A Non-Perturbative Description for Twists of Classical Field Theories

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1 Introduction

The idea of *twisting* a field theory goes back to Witten [Wit88a] (expanded on by Eguchi and Yang [EY90]). Witten showed that starting from an $N = 2$ supersymmetric gauge theory one could obtain a *topological theory* by considering the Q-cohomology of the observables with respect to a suitable odd symmetry Q satisfying $Q^2 = 0$. Physicists extended this idea to several other situations [VW94] [Yam88] [KW06] [Wit88b] [Wit11] , each of which yielded a "topological sector" of a supersymmetric field theory.

In this talk I'll describe a natural way of thinking about this idea of twisting, following work of Costello [Cos11a] that can be applied in a range of contexts encompassing but extending these examples. Furthermore I'll explain how, given the data describing a twist, to compute the twisted theory non-perturbatively, i.e. on the level of the full derived critical locus of an action functional. In particular this recovers interesting moduli spaces that occur in the geometric Langlands program as twists of $N = 4$ supersymmetric gauge theories (as one would expect from the work [KW06] of Kapustin and Witten). This is based on work in progress with Philsang Yoo.

2 The Idea of Twisting

I can describe twists in two possible frameworks for classical field theory. One is the *perturbative* framework used in the work of Costello and Gwilliam [CG14], where a theory is locally defined by an *elliptic moduli problem* or *elliptic L_∞ algebra* equipped with a pairing of a suitable degree. This kind of theory is amenable to quantisation by the renormalisation methods described in [Cos11b]. Alternatively we could use a global *perturbative* approach where the theory is locally defined by a derived stack with a -1-shifted symplectic pairing (in the sense of [PTVV13]). This definition is almost certainly too abstract for practical purposes. We can describe the twist in such a framework but we can't really say anything useful about it. However, this does allow us to twist interesting *examples* non-perturbatively and check that one obtains interesting, natural answers. The non-perturbative construction builds on the perturbative one, which was developed by Costello for 4d supersymmetric gauge theories [Cos11a].

Example 2.1. Some of the main motivating examples for us are the topological twists of $N = 4$ super Yang-Mills defined by Kapustin and Witten. There are two particularly interesting twists, called the A- and B-twists (the names are chosen so that their dimensional reductions to 2d are the A- and B-models with interesting targets). We prove that, if Σ is a compact complex curve, the moduli spaces of derived solutions to the equations of motion in the A- and B-twists on $\Sigma \times S^1$ (that is, the phase spaces in the theories obtained by dimensional reduction along Σ are given by

$$\mathrm{EOM}_A(\Sigma \times S^1) \cong T^*(\mathrm{Bun}_G(\Sigma \times S^1))_{\mathrm{dR}}$$
$$\text{and } \mathrm{EOM}_B(\Sigma \times S^1) \cong T^* \mathrm{Loc}_G(\Sigma \times S^1)$$

respectively. One can compute the Hilbert spaces associated to these phase spaces by geometric quantisation, obtaining $\Omega^\bullet(\mathrm{Bun}_G(\Sigma \times S^1)) \cong \mathrm{HH}_*(\text{D-mod}(\mathrm{Bun}_G(\Sigma)))$ and $\mathcal{O}(\mathrm{Loc}_G(\Sigma \times S^1)) \cong \mathrm{HH}_*(\mathrm{QCoh}(\mathrm{Loc}_G(\Sigma)))$. Furthermore one can analyse the operations on these spaces given by bordisms, for instance in the B-model case one obtains algebraic string topology operations. This recovers structures expected if we are to have compatibility with geometric Langlands, as Kapustin and Witten leads us to expect.

Let's recall a definition of a classical perturbative field theory before continuing.

Definition 2.2. An *elliptic L_∞ algebra E* on a manifold X is a local L_∞ algebra which is elliptic as a cochain complex. A *perturbative classical field theory* is an elliptic L_∞ algebra E equipped with a non-degenerate, invariant, symmetric bilinear pairing

$$\langle -,- \rangle \colon E \otimes E[3] \to \mathrm{Dens}_X$$

where Dens_X denotes the bundle of densities on X. Here *invariant* means that the induced pairing on the sheaf of compactly supported sections

$$\int_X \langle -,- \rangle \colon \mathcal{E}_c \otimes \mathcal{E}_c \to \mathbb{C}$$

is invariant.

In particular, such objects arise by the classical BV formalism, by taking polyvector fields on the space Φ of fields with its classical differential ι_{dS}. In other words, the derived critical locus of action always has a -1 shifted symplectic structure, and here E is the -1-shifted tangent bundle at a point.

The data required to twist a classical field theory is the action of a certain supergroup. Define a supergroup

$$H = \mathbb{C}^\times \ltimes \Pi\mathbb{C}$$

where $\mathbb{C}^\times$ acts with weight 1. This group arises as the group of automorphisms of an odd complex line.

Definition 2.3. *Twisting data* for a classical field theory Φ on a manifold X is a local action (α, Q) of H on $\Phi(U)$ for all U. That is, in the perturbative case Φ is a sheaf of L_∞ algebras with H-module structure, and in the non-perturbative case Φ is a sheaf of derived stacks with H-action. In our notation, α is a $\mathbb{C}^\times$ action, and Q is an odd infinitesimal symmetry with α-weight 1.

Remark 2.4. It's probably useful to recall what this means in the perturbative case. If $\mathcal{E}$ is an L_∞ algebra and G is an algebraic supergroup, write $C^\bullet(G)$ for the algebra of cochains on BG, and $C^\bullet(G)^\natural$ for the graded algebra obtained by forgetting the differential. A *G-module structure* is a $C^\bullet(G)^\natural$-module structure on $\mathcal{E}$, inducing an action on the Chevalley-Eilenberg cochains $C^\bullet(\mathcal{E})$, so that the L_∞ structure maps ℓ_n are $C^\bullet(G)$-linear. If furthermore $\mathcal{E}$ is equipped with an invariant pairing, we ask for this pairing to define a $C^\bullet(G)^\natural$-module map $\mathcal{E} \to \mathcal{E}^!$.

In natural examples we obtain this action from a local *supersymmetry action*, that is from the local action of a suitable supersymmetry algebra on Φ. We obtain twisting data by choosing a supercharge Q satisfying $Q^2 = 0$, and a copy of $\mathbb{C}^\times$ inside the group of R-symmetries so that Q has weight one. Such a $\mathbb{C}^\times$ always exists – and the choice of one doesn't really affect the eventual result – so the choice of Q is the important thing. As we'll see, judicious choice of Q yields theories with strong symmetry properties, for instance topological theories, as in the examples from the physics literature motivating this whole machine.

3 Twisting Perturbative Theories

Now, what does it actually *mean* to twist a theory, having chosen such data? We'll begin by discussing and motivating the perturbative picture. We use the following simple fact about the supergroup H.

Lemma 3.1. There is an equivalence of categories

$$F \colon \{\text{(super) vector spaces with an action of } H\} \to \{\text{(super) cochain complexes}\}.$$

This is easy to see: the $\mathbb{C}^\times$ action yields a grading by weight, and a generator for the $\Pi\mathbb{C}$ action yields a differential of degree one. With this in mind we can easily define the twist of a classical perturbative field theory.

Definition 3.2. The *twist* $\mathcal{E}^Q$ of a classical perturbative field theory $\mathcal{E}$ with respect to twisting data (α, Q) is the new sheaf of cochain complexes obtained by applying the functor F to $\mathcal{E}$ locally to obtain a double complex, then taking the total complex.

It's easy to see that $\mathcal{E}^Q$ is still a classical field theory. The twisted theory inherits an L_∞ structure from $\mathcal{E}$ since the H-action respects the L_∞ structure. The complex remains elliptic because the new cochain structure descends to the complex $\pi^*\mathcal{E}$ of symbols, where π is the projection from the cotangent bundle of spacetime, and the total complex with respect to Q and the original differential remains exact (since the spectral sequence of the double complex is contractible at the E_2 page). The invariant pairing on $\mathcal{E}^Q$ is the one inherited from the pairing on $\mathcal{E}$: it's easy to see this is well-defined, and remains nondegenerate and the correct degree because the H-action was compatible with the pairing.

Where does this definition come from? Well, it's actually fairly natural, especially when our twisting data comes from a local supersymmetry action. One might come to it from a line of reasoning like the following.

- The real motivation behind twisting is that one expects, given an odd square zero symmetry Q (i.e. an action of $\Pi\mathbb{C}$), after passing to the Q-invariants of a classical theory, all Q-exact infinitesimal symmetries – those of form $X = [Q, Y]$ – will act trivially. For instance, when Q is a square zero supercharge in a supersymmetry algebra, there will be a family of Q-exact translations which will act trivially in the twisted theory. It's always possible (in even dimensions) to find supercharges Q with a half-dimensional space of Q-exact translations, in this case the the twisted theory will be *holomorphic*. In many examples it's possible to find supercharges Q so that *all* translations are Q-exact, in which case the twisted theory will be *topological*.
- This is all well and good, but when one *actually* computes the (derived) Q-invariants one obtains a family of classical field theories over the space $B(\Pi\mathbb{C})$, that is a module over $\mathbb{C}[[t]]$ where t is a fermionic degree 1 parameter. One really wants to restrict interest to a generic fibre of this family.
- To do this we restrict to the odd formal *punctured* disc, or equivalently invert the parameter t, then take invariants for an action α of $\mathbb{C}^\times$ for which t has weight 1, thus extracting a "generic" fibre instead of the special fibre at 0. It's important to restrict to the formal punctured disc, since not all these invariant fields extend across zero: if we just took $\mathbb{C}^\times$ invariants in $\mathcal{E}[[t]]$ we'd obtain elements of $\mathcal{E}$ of the form ϕt^k where ϕ had weight $-k$. In particular we'd find ourselves throwing away everything of positive $\mathbb{C}^\times$ weight in $\mathcal{E}$.
- Now, this procedure is exactly the same as the definition we gave above. We specify twisting data (α, Q). Taking derived Q invariants corresponds to taking the complex $\mathcal{E}[[t]]$ with differential $d_\mathcal{E} + tQ$. Inverting t and taking invariants under the action α is then the same as adding the α weight to the original grading, and adding Q to the original differential, just as in our definition.

Now, we can turn the "expectation" from the first bullet point into a genuine statement.

Proposition 3.3. Suppose twisting data (α, Q) comes from the action of a supersymmetry algebra $\mathcal{A}$. The action of the Chevalley-Eilenberg cochains $C^\bullet(\mathcal{A})$ on the theory $\mathcal{E}$ defines an action of $C^\bullet((H^\bullet(\mathcal{A}), Q)$ on the twisted theory $\mathcal{E}^Q$, where we think of Q as a fermionic endomorphism of cohomological degree 0 acting on $\mathcal{A}$, and hence on $C^\bullet(\mathcal{A})$. Furthermore the action of the Poincaré algebra factors through the action of this algebra.

Remark 3.4. In particular, this tells us that Q-exact translations act trivially in the twisted theory.

Proof. We use the fact that, since $\mathcal{A}$ acts by symmetries, $[A, B](\phi) = A(B(\phi)) - B(A(\phi))$. Let ϕ and $\phi + Q\psi$ be equivalent fields in $\mathcal{E}^Q$, and let $A \in C^1(\mathcal{A})$ be a symmetry. The action of A on $\phi + Q\psi$ is by

$$\begin{aligned} A(\phi + Q\psi) &= A\phi + AQ\psi \\ &= A\phi + QA\psi - [Q, A]\psi \\ &= A\phi - [Q, A]\psi \end{aligned}$$

since $QA\psi = 0$ in $\mathcal{E}^Q$. This expression in turn equals $A\phi$ up to Q-exact elements of the supersymmetry algebra, so we have a well-defined action of the Q-cohomology.

Now, let $A = [Q, \lambda] \in C^1(\mathcal{A})$ be a Q-exact symmetry. The action of A on a field ϕ in $\mathcal{E}^Q$ is by

$$\begin{aligned} A\phi &= [Q, \lambda]\phi \\ &= Q\lambda\phi - \lambda Q\phi \\ &= 0 - 0 \end{aligned}$$

since $Q\lambda\phi$ and $Q\phi$ vanish in $\mathcal{E}^Q$. □

Example 3.5. The examples we're most interested in are those coming from $N = 4$ supersymmetric gauge theories in four dimensions, and where the twisting data comes from the supersymmetry algebra. There are several possible linearly independent topological supercharges here, i.e. supercharges Q for which all translations are Q-exact, but we're most interested in the so-called A- and B-twists.

4 Twisting Non-Perturbative Theories

Now, we'd like to be able to start with a classical field theory from a non-perturbative point of view – a shifted symplectic derived stack obtained as the derived critical locus of an action functional – and define its twist with respect to an action of the supergroup H. We already know how to do this formally locally near every classical point, so the challenge is to somehow glue these local perturbative twists together into a global twist. We'll do this in two steps; first we'll prove that the twisted theory is totally *determined* by its perturbative data, then we'll describe two constructions that show that a global object with the right formal data *exists*.

In order to prove any kind of uniqueness, let's describe exactly how to go from the global derived stack to a family of perturbative field theories. We'll need some language from derived algebraic geometry.

Definition 4.1. A *prestack* $\mathcal{X}$ is a functor of ∞-categories

$$\mathcal{X} \colon \mathrm{cdga}_{\leq 0} \to \mathrm{sSet}$$

from commutative dgas in non-positive degrees to simplicial sets. If the functor $\mathcal{X}$ factors through the inclusion $\mathrm{Set} \hookrightarrow \mathrm{sSet}$ of discrete simplicial sets then we call it a *prescheme*. We denote by $H^0(\mathcal{X})$ the left Kan extension of a prestack $\mathcal{X}$ along the functor $H^0 \colon \mathrm{cdga}_{\leq 0} \to \mathrm{cRing}$.

Definition 4.2. If $\mathcal{X}$ is a prestack, its *k-shifted tangent space* is the prestack $T[k]\mathcal{X}$ given by $T[k]\mathcal{X}(R) = \mathcal{X}(\tau_{\leq 0}(R \otimes \mathbb{C}[\varepsilon]))$, where ε is a parameter of degree $-k$ satisfying $\varepsilon^2 = 0$, and where $\tau_{\leq 0}$ is the cohomological truncation. The shifted tangent space naturally maps to $\mathcal{X}$, and we write $T_p[k]\mathcal{X}$ for the pullback under the inclusion of a closed point $p \in H^0(\mathcal{X})$.

Remark 4.3. Now, we'll need to impose some technical assumptions. By a *derived stack* we'll mean a derived Artin stack which is locally of finite presentation and k-geometric for some k in the sense of [TV08], [Gai12]. In particular, under these conditions the tangent space $T\mathcal{X}$ is neatly compatible with the cotangent complex $\mathbb{L}_\mathcal{X}$ as in [TV08] 1.4.1, and shifted p-forms admits a description in terms of the shifted tangent space, as in [PTVV13] 1.14.

Definition 4.4. If $\mathcal{X}$ is a derived stack then its shifted tangent complex $T_\mathcal{X}[-1]$ defines an *L_∞-space* over $\mathcal{X}$. We can define this in terms of its Chevalley-Eilenberg cochain complex, which will be a module over $\Omega^\natural_\mathcal{X} = \widehat{\mathrm{Sym}}_{\mathcal{O}_\mathcal{X}}(\mathbb{L}_\mathcal{X}[-1])$ with underlying graded vector space

$$\begin{aligned} \widehat{\mathrm{Sym}}_{\Omega^\natural_\mathcal{X}}((\mathbb{T}_\mathcal{X}[-1])^\vee[-1]) &= \widehat{\mathrm{Sym}}_{\Omega^\natural_\mathcal{X}}(\mathbb{L}_\mathcal{X}) \\ &= \widehat{\mathrm{Sym}}_{\mathcal{O}_\mathcal{X}}(\mathbb{L}_\mathcal{X} \oplus \mathbb{L}_\mathcal{X}[-1]) \end{aligned}$$

and degree 1 differential given by the sum of two terms: the derivation extended from the identity map $\mathbb{L}_\mathcal{X} \to \mathbb{L}_\mathcal{X}[-1]$, and the derivation extended from the dual to the Lie bracket $T_\mathcal{X}[1] \otimes T_\mathcal{X}[1] \to T_\mathcal{X}[1]$. For this to define an

L_∞ space structure we just need the differential to vanish on $\mathrm{Sym}^0_{\Omega^\natural_{\mathcal{X}}}(\mathbb{L}_{\mathcal{X}})$ modulo the ideal $\Omega^{\geq 1}_{\mathcal{X}} = \widehat{\mathrm{Sym}}^{\geq 1}(\mathbb{L}_{\mathcal{X}}[-1])$, which is manifestly the case.

We can now state in what sense global twists are unique.

Theorem 4.5. Let X and Y be preschemes such that $H^0(X)$ and $H^0(Y)$ are ordinary schemes. Suppose we're given isomorphisms $f_X\colon C \to H^0(X)$ and $f_Y\colon C \to H^0(Y)$ for a fixed scheme C, and suppose there exists a quasi-isomorphism $f_X^*T[-1]X \to f_Y^*T[-1]Y$ as sheaves of L_∞ algebras. Then there is a canonical isomorphism of prestacks $f\colon X \to Y$.

Furthermore if X and Y admit G actions for a group scheme G such that the equivalence between $f_X^*T[-1]X$ and $f_Y^*T[-1]Y$ is G-equivariant then f descends to an isomorphism of prestacks $X/G \to Y/G$.

Proof. Define a sheaf $\mathcal{F}$ of sets on the small Zariski site over C by

$$\mathcal{F}(U) = \left\{ \text{Isomorphism } X_U \to Y_U \text{ over } C \text{ such that the diagram } \begin{array}{ccc} TX_U & \longrightarrow & TY_U \\ \downarrow & & \downarrow \\ TX & \longrightarrow & TY \end{array} \text{ of sheaves over } C \text{ commutes} \right\}$$

where X_U is the pullback of X along an immersion $g\colon U \to C$. We'll show that $\mathcal{F}(U)$ is a single point for all U, thus in particular there is a unique isomorphism $X \to Y$ of prestacks over C compatible with the given equivalences of shifted tangent spaces. Since $\mathcal{F}$ is a sheaf it suffices to show $\mathcal{F}(U)$ is trivial when U is affine, and indeed since triviality can be checked at the level of stalks it suffices to show triviality when $U = \operatorname{Spec} R$ is affine local.

First suppose R is local Artinian. Then we can recover X_U from its deformation theory. Indeed, we have an isomorphism of prestacks $\mathrm{MC}_{T_p[-1]X_U} \cong X_U$. A priori this is only an isomorphism of formal moduli problems, but X_U can be determined as a prestack from its values on local Artinian rings. If S is any cdga in degrees ≤ 0 then then $X_U(S) = X_U(S_k)$, where S_k is the ring generated by the negative degree elements and the elements s in degree zero satisfying $s^k = 0$. The integer k here is the smallest integer such that $\mathfrak{m}^k = 0$ and $\mathfrak{m}_p^k = 0$ where $\mathfrak{m}$ is the maximal ideal of R, $\mathfrak{m}_p$ is the maximal ideal of R_p, and R_p is the local ring of X at p. To see this, observe that

$$X_U(S) = X(S) \times_{C(S)} \mathrm{Hom}(R, S)$$

and $\mathrm{Hom}(R, S) \cong \mathrm{Hom}(R, S_k)$. This in turn is isomorphic by base change to

$$\mathrm{Hom}(R_p, S) \otimes_{\mathrm{Hom}(H^0(R_p), S)} \mathrm{Hom}(R, S),$$

because the map $\operatorname{Spec} R \to C$ factors through Spec of the local ring $H^0(R_p)$, which is in turn the same as

$$\mathrm{Hom}(R_p, S_k) \otimes_{\mathrm{Hom}(H^0(R_p), S_k)} \mathrm{Hom}(R, S_k)$$

by definition of k, as required.

Now, the natural map of L_∞ algebras $T_p[-1]X_U \to T_p[-1]X$ is an inclusion of subcomplexes (and similarly for Y). To see this, $g\colon U \to C$ is an open immersion, thus by base change so is the map $X_U \to X$, and so is $H^0(X_U) \to H^0(X)$. Thus the induced map $H^1(T_p[-1]X_U) \to H^1(T_p[-1]X)$ is an injection, and the restriction of the map $T_p[-1]X_U \to T_p[-1]X$ to the piece in degrees ≥ 2 is an isomorphism, so the whole map is an inclusion of subcomplexes. In particular there is a unique isomorphism $T_p[-1]X_U \to T_p[-1]Y_U$ compatible with the isomorphism $T_p[-1]X \to T_p[-1]Y$. Applying the Maurer-Cartan functor then yields the required unique isomorphism $X_U \to Y_U$.

If R is now any local ring with maximal ideal $\mathfrak{m}$, consider the filtration by local Artinian rings $R/\mathfrak{m}^k$. The completion $\hat{R}$ arises as the limit where $k \to \infty$, and so since the functor $\mathcal{F}$ preserves limits (colimits in schemes, so limits in rings) we have that $\mathcal{F}(\operatorname{Spec} \hat{R})$ is trivial. Now use the fact that the map $R \to \hat{R}$ is faithfully flat to conclude triviality for $\operatorname{Spec} R$ by descent.

To complete the proof, we only need to observe that the isomorphism $X_U \to Y_U$ is G-equivariant when $U = \operatorname{Spec} R$ and R is a local Artinian algebra with an action of G: this implies G-equivariance for all U. To see this, by hypothesis we know that the map $T_p[-1]X \to T_p[-1]Y$ is G-equivariant, thus so is the map on subcomplexes $T_p[-1]X_U \to T_p[-1]Y_U$, thus – applying the Maurer-Cartan functor – so is the isomorphism $X_U \to Y_U$ as required. □

Thus any prestack whose shifted tangent complex agrees with the perturbative twist of a classical field theory is necessarily unique. But does it exist? There are two possible constructions we could give on the level of prestacks. I'll explain one in detail and sketch the other, but I won't prove that they define any sort of nice derived stacks in general (although we certainly know that they have nice tangent complex data).

Construction 4.6. Let $\mathcal{X} = X/G$ be a -1-shifted symplectic derived stack with twisting data (α, Q). We can define a new prestack by taking the left Kan extension of the functor of points of $\mathcal{X}$ along the functor

$$F\colon \{\text{super cdgas in degrees } \leq 0 \text{ with an action of } H\} \to \{\text{super cdgas in degrees } \leq 0\}$$

given by taking the total complex with respect to the original cochain complex structure and the cochain complex structure obtained from the H action, then taking the cohomological truncation in degrees ≤ 0. That is, define $\mathcal{X}^Q = \mathrm{LKE}_F(\mathcal{X})$.

We need to check that $\mathcal{X}^Q$ has the right shifted tangent complex (i.e. the one obtained by applying the functor F to the complex $T[-1]\mathcal{X}$). Choose a point $p \in H^0(\mathcal{X})$. Associated to the formal neighbourhood around p there is a formal derived moduli problem which we'll denote $\mathcal{X}_p$, which arises by applying the Maurer-Cartan functor to the L_∞ algebra $L = T_p[-1]\mathcal{X}$. We must show that the triangle

$$\begin{tikzcd}
\text{H-Art}_{\leq 0} \arrow[r, "\mathcal{X}_p"] \arrow[d, "F"] & \text{sSet} \\
\text{Art}_{\leq 0} \arrow[ur, "\mathcal{X}_p^Q"'] &
\end{tikzcd}$$

can be filled in by a natural transformation making it into a left Kan extension diagram. To see this we split the diagram up, and deal with the functor as a composite of several steps. First we decompose the functors $\mathcal{X}_p$ and $\mathcal{X}_p^Q$ as

$$\begin{tikzcd}
\text{H-Art}_{\leq 0} \arrow[r, "L \underset{H}{\otimes} -"] \arrow[d, "F"] & sL_\infty\text{-alg} \arrow[r, "MC"] & \text{sSet} \\
\text{Art}_{\leq 0} \arrow[ur, "F(L)\otimes -"'] & &
\end{tikzcd}$$

where sL_∞-alg denotes the category of simplicial L_∞ algebras. The functor $L \otimes_H -$ sends R to the simplicial L_∞-algebra with n-simplices $L \otimes_H \mathfrak{m}_R \otimes \Omega^\bullet(\Delta^n)$, and similarly for $F(L) \otimes -$ (where we're abusing notation to write $F(L)$ for the L_∞ algebra obtained by taking the total complex of L with respect to its original cochain complex structure and the one obtained from the H-action). We also modify the L_∞ structure using the multiplication structure on $\mathfrak{m}_R$. The functor MC sends a simplicial L_∞ algebra to the simplicial set obtained by taking degree one solutions to the Maurer-Cartan equations at each simplicial degree. We'll show that this functor sends suitable left Kan extensions to left Kan extensions, so that it will suffice to check that the triangle with target sL_∞-alg is a Kan extension diagram.

To do this, we observe that the colimit computing the Kan extension along F is sifted, so it suffices to check that MC preserves sifted colimits. The category $\text{Art}_{\leq 0}$ admits finite coproducts, thus so does the comma category $F \downarrow S$ for each $S \in \text{Art}_{\leq 0}$. In particular this comma category is sifted, so the left Kan extension along F is computed by sifted colimits.

To check that MC preserves sifted colimits it suffices to check that it preserves filtered colimits and reflexive coequalizers because the target category – sSet – admits all finite colimits. We'll check this on the level of individual simplices, i.e. for the functor $\text{MC}\colon L_\infty\text{-alg} \to \text{Set}$ (where we're abusing notation by giving it the same name). We'll check this functor preserves filtered colimits first. Filtered colimits of L_∞ algebras are the filtered colimits of the underlying cochain complexes with suitably determined L_∞ structure, that is to say we can think of an element of the filtered colimit along a filtered diagram J as an equivalence class of elements of the coproduct indexed by elements of J, where two elements are equivalent if they differ by a pair $(v, f(v))$ where $f : V_1 \to V_2$ is the image of a morphism in J and $v \in V_1$. Any such element is equivalent to an element which is zero in all but one position. Maurer-Cartan elements of this colimit are thus equivalence classes in the disjoint union $\sqcup_{j\in J}\mathrm{MC}(V_j)$ indexed by J, where two elements are equivalent if they differ by a pair $(x, f(x))$ where $f\colon \mathrm{MC}(V_1) \to \mathrm{MC}(V_2)$ is the image of a morphism in J. That is to say, Maurer-Cartan elements of the filtered colimit are elements of the filtered colimit of sets.

We must also check that MC preserves reflexive coequalizers. We have a sketch argument here, but it isn't written up yet. Completing this will prove that MC preserves sifted colimits, and in particular to check that our diagram is a left Kan extension it suffices to check it for the diagram into simplicial L_∞ algebras, since if so this left Kan extension is preserved (indeed, on the nose, not just up to homotopy).

Instead of applying the Maurer-Cartan functor we could apply the forgetful functor $U\colon sL_\infty\text{-alg} \to \mathrm{sSet}$. Clearly if the new triangle with target sSet is a left Kan extension diagram then the triangle with target sL_∞-alg is too, since the former satisfies a strictly stronger universal property (saying that the natural transformation is universal for all categories mapping to sSet, not just those which factor through simplicial L_∞-algebras). This diagram now factors as follows.

$$\begin{tikzcd}
\text{H-Art}_{\leq 0} \arrow[r] \arrow[d, "F"] & \text{H-Vect}_{\leq 0} \arrow[r, "U(L\underset{H}{\otimes}-)"] \arrow[d, "F'"] & \mathrm{sSet} \\
\mathrm{Art}_{\leq 0} \arrow[r] & \mathrm{Vect}_{\leq 0} \arrow[ur, "U(F(L)\otimes -)"'] &
\end{tikzcd}$$

where $\mathrm{Vect}_{\leq 0}$ denotes super cochain complexes in non-positive degrees, and where the horizontal arrows send an algebra R to its maximal ideal $\mathfrak{m}_R$. The left-hand square is cartesian, so for the required triangle to define a left Kan extension is suffices to check that the right-hand triangle in this diagram defines a left Kan extension.

To do this, we use the fact that representable functors $[A, -]$ left Kan extend to representable functors of form $[F'A, -]$. By the hom-tensor adjunction the functors $U(L\underset{H}{\otimes}-)$ and $U(F(L)\otimes-)$ are representable by $\mathrm{Hom}_{H\text{-Vect}}(L, \mathbb{C})$ and $\mathrm{Hom}_{\mathrm{Vect}}(L, \mathbb{C})$ respectively, and the latter is F' of the former, so this triangle is a left Kan extension, completing the proof.

Construction 4.7. Alternatively, we could define the global twist as a prestack in a different way. Again, assume that $\mathcal{X} = X/G$ is defined as the global quotient of a derived scheme by a group scheme, so in particular $T_p[-1]\mathcal{X}$ is concentrated in non-negative degrees for all p. Suppose further that $\mathcal{X}$ satisfies a finite type condition, in that the L_∞ operations ℓ_k for $T_p[-1]\mathcal{X}$ vanish for all p if k is sufficiently large. By taking the naïve truncation of the L_∞ space $T[-1]\mathcal{X}$ in degrees ≥ 1 we obtain a *nilpotent* L_∞ space isomorphic to $T[-1]X$. Using this nilpotence condition we can use the Maurer-Cartan functor to define a genuine prestack (or slightly stronger, a derived stack in the sense of [GG14]) by

$$\mathcal{X}^Q(R) = \mathrm{MC}(T[-1]X \otimes R)/G(R)$$

for R any cdga in degrees ≤ 0. More precisely this defines a sheaf of derived stacks over $H^0(X)$ by applying the construction to the L_∞ algebra of sections of $T[-1]X$ over an open set $U \subseteq H^0(X)$.

5 Examples From $N = 1$ and $N = 4$ Gauge Theories

Example 5.1. First, let's briefly talk about the $N = 1$ super twistor theory, i.e. Holomorphic Chern-Simons on super twistor space. Twistor space is just $\mathbb{PT} \cong \mathbb{CP}^3 \setminus \mathbb{CP}^1$. This theory admits twisting data from a section

$Q \in \Gamma((\mathbb{PT}; \mathcal{O}(1))$, since the BV complex is

$$\Omega^{0,\bullet}(\mathbb{PT}; (\Pi\mathcal{O}(-1) \oplus \mathcal{O}) \otimes \mathfrak{g}_P \oplus (\mathcal{O} \oplus \Pi\mathcal{O}(1)) \otimes K_{\mathbb{PT}} \otimes \mathfrak{g}_P^*)$$

and we give $\mathcal{O}(k)$ the $\mathbb{C}^\times$-weight k. The twist is therefore

$$\begin{aligned}&\Omega^{0,\bullet}(\mathbb{PT}; (\mathcal{O}(-1)[1] \to \mathcal{O}) \otimes \mathfrak{g}_P \oplus (\mathcal{O} \to \mathcal{O}(1)[-1]) \otimes K_{\mathbb{PT}} \otimes \mathfrak{g}_P^*)\\ &\cong \Omega^{0,\bullet}(\mathbb{PT}; \mathcal{O}_{Z(Q)} \otimes \mathfrak{g}_P \oplus \mathcal{O}_{Z(Q)}^\vee \otimes K_{\mathbb{PT}} \otimes \mathfrak{g}_P^*)\end{aligned}$$

where $Z(Q)$ is the zero locus of the section Q. Under dimensional reduction to $\mathbb{R}^4$ say we receive the cotangent theory to holomorphic G-bundles, near the holomorphic bundle isomorphic to $P|_{Z(Q)}$, under the projection $Z(Q) \to \mathbb{R}^4$.

Example 5.2. I'll conclude by discussing one of the motivating examples mentioned in the first section. I won't discuss any twists of the full $N = 4$ theory, but rather I'll discuss a simpler example where we start from the *holomorphically twisted* $N = 4$ theory and perform a *further* twist to obtain a topological theory.

The (complexified) $N = 4$ four-dimensional supersymmetry algebra looks like

$$(\mathfrak{sl}(2;\mathbb{C})_- \oplus \mathfrak{sl}(2;\mathbb{C})_+ \oplus \mathbb{C}^4 \oplus \mathfrak{sl}(4;\mathbb{C})_R) \oplus \Pi(S_+ \otimes W \oplus S_- \otimes W^*)$$

where

- $\mathfrak{sl}(2;\mathbb{C})_+ \oplus \mathfrak{sl}(2;\mathbb{C})_-$ is the complexification of $\mathfrak{so}(4) \cong \mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_-$, with two-dimensional complex semispin representations $S_\pm$.
- $\mathfrak{sl}(4;\mathbb{C})$ also acts on $\mathbb{C}^4$ by its fundamental representation.
- W is a four dimensional complex vector space acted on by the fundamental representation of $\mathfrak{sl}(4;\mathbb{C})_R$ (the R-symmetry algebra).
- there is an additional bracket from the vector valued pairing $\Gamma \colon S_+ \otimes S_- \to \mathbb{C}^4$, and the evaluation pairing $W \otimes W^* \to \mathbb{C}$.

We specify a basis for the space of supercharges by choosing bases

$$\begin{aligned}S_+ &= \langle \alpha_1, \alpha_2 \rangle\\ S_- &= \langle \alpha_1^\vee, \alpha_2^\vee \rangle\\ W &= \langle e_1, e_2, f_1, f_2 \rangle\\ W^* &= \langle e_1^*, e_2^*, f_1^*, f_2^* \rangle.\end{aligned}$$

The supercharge

$$Q_{\mathrm{hol}} = \alpha_1 \otimes e_1$$

(for example) squares to zero and has half-dimensional image in the translations, so after choosing an action α of $\mathbb{C}^\times$ so that it has weight 1, defines a *holomorphic* twist. We can consider the action of a $\mathbb{CP}^1$-family of *topological* supercharges extending Q_{hol}, given by

$$Q_{(\lambda:\mu)} = Q_{\mathrm{hol}} + (\lambda(\alpha_1^\vee \otimes f_1^* - \alpha_2^\vee \otimes f_2^*) + \mu(\alpha_2 \otimes e_2)), \text{ for } (\lambda : \mu) \in \mathbb{CP}^1.$$

To phrase it another way, the supercharges $Q_{(\lambda:\mu)} - Q_{\mathrm{hol}}$ are square-zero supercharges in the holomorphically twisted theory. Checking that these are indeed topological is an easy calculation, e.g. of the Q_{hol}-cohomology of the supersymmetry algebra. We'll be most interested in the cases where $(\lambda : \mu) = (0 : 1)$ and $(1 : 0)$, which we call the *A-twist* Q_A and the *B-twist* Q_B respectively.

Let's take the classical BV complex in the holomorphic twist as given, and compute the B-twisted complex. The underived moduli space of solutions to the equations of motion is given by the moduli stack of G-Higgs bundles. The classical BV complex in the holomorphically twisted theory, near a classical point (P, ϕ), is given by

$$\Omega^{\bullet,\bullet}(X; \mathfrak{g}_P \oplus \mathfrak{g}_P^*[1])[1]$$

with differential $\overline{\partial} + [-, \phi]$, and where $\Omega^{p,q}$ is naturally (before shifts) in cohomological degree q and superdegree $p \mod 2$. This theory is defined for any complex surface X. The action of the B-supercharge Q_B is by the vector field ∂, so the B-twisted theory has classical BV complex

$$\Omega^\bullet(X; \mathfrak{g}_P \oplus \mathfrak{g}_P^*[1])[1]$$

with the de Rham differential.

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