ALGEBRAIC STRUCTURES ON THE MODULI SPACE OF CURVES FROM REPRESENTATIONS OF VERTEX OPERATOR ALGEBRAS

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ABSTRACT. These are notes with my lectures at the University of Massachusetts, Amherst summer school on the Physical Mathematics of Quantum Field Theory.

INTRODUCTION

The moduli space $\overline{\mathrm{M}}_{g,n}$ of n-pointed (Deligne-Mumford) stable curves of genus g provides a natural environment in which we may study smooth curves and their degenerations. These spaces, for different values of g and n, are related to each other through systems of tautological maps. Algebraic structures on $\overline{\mathrm{M}}_{g,n}$ often reflect this, being governed by recursions, and amenable to inductive arguments. For these and other reasons, such structures are generally easier to work with. Sheaves on the moduli space of curves given by representations of vertex operator algebras (VOAs for short) exemplify this.

VOAs generalize commutative associative algebras as well as Lie algebras, and have played deep and important roles in both mathematics and mathematical physics. For instance, in understanding conformal field theories, finite group theory, and in the construction of knot invariants and 3-manifold invariants. Given nice enough VOAs and categories from which modules are selected, sheaves of coinvariants behave functorially with respect to these tautological maps.

In the first lecture, I will introduce the moduli spaces of curves that are involved in the construction, and also some of the questions that the sheaves may help answer. In lecture two I will introduce vertex operator algebras, and their modules giving some examples. In lecture three I will describe the sheaves of coinvariants and dual sheaves of conformal blocks, describing some of their important features. In the last lecture I will discuss a number of open problems.

1. LECTURE 1: THE MODULI SPACE OF CURVES AND VERTEX OPERATOR ALGEBRAS

A moduli space is a variety (or a scheme or a stack) that parametrizes some class of objects. The general moduli/parameter spaces philosophy goes something like the following:

- Objects X (like varieties with properties in common) can often correspond points in a moduli space $\mathcal{M}$. By studying $\mathcal{M}$ one can learn about X.
- Points $[X] \in \mathcal{M}$ with *good* properties often form a large (dense) open subset of $\mathcal{M}$.
- Points $[X] \in \mathcal{M}$ that don't have good properties occupy closed (proper) subsets of $\mathcal{M}$. The worse these points are, the smaller their ambient environment.

Today we will apply this philosophy to $\overline{\mathrm{M}}_g$, the moduli space of n-pointed Deligne-Mumford stable curves of genus $g \geq 2$. One dimensional algebraic varieties, arguably the simplest objects one studies, can be better understood as points on moduli spaces of curves. As curves arise in many contexts, moduli of curves are meeting grounds where a variety of techniques are applied in concert. In algebraic geometry, moduli of curves are particularly important: they help us understand smooth curves and their degenerations, and as special varieties, they have been one of the chief concrete, nontrivial settings where the nuanced theory of the minimal model program has been exhibited and explored [HH09, HH13, AFSvdW17, AFS17a, AFS17b]. They have also played a principal role as a prototype for moduli of higher dimensional varieties [KSB88, Ale02, HM06, HKT06, HKT09, CGK09].

It is not uncommon to refer to certain varieties as combinatorial: these include toric varieties: like projective space, weighted projective spaces, and certain blowups of those, Grassmannian varieties, or even more generally homogeneous varieties. These all come with group actions, and combinatorial data encoded in convex bodies keeps track of their important geometric features. Certain varieties like moduli of curves, have combinatorial structures reminiscent of varieties that are more traditionally considered to be combinatorial. As a result, various analogies have been made between them and the moduli of curves. Such comparisons have led to questions and conjectures, surprising formulas, and even arguments that have been used to detect and to prove some of the most important and often subtle geometric properties of the moduli space of curves.

As we shall see today, by looking at loci of curves with singularities, we are led to the study of moduli spaces $\overline{\mathrm{M}}_{g,n}$, parametrizing stable n-pointed curves of genus g. We'll also see that these spaces, for different g and n, are connected through tautological clutching and projection morphisms, give the system and these spaces a rich combinatorial structure. Algebraic structures on $\overline{\mathrm{M}}_{g,n}$ reflect this, and are often governed by recursions, and amenable to inductive arguments. Consequently, many questions can be reduced to moduli spaces of curves of smaller genus and fewer marked points. Problems about curves of positive genus can often be reduced to the smooth, projective, rational variety $\overline{\mathrm{M}}_{0,n}$, which can be constructed in a simple manner as a sequence of blowups of projective spaces. Today we will talk about this.

As you can see from other more complete surveys [Har84, Far09, Abr13, Coş17], this is a long studied subject with many points of focus!

1.1. **Why moduli?** The basic objects of study in algebraic geometry are varieties (or schemes or stacks). Zero sets of polynomials give algebraic varieties. The simplest are lines, which as can be seen in the picture below, taken together form varieties:

When learning about algebraic geometry, one typically starts with affine varieties, which in their simplest form are the zero sets of polynomials in some number of variables. Soon we learn that it is useful to homogenize those polynomials so we can study projective varieties for which there is more theory available. For instance, zero sets of degree d polynomials define curves in the affine plane, and homogeneous polynomials of degree d in three variables determine curves in $\mathbb{P}^2$, which

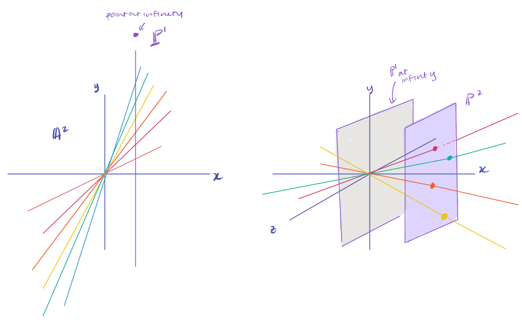


FIGURE 1. Imagining projective lines and spaces.

we can classify according to their genus

$$g = \frac{d(d-1)}{2}.$$

The genus of a curve is an invariant: If two curves have different genera, they can't be isomorphic. As some of you will discuss in the problem sessions, there are more geometric ways to define this number. For instance, the genus of a smooth curve C is

$$g = \dim \mathrm{H}^0(C, \omega_C) = \dim \mathrm{H}^1(C, \mathcal{O}_C),$$

where ω_C is the sheaf of regular 1-forms on C. If defined over the field of complex numbers, we may consider C as a Riemann surface, and the algebraic definition of genus is the same as the topological definition.

FIGURE 2. Picturing genus.

The simplest examples of plane curves have genus zero. These can be obtained as zero sets of conics in two variables:

$$f_{\alpha_\bullet}(x_1, x_2) = \sum_{j,k \geq 0, j+k \leq 2} \alpha_{jk} x_1^j x_2^k.$$

or as homogeneous polynomials of degree 2:

$$F_{a_\bullet}(x_0 : x_1 : x_2) = \sum_{\substack{i,j,k \geq 0 \\ i+j+k=2}} a_{ijk} x_0^i x_1^j x_2^k.$$

Note that the element

$$a_\bullet = [a_{200} : a_{110} : a_{101} : a_{020} : a_{011} : a_{002}] \in \mathbf{P}^5$$

determines the zero set $Z(F_{a_\bullet}) \subset \mathbf{P}^2$. In other words, there is a 5 dimensional family of rational curves. If we ask for only those curves that pass through a fixed set of points, say

$$p^1 = [1:0:0],\ p^2 = [0:1:0],\ p^3 = [0:0:1],\ \text{ and }\ p^4 = [1:1:1],$$

then since every point imposes a linear condition on the coefficients, we obtain a one dimensional family of 4-pointed rational curves.

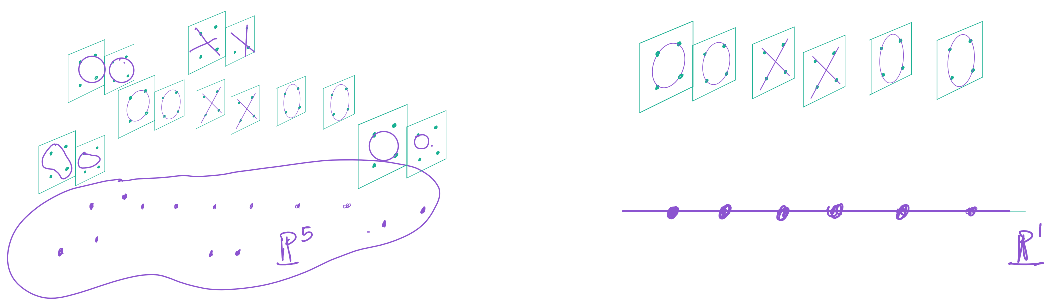


FIGURE 3. Families of 4-pointed rational curves.

Plane curves of genus 1 can be obtained as zero sets of cubic polynomials, and we can write down the general curve of genus 2 using the equation:

$$x_2^2 = x_1^6 + a_5x_1^5 + a_4x_1^4 + \cdots + a_1x_1 + a_0.$$

In other words, a point $(a_0, \ldots, a_5) \in \mathbf{A^6}$ determines a curve of genus 2, and there is a family of curves parametrized by an open subset of $\mathbf{A^6}$ that includes the general smooth curve of genus 2. As the coefficients change, the curves will sometimes have singularities.

1.2. **Moduli of curves.**

Definition 1.1. M_g *is the moduli space of smooth curves of genus g, the variety whose points are in one-to-one correspondence with isomorphism classes of smooth curves of genus* $g \geq 2$.

As smooth curves degenerate to curves with singularities, even if we just care about families of smooth curves it is useful to work with a a compactification of M_g – a proper space that contains M_g as a (dense) open subset. Such a space will necessarily parametrize curves with singularities.

We will consider the compactification $\overline{\mathrm{M}}_g$ whose points correspond to Deligne-Mumford stable curves of genus g. There are a number of choices of compactifications of M_g, and some of these receive birational morphisms from $\overline{\mathrm{M}}_g$ while others just receive rational maps from $\overline{\mathrm{M}}_g$. A few examples are given in the Appendix.

Definition 1.2. *A* ***stable curve*** C *of (arithmetic) genus g is a reduced, connected, one dimensional scheme such that*

(1) C has only ordinary double points as singularities.
(2) C has only a finite number of automorphisms.

Remark 1.1. *That C has finitely many automorphisms comes down to two conditions: (1) if C_i is a nonsingular rational component, then C_i meets the rest of the curve in at least three points, and (2) if C_i is a component of genus one, then it meets the rest of the curve in at least one point.*

Definition 1.3. $\overline{\mathrm{M}}_g$ *is the moduli space of stable curves of genus g, the variety whose points are in one-to-one correspondence with isomorphism classes of stable curves of genus* $g \geq 2$*.*

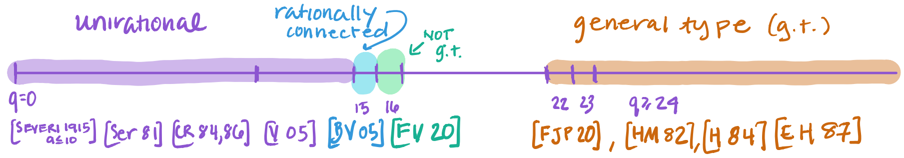
That such a variety $\overline{\mathrm{M}}_g$ exists is nontrivial. This was proved by Deligne and Mumford who constructed $\overline{\mathrm{M}}_g$ using Geometric Invariant Theory [DM69]. In the second lecture we will see Keel's construction of the space $\overline{\mathrm{M}}_{0,n}$.

This variety $\overline{\mathrm{M}}_g$ has the essential property that given any flat family $\mathcal{F} \to B$ of curves of genus g, there is a morphism $B \to \overline{\mathrm{M}}_g$, that takes a point $b \in B$ to the isomorphism class $[\mathcal{F}_b] \in \overline{\mathrm{M}}_g$ represented the fiber $\mathcal{F}_b$.

1.2.1. *How can studying* $\overline{\mathrm{M}}_g$ *tell us about curves?* Earlier we considered a family of curves parametrized by an open subset of $\mathbf{A}^6$, that included the general smooth curve of genus 2. Generally, if there is a family of curves parametrized by an open subset of $\mathbf{A}^{N+1}$ that includes the general curve of genus g, then one would have a dominant rational map from $\mathbf{P}^N$ to our compactification $\overline{\mathrm{M}}_g$. In other words, $\overline{\mathrm{M}}_g$ would be unirational. This would imply that there are no pluricanonical forms on $\overline{\mathrm{M}}_g$. Said otherwise still, the canonical divisor of $\overline{\mathrm{M}}_g$ would not be effective.

On the other hand, one of the most important results about the moduli space of curves, proved almost 40 years ago, is that for $g >> 0$ the canonical divisor of $\overline{\mathrm{M}}_g$ lives in the interior of the cone of effective divisors (for $g = 22$ and $g \geq 24$, by [EH87, HM82], and for by $g = 23$ [Far00]). Once the hard work was done to write down the classes of the canonical divisor, and an effective divisor called the Brill-Noether locus, to prove this famous result, a very easy combinatorial argument can be made to show that the canonical divisor is equal to an effective linear combination of the Brill-Noether and boundary divisors when the genus is large enough.

The upshot is that by shifting focus to the geometry of the moduli space of curves, we learn something basic and valuable about the existence of equations of smooth general curves. Nevertheless, basic open questions remain. First, our current understanding of such questions is incomplete – it can be summarized in the following picture:



So there is a gap in our understanding of the "nature" of $\overline{\mathrm{M}}_g$. On the other hand, even for those g for which we know the answer, there are still problems to solve. For instance if $\overline{\mathrm{M}}_g$ is known to be of general type, one can consider the canonical ring

$$\mathrm{R}_{\bullet} = \bigoplus_{m\geq 0} \Gamma(\overline{\mathrm{M}}_g, m\,\mathrm{K}_{\overline{\mathrm{M}}_g}),$$

which is now known to be finitely generated by the celebrated work of [BCHM10]. In particular, the canonical model $\mathrm{Proj}(\mathrm{R}_{\bullet})$, is birational to $\overline{\mathrm{M}}_g$.

It is still an open problem to construct this model, and efforts to achieve this goal have both furthered our understanding of the birational geometry of the moduli space of curves, as well as giving a highly nontrivial example where this developing theory can be experimented with and better understood.

Remark 1.2.2. *We have described moduli spaces of curves as projective varieties. But in doing so we gloss over some of what makes them moduli spaces. There is a functorial way to describe moduli spaces which leads to their study as stacks.*

1.2.3. *A stratification.* As we have seen in the examples above, even if we are only interested in smooth curves, we are naturally led to curves with singularities, and when considering curves with nodes, one is naturally led to curves with *marked points*.

The moduli space $\overline{\mathrm{M}}_g$ is a (3g-3)-dimensional projective variety. The set $\delta^k(\overline{\mathrm{M}}_g) = \{[C] \in \overline{\mathrm{M}}_g |\ C$ has at least k nodes$\}$ has codimension k in $\overline{\mathrm{M}}_g$. If $k = 1$, these loci have codimension one, and the boundary is a union of components:

(1) The component Δ_{irr} can be described as having generic point with a nonseparating node; the closure of the locus of curves whose normalization is a curve of genus $g-1$ with 2 marked points.
(2) Components $\Delta_{g_1} = \Delta_{g_2}$ are determined by partitions $g = g_1 + g_2$. These loci can be described as having generic point with a separating node – the closure of the set of curves whose normalization consists of 1-pointed curves of genus g_1 (and g_2).

We may picture generic elements in these sets, and their normalizations, as follows:

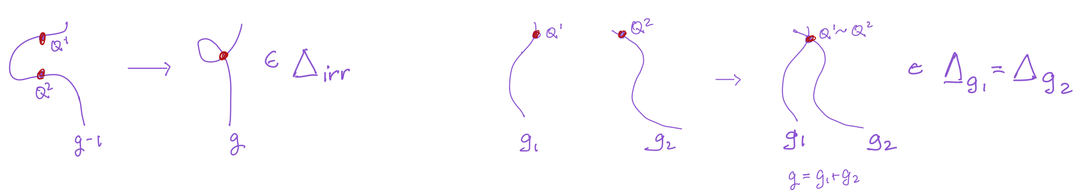


FIGURE 4. Clutching maps.

1.3. **Moduli of pointed curves.** By $\mathrm{M}_{g,n}$ we mean the quasi-projective variety whose points are in one-to-one correspondence with isomorphism classes of smooth n-pointed curves of genus $g \geq 0$. By the compactification $\overline{\mathrm{M}}_{g,n}$, we mean the variety whose points are in one-to-one correspondence with isomorphism classes of stable n-pointed curves of genus $g \geq 0$.

Definition 1.4. *A **stable n-pointed curve** is a complete connected curve C that has only nodes as singularities, together with an ordered collection p_1, p_2, …, $p_n \in C$ of distinct smooth points of C, such that the $(n+1)$-tuple $(C; p_1, \dots, p_n)$ has only a finite number of automorphisms.*

To get a sense of its combinatorial structure, we note that the moduli space is stratified by the topological type of the curves being parametrized. As we did last time in the case $n = 0$, we may describe these components of the boundary of $\overline{\mathrm{M}}_{g,n}$ as $\delta^k(\overline{\mathrm{M}}_{g,n}) = \{[(C, P^\bullet)] \in \overline{\mathrm{M}}_{g,n} | \ C$ has at least k nodes$\}$ in $\overline{\mathrm{M}}_{g,n}$ (a space of dimension $3g - 3 + n$). The locus $\delta^k(\overline{\mathrm{M}}_{g,n})$ has codimension k and is a union of irreducible components.

For instance, if $k = 1$, this codimension one locus is a union of components:

(1) Δ_{irr} has generic point a nonseparating node; the closure of the locus of curves whose normalization is a curve of genus $g - 1$ with $n + 2$ marked points.
(2) $\Delta_{g_1,N_1} = \Delta_{g_2,N_2}$ are determined by partitions $g = g_1 + g_2$ and $\{P^1, \dots, P^n\} = N_1 \cup N_2$, with generic point a separating node – the closure of the set of curves whose normalization consists of pointed curves of genus g_1 (and g_2) with marked points in the set N_1 (and N_2)

As before, one can describe the components Δ_{irr} and Δ_{g_1,N_1} as the images of attaching maps from moduli spaces of stable curves with smaller genus (or with fewer marked points):

$$\overline{\mathrm{M}}_{g-1,n+2} \longrightarrow \Delta_{irr} \subset \overline{\mathrm{M}}_{g,n}, \quad \text{and} \quad \overline{\mathrm{M}}_{g_1,n_1+1} \times \overline{\mathrm{M}}_{g_2,n_2+1} \longrightarrow \Delta_{g_1,N_1} = \Delta_{g_2,N_2} \subset \overline{\mathrm{M}}_{g,n}$$

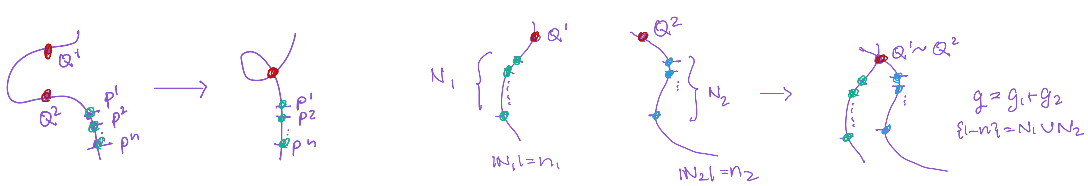


FIGURE 5. Tautological clutching maps.

There are also tautological point dropping maps, and using them we obtain $n + 1$ families of stable n-pointed curves parametrized by $\overline{\mathrm{M}}_{g,n}$

$$\pi_j : \overline{\mathrm{M}}_{g,n+1} \to \overline{\mathrm{M}}_{g,n}, \quad s_i : \overline{\mathrm{M}}_{g,n} \to \overline{\mathrm{M}}_{g,n+1}, \quad i \in \{1, \dots, n+1\} \setminus \{j\}$$

where π_j is the map that drops the j-th point, and s_i is the section that takes an n-pointed curve $(C; \vec{p})$ and at the i-th point attaches a copy of $\mathbb{P}^1$ labeled with two additional points p_i and p_{n+1}.

1.3.1. *Comparing* $\overline{\mathrm{M}}_{0,n}$ *with moduli spaces of higher genus curves.* The space $\overline{\mathrm{M}}_{0,n}$ has some advantages over $\overline{\mathrm{M}}_{g,n}$ for $g > 0$, for several reasons, three of which are easy to state. First $\overline{\mathrm{M}}_{0,n}$ is a fine moduli space (it parametrizes pointed curves with no nontrivial isomorphisms), unlike $\overline{\mathrm{M}}_{g,n}$ for $g > 0$, which parametrizes curves with non-trivial automorphism. Second, $\overline{\mathrm{M}}_{0,n}$ is smooth, whereas $\overline{\mathrm{M}}_{g,n}$ for $g > 0$ has singularities. So there are tools like intersection theory that are easier to carry out. Third, $\overline{\mathrm{M}}_{0,n}$ is rational (unlike $\overline{\mathrm{M}}_{g,n}$ for $g >> 0$), making some arguments easier.

There are a number of constructions of $\overline{\mathrm{M}}_{0,n}$, giving one different perspectives about the space, and tools to work with it. For instance, Kapranov showed $\overline{\mathrm{M}}_{0,n}$ is a Hilbert (or Chow quotient) of Veronese curves and can be seen as a quotient of a Grassmannian. There are at least four ways to construct the space as a sequence of blowups. Finn Knudsen was first to observe this, showing that $\overline{\mathrm{M}}_{0,n+2}$ could be constructed as a sequence of blowups of $\overline{\mathrm{M}}_{0,n+1} \times_{\overline{\mathrm{M}}_{0,n}} \overline{\mathrm{M}}_{0,n+1}$ (this product is not smooth), along non-regularly embedded subschemes. Keel improved this, giving an alternative construction of $\overline{\mathrm{M}}_{0,n}$ as a sequence of blowups of smooth varieties along smooth co-dimension 2 sub-varieties. The first case where we see anything interesting is for the 2-dimensional space $\overline{\mathrm{M}}_{0,5}$ which is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \cong \mathbb{P}^1 \times_{pt} \mathbb{P}^1$.

As a result of his construction, Keel in [Kee92] showed that Chow groups and homology groups are canonically isomorphic, giving recursive formulas for the Betti numbers, and an inductive recipe for the basis of Chow rings, which he shows are quotients of polynomial rings (he gives the generators and the relations). As an example, we know from Keel that there are $2^{n-1} - \binom{n}{2} - 1$ numerical (or linear, or algebraic) equivalence classes of codimension 1 classes (divisors) on $\overline{\mathrm{M}}_{0,n}$. One may also use the projection maps and facts about $\overline{\mathrm{M}}_{0,4}$, which is isomorphic to $\mathbb{P}^1$ to deduce numerical equivalences of divisors on $\overline{\mathrm{M}}_{0,n}$ for all n. For instance, since $\mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$, one has that on $\overline{\mathrm{M}}_{0,4}$, all boundary divisor classes are equivalent. So in particular,

$$\delta_{ij} \equiv \delta_{ik} \equiv \delta_{i\ell}, \ \text{ for } \{i,j,k,\ell\} = \{1,2,3,4\}.$$

One can show, using the point dropping maps, that for $n \geq 4$, on $\overline{\mathrm{M}}_{0,n}$,

$$\sum_{I \subset \{ijk\ell\}^c} \delta_{ij \cup I} \equiv \sum_{I \subset \{ijk\ell\}^c} \delta_{ik \cup I} \equiv \sum_{I \subset \{ijk\ell\}^c} \delta_{i\ell \cup I}, \ \text{ for any four indices } \{i,j,k,\ell\} \subset \{1,\ldots,n\}.$$

1.4. **Moduli of stable coordinized curves.** $\widehat{\mathcal{M}}_{g,n}$ is the stack parametrizing families of tuples $(C, P_\bullet, t_\bullet)$, where $(C, P_\bullet = (P_1, \ldots, P_n))$ is a stable n-pointed genus g curve, and $t_\bullet = (t_1, \ldots, t_n)$ with t_i a formal coordinate at P_i, for each i. A description of $\widehat{\mathcal{M}}_{g,n}$ over the locus of smooth curves is given in [ADCKP88] and [FBZ04, §6.5], and over the locus parametrizing stable curves with singularities in [DGT21, §2]. By dropping the coordinates, there is a projection $\widehat{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. A group scheme $(\mathrm{Aut}\, \mathcal{O})^{\oplus n}$, defined below, acts transitively on fibers by change of coordinates, giving $\widehat{\mathcal{M}}_{g,n}$ the structure of an $(\mathrm{Aut}\, \mathcal{O})^{\oplus n}$-torsor over $\overline{\mathcal{M}}_{g,n}$. Moreover, $\mathrm{Aut}\, \mathcal{O} = \mathbb{G}_m \ltimes \mathrm{Aut}_+ \mathcal{O}$, and the projection factors as a composition of an $(\mathrm{Aut}_+\mathcal{O})^{\oplus n}$-torsor and a $\mathbb{G}_m^{\oplus n}$-torsor:

(1)
$$\begin{tikzcd} \widehat{\mathcal{M}}_{g,n} \arrow[rr, "(\mathrm{Aut}\,\mathcal{O})^{\oplus n}"] \arrow[dr, "(\mathrm{Aut}_+\mathcal{O})^{\oplus n}"'] & & \overline{\mathcal{M}}_{g,n} \\ & \overline{\mathcal{J}}_{g,n}. \arrow[ur, "\mathbb{G}_m^{\oplus n}"'] & \end{tikzcd}$$

Here $\overline{\mathcal{J}}_{g,n}$ denotes the stack parametrizing families of pointed curves with first order tangent data. Closed points in $\overline{\mathcal{J}}_{g,n}$ are denoted $(C, P_\bullet, \tau_\bullet)$, where $(C, P_\bullet)$ is a stable n-pointed curve of genus g, and $\tau_\bullet = (\tau_1, \ldots, \tau_n)$ with τ_i a non-zero 1-jet at a formal coordinate at P_i, for each i.

Remark 1.4.1. *The sheaf of coinvariants on $\overline{\mathcal{M}}_{g,n}$ is defined first on $\widehat{\mathcal{M}}_{g,n}$, shown to descend to $\overline{\mathcal{J}}_{g,n}$, and then if conditions are right, shown to descend to $\overline{\mathcal{M}}_{g,n}$ using Tsuchimoto's method, as described carefully* [DGT22, §8]. *At the moment, the complete description of the descent is given only in case the conformal dimensions of modules are rational numbers. We are working on descent in greater generality, but for this reason, I will point out assumptions on V or categories of V-modules where this condition is known to hold.*

The group schemes discussed above represent functors. For instance, $\mathrm{Aut}\,\mathcal{O}$ represents the functor which assigns to a $\mathbb{C}$-algebra R the group

$$\mathrm{Aut}\,\mathcal{O}(R) = \left\{ z \mapsto \rho(z) = a_1 z + a_2 z^2 + \cdots \mid a_i \in R,\ a_1 \text{ a unit} \right\}$$

of continuous automorphisms of the algebra $R[\![z]\!]$ preserving the ideal $zR[\![z]\!]$. The group law is given by composition of series: $\rho_1 \cdot \rho_2 := \rho_2 \circ \rho_1$. The subgroup scheme $\mathrm{Aut}_+\mathcal{O}$ of $\mathrm{Aut}\,\mathcal{O}$ represents the functor assigning to a $\mathbb{C}$-algebra R the group:

$$\mathrm{Aut}_+\mathcal{O}(R) = \left\{ z \mapsto \rho(z) = z + a_2 z^2 + \cdots \mid a_i \in R \right\}.$$

To give more details about the actions, for a smooth curve C, let $\mathscr{A}ut_C$ be the smooth variety whose set of points are pairs (P, t), with $P \in C$, $t \in \widehat{\mathscr{O}}_P$ and $t \in \mathfrak{m}_P \setminus \mathfrak{m}_P^2$, a formal coordinate at P. Here $\mathfrak{m}_P$ is the maximal ideal of $\widehat{\mathscr{O}}_P$, the completed local ring at the point P. There is a simply transitive right action of $\mathrm{Aut}\,\mathcal{O}$ on $\mathscr{A}ut_C \to C$, given by changing coordinates:

$$\mathscr{A}ut_C \times \mathrm{Aut}\,\mathcal{O} \to \mathscr{A}ut_C, \qquad ((P,t), \rho) \mapsto (P, t \cdot \rho := \rho(t)),$$

making $\mathscr{A}ut_C$ a principal $(\mathrm{Aut}\,\mathcal{O})$-bundle on C. A choice of formal coordinate at P gives a trivialization

$$\mathrm{Aut}\,\mathcal{O} \xrightarrow{\simeq_t} \mathscr{A}ut_P, \qquad \rho \mapsto \rho(t).$$

If C is a *nodal* curve, then to define a principal $(\mathrm{Aut}\,\mathcal{O})$-bundle on C one may give a principal $(\mathrm{Aut}\,\mathcal{O})$-bundle on its normalization, together with a gluing isomorphism between the fibers over the preimages of each node. For simplicity, suppose C has a single node Q, and let $\widetilde{C} \to C$ denote its normalization, with Q_+ and Q_- the two preimages of Q in $\widetilde{C}$. A choice of formal coordinates $s_\pm$ at $Q_\pm$, respectively, determines a smoothing of the nodal curve C over $\mathrm{Spec}(\mathbb{C}[\![q]\!])$ such that, locally around the point Q in C, the family is defined by $s_+s_- = q$. One may identify the fibers at $Q_\pm$ by the gluing isomorphism induced from the identification $s_+ = \gamma(s_-)$:

$$\mathscr{A}ut_{Q_+} \simeq_{s_+} \operatorname{Aut}\mathcal{O} \xrightarrow{\cong} \operatorname{Aut}\mathcal{O} \simeq_{s_-} \mathscr{A}ut_{Q_-}, \qquad \rho(s_+) \mapsto \rho \circ \gamma(s_-),$$

where $\gamma \in \operatorname{Aut}\mathcal{O}$ is the involution defined as

$$\gamma(z) := \frac{1}{1+z} - 1 = -z + z^2 - z^3 + \cdots .$$

This may be carried out in families, and the identification of the universal curve $\overline{\mathcal{C}}_g \cong \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ leads to the definition of the principal $(\operatorname{Aut}\mathcal{O})$-bundle $\widehat{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_{g,1}$ (see [DGT21] for details).

INDEX

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