ is the pullback $\pi^\dagger$ under the projection map of stacks

$$X/H \to X/G.$$

It has a right adjoint π_*, and a left adjoint $\pi_!$ (since π is proper,) which we will denote

$$\begin{aligned}\mathrm{Ind}_H^G &= \pi_!\colon D_H(X) \to D_G(X)\\ \text{and } \mathrm{coInd}_H^G &= \pi_*\colon D_H(X) \to D_G(X)\end{aligned}$$

respectively.

We should be able to recover the definitions given above in the case where G is finite by taking X to be a point, and applying the 0^{th} cohomology to the result.

There is an alternative description of these functors, given in [15] for instance. Consider the diagram

$$\begin{tikzcd} & G \times X \arrow[r, "q"] \arrow[dl, "\alpha"'] & G/H \times X \arrow[dr, "\pi_2"] & \\ X & & & X \end{tikzcd}$$

where α is the (inverse) action map $(g, x) \mapsto g^{-1} \cdot x$, π_2 is projection onto the second factor, and q is the quotient $(g, x) \mapsto (gH, x)$. We can use this to define the coinduction functor. For any $\mathcal{S} \in D_H(X)$, there is a unique $\widetilde{\mathcal{S}} \in D_G(G/H \times X)$ such that $\alpha^*(\mathcal{S}) = q^*(\widetilde{\mathcal{S}})$. Then

$$\mathrm{coInd}_H^G(\mathcal{S}) = (\pi_2)_*(\widetilde{\mathcal{S}}).$$

We should really prove that this agrees with the previous definition. For instance, if we check it defines a right adjoint to the restriction functor then the two definitions must agree by uniqueness of right adjoints. (TODO: Prove it.)

4.2 Beilinson-Bernstein Localisation

The Beilinson-Bernstein correspondence gives an equivalence between the category of representations of $\mathfrak{g}$ and a suitable category of D-modules on the flag variety G/B. We will give this correspondence, and interpret it in terms of the finite Hecke category $D(B\backslash G/B)$. We follow the discussion of [10] section 11.

Let T be a fixed maximal torus of G contained in a Borel B as usual. The torus T can be recovered by quotienting B by its unipotent radical N. Fix λ a positive root of $\mathfrak{g}$. We can build a line bundle $\mathcal{L}(\lambda)$ on the flag variety from this root by the following construction. The root λ describes a one-dimensional representation of T, and hence a one-dimensional representation ρ_λ of B by pulling back under the projection

$$B \twoheadrightarrow B/N = T.$$

Let $\mathcal{L}(\lambda)$ be the associated line bundle to the canonical B-bundle on G/B under this representation. The Borel-Weil-Bott Theore describes the representations of G given by the cohomology groups of this line bundle.

In other words, notice that G/N has the structure of a T-bundle over G/B. Inside $\mathcal{O}_{G/N}$, we have subsheaves according to the behaviours of functions in the fibre direction: let $\mathcal{O}^\lambda_{G/N}$ be the subsheaf of functions transforming according to the root λ in the fibre direction. Then $\mathcal{L}(\lambda)$ is the pushforward of this sheaf under the projection.

Let D_λ denote the sheaf of twisted differential operators on this line bundle $\mathcal{L}(\lambda)$. More precisely

Definition 4.4. If $\mathcal{V}$ is a locally free $\mathcal{O}$-module on a smooth variety X, the sheaf $D_X^{\mathcal{V}}$ of *$\mathcal{V}$-twisted differential operators* is defined inductively to be the following filtered sheaf of rings:

$$\begin{aligned} F_0(D_X^{\mathcal{V}}) &= \mathbb{C}_X \subseteq \mathcal{E}nd_{\mathbb{C}_X}(\mathcal{V}) && p = 0 \\ F_p(D_X^{\mathcal{V}}) &= \{\phi \in \mathcal{E}nd_{\mathbb{C}_X}(\mathcal{V}) \colon \phi f - f\phi \in F_{p-1}(D_X^{\mathcal{V}})\} && p > 0. \end{aligned}$$

If $\mathcal{V} = \mathcal{O}$, we recover the usual definition of D_X. We write D_λ for $D_{G/B}^{\mathcal{L}(\lambda)}$.

We can define a map of rings from $U(\mathfrak{g})$ to the global sections of this sheaf as follows. Let's first describe the untwisted story, following the notes [9] of Gaitsgory. In general, whenever G acts on a smooth algebraic variety X, we have an inclusion of Lie algebras by differentiating the action

$$\mathfrak{g} \hookrightarrow \Gamma(X, TX)$$

and hence a map of algebras

$$U(\mathfrak{g}) \to \Gamma(X, D_X).$$

So $\Gamma(X, D_X)$-modules are naturally also $U(\mathfrak{g})$-modules, i.e. we can view the global section functor as going

$$\Gamma\colon D_X\text{-mod} \to \mathfrak{g}\text{-mod}$$

on the underived level at least. For twisted modules, we would do the same thing, except we replace $\mathcal{O}$ with an equivariant vector bundle $\mathcal{V}$ in the differentiation. I'll omit the details, but we get a map of algebras $U(\mathfrak{g}) \to \Gamma(X, D_X^{\mathcal{V}})$.

We can construct a left adjoint Loc to this functor Γ, a "localisation" functor, i.e.

$$\mathrm{Loc}\colon \mathfrak{g}\text{-mod} \to D_X^{\mathcal{V}}\text{-mod}.$$

We can understand the localisation functor as just a tensor product: we send a $\mathfrak{g}$-module M to the base change

$$\underline{M} \otimes_{\underline{U(\mathfrak{g})}} D_X^{\mathcal{V}}$$

where underline denotes the constant sheaf with given fibre, and where $U(\mathfrak{g})$ acts on $D_X^{\mathcal{V}}$ by the algebra map described above.

In particular, when X is the flag variety G/B, we have maps of algebras for each root

$$\Phi_\lambda\colon U(\mathfrak{g}) \to \Gamma(X, D_\lambda).$$

Let's introduce some notation. For a root λ, let $\mathrm{Mod}_{qc}(D_\lambda)$ denote the abelian category of modules over the sheaf of twisted differential operators D_λ, quasi-coherent as $\mathcal{O}$-modules. Let $\mathrm{Mod}_{qc}(\mathfrak{g})$ denote the category of quasi-coherent $U(\mathfrak{g})$-modules. An object in this category is said to have *central character* χ_λ if the restriction of the action to the centre $\mathfrak{z}(\mathfrak{g})$ acts by the character

$$\begin{aligned} \chi_\lambda\colon \mathfrak{z}(\mathfrak{g}) &\to k \\ z &\mapsto \gamma(z)(\lambda) \end{aligned}$$

where γ is the *Harish-Chandra homomorphism* $\mathfrak{z}(\mathfrak{g}) \to U(\mathfrak{t})$. Denote the category of $U(\mathfrak{g})$-modules with central character χ_λ by $\mathrm{Mod}_{qc}(\mathfrak{g}, \chi_\lambda)$.

Theorem 4.5 (Beilinson-Bernstein)**.** The maps of algebras Φ_λ are surjective for every root λ. If λ is regular and antidominant, then the induced map

$$\mathrm{Mod}_{qc}(\mathfrak{g}, \chi_\lambda) \to \mathrm{Mod}_{qc}(D_\lambda)$$

is an equivalence of categories. For instance, if $\lambda = 1$, then the right-hand side is $D(G/B)$, and the left-hand side is all quasi-coherent $U(\mathfrak{g})$-modules with the centre acting trivially.

Here a weight being *regular antidominant* means that for all positive roots $\alpha \in \Delta^+$,

$$\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}_{\geq 0}.$$

We should be clear about what the functors are in each direction. We've spelled this out in the untwisted case only: To a $D_{G/B}$-module, we take its global sections, and use the action of G on the flag variety to give this a $U(\mathfrak{g})$-module structure. We would have to check that it has trivial central character. In the other direction, we can take a $U(\mathfrak{g})$-module, and use the action to base change to $D_{G/B}$. The functors would have to be modified if we picked a non-trivial central character.

Thus, in order to understand representations of G, or equivalently $U(\mathfrak{g})$-modules, we only need to understand D_λ-modules for various weights λ: i.e. modules for twisted differential operators on the flag variety G/B. These categories came up before. For instance, we're just talking about the category $D(G/B)$, which is controlled by its endomorphism category: the finite Hecke category $\boldsymbol{\mathcal{H}}$. In the twisted case, as we will discuss, twisted D-modules coincide with monodromic D-modules on G/N, so the endomorphism category of the sum of *all* the twisted categories is (at least heuristically) the full twisted Hecke category. This is why the Hecke categories were so important for us to understand: they contain the entire representation theory of the group G.

Remark 4.6. The decategorified analogue of this is the fact – as described in the introduction – that all representations of a finite group arise as summands of the representations obtained by induction from the trivial representation on subgroups. In particular, they all arise as summands of the regular representation.

4.3 Beilinson-Bernstein and Twisted Hecke Categories

We've already introduced the finite Hecke category $\mathcal{H}$ and its twisted analogues $\mathcal{H}_\lambda$. In the untwisted case, the finite Hecke category describes endomorphisms of the category of D-modules on the flag variety G/B, which, as described above, is equivalent to the category of representations of $\mathfrak{g}$ with trivial central character. What is the analogous statement for other characters? Recall that G/N is naturally a T-bundle over G/B, and that we say a D-module on this total space is λ-*monodromic* if T acts locally finitely with (TODO: generalised?) monodromy λ.

Proposition 4.7. The category D_λ of λ-twisted differential operators on the flag variety G/B is equivalent to the category of λ-monodromic D-modules on G/N.

Proof. This is discussed in section 2.5 of [2]. We'll describe functors between the two categories (TODO: ...) □

4.4 The Horocycle Transform

The horocycle transform is the integral transform giving us a notion of "trace" of a representation, or more precisely an element of the Hecke category. By a natural push-pull construction similar to the one we defined on finite groups in 1.1 we give a functor from the Hecke category to the category of adjoint equivariant sheaves on the group G.

Definition 4.8. The *horocycle transform* (for trivial central character) is the functor $H = p_* q^!$ given by pushing and pulling along the correspondence

$$\begin{tikzcd} & G/_{ad}\,B \arrow[dl, "p"'] \arrow[dr, "q"] & \\ G/_{ad}\,G & & B\backslash G/B. \end{tikzcd}$$

Equivalently, we could look at G-equivariant sheaves on the following correspondence:

$$\begin{tikzcd} & G \times G/B \arrow[dl] \arrow[dr] & \\ G & & G/B \times G/B \end{tikzcd}$$

where G acts diagonally on the products, and where the map $G \times G/B \to G/B \times G/B$ is via the action on the first factor, and projection on the second factor. Showing that the two correspondences are equivalent is not a difficult exercise. All one has to do is note that G-diagonal orbits on $G/B \times G/B$ are the same as points in $B\backslash G/B$ (indeed, this is true for any group and subgroup). Similarly G-diagonal orbits on $G \times G/B$ are the same as points of $G/_{ad}\,B$.

Remark 4.9. The name *horocycle transform* comes from a kind of *Radon transform* involving integrating over horocycles in a hyperbolic space. Radon transforms are a rather general notion of integral transform, between functions on a set X, and functions on a collection Y of subsets of X: one sends a function f on X to the function

$$\hat{f}(S) = \int_S f(x)dx \quad \text{for } S \in Y$$

given by integrating along the preferred subsets. Rather more interestingly, many such Radon transforms are invertible. How does the integral transform described above fit into this framework? Here the role of X is played by the group G, and the role of Y is played by the space $G/B \times G/B$, thought of as the variety of pairs of Borel subgroups in G. This Y is sometimes called the *horocycle space*. We can see why by looking at $SL(2)$, or more concretely at the *real* group $SL(2,\mathbb{R})$. In this case, the quotient G/B can be thought of as the boundary circle of the upper half plane: a homogeneous space for $PSL(2,\mathbb{R})$, i.e. a less refined quotient by a maximal compact. Thus the product $G/B \times G/B$ can be thought of as the collection of pairs of points on this boundary, and any two such points have a unique horocycle connecting them.

Definition 4.10. The *generalised horocycle transform* is the functor $H = p_* q^!$ given by pushing and pulling along the correspondence

$$\begin{tikzcd} & G/_{ad}\, B \arrow[dl, "p"'] \arrow[dr, "q"] & \\ G/_{ad}\, G & & (N\backslash G/N)\,/T. \end{tikzcd}$$

The *central character* of a monodromic sheaf $\mathcal{S} \in D(G/N)$ is identified as described in 3.5: we described a splitting of this category into a sum of categories of sheaves with generalised monodromy λ. The twisted Hecke category itself is generated by elements preserving various summands for various λ, so in a sense this horocycle transform is a sum of twisted horocycle transforms for different twists. (TODO: Describe alternatively with equivariant sheaves on $G \leftarrow G \times G/B \to (G/N \times G/N)/T$. Describe the map on the right.)

Example 4.11. In our discussion of the finite Hecke category, we discussed the sheaves $k_w \in \boldsymbol{\mathcal{H}}$ coming from the inclusions of the Bruhat tori for each Weyl group element. Applying the horocycle transform to these objects, we get the "Lusztig sheaves" K^w_k corresponding to the constant local system on the torus. We'll describe these sheaves in another way in section 4.5 below. Similarly, we produce the more general sheaves $K^w_{\mathcal{L}}$ by applying this functor to the elements of the twisted Hecke category given by a local system $\mathcal{L}$ on the torus over some Bruhat orbit.

This example suggests the real reason for the definitions we've made. We've been trying to define character sheaves by coming up with a categorified version of "trace" that we can apply to certain categorical representations to produce equivariant sheaves on the group. The argument we made via Beilinson-Bernstein implied that sufficiently rich notions of representations of G are supplied via the finite Hecke category and its twisted analogues. This horocycle transform is exactly the categorical trace we were looking for. Thus, summarising the discussion up to this point we have the following definition of a character sheaf:

Definition 4.12. The category Ch_0 of *unipotent character sheaves* on the group G is the full subcategory of $D(G/_{ad}\, G)$ cogenerated by the image of the horocycle transform H (i.e. the closure of this image under small colimits). (TODO: or finite?)

Similarly, the category Ch_λ of *character sheaves* with central character χ on G is the full subcategory of $D(G/_{ad}\, G)$ cogenerated by the image of the twisted horocycle transform H_λ. The category Ch of all character sheaves is the subcategory generated by all these images, or equivalently the sum of the categories Ch_λ.

Let's justify that last statement. Given an arbitrary character sheaf, we can decompose it into a sum of character sheaves with different central characters:

$$\mathcal{S} = \oplus_\lambda \mathcal{S}_\lambda$$

with $\mathcal{S}_\lambda \in \mathrm{Ch}_\lambda$. To see this, look at the action of the centre $\mathfrak{z}(\mathfrak{g})$ on $\mathcal{S}$, using the action of G to give $\mathcal{S}$ a $U(\mathfrak{g})$-module structure as in 4.2. (TODO: Elaborate on this.)

Finally, why is the horocycle transform a kind of categorical trace? We can see that it directly generalises the universal trace map for the endomorphisms of a representation of a finite group, as described in the introduction. In [3], Ben-Zvi and Nadler prove that the category Ch is naturally equivalent to the *categorical Hochschild homology* (abelianisation) of the Hecke category $\boldsymbol{\mathcal{H}}$ with *universal trace* given by the horocycle correspondence. What does this mean? One way of describing it is via the language of topological quantum field theory (TQFT). (TODO: Explain why in an (un)oriented TQFT taking values in a Morita category the Hochschild homology and cohomology are forced to be isomorphic, and arise as the object of $\mathrm{Mor}(1, 1)$ assigned to the circle. Then the disc gives a 2-bordism, hence a trace map.)

4.5 Lusztig's sheaves $K^w_{\mathcal{L}}$

In this section, we'll produce our first examples of character sheaves, which will actually turn out to be a family of generators for the category. Let $T \subseteq G$ be an embedded maximal torus. For each element $w \in W$, there is a translated torus wT contained in the Bruhat orbit BwB as described above in 3.1

In classical language, let $i_w \colon wT \to G$ be the inclusions of the Bruhat tori, one for each Weyl group element. Let $\mathcal{L}$ be a local system on the torus wT. Then we can define Lusztig's K-sheaves as follows. We can identify rank one local systems on the torus wT with irreducible B adjoint equivariant local systems on the Bruhat stratum BwB, by pullback under the inclusion $i \colon wT \hookrightarrow BwB$. The pullback $i^\dagger$ sends an irreducible B-equivariant object to a rank one local system on wT, is fully faithful and essentially surjective, so defines an equivalence. (TODO: justify this.) Suppose our local system $\mathcal{L}$ on wT is identified with $i^\dagger L$ for L a local system on BwB. Then define

$$\widetilde{K}^w_{\mathcal{L}} = (j_w)_! L$$

where j_w is the inclusion $BwB \hookrightarrow G$. This sheaf is equivariant for the B adjoint action by construction, so defines an element of $D_B(G)$.

Definition 4.13. The sheaves $K^w_{\mathcal{L}} \in D(G/_{ad}\,G)$ are defined to be the induction $\mathrm{coInd}^G_H \widetilde{K}^w_{\mathcal{L}}$ of these B-equivariant sheaves.

For technical reasons, we'll restrict attention to those local systems $\mathcal{L}$ on T which are fixed by w. That is, $w^*(\mathcal{L}) = \mathcal{L}$. This ensures that it really corresponds to a B bi-equivariant sheaf on BwB, rather than merely an adjoint equivariant object. (TODO: I'm not sure this is right. Rewrite this paragraph.)

Lemma 4.14. The sheaves $\widetilde{K}^w_{\mathcal{L}}$ generate the category of B-equivariant sheaves on G constructible with respect to the Bruhat stratification.

Proof. We can work with the sheaves $(j_w)_! L$ as an alternative construction of $\widetilde{K}^w_{\mathcal{L}}$, where L are B adjoint equivariant local systems on BwB. The pushforward gives a sheaf on G supported exactly on the relevant Bruhat orbit, and locally constant there, so certainly $\widetilde{K}^w_{\mathcal{L}}$ are Bruhat constructible. We can construct any Bruhat-constructible equivariant sheaf by iteratively taking shifts and cones of these "costandard objects", as described in [16] Proposition 4.3.1. for instance. □

We can rephrase this in more categorical language, in terms of the horocycle transform: for instance if $\mathcal{L}$ is trivial, the sheaves $K^w_{\mathcal{O}}$ correspond to lifts of the family k_w of generators for the finite Hecke category. First, we must note the ways in which this story can be generalised to include other parts of the representation theory of G. This will correspond to working with the unipotent horocycle transform only. (TODO: Complete this, explaining why this is an application of the horocycle transform machinery.)

Proposition 4.15. The category of character sheaves is generated by the collection of sheaves $K^w_{\mathcal{L}}$ for $w \in W$ and $\mathcal{L}$ a one-dimensional local system on T. More classically, the irreducible objects in the heart of the derived category of character sheaves are precisely the irreducible summands of the sheaves $K^w_{\mathcal{L}}$.

Proof. This is an immediate consequence of facts we've already observed. In particular, lemma 4.14 implies that (TODO: ...) □

4.6 Example: $SL_2(\mathbb{C})$

Let's try to compute the irreducible character sheaves for the group $G = SL_2(\mathbb{C})$. I'll use the language of perverse sheaves rather than regular holonomic D-modules. Let B be the group of upper triangular matrices in SL_2, containing the maximal torus $T \cong \mathbb{C}^\times$ of diagonal matrices, and with unipotent radical $N \cong \mathbb{C}$, the group of upper triangular matrices with 1 on the diagonal. The flag variety G/B is isomorphic to $\mathbb{P}^1$, and the action of $B \cong \mathbb{C}^\times \times \mathbb{C}$ by left multiplication has orbits $\mathring{\mathbb{A}}^1 \subseteq \mathbb{P}^1$ and $\{\infty\}$. Equivalently, the Bruhat decomposition of G has two strata: B and $G \setminus B$ corresponding to the elements of the Weyl group $W = \{e, w\}$.

We must compute and decompose the Lusztig sheaves K^e_λ, K^w_λ, where $\lambda \in \mathbb{C}^\times$. We'll start with the unipotent character sheaves, i.e. $\lambda = 0$. So we must apply the horocycle transform to the following two B-equivariant sheaves on $\mathbb{P}^1$:

$$\begin{array}{ll} \delta_\infty & \text{the skyscraper sheaf at } \infty \\ \mathbb{C}_{\mathbb{P}^1} & \text{the constant sheaf.} \end{array}$$

the first sheaf is the pushforward of the constant sheaf on the orbit $\{\infty\}$ (any pushforward). The latter comes from computing that the $*$ pushforward arises as an extension

$$\mathbb{C}_{\mathbb{P}^1} \to j_*(\mathbb{C}_{\mathbb{A}^1}) \to \delta_\infty[-1] \xrightarrow{+1},$$

where j is the inclusion of the orbit $\mathbb{A}^1$. We could instead have taken the ! pushforward (extension by zero), or the !$*$ pushforward (IC extension): they will all generate the same category in the end. The IC extension in particular is just this constant sheaf.

Consider now the G-equivariant horocycle correspondence

$$\begin{tikzcd} & G \times G/B \arrow[dl, "p"'] \arrow[dr, "q"] & \\ G & & G/B \times G/B \end{tikzcd}$$

Recall the maps are given by the projection $p = \pi_1 \colon G \times G/B \to G$, and the map $q = (\alpha, \pi_2) \colon G \times G/B \to G/B \times G/B$ where α is the action map. The skyscraper sheaf δ_∞ is the pushforward of the constant sheaf on $B\backslash B/B \subseteq B\backslash G/B$, or as a G-equivariant sheaf, the !-pushforward of the diagonal $G/B \overset{\Delta}{\hookrightarrow} G/B \times G/B$. So in this diagram we must push and pull this diagonal sheaf, and the constant sheaf on $G/B \times G/B$.

The constant sheaf is easy. We just get the constant sheaf on G with fibre $H^*(G/B) \cong \mathbb{C} \oplus \mathbb{C}[-2]$. This clearly decomposes into two irreducible objects: shifts of the constant character sheaf. For the diagonal, form the pullback square

$$\begin{tikzcd} & G \times G/B \arrow[dl, "p"'] \arrow[dr, "q"] & & \widetilde{G} \arrow[ll, hook'] \arrow[dr] & \\ G & & G/B \times G/B & & G/B \arrow[ll, hook', "\Delta"'] \end{tikzcd} .$$

We find the pullback of q along Δ is the projection map $\widetilde{G} \to G/B$, where

$$\widetilde{G} = \{(g, B') \in G \times G/B \colon g \in B'\}$$

is the Grothendieck-Springer space. The composite map $\pi \colon \widetilde{G} \to G$ is the Grothendieck-Springer map, restricting to the Springer resolution of the nilpotent cone. The sheaf K_0^e is the pushforward $\pi_! \mathbb{C}_{\widetilde{G}}$, so we must compute this.

Remark 4.16. In this discussion, we haven't used the specific fact that $G = SL_2$. The sheaf K_0^e is computed in such a way for any group. It is called the *Grothendieck-Springer sheaf*, and sometimes denoted $\mathcal{G}$.

So lets try to understand this Grothendieck-Springer sheaf: in particular, its decomposition into a sum of irreducible objects. First, consider the regular semisimple locus inside G, and its preimage $\pi|_{rs} \colon \widetilde{G}^{rs} \to G^{rs}$. This map is a W-fold cover for any G. Indeed, a regular semisimple element has centralizer a maximal torus in G by definition, and is contained inside precisely those Borels which contain that torus. It is easy to check this explicitly for SL_2. In this example we can also compute the Grothendieck-Springer fibres for other elements. The elements ± 1 are central, so contained in every Borel: the fibres over them are G/B. Finally, the unipotent elements are each contained in a unique Borel, since they have trivial centralizer.

This tells us the stalks of our sheaf $\mathcal{G}$:

- The stalk over ± 1 is $H^*(G/B) \cong \mathbb{C} \oplus \mathbb{C}[-2]$.

- The stalk over a unipotent element is $\mathbb{C}$.
- The stalk over a regular semisimple element is $\mathbb{C} \oplus \mathbb{C}$.

The pull back to each of these strata is a local system. On the central and unipotent elements it is the trivial local system, and on the regular semisimple elements it has monodromy corresponding to the regular representation of W (which is exhibited as a subgroup of the fundamental group by our explicit W-fold cover). To see this, we use the fact that the projection π is *small* [1], and apply a simple form of the *decomposition theorem*, which tells us in this case that we can compute $\mathcal{G}$ as an IC (minimal) extension

$$\pi_! \mathbb{C}_{\widetilde{G}} = IC((\pi|_{rs})_* \mathbb{C}_{\widetilde{G}_{rs}}).$$

This sheaf $\mathcal{G}$ splits into two irreducible pieces when restricted to the regular semisimple locus, corresponding to the two irreducible representations of $\mathbb{Z}/2\mathbb{Z}$. Taking the minimal extensions of these summands we decompose $\mathcal{G}$ into irreducible pieces:

$$\mathcal{G} = \mathbb{C} \oplus \mathcal{F}$$

where $\mathcal{F}$ has monodromy on the regular semisimple locus corresponding to the sign representation of W, has zero stalks on the unipotent stratum, and has stalk $\mathbb{C}[-2]$ at ± 1.

The story is similar but more complicated for non-trivial monodromy. Choose a general $\lambda \in \mathbb{C}^\times$. We're now considering the twisted horocycle transform

$$\begin{tikzcd} & G \times G/B \arrow[dl, "p"'] \arrow[dr, "q"] & \\ G & & (G/N \times G/N)/T \end{tikzcd}$$

where the map q is still given by the action on one component, and the projection on the other, i.e. $(g_1, g_2 B) \mapsto (g_1 g_2 N, g_2 N)T$. The space $(G/N \times G/N)/T$ is a T-torsor over $G/B \times G/B$, and we need to apply the horocycle transform to the sheaf on either the total space or the (trivial) torsor over the diagonal $G/B \subseteq G/B \times G/B$ with global monodromy in the fibre given by λ (Note: this diagonal is not the same as the diagonally embedded $G/N \hookrightarrow G/N \times G/N$ modulo the T action). Computing the pullback of q along the diagonal again, we produce the Grothendieck-Springer space $\widetilde{G} \subseteq G \times G/B$ as before.

As before, we'll first consider the Lusztig sheaves K_λ^w where w is the non-identity element of the Weyl group. This only makes sense if $\lambda = \pm 1$, as these are the only characters which are fixed by the Weyl group action on $\mathbb{C}^\times$ (by inversion). We've already seen K_1^w, and we'll say something about K_{-1}^w a bit later.

Now, we compute the analogue of the Grothendieck-Springer sheaves: the sheaves K_λ^e. As we saw, this corresponds to pulling back certain local systems on a T-torsor over G/B to sheaves on $\widetilde{G}$, and pushing forward along the Grothendieck-Springer map. If we take $\lambda = 1$ we recover the previous result. More generally, we are pulling back along the map $\widetilde{G}$ to the diagonal T torsor on G/B. The resulting sheaf on $\widetilde{G}$ is the local system with monodromy λ (note that $\widetilde{G}$ is homotopy equivalent to $G/B \times B$, so has the same fundamental group as $T = \mathbb{C}^\times$). To see this, we just note that the map $\widetilde{G}$ to the diagonal is an N-fibration, hence a homotopy equivalence.

We can see that the fibres of the pushforward sheaf are still computed by the cohomology of the Grothendieck-Springer fibres, but the monodromy on the strata will be different. As before we can apply the decomposition theorem

$$K_\lambda^e = IC((\pi|_{rs})_* \mathcal{L}_\lambda|_{rs})$$

where $\mathcal{L}_\lambda$ is the local system on $\mathcal{G}$ we produced above. Let's first consider the regular semisimple locus of G. As we know, $\widetilde{G}^{rs} \to G^{rs}$ is a W-fold cover. We can choose a representative for the generator for $\pi_1(\widetilde{G})$ contained within

[1]A map $f\colon X \to Y$ is called *semismall* if Y is stratified such that f restricts to a locally trivial fibration on strata, and for each stratum Y_α and $y \in Y_\alpha$, we have $\dim Y_\alpha + 2 \dim f^{-1}(y) \leq \dim X$. It is called *small* if the inequality is strict on all strata apart from the open stratum.

$\widetilde{G}^{rs}$, by choosing a representative path contained in $T \cap G^{rs}$, and taking any lift to a loop in $\widetilde{G}$. This also interacts with the generator of the copy of $W \subseteq \pi_1(G^{rs})$, so we have $\mathbb{Z} \subseteq \pi_1(\widetilde{G}^{rs})$, and $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} \subseteq \pi_1(G^{rs})$.

We should check which extension $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ arises in the fundamental group of G^{rs}. Let γ be the path we constructed in $T \cap G^{rs}$. Choose a basepoint $g \in G^{rs}$, and let δ be a path between the two preimages of g in $\widetilde{G}^{rs}$. Let γ_1 and γ_2 be lifts of γ to loops based at these two points. Then

$$\delta^{-1} \cdot \gamma_1 \cdot \delta \sim \gamma_2^{-1}.$$

Indeed, we can construct a homotopy in $\widetilde{G}^{rs}$ between these paths. Let B_1, B_2 be the two Borels containing the regular semisimple element g. To be precise, they are the upper triangular and lower triangular Borels respectively. The path δ can be taken to be a rotation

$$\delta_t(g, hB) = \left(\begin{pmatrix} \cos \pi t & -\sin \pi t \\ \sin \pi t & \cos \pi t \end{pmatrix} g, \begin{pmatrix} \cos \pi t & -\sin \pi t \\ \sin \pi t & \cos \pi t \end{pmatrix} hB \right)$$

postcomposed with a path linking $-g$ with g in B_2. Now, look at the path $\gamma_1 \cdot \delta$. This is homotopic to $\delta \cdot$ (a lift of γ along δ), but this lift is homotopic to γ_2^{-1} as required. (TODO: This is very sketchy. Rewrite more clearly.) Thus the extension is the affine Weyl group W_{aff}, which in this case is just the infinite dihedral group

$$\langle a, b | b^2 = baba = 1 \rangle.$$

Now, on the regular semisimple locus, K_λ^e is a 2d local system, given by pushing forward the local system on $\widetilde{G}^{rs}$ with monodromy λ. We can think of this as inducing the representation $\mathbb{C}_\lambda$ of $\mathbb{Z}$ to a 2d representation of $\mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}$. The resulting representation is the $\mathbb{Z}$-representation $\mathbb{C}_\lambda \oplus \mathbb{C}_{\lambda^{-1}}$ with $\mathbb{Z}/2\mathbb{Z}$ acting by swapping the factors. That is, we have monodromy

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ in the } \mathbb{Z} \text{ direction.}$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ in the } \mathbb{Z}/2\mathbb{Z} \text{ direction.}$$

Indeed, these matrices do satisfy the appropriate relations describing a representation of the group $\mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}$. These matrices commute if and only if $\lambda = \pm 1$. If this is not the case, this representation is irreducible, so its IC extension defines an irreducible character sheaf. We've already studied the case $\lambda = 1$, so the remaining case is $\lambda = -1$. In this case the local system splits as a sum of two rank one local systems with monodromies $(1, -1)$ and $(-1, -1)$. So taking their IC extensions we get a pair of character sheaves with central character -1, say $\mathcal{F}_+^{-1}$ and $\mathcal{F}_-^{-1}$. What do these various IC extensions look like? We have different behaviours on the two components of the unipotent cone.

Recall we can compute the IC extension by a sequence of $*$ pushforwards and truncations. Let $U^\pm$ be the two components of the unipotent cone, not including ± 1. If $u \in U^+$, then a neighbourhood of u in G^{rs} has fundamental group $\mathbb{Z}$. The restriction of the local system $\mathcal{F}_+^{-1}$ has monodromy 1 around this loop, and the restriction of $\mathcal{F}_-^{-1}$ has monodromy -1. If $u \in U^-$ instead, these monodromies are reversed. By computing the intersection cohomology of $\mathbb{C}$ with coefficients in these local systems on $\mathbb{C}^\times$, we see $\mathcal{F}_+^{-1}$ extends to a local system on U^+, and extends by zero across U^-. For $\mathcal{F}_-^{-1}$ the reverse occurs. We compute the stalks at ± 1 similarly: $\mathcal{F}_+^{-1}$ has stalk $\mathbb{C}$ at 1 and $\mathbb{C}[-2]$ at -1, and $\mathcal{F}_-^{-1}$ has the opposite. (TODO: Justify this, especially the monodromy statement.)

Now, the components $U^\pm$ both have fundamental group $\mathbb{Z}/2\mathbb{Z}$ (indeed, they are homotopy equivalent to the link of the singularity at ± 1, which is the unit tangent bundle on S^3, i.e. $\mathbb{RP}^3$). Can we produce character sheaves with non-trivial monodromy around U^+ or U^-? In fact we can. Let $\widetilde{U}^\pm$ denote the universal covers. Pushing forward the constant sheaf on one of these universal covers gives a rank two local system on $U^\pm$ with monodromy given by the regular representation of $\mathbb{Z}/2\mathbb{Z}$. Take the summand corresponding to the sign representation, and extend by zero to a sheaf on G. It is important to note that this extension by zero agrees with the IC extension to $\overline{U}$ for this non-trivial local system, therefore the resulting sheaf remains perverse. This would not be true if we took the trivial local system, in which case the IC extension would be the constant sheaf on $\overline{U}$. We'll call the two resulting sheaves $\mathcal{G}_+$ and $\mathcal{G}_-$. If we compute the singular support of $\mathcal{G}_+$ say, we find that it consists of conormal vectors to points $u \in U^+$:

$$\{(u, X) \in U^+ \times \mathfrak{g}^* \colon X \in N_u^* U^+\}.$$

The conormal vectors to the unipotent stratum are nilpotent, and thus the sheaves $G^\pm$ are also irreducible character sheaves, supported only on the unipotent locus. This last statement follows by checking it holds in the nilpotent cone in the Lie algebra, which is isomorphic to U^+ by the exponential map. The fibre of the normal bundle to the cone $\{x^2 - y^2 - z^2 = 0\}$ at a point (x, y, z) is spanned by $(x, -y, -z)$ which is also on the cone.

We know these sheaves should appear as summands of some Lusztig sheaf $K_{\mathcal{L}}^w$. Where do these character sheaves $\mathcal{G}_\pm$ appear? They must be summands of the sheaf K_{-1}^w for w the non-identity element of the Weyl group.

Let's explain the singular support calculation in some detail, beginning with a simpler computation.

Example 4.17. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ of the group $G = SL_2(\mathbb{R})$. Let $\mathcal{N} \subseteq \mathfrak{g}$ be the nilpotent cone. Let $\mathcal{L}_\lambda$ be the rank one local system on $\mathcal{N} \setminus 0$ with monodromy λ on each component (after choosing some orientation on the components, say). Let $j_! \mathcal{L}_\lambda$ be the extension by 0 of this sheaf to a sheaf on $\mathfrak{g}$: I'll describe the singular support of this sheaf.

At points $n \in \mathcal{N} \setminus 0$, we have

$$SS(j_! \mathcal{L}_\lambda) \cap T_n^* \mathfrak{g} = N_n^* \mathcal{N}$$

the conormal vectors to $\mathcal{N}$ at n. So it remains to compute $SS(j_! \mathcal{L}_\lambda) \cap T_0^* \mathfrak{g}$. We use the following definition of singular support (Kashiwara-Schapira Prop 5.1.1 3))

Definition 4.18. Let X be a vector space. Let $\mathcal{F} \in D^b(X)$ be a constructible complex. A covector $\xi \in T_0^* X$ is called *regular* for $\mathcal{F}$ if there exists a neighbourhood U of 0, an $\varepsilon > 0$ and a proper closed convex cone γ in X such that if $0 \neq v \in \gamma$,

$$\langle v, \xi \rangle < 0$$

(i.e. γ is contained in the half space determined by this condition) all such that the natural map

$$R\Gamma(\{x \colon \langle x, \xi \rangle \geq -\varepsilon\} \cap (\gamma + u); \mathcal{F}) \to R\Gamma(\{x \colon \langle x, \xi \rangle = -\varepsilon\} \cap (\gamma + u); \mathcal{F})$$

is an isomorphism for all $u \in U$. If this does not occur, ξ is called *singular*, and lies in the singular support.

Let's try to parse this definition in our specific situation. Fix a $\xi \in T_0^* \mathfrak{g}$. There will be two distinct cases: either the hyperplane $\{x \colon \langle x, \xi \rangle = 0\}$ contains a line in $\mathcal{N}$, or it doesn't. The crucial difference lies in how the hyperplane $L_\varepsilon = \{x \colon \langle x, \xi \rangle = -\varepsilon\}$ meets the nilpotent cone.

First suppose it doesn't. Choose γ not meeting $\mathcal{N} \setminus 0$ at all. Choose a ball U, so that translates of γ are either disjoint from $\mathcal{N} \setminus 0$, meet it in a closed disc, or meet it in a closed contractible neighbourhood of $0 \in \mathcal{N}$. Now choose some ε, and look at the $L_\varepsilon \cap (\gamma + u)$ for $u \in U$. In the three cases above, these intersections look like either the empty set, a contractible set, or a circle respectively. The resulting restriction maps then are clearly isomorphisms in the first two cases, but only an isomorphism in the last case if $\mathcal{L}_\lambda$ is a non-trivial local system. This is because $\{x \colon \langle x, \xi \rangle \geq -\varepsilon\}$ has two components, so H^0 is two dimensional. The same problem will occur for any choice of U, γ, ε.

Now, suppose the hyperplane meets the nilpotent cone in a line. Choose any U, ε, γ as above. Choose a $u \in U$ so that 0 lies in the interior of the shifted cone $\gamma + u$. Then $L_\varepsilon \cap (\gamma + u)$ meets $\mathcal{N} \setminus 0$ in a contractible set, but $\{x \colon \langle x, \xi \rangle \geq -\varepsilon\} \cap (\gamma + u)$ meets $\mathcal{N} \setminus 0$ in a neighbourhood of 0 in $\mathcal{N}$, so the restriction map cannot be an isomorphism for any local system $\mathcal{L}_\lambda$.

In summary, if $\mathcal{L}_\lambda$ is a constant local system, every covector at 0 is singular, but if $\mathcal{L}_\lambda$ is a non-constant local system, all covectors such that L_ε meets $\mathcal{N}$ in a line are singular, but the other covectors are regular. Upon identifying the tangent and cotangent bundles (via the killing form on $\mathfrak{g}$), this says the fibre of the singular support above 0 is precisely $\mathcal{N}$ itself.

Example 4.19. Now, instead let $\mathfrak{g}$ be the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, with nilpotent cone $\mathfrak{N}$. This cone has a single singular point at 0, which is a Morse singularity with link homotopy equivalent to $\mathbb{RP}^3$. Thus the punctured cone $\mathcal{N} \setminus 0$ has fundamental group $\mathbb{Z}/2\mathbb{Z}$. There are two rank one local systems on this space, which we will denote $\mathcal{L}_\pm$ (so $\mathcal{L}_+$ is the trivial local system, and $\mathcal{L}_-$ is the local system corresponding to the sign representation of $\mathbb{Z}/2\mathbb{Z}$). Take the extensions by zero of these sheaves to produce sheaves $p_!\mathcal{L}_\pm$ on $\mathfrak{g}$. What are their singular supports?

As before, we know the fibres of the singular support over points $n \in \mathcal{N} \setminus 0$

$$SS(j_!\mathcal{L}_\lambda) \cap T_n^*\mathfrak{g} = N_n^*\mathcal{N}$$

and must compute the fibre over 0. We'll use a similar method to before. So for a fixed $\xi \in T_0^*\mathfrak{g}$ let $L_\varepsilon = \{x \colon \langle x, \xi\rangle = -\varepsilon\}$ be the hypersurface corresponding to some $\varepsilon \geq 0$, and let $H_\varepsilon = \{x \colon \langle x, \xi\rangle \geq -\varepsilon\}$ be the half space bounded by this hypersurface. We allow $\varepsilon = 0$ to produce a hyperplane L_0 bounding a half space H_0. Again, there will be two cases: either L_0 meets the nilpotent cone $\mathcal{N}$ in a (complex) line, or it only meets it at 0. The latter is the generic case (transverse intersection). Identifying the covector ξ with a vector via the killing form, the former case occurs whenever this vector lies in the nilpotent cone.

We must investigate how the hypersurfaces L_ε meet the nilpotent cone in these two cases. (TODO: ...)

5 Singular Support of Character Sheaves

In this section, we'll describe a characterisation of character sheaves by their singular support, due to Mirković and Vilonen [15]. Roughly, a constructible sheaf on $G/_{ad}\,G$ is a character sheaf if and only if its singular support is contained in $G \times \mathcal{N} \subseteq T^*G$, where $\mathcal{N}$ denotes the nilpotent cone. We'll explain this, and make it precise, beginning with an introduction to the notion of the singular support of a constructible sheaf.

5.1 Singular Support

Let k be an algebraically closed field of characteristic zero. The assumption on the characteristic is important: there isn't a well-developed notion of singular support in positive characteristic. Let X be a smooth algebraic variety over k: we'll have $X = G$ in mind. For intuition, we'll often give geometric descriptions in the case where k is $\mathbb{C}$.

In this setting, we can identify the category of constructible sheaves on X with the derived category of D-modules on X by the Riemann-Hilbert correspondence. Precisely:

Theorem 5.1. If X is a smooth algebraic variety over a field k of characteristic 0, then there is an equivalence of triangulated categories

$$D_c^b(X) \cong D^b(D\text{-mod}_{\mathrm{rs}}(X)),$$

where $D\text{-mod}_{\mathrm{rs}}(X)$ is the category of holonomic D-modules on X with regular singularities. Furthermore, the equivalence commutes with the usual six functors $f^\dagger, f_*, f^!, f_!, \mathcal{H}om, \boxtimes$, where $f \colon X \to Y$ is a morphism of smooth varieties over k.

There is a natural t-structure on each category, exchanged by the Riemann-Hilbert correspondence (de Rham functor): the abelian category of *perverse sheaves* on X is equivalent to the abelian category of D-modules. Associated to a D-module or perverse sheaf, we can define the notion of its *singular support*, also called the *characteristic variety* or *microsupport*: a subvariety of the cotangent bundle T^*X.

First, we must recall the notion of a *good filtration* of a D-module M. This is a filtration that is suitably compatible with the canonical filtration on D_X by degree of a differential operator. Given an exhaustive increasing filtration $F_\bullet$ on M, we can form the associated graded module

$$\mathrm{gr}^F(M) = \bigoplus_{i=1}^{\infty} F_iM/F_{i-1}M.$$

This is a module over the sheaf $\mathrm{gr}^F(D_X) \cong \pi_* \mathcal{O}_{T^*X}$.

Definition 5.2. Let $(F_i)_{i\in\mathbb{N}}$ on M be an exhaustive increasing filtration on M such that

$$(F_i D_X)(F_j M) \subseteq F_{i+j} M.$$

$F_\bullet$ is called *good* if the associated graded module $\mathrm{gr}^F(M)$ is coherent.

Proposition 5.3. A quasi-coherent D_X-module M if coherent if and only if it admits a good filtration $F_\bullet$.

For a proof, see [10] Theorem 2.1.3.

Definition 5.4. The *singular support* of a D-module M on X is roughly the usual support of the associated graded module under a good filtration $F_\bullet$. More precisely, we extract a coherent $\mathcal{O}_{T^*X}$-module from $\mathrm{gr}^F M$ by

$$\widetilde{\mathrm{gr}}^F(M) = \mathcal{O}_{T^*X} \otimes_{\pi^{-1}\pi_*\mathcal{O}_{T^*X}} \pi^{-1}(\mathrm{gr}^F M)$$

where $\pi \colon T^*X \to X$ is the projection. Then the singular support of M, $SS(M)$ is the support of $\widetilde{\mathrm{gr}}^F M$.

It is fairly simple to prove that the singular support is independent of the choice of good filtration (see [10] D.3.1 for instance). One connects any two good filtrations by a finite chain of filtrations that are, pairwise, sufficiently close together that the associated gradeds have the same support.

If $M^\bullet$ is instead a *complex* of D-modules, one defines its singular support to be the union of the singular supports of its cohomology sheaves:

$$SS(M^\bullet) = \bigcup_{i=o}^{\infty} SS(H^i(M)).$$

In the case where $X = G$ is an algebraic group, the tangent and cotangent bundles TG and T^*G are trivialisable, naturally isomorphic to $G \times \mathfrak{g}$ and $G \times \mathfrak{g}^*$ respectively. In certain circumstances we will be able to identify these by choosing an isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$, but we'll always try to make it explicit when we're using such an identification. This trivialisation of the cotangent bundle gives us many canonical sub-bundles, by choosing subspaces of the Lie algebra $\mathfrak{g}^*$. Thus we can impose conditions on a D-module by requiring that its singular support lands within an appropriate sub-bundle of the cotangent bundle.

5.2 Induction and Singular Supports

Let G be a complex algebraic group acting on any complex algebraic variety, and let H be a subgroup. We have induction and coinduction functors from $D_H(X)$ to $D_G(X)$. What do they do to the singular supports of objects in these categories? We'll focus on the coinduction functor.

Lemma 5.5 ([15] 1.2)**.** If $\mathcal{S} \in D_H(X)$, then the singular support of the induced sheaf is given by

$$SS(\mathrm{coInd}_H^G(\mathcal{S}) = \overline{G \cdot SS(\mathcal{S})}.$$

Proof. We use the definition of induction given at the end of section 4.1. That is, consider the projection $\pi_2 \colon T^*(G/H \times X) \to T^*(G)$. We have, for $\mathcal{S} \in D_H(X)$

$$SS(\mathrm{coInd}_H^G \mathcal{S}) \subseteq \overline{\pi_2(SS(\widetilde{\mathcal{S}}))}$$

(a technical check: see [15] appendix B). Here $\widetilde{\mathcal{S}}$ is as in section 4.1. So we must understand this projection. First notice

$$\pi_2(SS(\widetilde{\mathcal{S}})) = \pi_2(SS(q^*\widetilde{\mathcal{S}})) = \pi_2(SS(\alpha^*\mathcal{S}))$$

where α is the action map and $q \colon G \times X \to G/H \times X$ is the projection, by definition of $\widetilde{\mathcal{S}}$, and after checking that the pullback under q does not affect the projection onto the second factor. Working fibrewise we can check $\pi_2(SS(\alpha^*\mathcal{S})) = G \cdot SS(\mathcal{S})$, as required.

□

5.3 The Mirković-Vilonen Characterisation

Let's first recall the definition of the *nilpotent cone* of an algebraic group G. Consider the map

$$\chi \colon \mathfrak{g} \to \mathfrak{c} = \mathfrak{g}//G$$

given by projection onto the GIT quotient. This should be thought of as sending a matric to the non-leading coefficients of its characteristic polynomial: indeed for $G = GL_n$, this is precisely what the map does. Later on, when we study the example of $\mathfrak{sl}_2$ in detail, this map will simply be the determinant map down to $\mathbb{C}$. The *nilpotent cone* inside $\mathfrak{g}$ is the fibre

$$\mathcal{N} = \chi^{-1}(0)$$

corresponding to the fact that a matrix is nilpotent if and only if its characteristic polynomial is t^n. It will be useful for later discussion to recall the notion of the *Springer resolution* of the nilpotent cone. Consider the flag variety G/B of the Lie group G or its Lie algebra $\mathfrak{g}$, i.e. the variety of Borel subgroups $B \subseteq G$, or equivalently Borel subalgebras $\mathfrak{b} \subseteq \mathfrak{g}$. (TODO: relation to G/B?). If $\mathfrak{b}_0$ is a fixed Borel subalgebra, there is a canonical $\mathfrak{b}_0$-bundle over G/B, namely

$$\widetilde{\mathfrak{g}} = \{(x,) \in \mathfrak{g} \times G/B \colon x \in \}.$$

There is an obvious projection $g \colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$ by forgetting the subalgebra. This is called the *Grothendieck-Springer resolution* of $\mathfrak{g}$, and its restriction to

$$s \colon \widetilde{\mathcal{N}} \to \mathcal{N}$$

is a resolution of singularities called the *Springer resolution*. The resolution $\widetilde{\mathcal{N}}$ is isomorphic to the cotangent bundle $T^*(G/B)$ via (TODO: Fill in.) We'll use this resolution in the next section.

Now, we're ready to state the main theorem of this section:

Theorem 5.6 ([15] Theorem 4.4)**.** An adjoint equivariant perverse sheaf on G is an irreducible character sheaf if and only if its singular support is contained in $G \times \mathcal{N}$.

Notice first of all that this makes sense in the case where $G = T$ is a torus. Then the adjoint action of T on itself is trivial, so we're just looking at perverse sheaves on T. What is the nilpotent cone here? Well, χ is just the identity, so $\mathcal{N} = 0$. So we're looking at complexes of constructible sheaves whose singular supports are contained in the zero section of the cotangent bundle. These are exactly the *local systems* on T, since a constructible sheaf which is a non-trivial combination of local systems on strata has singular support in conormal directions to some stratum. However, for a genuine (complex of) local systems, all non-zero codirections are regular for the sheaf at every point, so the singular support is just the zero section.

Proof. First we'll prove that character sheaves have nilpotent singular support. It suffices to prove that the induced sheaves $\mathrm{coInd}_B^G K_{\mathcal{L}}^w$ have nilpotent singular support, since the sheaves $K_{\mathcal{L}}^w$ come from a generating set of the Hecke category. We know that these sheaves are constructible with respect to the Bruhat stratification (see 4.14). Any Bruhat constructible sheaf has nilpotent singular support. To see this, we compute the conormal bundle to the Bruhat cell BwB in G: the fibre at w is computed to be

$$N_w^*(BwB) = \mathfrak{n} \cap ad(w)\mathfrak{n}$$

because it comprises the covectors normal to both B and wBw^{-1}. This is contained in $G \times \mathcal{N}$, so local systems on Bruhat strata, and more generally any Bruhat constructible sheaves have nilpotent singular support. Finally, apply 5.5 to see that coInd preserves the property of having nilpotent singular support: $G \times \mathcal{N}$ is fixed by the action of G, and closed.

Conversely, we must prove that an equivariant sheaf $\mathcal{F}$ with nilpotent singular support is a character sheaf. We can recover $\mathcal{F}$ as a summand of the sheaf

$$\mathrm{coInd}_B^G \mathrm{coInd}_{\{e\}}^N \mathcal{F} \cong \hat{H} \circ H\mathcal{F},$$

where H is the horocycle transform and $\hat{H}$ is its right adjoint, by computing the monad. Indeed, the functor $\hat{H} \circ H$ turns out to be precisely the functor of convolution with the Springer sheaf, given by pushing forward the constant sheaf under the Springer resolution. One of the summands of this convolution functor is the identity. We'll explain this below.

Now, $\mathcal{F}' = \mathrm{coInd}_{\{e\}}^N \mathcal{F}$ is Bruhat constructible by a similar argument to the above: we show its singular support is contained in the union of the conormal bundles to the Bruhat cells. It is N adjoint equivariant a fortiori, but also N-equivariant for the right multiplication action, thus it is $N \times N$-equivariant for the action

$$(u, v) \cdot g = ugv^{-1}.$$

Thus the fibre $SS(\mathcal{F}') \cap T_g^*(G)$ is orthogonal to the $N \times N$-orbit $T_g(NgN) = \mathfrak{n} + ad(g)\mathfrak{n}$. But $\mathcal{F}'$ is still nilpotent, thus the fibre is contained in $\mathcal{N}$, hence contained in $\mathfrak{n} \cap ad(g)\mathfrak{n}$. This establishes the claim, by our previous description of the Bruhat cells. Thus its induction to G is a character sheaf, as are all its summands, including $\mathcal{F}$. □

Thus it only remains to understand the monad $\hat{H} \circ H$ (see [15] Theorem 3.6.)

Proposition 5.7. Let $s \colon \widetilde{\mathcal{N}} \to \mathcal{N}$ be the Springer resolution as discussed above. Define the *Springer sheaf* to be the pushforward $\mathcal{S} = s_! \mathcal{O}_{\widetilde{\mathcal{N}}}$ of the structure sheaf (as a holonomic D-module; if we were working with constructible sheaves we'd use the constant sheaf). Then

$$\hat{H} \circ H(\mathcal{F}) \cong \mathcal{S} * \mathcal{F}$$

for all $\mathcal{F} \in D(G)$. Furthermore, this functor contains a shift of the identity functor as a direct summand.

It is worth remarking that the Springer and Grothendieck-Springer sheaves, while related, are not the same. In fact $\mathcal{S}$ is not generally a character sheaf, whereas $\mathcal{G}$ always is. On the level of Lie algebras, these two sheaves are Fourier dual.

Proof. We compute the composite $\hat{H} \circ H \colon D(G/_{ad}\, G) \to D(G/_{ad}\, G)$ as an integral transform itself. That is, we look at the horocycle transform as a functor on G-equivariant sheaves, and form the pullback

$$\begin{tikzcd}
 & & S \arrow[dl] \arrow[dr] \arrow[ddll, bend right, "q_1"'] \arrow[ddrr, bend left, "q_2"] & & \\
 & G \times G/B \arrow[dl] \arrow[dr] & & G \times G/B \arrow[dl] \arrow[dr] & \\
G & & (G/N \times G/N)/T & & G
\end{tikzcd}$$

where the pullback space S is $\{(g_1, g_2, B) \in G \times G \times G/B \colon g_1 B = g_2 B\}$. This functor is computed via the projection formula applied to π to be just

$$\begin{aligned}
(q_1)_!(q_2)^* \mathcal{F} &= ((p_2)_! \pi_! \pi^* p_1^*) \mathcal{F} \\
&= p_{2!}\,(p_1^* \mathcal{F} \otimes \pi_! \pi^* \mathcal{O}_{G \times G}) \\
&= p_{2!}\,(p_1^* \mathcal{F} \otimes \pi_! \mathcal{O}_S)
\end{aligned}$$

where $\pi \colon S \to G \times G/B$ is the projection onto the first and third factors, say. That is, we have an integral transform with kernel $\pi_! \mathcal{O}_S$. One can understand this sheaf via the G-equivariant of the projection π. This implies that

$$\pi_! \mathcal{O}_S = \mu^\dagger(s_!(\mathcal{O}_{\widetilde{\mathcal{N}}})$$

where μ is the multiplication map. This establishes the claim. To see that a shift of the identity functor lives inside this convolution as a summand, is to see that a shift of the monoidal unit δ – the skyscraper D-module at the identity – occurs as a summand of $\mathcal{S}$. This can be computed by viewing $\mathcal{O}_S$ as a constructible sheaf and applying the decomposition theorem of Beilinson, Bernstein and Deligne to the Springer resolution. This computation is worked out in [7] and is closely related to the calculation we did for the decomposition of the Grothendieck-Springer sheaf for SL_2. □

6 Character Sheaves on a Lie Algebra

In this section we'll discuss an analogous theory of character sheaves on a Lie algebra, as initially discussed in (TODO: reference, probably Lusztig). We use the notion of the horocycle transform to define objects very much like the character sheaves we've already discussed on a Lie algebra $\mathfrak{g}$, and – following the discussion of Mirković in [14] – prove an analogous characterisation of character sheaves by their singular support to 5.6 above. The primary advantage to working over a Lie algebra rather than a group is that one has a *Fourier transform* available. Fourier transforms turns statements about singular supports to statements about ordinary supports, so it is not so surprising to learn that character sheaves are dual to sheaves supported on a finite union of adjoint orbits. One would very much like to produce a descent theorem analogous to the one discussed in 2.4 for a torus, allowing one to relate character sheaves on a group to character sheaves on its Lie algebra.

6.1 The Fourier Transform

For simplicity, we'll work over the complex numbers. Let V be a vector space over $\mathbb{C}$. Let V^* denote the *real* dual space. Let $D_{\mathbb{R}_+}(V)$ denote the category of *conic* constructible sheaves, i.e. those that are equivariant with respect to the action of $\mathbb{R}_+$ on V by scaling: $\lambda \cdot v = \lambda v$. We can define a Fourier transform on this category, following Kashiwara-Schapira ([11] section 9.7). By π_1, π_2, we denote the two projections from the product $V \times V^*$ onto its factors.

Definition 6.1. The *Fourier transform functor* $F\colon D_{\mathbb{R}_+}(V) \to D_{\mathbb{R}_+}(V^*)$ is given by the integral transform

$$F(\mathcal{S}) = (\pi_2)_!((\pi_1)^*\mathcal{S} \otimes \mathbb{C}_N)$$

where $\mathbb{C}_P$ is the constant sheaf on the subspace (TODO: check this is right)

$$N = \{(v, f)\colon f(v) \leq 0\} \subseteq V \times V^*.$$

Why is this called a Fourier transform? It is really a microlocalisation of a more classical Fourier transform. (TODO: Spell this out.)

Using the Riemann-Hilbert correspondence 5.1, we can give a nice description of the Fourier transform of a constructible sheaf. Indeed, let V be an n-dimensional vector space over k, and let $\mathcal{S} \in D(V)$. Under the Riemann-Hilbert correspondence we identify $\mathcal{S}$ with a D-module on V, i.e. a module for the ring $D_n = k[z_1, \ldots, z_n, \partial_1, \ldots, \partial_n]$, where $[z_i, z_j] = [\partial_i, \partial_j] = 0$ and $[z_i, \partial_j] = \delta_{ij}$. Then the Fourier transform functor is given by

$$\begin{aligned} F\colon D(V) &\to D(V) \\ M &\mapsto FM \end{aligned}$$

where $FM = M$ as a k-vector space, but if $F(m) \in FM$, D_n acts by $z_i(F(m)) = \partial_i(m), \partial_i(F(m)) = -z_i(m)$. (TODO: Check signs, try to improve notation.)

Examples 6.2. We'll compute the Fourier transform of some constructible adjoint equivariant sheaves on $\mathfrak{g} = \mathfrak{sl}_2$. It is easy to compute the decomposition of $\mathfrak{g}$ into orbits for the adjoint action, by looking at Jordan type. Possible Jordan types for a two-by-two traceless matrix are

$$\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

for $\lambda \in \mathbb{C}^\times$. The orbits under the adjoint action are thus precisely the orbits for the map $\chi\colon \mathfrak{g} \to \mathbb{C}$, which in this case is just the determinant map, with the exception that the nilpotent cone splits into two orbits: $\{0\}$ and $\mathcal{N} \setminus \{0\}$.

Firstly, the skyscraper sheaf at the origin. We'll shift this into degree 3 (TODO: ? -3?) to make it perverse, i.e. to make it correspond to the skyscraper D-module at 0 under the Riemann-Hilbert correspondence. Now, we compute (TODO: ...)

6.2 The Horocycle Transform on a Lie Algebra

There is a natural analogue of the horocycle transform in this setting. We'll describe it, and compute its Fourier dual, following the description in [14]. Let $g\colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$ be the Grothendieck-Springer resolution described in 5.3, so $\widetilde{\mathfrak{g}}$ is the total space of the canonical $\mathfrak{b}$-bundle on G/B. Similarly, there is a canonical $\mathfrak{g}/[\mathfrak{b},\mathfrak{b}] = \mathfrak{g}/\mathfrak{n}$-bundle on G/B, where $\mathfrak{n} = [\mathfrak{b},\mathfrak{b}]$. Call the total space of this bundle $\widetilde{\mathfrak{g}/\mathfrak{n}}$.

Definition 6.3. Consider the correspondence

$$\begin{tikzcd} & \mathfrak{g} \times G/B \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathfrak{g} & & \widetilde{\mathfrak{g}/\mathfrak{n}} \end{tikzcd}.$$

Define the *horocycle transform* to be the composite $H = p_* q^! \colon D_G(\widetilde{\mathfrak{g}/\mathfrak{n}}) \to D_G(\mathfrak{g})$ on G-equivariant sheaves.

This is actually equivalent to a more natural functor, more closely analogous to the horocycle transform we're familiar with.

Lemma 6.4. The adjoint pair of functors

$$\begin{tikzcd} D_G(\widetilde{\mathfrak{g}/\mathfrak{n}}) \arrow[r, shift left, "s^\dagger \circ \mathrm{Res}_B^G"] & D_B(\mathfrak{g}/\mathfrak{n}) \arrow[l, shift left, "\mathrm{coInd}_B^G \circ s_*"] \end{tikzcd}$$

define an equivalence of categories, where s is the analogue of the Springer map in this setting. Furthermore, the diagram

$$\begin{tikzcd} & D_G(\mathfrak{g}) & \\ D_G(\widetilde{\mathfrak{g}/\mathfrak{n}}) \arrow[ur, "H"] \arrow[rr, shift left] & & D_B(\mathfrak{g}/\mathfrak{n}) \arrow[ul, "\mathrm{coInd}_B^G \circ \pi^\dagger"'] \arrow[ll, shift left] \end{tikzcd}$$

commutes.

Proof. (TODO: ...) □

Thus, the horocycle transform H is equivalent to the functor $\mathrm{coInd}_B^G \circ \pi^\dagger \colon D_B(\mathfrak{g}/\mathfrak{n}) \to D_G(\mathfrak{g})$. By abuse of notation, we will also refer to this functor as H from now on. This gives a natural definition of a character sheaf on a Lie algebra.

Definition 6.5. A sheaf on $\mathfrak{g}/\mathfrak{n}$ is called *monodromic* if (TODO: ...) This is the Fourier dual notion to having *support* contained in (TODO: ...)

Definition 6.6. The category of *character sheaves* on $\mathfrak{g}$ is the full subcategory Ch of $D_G(\mathfrak{g})$ generated under colimits (i.e. taking summands, shifts (TODO: ?)) by the images $H(\mathcal{S})$ of monodromic sheaves $\mathcal{S} \in D_B(\mathfrak{g}/\mathfrak{n})$.

Let's parse this definition, and compare it with the definition of a character sheaf on the algebraic group G. (TODO: ...)

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