Chris Elliott – Research Statement

My research takes place on the boundary between pure mathematics and theoretical physics. Some of the key ideas in modern high-energy physics – coming from the study of "supersymmetric quantum field theories" – have been strikingly influential in a wide range of research programs in representation theory, algebraic geometry, and differential geometry. However, many of these influential physical constructions do not stand on firm mathematical ground, meaning that they can only be applied on a heuristic or ad-hoc basis. The long term goal of my research program is to develop supersymmetric quantum field theory (QFT) into its own discipline of study within pure mathematics. Even glimpses of this mathematical theory have already had significant repercussions in algebraic and differential geometry, and in representation theory. A famous example is how Seiberg and Witten's work on supersymmetric gauge theory led to the theory of Seiberg–Witten invariants, which revolutionized 4-manifold theory.

While ambitious, the goal of bringing supersymmetric QFT into mathematics is genuinely achievable. I have been including the word "supersymmetric" in the summary above very deliberately. Although the development of a complete formalism for quantum field theory still seems out of reach for mathematicians, more progress can be made when we restrict attention to the class of supersymmetric theories, which carry a great deal of extra structure. For example, we can work with "sectors" of a supersymmetric theory that are fixed by a choice of symmetry (the output of this procedure is usually called a "twist"). Studying these sectors is much more approachable using modern mathematical machinery, but the output still retains all the necessary information to be useful in pure mathematics. Indeed, in joint work with Yoo, I have shown how the new language of derived algebraic geometry allows one to cast into a sharp, mathematical form a proposal of Witten and Kapustin, which relates certain supersymmetric 4-dimensional gauge theories to the geometric Langlands program, an endeavor at the heart of contemporary representation theory.

In order to understand these supersymmetric quantum field theories, I apply a collection of tools that can broadly be referred to as "higher algebra", including ideas coming from derived geometry, homotopy theory and homotopical algebra, and the theory of ∞-categories. My work does not fit neatly into any one field – I am not exactly an algebraic topologist, an algebraic geometer or a representation theorist – but I engage with mathematicians across a number of different fields. For example, the sorts of ideas that I think about engage with representation theory (via Kapustin and Witten's approach to the geometric Langlands correspondence [KW07], the theory of symplectic duality introduced by Braden, Licata, Proudfoot and Webster [Bra+16; Bul+16] and the related development of Coulomb branches by Braverman, Finkelberg and Nakajima [BFN18]), with algebraic geometry (for instance, with Kontsevich's homological mirror symmetry program [Kon95]) and with differential geometry (through gauge theory and its application to manifold invariants, as in the theory of Donaldson and Seiberg–Witten invariants [SW94; Wit94] and the Donaldson–Thomas theory of threefolds [DT98].)

Collaboration and communication with theoretical physicists is also an essential part of this work, and has significantly furthered my research program so far. When I was a post-doc at the IHÉS I worked with the physicist Vasily Pestun, studying integrable systems related to 3d and 4d supersymmetric gauge theory [EP19]. My research goals related to QFT have allowed me to serve as a kind of ambassador between mathematicians and physicists, organizing several conferences, as well as smaller scale seminars, with the explicit goal of creating a dialogue between mathematicians and physicists whose interests overlap. In these conferences, as well as in seminars and classes in my current role at the University of Massachusetts Amherst, I have been bringing together graduate students in mathematics and physics, in order to foster this kind of interdisciplinary thinking early in students' careers.

The development of the fundamental building blocks of this theory of supersymmetric QFT comprises my current work and my near-term research objectives. The key tools I will need to construct these building blocks, coming from higher algebra, are already in place, as a consequence of my prior work and others in the field. Having laid this groundwork, I can now systematically use these tools to realize the foundation of the theory that I have envisioned. I will summarize some of this prior work, and my proposals for their application, in the remaining part of the introduction, and then give a more detailed account in the following sections. Each numbered entry refers to one thread within the study of supersymmetric QFT.

1. (**Classical local**) Many supersymmetric QFTs arise by quantizing a supersymmetric *classical* field theory. Roughly speaking, this means we study the solutions to a system of differential equations on a manifold. I study these classical systems from the point of view of higher algebra, using the language of the classical BV (Batalin–Vilkovisky) formalism. For example, one can model a classical system locally near a fixed solution either as a formal moduli problem, or equivalently as an dg Lie or L_∞ algebra. This language is very practical for performing actual computations. For example, I was able to produce a complete classification of supersymmetric Yang–Mills theories, and all of their twists, in all dimensions from this point of view [ESW22]. One of my medium-term research goals is to extend this work to theories of supergravity – I developed the study of some of the theories that I expect to arise in joint work with Williams [EW21] (see also Section 5.)

2. (**Classical global**) One can take an even richer approach to the study of classical field theories, in which one studies the whole moduli space of solutions to the equations of motion. There is a powerful toolkit in which these moduli spaces can be studied, known as "derived geometry" – this setting makes manifest structures on these moduli spaces that would not otherwise be visible, such as shifted analogues of symplectic structures. The derived language is currently most well developed in the world of algebraic geometry, which is also a rich setting for mathematical applications (for instance to the geometric Langlands program). In some of the most fundamental examples, those related to geometric Langlands, we are able calculate these derived moduli spaces of solutions, as I demonstrated in work with Philsang Yoo [EY18]. This provides a resolution to the previously unanswered question of how to incorporate algebraic structures into the gauge theoretic approach to the geometric Langlands program. Describing more general twisted supersymmetric QFTs from this point of view will be an important ingredient underlying some of the threads I describe in points 3–5 below.

3. (**Quantum local**) Now, we can begin to quantize these classical field theories. Using the local language for field theory from point 1 there is an effective family of techniques for constructing these quantizations, developed by Costello and Gwilliam in [Cos11; CG16; CG18b]. Using these techniques I can construct the quantum algebra of observables in any of the twisted supersymmetric QFTs. For one of the most important families of examples – those related to the geometric Langlands program – we made this construction in [EGW21]. These techniques are broadly applicable, and can be generalized to a very broad family of supersymmetric field theories. I proved in work with Safronov [ES19] that this procedure can be interpreted in the language of $\mathbb{E}_n$-algebras and factorization homology, widely studied in algebraic topology.

4. (**Quantum global**) Even better, one can combine the ideas from points 2 and 3. That is, quantize a classical field theory locally at every point in the moduli space of solutions, producing a family of quantum field theories over the moduli space. For example, in [EG21] we showed that the phenomenon of symmetry breaking in quantum field theory can be realized very naturally from this point of view. By applying this procedure to twists of supersymmetric field theories, we will be able to describe quantizations of all sorts of interesting moduli spaces – I will discuss some possible applications to manifolds with special holonomy in Section 1.

5. (**Quantum states**) So far I have been discussing the deformation quantization of observables in quantum field theories. There is a parallel theory of states in a quantum field theory, associated with geometric quantization. In my work with Philsang Yoo [EY19; EY20] I investigated this geometric quantization in a higher categorical setting, as developed by Safronov [Saf20], and its connections to the quantum geometric Langlands correspondence. In particular we explained how the theory of singular support in geometric Langlands arises in quantum field theory, which itself has an interesting interpretation in terms of the interaction between states and observables. One part of my research program is to understand this interpretation more precisely.

6. (**Defects**) Finally, there is a very rich family of structures that have been little explored in this context so far, the study of defects in supersymmetric quantum field theories associated to a submanifold of an n-manifold. Using a formalism for defects I have developed in work with Contreras and Gwilliam [CEG22] we can extend our quantum field theories to sheaves parameterized by a configuration space labelling defects. This allows for the consideration of genuinely non-perturbative phenomena using the language of factorization algebras and derived geometry.

1 Twisted Gauge Theory

In joint work with Safronov and Williams [ESW22], I gave a complete classification and description of supersymmetric twists of Yang–Mills gauge theories using formal derived geometry, at the classical level. For each super Yang–Mills theory, and each square-zero odd symmetry Q, we describe a concrete model for the twist by Q: these theories are typically realized as generalizations of Chern–Simons or BF theory.

Extending these to the quantum level requires the analysis of *anomalies* for these classical theories. In dimensions ≤ 8 these twisted theories are all 1-loop exact, meaning that computing the quantization requires only a single calculation involving Feynman diagrams with 1 loop. The local observables in a quantum field theory can be described using the language of factorization algebras. In the case of a topological twist, however, in joint work with Safronov [ES19] I showed that the local observables can be defined more concretely as an $\mathbb{E}_n$-algebra: an algebra over the operad of little disks. These $\mathbb{E}_n$ algebras are mathematically interesting objects: for small n an $\mathbb{E}_n$ structure can be thought of as Koszul dual to something like a quantum group, possibly with one or more spectral parameters. More specifically, we proved the following.

Theorem 1.1 ([ES19])**.** The algebra of local operators in a topologically twisted quantum field theory in dimension n can be canonically equipped the structure of an $\mathbb{E}_n$-algebra whenever the natural map extending observables defined on a ball of radius r to a ball of radius $R > r$ is a quasi-isomorphism. This condition is satisfied for all twists of supersymmetric Yang–Mills theories with matter in all dimensions.

The classification of twists includes familiar entries, such as Donaldson–Witten theory in dimension 4, as well as less familiar entries. For instance, there is a topological twist in dimension 7 that depends on a G_2-structure, and a topological twist in dimension 8 defined on a $\mathrm{Spin}(7)$-structure. The technique of factorization homology allows us to study the global observables on n-manifolds with special holonomy. One of my near-term research aims is the following.

Objective 1.2. Where the anomaly vanishes, compute the filtered $\mathbb{E}_n$-algebra of quantum observables in all topologically twisted supersymmetric field theories in dimensions ≤ 8. Use factorization homology to describe the algebra of global observables on the most general possible class of compact n-manifold in each case.

In order to compute the factorization homology on manifolds with interesting holonomy it is necessary to construct the topological quantum field theory in an appropriately equivariant way. In particular, it is necessary to show that an *anomaly* for this equivariant structure (the framing anomaly) vanishes. I studied these framing anomalies in a general topological context in work with Gwilliam [EG22], and therefore the necessary tools for computing this factorization homology are already in place.

Example 1.3. One of the most interesting examples occurs in dimension 4, where maximal super Yang–Mills theory has a rich family of twists studied, amond others, by Kapustin and Witten. In work with Gwilliam and Williams [EGW21] I constructed the framed $\mathbb{E}_4$ algebras associated to these theories and began the study of the factorization homology on oriented 4-manifolds.

Note that in many interesting examples the $\mathbb{E}_n$-algebra of observables will be contractible, it will only have interesting cohomology when considered as a filtered algebra, or alternatively as an explicit deformation of the algebra of observables in another twisted theory. These theories become interesting when we consider local observables in families parameterized by the classical moduli space of solutions to the equations of motion (or subspaces thereof: the space of solutions to the “equations of unbroken supersymmetry”, as recently studied in [SW21]). For example, the 7- and 8-dimensional twisted theories described above, when considered in families in this way, will describe quantizations of the moduli spaces of G_2-monopoles and $\mathrm{Spin}(7)$-instantons respectively.

2 Defects and Nonperturbative Aspects

The concept of a *defect* for a classical or quantum field theory can be investigated using the language of the BV formalism and of factorization algebras using an approach I have pursued in work with Ivan Contreras and Owen Gwilliam [CEG22]. Somewhat informally, one can analyze defects using the following framework.

Definition 2.1. A *defect* for a classical field theory $\mathcal{E}$ on a manifold M, along a submanifold $D \subseteq M$, is a choice of tubular neighborhood U for D together with a boundary condition for $\mathcal{E}$ along ∂U in the sense of Rabinovich [Rab21].

In the setting where the theory $\mathcal{E}$ is topological in the directions normal to D, this notion does not depend on the choice of tubular neighborhood. In contrast, in the non-topological case a more refined notion would involve an equivalence class of boundary conditions defined for increasingly small neighborhoods of D.

This theory of defects will allow me to begin to describe *non-perturbative aspects* of classical theory in a way not previously accessible in the formalism of factorization algebras. The main idea here is to construct quantum field theories in families over a parameter space that captures non-perturbative classical aspects of the field theory. The first natural example here involves defects. I will be able to describe families of field theories parameterized by a configuration space of defects. One of the first examples is foundational, but should be closely related to familiar examples for experts in vertex algebras.

Objective 2.2. Quantize the theory of a free boson on $\mathbb{C}$ in the presence of a finite set of point defects, labelled by integers. Realize this quantization as a sheaf of vertex algebras over the labelled configuration space of points in $\mathbb{C}$. Study the relationship between the cohomology of this sheaf and the lattice vertex algebra.

Another exciting example to investigate is that of Yang–Mills theory on $\mathbb{R}^4$, together with line defects along a set of parallel lines. The local quantum observables of this theory will produce a sheaf of factorization algebras over a labelled configuration space of points in $\mathbb{R}^3$. This situation was studied by [AST13]. Specifically, the labels are given by elements of $Z(G) \times Z(G^\vee)$ where G is the gauge group of the theory and $G^\vee$ is its Langlands dual. In particular, even though the perturbative theory is only sensitive to the Lie algebra, the family of theories over the configuration space depends on the specific form of the group.

Objective 2.3. Fix a reductive Lie algebra $\mathfrak{g}$ with adjoint form G_{ad}, and dual Lie algebra $\mathfrak{g}^\vee$ with simply adjoint form $\widetilde{G}^\vee_{\mathrm{ad}}$, and let $\mathcal{E}_{\mathrm{ad}}$ denote the sheaf over $\mathrm{Conf}(\mathbb{R}^3)$ of quantum observables of G_{ad}-Yang–Mills theory with dyonic line defects whose charges lie in the product of the root and coroot lattices. Identify the obstruction to descending along the map

$$\mathrm{Conf}(\mathbb{R}^3) \to \mathrm{Conf}(\mathbb{R}^3)^{\pi \times \pi^\vee},$$

where G', $G'^\vee$ are forms of $\mathfrak{g}$ and $\mathfrak{g}^\vee$, $\pi = Z(G')$ and $\pi^\vee = Z(G'^\vee)$, with a higher discrete gauge theory with discrete gauge group $\pi \times \pi^\vee$.

3 4d Gauge Theory and the Geometric Langlands Correspondence

I can extend this local understanding of twisted supersymmetric field theories considerably, focusing on a specific family of examples: twists of 4-dimensional field theories with maximal supersymmetry (referred to as $\mathcal{N} = 4$ supersymetry). This theory has a rich family of possible twists, including a $\mathbb{CP}^1$-family of topological twists whose relationship to geometric representation theory was first studied by Kapustin and Witten [KW07]. I studied these twists, and their connection to the quantum geometric Langlands correspondence, in a series of papers joint with Philsang Yoo [EY18; EY19; EY20]. We showed the following.

Theorem 3.1. The category $D_\kappa(\mathrm{Bun}_G(C))$ of κ-twisted D-modules arises by quantizing the moduli stack associated to a Kapustin–Witten twist of 4d $\mathcal{N} = 4$ gauge theory, where κ coincides with the "fundamental parameter" $\Psi \in \mathbb{CP}^1$ of Kapustin and Witten. If $\kappa \to \infty$, by considering boundary conditions compatible with a canonical choice of vacuum, the same quantization procedure produces the category $\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Flat}_G(C))$ of ind-coherent sheaves on the moduli stack of flat G-bundles with nilpotent singular support, as introduced by Arinkin and Gaitsgory.

Using this previous work, and Objective 1, I intend to develop a much more sophisticated understanding of these theories that can be applied to the geometric Langlands program. Using techniques recently developed by Gwilliam, Rabinovich and Williams [GRW20], given a stratified manifold with boundary, and the data of a classical supersymmetric field theory with boundary conditions and defects, I can compute the local observables in a twist of the whole configuration. The result will be a *stratified* factorization algebra.

Going even further, we can study *line defects* in Kapustin–Witten theory and its boundary theories. In work with Contreras and Gwilliam [CEG22] I introduced defects into the factorization algebraic description of quantum field theory. In current work in progress we are systematically studying examples that include twists of the Wilson and 't Hooft lines studied in 4d gauge theory.

Objective 3.2. In each of the Kapustin–Witten twists, compute:

1. The 4d stratified factorization algebras associated to Gaiotto corner configurations on $\Sigma \times \mathbb{R}^2_{\geq 0}$.
2. The category of line defects in the 3d boundary theories, and natural functors from these categories to the category of modules for the vertex algebra at the corner.

As described in the previous section, I computed and analyzed the $\mathbb{E}_4$-algebras in the interior of the corner in joint work with Gwilliam and Williams [EGW21], along with their factorization homology on curved 4-manifolds.

Let me elaborate on the meaning of these constructions, and their significance for geometric representation theory. Recent work of Gaiotto and Frenkel–Gaiotto [FG20] has analyzed the Kapustin–Witten twists of boundary conditions for 4d supersymmetric gauge theory, and applied this to the quantum geometric Langlands program at irrational level. I will first compute the stratified factorization algebra on a 4-manifold with boundary describing the observables in one of Kapustin–Witten's twists in the interior, and the observables in a twist of a supersymmetric boundary theory on the boundary.

Next, consider a 4-manifold with corners of the form $\Sigma \times \mathbb{R}^2_{\geq 0}$, where Σ is a Riemann surface. Work of Gaiotto–Rapčák [GR19] and Frenkel–Gaiotto describes field theories on such a corner configuration, where we consider a 4d twist of $\mathcal{N} = 4$ supersymmetric gauge theory on the interior, a twisted 3d boundary theory on each face of the boundary, and a 2d interface theory at the corner. This corner theory is not topological, but only holomorphic. When $\Sigma = \mathbb{R}^2$, the stratified factorization algebra will comprise an $\mathbb{E}_4$ algebra in the interior, a pair of $\mathbb{E}_3$-modules associated to the boundary walls, and a vertex algebra with the structure of a bimodule for the $\mathbb{E}_3$-algebras at the corner as indicated in Figure 1.

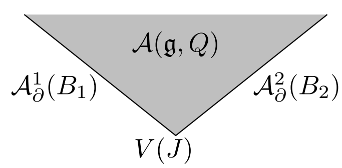

Figure 1: Illustration of the stratified factorization algebra data associated to a corner configuration on $\mathbb{R}^2 \times \mathbb{R}^2_{\geq 0}$. Here $\mathcal{A}$ is an $\mathbb{E}_4$-algebra constructed as the quantum algebra of observables in the interior, depending on a reductive Lie algebra $\mathfrak{g}$ and a topological twist Q; $\mathcal{A}^1_\partial$ and $\mathcal{A}^2_\partial$ are $\mathbb{E}_3$ algebras with an action of $\mathcal{A}$ associated to a pair of 3d boundary theories (B_1 and B_2); and V is a vertex algebra with the structure of a $(\mathcal{A}^1_\partial, \mathcal{A}^2_\partial)$-bimodule associated to a holomorphic junction J between the two boundary conditions.

Finally, as well as local operators, I can compute *line operators* in each of the 3d boundary theories. The collection of line operators in a 3d topological field theory has the structure of a braided monoidal dg-category.

Frenkel and Gaiotto proposed a description of integral kernels for the quantum geometric Langlands correspondence at irrational level using these ideas. Using my work with Yoo [EY19], I will also be able to address the rational level case. As my prior work shows, in the rational level case we obtain the correct categories of sheaves for the geometric Langlands program by considering the action of the algebra of local observables on the category of boundary conditions. The moduli space of *vacua* is the spectrum of the algebra of local observables. In the geometric Langlands program there is a correction term to the most straightforward statement, due to Arinkin and Gaitsgory [AG15]. I showed that this correction term arises in gauge theory by considering the localization at a point in the moduli space of vacua.

Another of my research objectives extends this idea further by using an interpretation that is also applicable to the topological field theories studied in the homological mirror symmetry community. Work of Johnson-Freyd and Scheimbauer [JFS17] gave a functorial model for relative topological field theories. It is well-known that the B-model with target $\mathcal{X}$, viewed as a functorial field theory assigning a category $\mathrm{QCoh}(\mathcal{X})$ (or the category of ind-coherent sheaves $\mathrm{IndCoh}(\mathcal{X})$), only defines a fully extended functorial field theory if $\mathcal{X}$ is smooth and proper – this excludes the 2d B-model coming from compactification of the 4d Kapustin–Witten B-twist along a Riemann surface C, where $\mathcal{X}$ is the moduli stack $\mathrm{Flat}_G(C)$ of flat G-bundles on C. On the other hand, we *can* associate a fully extended functorial field theory to this 2d B-model, which we call the theory of observables. This is the functorial field theory valued in a Morita category of $\mathbb{E}_2$ algebras that assigns the algebra of local observables to the point. I will use this to construct the Kapustin–Witten B-twist as a relative field theory. This can then be extended to more general Kapustin–Witten twists using the category of renormalized D-modules developed by Gaitsgory.

Objective 3.3. For a general class of derived stacks $\mathcal{X}$, show that the category $\mathrm{IndCoh}(\mathcal{X})$ defines an extended TQFT *relative* to its theory of observables. That is, the following natural transformation is relatively dualizable:

$$\begin{tikzcd} \mathrm{Bord}_2 \arrow[r, "\mathrm{Obs}"] \arrow[rr, bend right, "\mathrm{triv}"', ""{name=T}] & \mathbb{E}_2\text{-Alg} \arrow[r, "\text{-mod}"] \arrow[d, phantom, "\Downarrow \mathrm{IndCoh}(\mathcal{X})"] & \mathrm{dgCat}. \\ & {} & \end{tikzcd}$$

Show that in the case $\mathcal{X} = \mathrm{Flat}_G(C)$ arising by dimensional reduction from the 4d Kapustin–Witten B-twist, the choice of a vacuum defines a second relative field theory, and therefore by composing the two we can define a fully extended functorial TQFT. Prove a corresponding statement for more general 4d twists, starting with a category of renormalized D-modules.

4 5 and 6d Twisted Gauge Theory and Integrable Systems

In joint work with Vasily Pestun [EP19], I studied moduli spaces arising either from twists of 5 dimensional super Yang–Mills theories, or as the Coulomb branches of 4-dimensional quiver gauge theories of ADE type. Our work concerns the following moduli space (versions of which have been studied previously by Hurtubise-Markman [HM02], Bouthier [Bou14] and Frenkel-Ngô [FN11]).

Definition 4.1. The moduli space of *multiplicative G-Higgs bundles* on a curve C consists of pairs (P, ϕ) where P is a principal G-bundle on C and ϕ is a meromorphic automorphism of P. We fix the locations of the poles of ϕ at a divisor $D = \{z_1, \ldots, z_k\}$. We can also fix the local behaviour near the poles – controlled by a dominant coweight $\omega^\vee_{z_i}$ of G at each puncture. Denote these moduli spaces by $\mathrm{mHiggs}_G(C, D)$ (without fixed local behaviour) and $\mathrm{mHiggs}_G(C, D, \omega^\vee)$ (with fixed local behaviour where $\omega^\vee$ denotes a k-tuple of dominant coweights) respectively.

We focus our attention on the following rational/trigonometric/elliptic trichotomy:

- (Rational) $C = \mathbb{CP}^1$ and we fix a framing at the point ∞.
- (Trigonometric) $C = \mathbb{CP}^1$ and we fix a B_+ reduction at ∞ and a B_--reduction at 0 so that the respective induced T-reductions coincide.
- (Elliptic) $C = E$ is an elliptic curve (with no additional decorations).

In these cases the moduli space – like the ordinary Hitchin system – has the structure of an algebraic integrable system which we can naturally describe using the theory of shifted Poisson and coisotropic structures [Cal+17; MS18]. In particular it has an algebraic symplectic structure. If one doesn't fix dominant coweights at the punctures the full infinite-type moduli space has a Poisson structure and the moduli spaces with fixed coweights are symplectic leaves.

A theorem of Charbonneau and Hurtubise [CH10] (for GL_n) and Smith [Smi16] (for general G) tell us that the moduli space of multiplicative G-Higgs bundles (or rather its polystable locus) is analytically isomorphic to the moduli space of G-monopoles on $C \times S^1$. In the rational case this moduli space of periodic monopoles can be realized as a hyperkähler quotient. In particular it is holomorphic symplectic. In this case Pestun and I prove the following.

Theorem 4.2 ([EP19]). In the rational case, the isomorphism identifying the moduli space of periodic monopoles and the moduli space of multiplicative Higgs bundles is compatible with the holomorphic symplectic structures on both sides. The holomorphic symplectic structure on the multiplicative Higgs moduli space can be identified with the pullback of the Poisson Lie structure under the map $\mathrm{mHiggs}_G^{\mathrm{fr}}(\mathbb{CP}^1, D) \to G_1[[z^{-1}]]$ given by restriction to a formal neighbourhood of the framed point ∞. For $G = \mathrm{GL}_n$ the symplectic leaves coincide with the symplectic leaves classified by Shapiro [Sha16].

Our equivalence promotes the holomorphic symplectic structure on $\mathrm{mHiggs}_G^{\mathrm{fr}}(\mathbb{CP}^1, D, \omega^\vee)$ to a hyperkähler structure. We can identify the holomorphic symplectic space obtained by rotating to a point q in the twistor sphere with the moduli space of *q-connections*: principal G-bundles P equipped with a meromorphic isomorphism $P \to q^*P$ from P to its translate.

This work is motivated in part by the work of Nekrasov and Pestun [NP12], which implies that moduli spaces of multiplicative Higgs bundles arise as the Seiberg-Witten integrable system associated to ADE quiver gauge theories. In particular they should admit natural hyperkähler structures. Nekrasov and Pestun also conjectured that the quantization of these moduli spaces should be closely related to the Yangian – this conjecture is verified by our Poisson map to the Poisson Lie group that quantizes to give the Yangian algebra. The moduli spaces $\mathrm{mHiggs}_G^{\mathrm{fr}}(\mathbb{CP}^1, D, \omega^\vee)$ quantize to give representations of the Yangian first studied by Gerasimov, Kharchev, Lebedev and Oblezin [Ger+05].

The ordinary Hitchin system and the multiplicative Hitchin system themselves comprise two parts of another rational/trigonometric/elliptic trichotomy; one can additionally consider *elliptic* Hitchin systems, given by maps into the moduli stack $\mathrm{Bun}_G^{\mathrm{ss}}(E)$ of semistable G-bundles on an elliptic curve. These moduli stacks arise either from twists of 6-dimensional super Yang–Mills theories, or as the Coulomb branches of 4d quiver gauge theories of affine ADE type. Let us consider the case where $G = \mathrm{GL}_n$. In complex structure J, via the Nahm transform we can identify the component of the elliptic Hitchin system of degree k with a moduli space of flat bundles on a punctured torus $\mathrm{Flat}_{\mathrm{GL}_k}(T^2, \{p_1, \dots, p_n\})$. These will describe symplectic leaves in the double loop group $\widehat{\widehat{\mathrm{GL}}}_n$. The following research goal is interesting even if $n = 2$.

Objective 4.3. Quantize this description in order to obtain a new geometric description of the quantum toroidal algebra $U_{q,t}(\widehat{\widehat{\mathfrak{gl}}}_n)$, generalizing Schiffmann-Vasserot's identification for $n = 1$ of the quantum toroidal algebra and the stable limit of the spherical DAHA [SV13].

5 Twisted Supergravity and Twisted Holography

While widely studied in physics, the mathematical study of supersymmetric gravity theories is much less developed than that of gauge theory; until now, almost all supergravity theories have been viewed as mathematically intractible. My techniques, however, allow me to take a similar approach to the one discussed for gauge theories above, and compute the supersymmetric twists of these sophisticated supergravity theories. While the interpretation of this procedure in the context of gravity is slightly different, many of the same techniques can be applied.

There is a maximal supergravity theory defined on flat 11-manifolds. Supersymmetric theories in 11-dimensions admit several inequivalent twists, which I can classify using a result of Igusa [Igu70], including one defined on the product of a Calabi-Yau 5-fold and a line, and one defined on the product of a G_2-manifold and a complex surface. The former twist has been conjecturally described by Costello and Li [CL16] in terms of Kodaira–Spencer, or BCOV, theory.

In work with Williams [EW21]I studied 4d and 5d theories, which we conjecture are given by suitable twists of minimal supergravity in these dimensions. These theories can be obtained from Costello and Li's conjectural description by compactification. These theories are very interesting in their own right. For example, we proved the following result.

Theorem 5.1 ([EW21])**.** There is an interacting QFT that can be defined on $\mathbb{C}^{2n} \times \mathbb{R}^m$ for any n and m, involving the natural holomorphic symplectic form on $\mathbb{C}^{2n}$. This theory is 1-loop exact, and it has a 1-loop anomaly given by a class in the Gelfand–Fuchs cohomology of the Lie algebra of Hamiltonian vector fields on $\mathbb{C}^{2n}$. This class vanishes if $m \leq 6$. The resulting factorization algebra of quantum local observables is not locally constant, but the action of the group $\mathbb{R}^{4n+m}$ of all translations is homotopically trivialized.

In my research going forward, I intend to investigate the problem of the classification and computation of all possible twists of supergravity theories, using the same techniques as used in my work with Safronov and Williams [ESW22]. While supergravity theories have much more complicated action functionals than Yang–Mills theories, performing the twist of the underlying free theory is a matter of homological algebra: one can write down the BV complex of the underlying free theory starting from the data of the quadratic term in the Taylor expansion of the action functional around a fixed classical solution. While the higher order terms in this Taylor expansion are typically very complicated for supergravity theories, after twisting, I expect to be able write down a "smaller" quasi-isomorphic complex with much fewer fields. The higher order terms can be computed using homotopy transfer along this quasi-isomorphism, using explicit formulas as given, for example, by Loday and Vallette [LV12, Theorem 10.3.9] to show that many terms in the complicated supergravity interaction no longer contribute after twisting.

Objective 5.2. Prove the Costello–Li conjecture, as well as our 4- and 5-dimensional conjectures, by computing all twists of 11-dimensional supergravity.

The 4- and 5-dimensional computations are current work in progress with Brian Williams and Ingmar Saberi.

5.1 The Pure Spinor Formalism

A separate aspect of my work addresses the direct construction of supersymmetric field theories, including the construction of 11-dimensional supergravity theory. The interaction in 11-dimensional supergravity theory is, on its face, somewhat opaque. My work on the pure spinor formalism will allow for a much more intuitive realization of the origin of this interaction in the BV formalism.

In joint work with Ingmar Saberi and Fabian Hahner I developed a "derived" version of the pure spinor formalism. The pure spinor formalism allows for the construction of supersymmetric field theories starting from the datum of a coherent sheaf on the nilpotence variety $\mathcal{N}$: the space of square-zero elements of the supersymmetry algebra. This classical approach produces many interesting examples, but does not work universally. Our derived approach, however, provides an *equivalence of dg-categories*, hence a systematic construction of all examples.

Theorem 5.3 ([EHS22])**.** Let $\mathfrak{n}$ be a supertranslation algebra acted on by a Lie algebra $\mathfrak{g}_0$. Let $\mathfrak{g} = \mathfrak{g}_0 \ltimes \mathfrak{n}$ be the associated supersymmetry algebra. There is an equivalence of dg-categories

$$\mathrm{Mult}_{\mathfrak{g}} \leftrightarrows \mathrm{Coh}(\mathrm{Spec}\, \mathrm{C}^\bullet(\mathfrak{n}))^{\mathfrak{g}_0}$$

between $\mathfrak{g}$-multiplets – affine dg vector bundles over the even part of $\mathfrak{n}$ with a geometric action of $\mathfrak{g}$ – and $\mathfrak{g}_0$-equivariant dg-modules over the Chevalley–Eilenberg algebra $C^\bullet(\mathfrak{n})$.

We think of $\mathrm{Spec}\, \mathrm{C}^\bullet(\mathfrak{n})$ as a derived enhancement of the nilpotence variety $\mathcal{N}$, which can be identified with $\mathrm{Spec}\, \mathrm{H}^0(\mathfrak{n})$.

In order to fully take advantage of this result, one of my research objectives is to extend it to capture not merely multiplets, but genuine supersymetric classical field theories. In the BV formalism, to obtain a classical supersymmetric field theory from a multiplet we need two additional pieces of data. First, a shifted symplectic pairing of degree -3, and second, a dg Lie (or more generally, L_∞) structure.

Objective 5.4. Characterize the image of the category of classical field theories under the derived pure spinor equivalence, as a category of coherent sheaves on the derived nilpotence variety equipped with additional structure.

The main step needed to achieve this objective is to identify an appropriate "convolution" monoidal structure on the category of coherent sheaves. The additional structures needed to define classical field theories are built using the tensor product of multiplets, and the pure spinor equivalence is not monoidal for the standard monoidal structures on the two sides. Instead the tensor product of multiplets corresponds to a novel convolution monoidal structure on the right-hand side, which we will identify.

With this objective in hand, we can apply it to the construction of supersymmetric field theories in a systematic way. For example, the 11-dimensional supergravity theory arises, under the pure spinor equivalence, from a shift of the structure sheaf of the appropriate derived nilpotence variety $\mathcal{N}_{11}$. With the previous objective in hand, the pure spinor formalism will lead to a more natural description of the 11-dimensional supergravity interaction.

Objective 5.5. Describe structures on $\mathcal{N}_{11}$ that produce interactions – i.e. an L_∞ structure – on the corresponding supergravity multiplet.

Even better, this will allow for a more direct computation of the twists of supergravity theories, by working on the coherent sheaf side of the correspondence and computing the appropriate deformation in that setting.

5.2 Twisted Holography

With both descriptions for twisted gauge theory and twisted supergravity in hand, I will be able to investigate twisted examples of the AdS/CFT, or holographic, duality between such pairs of theories. A proposal for how this duality behaves after twisting was recently made by Costello [Cos17; CG18a; CP20], in terms of the idea of Koszul duality. Consider $\mathbb{R}^n \subseteq \mathbb{R}^{10}$, where $\mathbb{R}^n$ is thought of as the worldsheet of a stack of N D_{n-1}-branes. Using what the research described so far, I will then be able to construct two $\mathbb{E}_n$ algebras.

1. The algebra $\mathcal{A}_{\mathrm{gauge}}$ of local operators in a topological or partially topological twist of supersymmetric gauge theory on $\mathbb{R}^n$ for the gauge group GL_N, in the limit $N \to \infty$.
2. The algebra $\mathcal{A}_{\mathrm{gravity}}$ of local operators in a topological or partially topological twist of supergravity on $\mathbb{R}^{10} \setminus \mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}_{>0} \times S^{9-n}$.

For twisted supergravity theories on $\mathbb{R}^{10}$, we can study *defect* theories along $\mathbb{R}^n$, using the language of my work [CEG22] discussed above. As a result the latter algebra, $\mathcal{A}_{\mathrm{gravity}}$ will naturally act on the algebra of observables in any defect theory. The idea of twisted holography is to characterize the universal defect theory as the Koszul dual algebra $\mathcal{A}^!_{\mathrm{gravity}}$. The twisted holography conjecture identifies this algebra with a deformation of the algebra $\mathcal{A}_{\mathrm{gauge}}$.

The idea of twisted holography fits into an existing research program, but so far there has been little research from the "ground up", starting with the full supergravity and supersymmetric gauge theories. Instead, existing research has largely begun with conjectural models for the twisted theories and performed compatibility checks to see that the calculations "make sense" from the point of view of the full supersymmetric theories, but without engaging with these theories in their entirety.

Objective 5.6. Construct twisted holography dualities in the case $n = 4$ by realizing the algebra of local operators in twisted supergravity theory as a deformation of the Koszul dual to the algebra of local operators in 4d $\mathcal{N} = 4$ gauge theory.

This result will have applications to the theory of quantum groups. For example, there is a 4d partially topological twist, where the Koszul dual to the algebra of local observables can be identified with a quantum deformation of the enveloping algebra $U(\mathfrak{gl}_N[\varepsilon][z])$, where ε is a parameter of degree 1. This story parallels work of Costello, and later Costello, Witten and Yamazaki, who studied the Koszul dual to the algebra of local operators in a twist of 4d $\mathcal{N} = 1$ gauge theory, and proved that it recovered the Yangian quantum group. Using twisted holography, we will identify a deformation of this quantum group, in the large N limit, with the observables in twisted supergravity theory. This gives the quantum group a lot of additional structure, for example, it will canonically act on the category of line operators in the twisted supergravity theory, or on the algebra of observables in the twist of any supersymmetric boundary theory.

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