Math 131-H – Honors Calculus I – Lesson Plans

Fall 2019

1. Tuesday September 3rd
   (a) Introduce myself and go through the syllabus
   (b) Mention Moodle and the course webpage: `https://people.math.umass.edu/~celliott/Math_131H_Fall_2019.html`.
   (c) Hand out index cards, asking for name, pronouns, year (Freshman, Sophomore, etc), major (if you know), what you're hoping to get out of this class.
   (d) Introduce the class by talking about a few classical problems having to do with *change* and *motion*: finding speed from position or the other way around, finding areas and volumes, approximating a graph by a line (of course, the first and third are the same).
      - First classical problem: the rate of change of a function.
      - For instance, here's a practical example, suppose $h(t)$ (a function) represents the height of a falling weight at time t (draw a picture). What is the speed of the weight at some specific time t_0? This is an example of a *rate of change*: in this case, the speed represents the instantaneous change of the height function $h(t)$ at the time $t = t_0$. Notation: $\mathrm{d}h/\mathrm{d}t(t_0)$, or $h'(t_0)$.
      - There are other physical examples of rates of change. For example, image a tank full of a liquid, which is draining out through a pipe. The *flow rate* through the pipe is the rate of change of the volume of liquid in the tank: $\mathrm{d}V/\mathrm{d}t$.
      - Force: rate of change of momentum $F = \mathrm{d}p/\mathrm{d}t$.
      - Second classical problem is from geometry: approximating a curve by a straight line.
      - Idea: the *tangent line* to a curve at a point is the line that best approximates the curve – it should be moving in the "same direction". If you zoom in on the curve more and more, the tangent line should better and better approximate the curve (draw a picture).
      - These two problems are actually the same! Express our curve as the graph of a function: say $y = f(x)$. The slope of the tangent line at a point $x = x_0$ is the instantaneous rate of change of the height y of the curve with respect to the horizontal position x.
      - Moral: we can understand both physical ideas like speed, and geometric ideas like tangent lines by thinking about functions.
      - Special case: you can find when a function is maximized or minimized by looking where the derivative is zero: very useful in optimization!
      - So how to we actually find this rate of change? Well we can approximate the tangent line by chords, also called secant lines (draw a picture). Need the idea of a *limit*, which we'll introduce next time.

2. Thursday September 5th

   (a) Limits, section 2.2 of the textbook.

   (b) Start by saying the idea of a limit: useful when a function is not defined at some value, but it is defined *near* that value.

   (c) We'll define limits from the left and from the right informally.

   (d) The definition we'll give will be like:

   Definition 0.1. We say the limit of $f(x)$ from the left at $x = x_0$ is equal to L if we can make the value of $f(x)$ arbitrarily close to L by making $x < x_0$ sufficiently close to x_0. Remark that the function f must be defined in some region of $x < x_0$, but it doesn't need to be defined at x_0.

   (e) Likewise from the right. When they coincide we just say the limit. Notation.

   (f) Note that this is the informal definition, we'll state the definition more formally next week. See also analysis, taught here as Math 523H. Anecdote. In real life functions are usually pretty nice, but abstractly functions can be really weird. E.g. indicator function of $\mathbb{Q}$.

   (g) Different behaviour, with examples: all the limit as $x \to x_0$.

      i. Limit exists and is finite (the function need not be defined at x_0, e.g. could do $(x^2 - 1)/(x - 1)$ at $x_0 = 1$, away from that point equals $x + 1$, so limit will be 2.)

      ii. Another example: $\sin(x)/x$. We'll prove this later, but you can verify using a calculator now.

      iii. Left and right limits both exist, but are different: Heaviside step function. Another example: the sign function $\operatorname{sgn}(x) = x/|x|$, $\operatorname{sgn}(0) = 0$.

      iv. Limit does not exist, say $x \to \pm\infty$ from one side or the other (vertical asymptote). A good example is $1/x^2$ (where the left and right limits coincide, and $1/x$ (where they do not).

      v. Limit does not exist, nor is there an asymptote, e.g. $\sin(\pi/2x)$. Can consider reciprocals of odd or of even integers.

   (h) Might move on to state the limit laws: e.g. addition, multiplication, quotient, powers. Works for left, right and total limits. Maybe sketch a proof of one of them with pictures? For instance, addition, for the left limit.

3. Tuesday September 10th

   (a) Continue Section 2.3 as necessary, then start with Section 2.4: the rigorous definition of a limit.

   (b) Start with our final result on the properties of limits: the squeeze theorem.

   Theorem 0.2. If $f(x) \le g(x) \le h(x)$ for all x (in a neighbourhood of x_0), and $\lim f(x) = \lim h(x) = L$ as $x \to x_0$, then $\lim g(x) = L$ at x_0 also.

   (c) An example: $x^2 \sin(1/x)$. It's bounded by $\pm x^2$, so the limit at $x = 0$ is zero.

   (d) An interesting example: $\sin(x)/x$.

   Proof. First, note (Lemma) that $\tan(x) \ge x \ge \sin(x)$ in the range $0 < x < \pi/2$. We see this using a picture: arc of a circle with angle x, radius 1. Draw the chord, and the vertical tangent. Get 3 regions with area $\tan(x)/2 \ge x/2 \ge \sin(x)/2$. Take the reciprocal, and multiply by $2\sin(x)$, and we get $\cos(x) \le \sin(x)/x \le 1$, to which we can apply squeeze. □

   (e) Now move on to the rigorous definition of a limit. Recall under what circumstances, and for what reasons, rigorous definitions are important in math. Here part of the motivation is that arbitrary functions are weird: generality!

   (f) Recall the definition we gave last time, and highlight where it's imprecise.

   Definition 0.3. We say the limit of $f(x)$ from the left at $x = x_0$ is equal to L if we can make the value of $f(x)$ arbitrarily close to L by making $x < x_0$ sufficiently close to x_0.

   It's imprecise where we say "arbitrarily close" and "sufficiently close".

   (g) So, start making it more precise. Begin with "arbitrarily close to". Choose an ε. Then "sufficiently close to". That means finding δ so that $|x - x_0| < \delta$.

   Definition 0.4. We say the limit of $f(x)$ from the left at $x = x_0$ is equal to L if for any number $\varepsilon > 0$, there exists some $\delta > 0$ so that we can make $|f(x) - L| < \varepsilon$ by requiring that $x_0 - x < \delta$.

   (h) Say how the definition changes for the limit from the right, and from both sides.

   (i) Use this definition to prove the limit of $f(x) = x$ at x_0 is x_0.

   (j) Prove a limit law: addition law is a good example. Find δ_1, δ_2 making the two summands each within $\varepsilon/2$ of the limit, then take δ to be the minimum of δ_1 and δ_2.

   (k) Precise definition of $\pm\infty$ limit. Replace $|f(x) - L| < \varepsilon$ by $f(x) > N$.

   (l) Could do an example: $f(x) = \frac{1}{x}$, limit from the right: $x < \delta \implies \frac{1}{x} > \frac{1}{\delta}$, so set $\delta = 1/N$.

4. Thursday September 12th

   (a) Section 2.5: definition of continuity.

   (b) Intuitive idea: can draw a graph without lifting your pen.

   (c) Definition: limit is equal to the value.

   Definition 0.5. A function $f(x)$ is continuous at a point x_0 if the limit $\lim_{x\to x_0} f(x)$ equals the value x_0

   (d) Note that, in particular, the function has to be defined at x_0, *and* the limit has to exist.

   (e) First examples: constant function $f(x) = c$ and linear function $f(x) = kx$ are continuous. Also trig, exponential and $\log$ are continuous where they are defined. Note that e.g. $\tan(x)$ is not continuous at $\pi/2$ etc, vertical asymptotes.

   (f) Limit laws revisited: state as a theorem. They say in particular that the sum, difference, product, quotient, powers etc of continuous functions are continuous.

   (g) Consequence: polynomials, rational functions are continuous (prove this).

   (h) Limit laws for a general continuous function.

   Theorem 0.6. Say $f(x)$ is a continuous function. Let $g(x)$ be another function, and suppose $\lim_{x\to x_0} g(x) = L$. Then $\lim_{x\to x_0} f(g(x)) = f(L)$.

   (i) More complicated examples. First $f(x) = 1$ if $x = 1/n$, 0 otherwise. Check continuity in different regions: $x > 1$ and $x < 0$, $x = 1/n$, $0 < x < 1$ not equal to $1/n$, $x = 0$ (compute from left and right).

   (j) Now $f(x) = 1/n$ if $x = 1/n$, 0 otherwise. Mostly this is the same, except at 0 (use the squeeze theorem).

   (k) I ended up ad-libbing something about limits at infinity.

5. Tuesday September 17th
   (a) Announce that there may be an emergency system test.
   (b) Homeworks are due, hand them in at the end of class.
   (c) We'll continue with calculating limits at infinity (§2.6) – review the definition.
   (d) We can work out many examples using the limit laws.
      i. The limit of $1/x^n$ for any positive integer n.
      ii. A complicated example like $(2x^3 + x + 17)/(x^3 + 10x^2 + 4x + 1)$. Solve by dividing top and bottom by x^3 then using the limit laws. What do we learn: it only depends on the largest power in the numerator and denominator.
      iii. Remark: why did we have to divide through numerator and denominator? To apply the limit laws can't have one of the limits being infinity: since infinity is not a number we can't manipulate it.
      iv. Another example: say $(x^3 + 7)/(2x^4 + x^2 + 1)$. We'll likewise find that the limit is zero. If the power on the bottom is larger than the power on the top, the limit will be zero.
      v. Opposite behaviour, something like $(x^5 + x^2)/(x^3 + 8x - 1)$. We'll find that the limit is infinity.
      vi. Some other examples: $e^{1/x}$: use the general limit law for continuous functions.
      vii. Even combine them: $e^{(2x^2+1)/(3x^2-1)}$: limit will be $e^{2/3}$.
      viii. An example with roots: $(\sqrt{x^4 - 13}/(\sqrt{3x^4 + x^2 = x})$. Divide numerator and denominator by x^2 here, we'll get $1/\sqrt{3}$ I think.
      ix. Note $1/x^r \to 0$ for any positive rational r.
   (e) In the second part of the class, cover the definition of the derivative as a limit. Start by recalling the secant line story, to find the tangent line as a limit.
   (f) Recall how to find the slope of a line through any two points $(x_1, y_1), (x_2, y_2)$.

      Definition 0.7. The slope of the tangent line of f at $x = x_0$ is

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

   (g) So the equation of the tangent line is $y = mx + f(x_0)$.
   (h) Define the derivative of a function f.
   (i) Examples: a parabola. First find the tangent line at a point in a specific example. Then a hyperbola, likewise. Next time we'll talk about the derivative as a function.
   (j) Example: metal ball on a smooth table, initially with a magnetic field so that it's moving with a specific trajectory given by a function (say the field strength is linear, so it's moving in a parabola), turn it off and see what the trajectory is. For instance, if at time t the coordinates on the table are $(x, y) = (t, 1 + 10t^2)$, and the field is turned off at time t_0, find the position at a later time $t_0 + t_1$.

6. Thursday September 19th

   (a) Continue our introduction to derivatives with Section 2.8, on the derivative as a function.

   (b) Define the derivative as a function.

   (c) How do we calculate it? Well, we've done two or three examples already: review $1/x$ example.

   (d) Prove the derivative of x^n is what it should be: let $h = x - x_0$, then factor out $(x - x_0)$ from $(x - x_0)^n = (x - x_0)(x^{n-1} + x_0 x^{n-2} + \cdots + x_0^{n-1})$. Alternatively, use the binomial theorem.

   (e) Define differentiability: the limit corresponding to the derivative exists.

   (f) Examples: a discontinuous function (e.g. isolated singularity), the absolute value function.

   (g) Another good example: tangent line becomes vertical: $f(x) = \sqrt[3]{x}$: defined everywhere, and continuous everywhere.

   (h) In these three examples, plot the derivative: it'll have a removable discontinuity, a jump, and a vertical asymptote.

   (i) Prove the following:

       Theorem 0.8. If a function is differentiable at x_0 then it is continuous at x_0.

       To do it, write down the definition of the derivative, multiply the limit by h, it goes to zero, which implies the limit is $f(x_0)$.

   (j) Around here, can I give them an example?

   (k) Higher derivatives as functions: k-times differentiable functions. Introduce the notation.

   (l) Position, speed, acceleration, jerk, (higher derivatives rarely appear but are sometimes amusingly called snap, crackle and pop.)

   (m) Example: position of an object at time t given by $p(t) = 10 - 8t^2 - t^3$, find derivatives, and therefore describe the motion (constant negative jerk, so rate of deceleration is increasing).

7. Tuesday September 24th

   (a) Continuing with Section 3.1.
   (b) Derivative of an arbitrary exponential function a^x, then more generally a^{cx}. We find that the rate of change of the function is proportional to the function, with factor of proportionality given by a limit depending on a and c. That is, the function satisfies $f'(x) = kf(x)$. This is an example of a *differential equation*.
   (c) Picking out the number e. It's the number a where $\lim_{h\to 0}(a^h - 1)/h = 1$.
   (d) Manipulate: $\lim_{n\to\infty} n(e^{1/n} - 1) \to 1$, so $\lim_{n\to\infty} e^{1/n} - (1/n + 1) \to 0$, so e is the limit of $(1 + 1/n)^n$ as $n \to \infty$: this is the formula appearing in the compound interest formula.
   (e) In terms of e we can find this limit for any a using rules for exponentials: $a^x = e^{x \log a}$.
   (f) Move on to Section 3.2: the product and quotient rules.
   (g) Introduce this notation Δu, Δx etc for the change in $u(x)$, x, and then rewrite the definition of the limit corresponding to $\lim_{\Delta x\to 0} \Delta u/\Delta x$.
   (h) Geometric intuition for the product rule: draw a rectangle with sides $u + \Delta u$ and $v + \Delta v$. Divide by Δx and take the limit as Δx becomes small: the square term goes to zero.
   (i) Prove the product rule (also known as the Leibniz rule). Just means making the above argument precise: compute $\Delta(uv)/\Delta(x)$ then take the limit. Here $\Delta(uv) = u(x + h)v(x + h) - u(x)v(x) = (u + \Delta u)(v + \Delta v) - uv$.
   (j) Examples: polynomial times e^x. Do x^4e^x, first few derivatives.
   (k) Prove the quotient rule. Calculate $\Delta(u/v) = u(x + h)/v(x + h) - u(x)/v(x) = (u + \Delta u)/(v + \Delta v) - u/v$. Expand it out into a single fraction.
   (l) How to remember the sign: set one function to be zero.
   (m) Application: derivative of $\frac{1}{x^n}$.
   (n) Examples: rational functions. Do a cubic over a quadratic.
   (o) Another example: e^{kx} times the same rational function, if we like we can do it in two different ways, where the exponential is either in the numerator or the denominator.

8. Thursday September 26th

   (a) Return homework 1.

   (b) Differentiation of trig functions, §3.3 in the textbook.

   (c) Start by filling in some points on the graph of the derivative of $\sin(x)$: it's periodic, and we can see where it's positive and negative.

   (d) Calculate explicitly using the angle sum identities, and the limit $\sin(x)/x \to 1$ as $x \to 0$. More specifically, $\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$

   (e) We also need the limit of $(\cos(h) - 1)/h$: to do this multiply numerator and denominator by $\cos(h) + 1$, then factor out $\sin(h)/h$.

   (f) Do the same with cosine: limit of $(\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x))/h$, same method works.

   (g) More generally, differentiate $\sin(kx)$ and $\cos(kx)$. Can do it using limits (with a substitution), or using the chain rule. Say the differential equation they satisfy.

   (h) All higher derivatives can now easily be worked out: it's periodic! Comment on the differential equation they satisfy.

   (i) Now use the quotient rule to work out the derivatives of the other main trig functions.

   (j) Example: we can find the second derivative of tan using the product rule.

   (k) Another example: $(\tan(x) - 1)/\sec(x)$ in two ways: directly using quotient rule, and by rewriting as $\sin(x) - \cos(x)$.

   (l) Applications: simple harmonic motion. Have a mass m on a spring. Force is negative, proportional to how stretched the string is, so, say position $p(t)$ satisfies the following:

   $$mp'' = -kp,$$

   where $k > 0$ is the spring constant. One can then solve the equation: it obeys a sinusoid. Can work out the constants using the initial conditions.

   (m) (Maybe an exercise for the group). Compute the derivative of $e^{cx}\sin(x)$: get $e^{cx}(\cos(x) + c\sin(x))$: differentiate again, get $e^{cx}((c^2-1)\sin(x) + 2c\cos(x))$. This also solves a differential equation, specifically

   $$f'' = 2cf' - (1 + c^2)f.$$

   This describes some kind of damped harmonic motion: with a force like friction proportional to the speed (think c negative, otherwise it's reinforced).

   (n) Powers of $\sin(x)$: differentiate the first few, then do $\sin^n(x)$ by induction. Leads into the discussion of the chain rule.

9. Tuesday October 1st

   (a) Collect Homework 2. Then make an announcement on the exam and review session (Wednesday October 2nd from 6-8pm, in room GSMN (Goessmann) 51). We'll also have a review class on Thursday.

   (b) Topic will be the chain rule: §3.4 in the textbook.

   (c) Idea in terms of rates of change: say we have a function given as a composite $f(x) = g(u(x))$. The rate of change of f with respect to x should account for both the rate of change of g with respect to u and of u with respect to x. Doubling *either* one should double the overall rate of change, so we guess the overall rate of change is the product:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}g}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

   (d) State the chain rule: rewriting in terms of f'. Use the notation $f = g \circ u$.

   (e) Prove it: it's a bit harder than the proofs we've already seen. You need to come up with a solution to a somewhat tricky problem. This won't be examined.

Proof. To start with, we would like to write the following.

$$\begin{aligned}(g \circ u)'(x_0) &= \lim_{h\to 0} \frac{g(u(x_0+h)) - g(u(x_0))}{h} \\ &= \left(\lim_{h\to 0} \frac{g(u(x_0+h)) - g(u(x_0))}{u(x_0+h) - u(x_0)}\right)\left(\lim_{h\to 0}\frac{u(x_0+h)-u(x_0)}{h}\right).\end{aligned}$$

This is the idea of the proof, and the reason why the chain rule should be true! The only problem is: what if there are some values of h where $u(x+h) - u(x) = 0$? If so, then the above thing doesn't make sense, because we'd be dividing by zero.

We can get around this with a trick: we just define a new function:

$$F(x) = \begin{cases} \frac{g(u(x)) - g(u(x_0))}{u(x)-u(x_0)} & u(x) \neq u(x_0) \\ g'(u(x_0)) & u(x) = u(x_0). \end{cases}$$

Then

$$\begin{aligned}(g \circ u)'(x_0) &= \lim_{h\to 0} \frac{g(u(x_0+h)) - g(u(x_0))}{h} \\ &= \left(\lim_{h\to 0} F(x_0+h)\right)\left(\lim_{h\to 0}\frac{u(x_0+h)-u(x_0)}{h}\right)\end{aligned}$$

(it either is equal to the first line if $u(x_0+h) \neq u(x_0)$, or both the first and second lines equal 0). We just need to calculate $\lim_{h\to 0} F(x_0+h)$ to complete the proof. In fact, we just need to notice that F is continuous at x_0, because g is differentiable at x_0 (that's the definition of differentiability), so the limit is the value $F(x_0)$. $\square$

   (f) Ok, let's move on to lots of examples.

      i. Simple example: $\sqrt{x^2+1}$.

      ii. One we already saw: $\sin(x)^n$. Alternatively, do it in the other order: $\sin(x^n)$. We can also get $\sin(kx)$, e^{kx} etc back from this, and work out what happens to trig functions like $\cot(kx)$ (it's $-k\mathrm{cosec}^2(kx)$.)

      iii. More than two functions in the chain, e.g. $e^{\tan(x^3)}$.

      iv. A new way of seeing the quotient rule from the product rule.

      v. A new way of calculating the derivative of $x^{1/n}$: we have $x = (x^{1/n})^n$, so the chain rule says $1 = \frac{\mathrm{d}x^{1/n}}{\mathrm{d}x} n(x^{1/n})^{n-1}$, then we can rearrange.

      vi. You can use this for any inverse function, e.g. $\arcsin(x)$. $1 = \frac{\mathrm{d}\arcsin(x)}{\mathrm{d}x}\cos(\arcsin(x))$. By the way, you can then simplify $\cos(\arcsin(x)) = \sqrt{1 - \sin(\arcsin(x))^2} = \sqrt{1-x^2}$. We'll do a bunch more examples next time (though we can likewise find the derivative of $\arccos$ being $-1/\sqrt{1-x^2}$, and of $\arctan$ being $1/(1+x^2)$ if we have time.

      vii. One harder one: $\mathrm{arcsec}(x)$. Use $\sec' = \tan\sec$ and $\tan^2 = \sec^2 - 1$. Must take care with which square root we're taking at the end: want to always get a positive number (graph $\mathrm{arcsec}(x)$ to see why).

10. Thursday October 3rd

    (a) Today is a review class, covering Chapter 2, and 3.1–3.2 in the textbook.

    (b) Start with information on the exam: no notes or calculator, one hour long, show up 5 mins early please if possible. Make sure you understand how to solve the problems on the practice exam! If you do then the midterm will be fine.

    (c) The main topics we've covered are: the definition of the limit, calculating limits using the limit laws, the squeeze theorem, the limit definition of the derivative, calculating derivatives using the power law, exponentials, and the product and quotient rule, and continuity and differentiability. Let's take them one at a time.

        i. The definition of the limit.
        ii. Calculating limits.
        iii. The definition of the derivative.
        iv. Rules for derivatives.
        v. Continuity and differentiability.

11. Tuesday October 8th

  (a) Return exams and homeworks.
  (b) Comment on midterm: simplification of e^{6x+12}/e^{x+11}.
  (c) Tell students about library reserve.
  (d) Topic for today: implicit differentiation. Compare to Section 3.5 in the textbook.
  (e) Idea: we know now how to find the tangent line to the graph of a function, but not all curves are graphs?
  (f) To explain this, recall what the graph of a function means. It means we consider all the points (x_0, y_0) in the plane where $y_0 = f(x_0)$. Write in set-builder notation as $\{(x, y) \colon y = f(x)\}$.
  (g) Graph satisfy the *vertical line property*: at most one y-value for every x-value.
  (h) For a curve that isn't a graph, consider the unit circle. It clearly doesn't satisfy the vertical line property. We can describe the circle as the set of points where $x^2 + y^2 = 1$. So it's described by an equation in x and y, but not one of the form $y = f(x)$
  (i) Note though that half circles are: $y = \pm\sqrt{1 - x^2}$. You can always "locally" write the curve as the graph of a function.
  (j) Do another example, say a lemniscate. Lemniscate means "figure 8", and there are a bunch of equations that define them. We can do e.g. the "Lemniscate of Bernoulli" $(x^2 + y^2)^2 = 2(x^2 - y^2)$ (by the way, it's the set of points where the product of the distance from 1 and from -1 is 1). Use the notation I use in my homework.
  (k) With curves like this, use implicit differentiation to find the slopes of the tangentline. Uses the *chain rule* E.g. do the circle: get $2x + 2y(y') = 0$, so $y' = -x/y$. Now you can plug in a point (x_0, y_0) to find the slope of the tangent line, e.g. do $(1/3, 1/4)$, get slope $-4/3$, and equation $y - 1/4 = -4/3(x - 1/3)$ or $y = -4/3x + (4/9 - 1/4)$.
  (l) Do the lemniscate: $2(x^2 + y^2)(2x + 2yy') = 4x - 4yy'$, so $2x(2 - 2x^2 - 2y^2) = 2yy'(2 + 2x^2 + 2y^2)$, and $y' = (x/y)(1 - x^2 - y^2)/(1 + x^2 + y^2)$.
  (m) Solve to find the points where the slope is vertical or horizontal.
  (n) Some other example: cardioid $(x^2+y^2)^2+4x(x^2+y^2) = 4y^2$, an example of a limaçon, or more generally a trochoid. Alternatively maybe Lamé's quartic $x^4 + y^4 = 1$.
  (o) In Stewart they do the Folium of Descartes: $x^3 + y^3 = 6xy$. Differentiate: $3x^2 + 3y^2y' = 6xy' + 6y$, so $y'(6x - 3y^2) = 3x^2 - 6y$. Can find where the tangent line is horizontal, vertical.
  (p) Another method of differentiating inverse functions: let's do $\operatorname{arcsec}(x)$ again. We say $y = \operatorname{arcsec}(x)$, so $x = \sec(y)$. That means $1 = y' \sec(y)\tan(y)$, so $(y')^2 = (\sec^2(y)\tan^2(y))^{-1} = (x^2(1 - x^2))^{-1}$,so $y' = (|x|\sqrt{1 - x^2})^{-1}$ as we saw before.

12. Thursday October 10th

   (a) Survey link `https://tinyurl.com/Elliott-Math131H`.
   (b) First 20 minutes of the class, MAP survey.
   (c) Topic for the remainder – hyperbolic trig functions – compare to §3.11 in the textbook. We're taking the topics a bit out of order, and will come back to earlier parts of §3 next week.
   (d) Another reminder, no class next Tuesday due to the holiday, and Monday schedule.
   (e) Recall *circular* trig functions: coordinates of a point on the circle of angle x are $\sin(x)$ and $\cos(x)$. Note that the area of the sector is $x/2$ (recall why).
   (f) Hyperbolic trig functions are similar, but instead of the circle $x^2 + y^2 = 1$, use the regular hyperbola $x^2 - y^2 = 1$. Show why this is just the usual hyperbola, rotated (set $u = x - y$, $v = x + y$ and draw the u and v axes).
   (g) We can give a formula for $\sinh$ and $\cosh$:
   $$\sinh(x) = (e^x - e^{-x})/2, \quad \cosh(x) = (e^x + e^{-x})/2.$$
   (h) Define the other hyperbolic trig functions. Graph them.
   (i) Why are they especially nice? Differential equations again! Compute the derivatives.
   (j) Differentiate $\tanh$ (use identity that will be on the worksheet), then an inverse function, say $\operatorname{arctanh}$. Get $1/(1 - x^2)$.
   (k) Catenary! Hanging ropes/cables, and also arches, have the equation of a catenary – most stable shape. The catenary is defined by $y = \cosh(ax)$ for some constant a. Where does this come from? Well, it's from a differential equation!
   (l) Consider the forces on a piece of the cable, from the node (lowest point) – which we'll say is at $(0, 0)$ to some other point $P = (x, y)$.
   (m) There's a downward force due to gravity, which is proportional to the length of the cable, say $\delta g s$ where s is the length and δ the density.
   (n) There's a horizontal tension force at the node, call the amplitude of this force T_0, and there's a diagonal tension at P, say of amplitude T. This splits into $T\sin(\theta)$ vertically, and $T\cos(\theta)$ horizontally. So, because the forces must balance to be in equilibrium:
   $$\begin{aligned} T\sin(\theta) &= \delta g s \\ T\cos(\theta) &= T_0 \\ \text{so } \tan(\theta) &= \frac{\delta g}{T_0} s. \end{aligned}$$
   But $\tan(\theta)$ is just $\frac{\mathrm{d}y}{\mathrm{d}x}$! So
   $$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\delta g}{T_0} s \\ \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \frac{\delta g}{T_0} \frac{\mathrm{d}s}{\mathrm{d}x} \\ &= \frac{\delta g}{T_0} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}. \end{aligned}$$
   Using a formula for the rate of change of the length of the curve. You can solve this equation, if $a = \frac{\delta g}{T_0}$, by setting $y = \cosh(ax)$.

13. Thursday October 17th

    (a) Collect homework 3.

    (b) Topics for today: Sections 3.8 and 3.9.

    (c) Exponential growth and decay: this means that the rate of change $\mathrm{d}f/\mathrm{d}t$ of a function $f(t)$ is proportional to its value. That is, $f(t)$ satisfies the differential equation $\frac{\mathrm{d}f}{\mathrm{d}t} = kf(t)$, for a constant k which we call the *relative growth rate*. Growth means k is positive, decay means k is negative. We already know which functions solve this equation: a general solution looks like Ae^{kt} for some constant A.

    (d) How do you find A? Initial condition: $A = f(0)$.

    (e) How do you fit to an exponential model? Let's do an example on population growth: this is an example in the textbook. Population of the world in 1950: 2.56 billion, and in 1960: 3.04 billion. Try to fit to an exponential. So let t be the number of years after 1950. We assume that the number of people $P(t)$ is Ae^{kt} for $A = 2.56$ billion, and some k which we need to find. So our model says, in billions of people:

    $$3.04 = P(10) = 2.56 \times e^{10k},$$

    so $k = \frac{1}{10}\log(3.04/2.56)$, and that logarithm approximately equals 0.17, so set $k = 0.017$. By the way, let's test the model for the year 2019, so $t = 69$. We find $P(69) = 2.56 \times e^{1.173} = 8.27$ billion people. The real population is about 7.7 billion, so our model over-estimated, but not by too much.

    (f) Doubling time / half-life. If k is positive then the value of $f(t)$ is growing in time: you can find the time it takes for $f(t)$ to *double*. A feature of exponential growth is that it will always take the same amount of time to double. Why is that? Well:

    $$2f(t) = 2Ae^{kt} = e^{\log(2)}Ae^{kt} = Ae^{k(t+\log(2)/k)}.$$

    So in particular, we double whenever t increases by $\log(2)/k$. E.g. in our example that's about every 41 years.

    (g) If k is negative, then instead the value of $f(t)$ is shrinking, and it will *halve* every time t increases $\log(2)/k$. This is called the *half-life*.

    (h) One more example: radioactive decay (this is where you usually hear the term half-life). The idea is that you have a sample of radioactive isotopes: each atom has a fixed probability of decaying in each time period, so the number of undecayed isotopes is modelled by an exponential with negative k. You can find k in terms of the half-life, or vice versa.

    (i) Second topic for today: related rates (§3.9). As the name suggests, this is when two rates of change are *related* to one another, and so you can find one in terms of the other. Generally this is an application of the chain rule.

    (j) It's easiest to explain this in an example. So let's say that we have a ladder of length L leaning against a wall, and say the base of the ladder is being pulled away from the wall at a constant rate, say 3 ms^{-1}. At what rate is the top of the ladder sliding down the wall? Well, let's write x and y for the position of the base and the top of the ladder (picture). Then Pythagoras's theorem tells us

    $$\begin{aligned} x^2 + y^2 &= L^2 \\ \implies \frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) &= 0 \\ \implies 2x \cdot x + 2y \cdot y &= 0. \end{aligned}$$

    We know $\cdot x = 3$, so $\cdot y = -3x/y$. For instance, suppose $L = 5$, let's find the rate of change when $x = 4$. So at that time $y = 3$, so $\cdot y = -4\mathrm{ms}^{-1}$ (downward motion).

    (k) We could do a more complicated example where the force on the ladder is constantly increasing, so say $\cdot x = 2x$ say. We could even do a general function $f(t)$!

    (l) Ideal gas law: $PV = RT$ for 1 mole of a gas, R is the "ideal gas constant", P is pressure, V is the volume and T is the temperature. Suppose we have gas in a piston, so the volume can change. Say the temperature is held constant but the pressure is increasing at a constant rate x (or, another example, like $\log(t)$): what happens to the volume? Get $\cdot V = cV/P = cV^2/RT$. Give V at some time, can determine $\cdot V$ (can even solve the ODE to get $x = 1/(\mathrm{const} - kt)$ for $k = c/RT$).

14. Tuesday October 22nd

   (a) Topic: Section 4.1: maxima and minima.

   (b) Begin with the definition of a global maximum and minimum on an interval (open or closed, can be infinite), and then of a local maximum and minimum. Draw some pictures.

   (c) Note that global maxima / minima don't always exist (examples, including $\tan(x)$). On the other hand, maxima and minima do exist on *closed* intervals, at least for continuous functions.

   Theorem 0.9 (Extreme Value Theorem)**.** If a function f is continuous on a closed interval $[a, b]$, then f has a global minimum and a maximum value somewhere in the interval (maybe at one of the end points).

   (d) I won't prove this, but here's the idea. First of all, your function had better not get arbitrarily big: if you did, that would contradict continuity! So there must be a "least upper bound" to the values that the function takes, and the values get arbitrarily close to this value. Continuity guarantees that you actually hit it.

   (e) By the way, you can use this idea to make sense of "continuity" of functions in much more general contexts than the real line. This leads to the definition of a continuous function on a topological space.

   (f) Examples: 1) draw a picture. 2) Non-example: $\tan(x)$ doesn't apply because it's not continuous at eg $\pi/2$. This also shows why the interval needs to be closed, otherwise there might be an asymptote at the end point!

   (g) How do we find local maxima and minima? Well, we can use differentiation. The idea is that at a max/min the tangent line should be horizontal. State it as follows.

   Theorem 0.10 (Fermat)**.** If c is a local min or max, and $f'(c)$ exists (no corners!) then $f'(c) = 0$.

   (h) Points where $f'(c) = 0$ are called *critical points*. By the way note that the opposite isn't true: we can have $f'(c) = 0$ without being a local min or max (*inflection point*, for instance $f(x) = x^3$ at $x = 0$.

   Proof. Focus on maxima – it's the same for minima. The idea is the following picture. If the slope of the tangent line is not zero, you can go "up" by a small amount in some direction, and increase the value, so you aren't a maximum. We make this idea precise using limits. So, being a local maximum means that, for small enough h, $f(c+h) \leq f(c)$. Or, in other words, $f(c+h) - f(c) \leq 0$. Now, let's divide by h. If h is positive that means $(f(c+h) - f(c))/h \leq 0$, so taking the limit as $h \to 0$, $f'(c) \leq 0$. If h is negative, then when we divide by it we change the direction of the inequality, so $(f(c+h) - f(c))/h \geq 0$, and taking the limit as $h \to 0$, $f'(c) \geq 0$. Therefore $f'(c) = 0$. (Draw a picture as I explain this). □

   (i) In practice, for continuous functions, how do we find the global maxima and minima on a closed interval, which we know exist? Well, we take all the critical points, as well as the end points, and see which is biggest, and which is smallest! By the way, we know that one them *is* biggest using the extreme value theorem.

   (j) Example: let's let $f(x) = x - 2\sin(x)$. We'll look at the interval $[0, 2\pi]$. At the end-points we have $f(0) = 0, f(2\pi) = 2\pi$. Now, differentiate to find the critical points. We have $f'(x) = 1 - 2\cos(x)$, which is zero when $\cos(x) = 1/2$. So $x = \pi/3$ or $5\pi/3$ (half an equilateral triangle). $f(\pi/3) = \pi/3 - \sqrt{3}$, and $f(5\pi/3) = 5\pi/3 + \sqrt{3}$. Now, using $\sqrt(3) > 1/7 > \pi/3$, it's clear what the minimum and maximum are.

15. Tuesday October 29th

(a) Section 1: start with reviewing the feedback. Key points: homework review, speaking more loudly (ask me to speak up), writing extra large on the two side boards, stating theorem/page numbers in the textbook for main results.

(b) Section 2: hand out cards asking for feedback.

(c) Collect hwk 4 and Return hwk 3 graded. Also remind of the date and content of the next midterm. It's going to cover 3.3 to 4.5 in the textbook.

(d) Review hwk 3.

(e) Topic for today: the mean value theorem, section 4.2. (p282 CHECK)

Theorem 0.11 (The Mean Value Theorem (Cauchy))**.** If a function $f(x)$ is differentiable on a closed interval $[a, b]$, then there is some point c in the interval where

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

That is, the tangent line at c is parallel to the secant line (draw a picture). The slope of the secant line is the *average* or *mean* value for the slope of the tangent line.

(f) Example: average speed traps.

(g) Remark: you need to be differentiable, continuous is not enough. For instance, consider a locally linear function on $[0, 1]$ which is 0 for $x < 1/2$, and given by $2x - 1$ for $x \geq 1/2$. This doesn't obey the mean value theorem.

(h) Application: suppose two functions f and g have the same derivative: $f' = g'$. So the derivative of $f - g$ is zero. By the mean value theorem, the slope of the secant line on any interval must be zero, so $f - g$ must be *constant*.

(i) The mean value theorem is really important in calculus. Applications will include: 1) L'Hôpital's theorem, 2) The fundamental theorem of calculus, 3) Taylor series.

(j) We'll prove this theorem today. We start out with the case where $f(a) = f(b)$. This is called *Rolle's theorem*.

Theorem 0.12 (Rolle)**.** If $f(x)$ is differentiable on $[a, b]$ and $f(a) = f(b)$, then there is some c in the interval where $f'(c) = 0$ (a critical point).

Proof. We use the extreme value theorem, from last time. That says that f has a global minimum, and a global maximum somewhere in $[a, b]$. If both the global minima *and* the global maxima occur at the end-points, then the minimum and maximum are equal, and so the function must be constant (the statement is clear for constant functions). Otherwise, there must be a global minimum or maximum somewhere in the interior (a, b), say at the point c. Any global minimum/maximum is also a *local* minimum/maximum. We know from Fermat's theorem last time that this local min/max is a critical point. $\square$

(k) Now, we get the mean value theorem simply by rotating Rolle's theorem! (Draw a picture) More precisely, we might not get a function, so we shear instead: subtract off a linear function.

Proof of Mean Value Theorem. Start with a function $f(x)$. We're going to subtract a linear function in order to get a new function $g(x)$ with $g(a) = g(b)$. So define

$$g(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right).$$

(Draw a picture). If $f(x)$ is differentiable, then so is $g(x)$. We can check that Rolle's theorem applies: $g(a) = g(b) = 0$. If That means there is a point c where $g'(c) = 0$. Now we differentiate, we get

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

so if $g'(c) = 0$, then $f'(c)$ has the value we want. $\square$

(l) Example: finding that point for $f(x) = x^2$, on the interval $[0, 1]$, it happens at $x = 1/2$. On the interval $[-2, 3]$, slope is $5/3$ so it happens at $5/6$.

16. Tuesday October 31st

   (a) In Section 2, address comments from the cards. TODO

   (b) Today we'll discuss two sections in the textbook, section 4.3, and section 4.5 (we'll go back to 4.4 on Tuesday). There's an overall theme though: understanding functions, by understanding their geometry – i.e graphing them. One might say, this is one of the main themes of the whole class.

   (c) First, section 4.3: we'll understand more about different sorts of critical points. In particular, how can we tell the difference between the three types?

   (d) Recall that critical points are those points where the tangent line is horizontal. If $f'(x)$ is continuous, then in between the two critical points, the slope of the graph is either positive or negative, not zero. That means the function is either increasing or decreasing. You can use that to work out whether the critical points are maxima, minima or inflection points.

   (e) Caveat: this doesn't work if there are places where the function is not differentiable, e.g. corners, discontinuities, or vertical asymptotes. It also doesn't work for the first and last critical points. So in general, we need more information.

   (f) Definition of concavity: this tells us something about the local shape of the graph of a function. We say a function is concave upwards at x if, on an interval around x, the graph of $f(x)$ lies *above* its tangent lines. Likewise we say it is concave downwards if it lies *below* its tangent lines.

   (g) We can test concavity using the second derivative $f''(x)$. Being concave upwards means the slope $f'(x)$ is *increasing*, whereas downard means $f'(x)$ is *decreasing*. So f is concave upwards at a point x if $f''(x) > 0$ and downwards if $f''(x) < 0$. Where it *changes concavity*, we must have $f''(x) = 0$ (draw pictures).

   (h) Compare to what the sign of the zeroth and first derivative means.

   (i) Important remark: You can have $f''(x) = 0$ without changing concavity! We'll see this in examples.

   (j) Examples: 1) Say $f(x) = x^3$. Look at the graph, it is concave downwards for negative x and upwards for positive x. It changes concavity at $x = 0$. Indeed, the second derivative is $f''(x) = 6x$. 2) $f(x) = \sin(x)$. 3) $e^{1/x}$. There are no critical points, but you change concavity at $x = -1/2$ (from downward to upward). In the textbook, this is also called an inflection point.

   (k) $f''(x) = 0$. Do three examples: $x^3, -x^4, x^4$. Here the second derivative is zero at $x = 0$ in all cases, so this is not enough to determine concavity!

   (l) So let's get back to classifying critical points. This is called the second derivative test. If you are concave upwards then you must be a local minimum, so a critical point is a local minimum when $f''(x) > 0$. Likewise if you are concave downwards then you must be a local maximum, so a critical point is a local maximum when $f''(x) < 0$. If you change concavity then you are an inflecion point. We can classify these, but it's a little harder.

   (m) General rule: which is the first n where $f^{(n)}(x) \neq 0$? If n is even, you are a local minimum if $f^{(n)}(x) > 0$ and a local max if < 0. If n is odd then you are an inflection point. Compare to our three examples above. You don't have to go beyond the third derivative very often, but one example might be $x^2 \cos(x)$.

   (n) So, we've learned how to classify our critical points: that tells us a lot about our function! Let's put everything we've learned together, and talk about how to describe a general function: by sketching it (distinguish from a more detailed graph: just want the key features).

   (o) Steps to sketching: 0) Look for symmetry: odd even functions – you'll cut your work in half! 1) Where is the function defined? Vertical asymptotes? What happens near them? 2) Identify the following special points of the function: a) intersections with the axes, b) critical points / inflection points, c) discontinuities/corners. 3) What happens to the function at $\pm$ infinity? Does it go to $\pm\infty$? Horizontal asymptote? Or does it oscillate?

   (p) For the rest of the class, I'm going to go over some examples. 1) Start with a rational function, say $(x^2-1)/(x^2-4x+4)$. Factor it, there's one vertical asymptote. There's one critical point, it's a minimum. 2) $f(x) = \log(4 - x^2)$. 3) Let's consider now $(x^2 - x - 2)/(x - 3)$. This will have what's called a *diagonal asymptote*: definition: there's a linear function where $f(x) - (mx + b) \to 0$ as $x \to \infty$. Typical example: rational function where the degree of the numerator is higher than the denominator by one. 4) e^x/x

17. Tuesday October 31st

    (a) Review exam info. Review session in GSMN 51, Weds 6-8.

    (b) Return hwk 4, and go over solutions.

    (c) Main topic today: l'hôpital's rule, section 4.4.

    (d) Do I first want to talk about diagonal asymptotes from last time? Maybe. The example would be $(x^2 - x - 2)/(x - 3)$. This will have what's called a *diagonal asymptote*: definition: there's a linear function where $f(x) - (mx + b) \to 0$ as $x \to \infty$. Typical example: rational function where the degree of the numerator is higher than the denominator by one.

    (e) Define an indeterminate form. Two types: $0/0$ and ∞/∞.

    (f) For the theorem, state it, with the clear caveats that we must start with an indeterminate form, and the limit after differentiating must exist (it's allowed to be infinite). Give some historical background (Guillaume de l'Hôpital, late c17).

    (g) Proof in simple case: where the limit is finite (so say $f'(x_0)$ and $g'(x_0)$ are finite). Say we're in the $0/0$ case, so we're finding the limit at x_0 and $f(x_0) = g(x_0) = 0$. Then

$$\begin{aligned}\lim_{x \to x_0} \frac{f(x)}{g(x)} &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} \\ &= \lim_{x \to x_0} \frac{(f(x) - f(x_0))/(x - x_0)}{(g(x) - g(x_0))/(x - x_0)} = \frac{f'(x_0)}{g'(x_0)}.\end{aligned}$$

    (h) First examples: recover the things we proved using the squeeze theorem: $\sin(x)/x$, $(\cos(x) - 1)/x$, $\log(1 + x)/x$ as $x \to 0$. Note: when we calculated the derivative of $\sin(x)$ we used this limit! So this doesn't give us a new proof of the first limit, that would be a circular argument.

    (i) Example: $\tan(x) - x)/x^3$: must apply the theorem twice ($\sec' = \sec\tan$).

    (j) Some more examples, now of type ∞/∞. Rational functions (gives a new method). Things like e^x/x^n.

    (k) Non-example: $\sin(x)/(1 - \cos(x))$ as $x \to \pi$. Not applicable here because denominator doesn't go to zero.

    (l) Other kinds of intederminate expressions: $\infty \times 0$, $\infty - \infty$, 0^0, ∞^0, 1^∞. etc. We can often put these into l'Hôpital form. The first is easy: just take the reciprocal of one of the terms!

    (m) So, for example, $x^n \log(x)$, as $x \to 0$. Write it as $\log(x)/x^{-n}$: the theorem says it goes to zero.

    (n) For powers, take logarithms: $h = f^g = e^{g \log(f)}$, so $\log(h) = g \log(f) = \log(f)/(1/g)$. We can now use the theorem.

    (o) Examples: x^x, $x^{1/x}$, $(4x + 1)^{\cot(x)}$, $\cos(x)^x$.

    (p) Finally, differences can often be rewritten as ratios. For instance $\sec(x) - \tan(x)$, $\cot(x) - 1/x = (x\cot(x) - 1)/x$. Evaluate the first part using the theorem, then apply the theorem (it's infinite).

18. Tuesday November 12th

    (a) Go over exam, mention last problem. Also first problem on the chain rule.

    (b) Reminder: homework due Thursday.

    (c) Office hour change Thursday: now 2:15 to 4:15.

    (d) Today's topic: the Newton-Raphson method, which is chapter 4.8 in the textbook. This is a technique for approximately finding *roots* of functions. That is, for a function $f(x)$, finding those points x where $f(x) = 0$ – the points where the graph intercepts the x-axis. For some types of function we know how to do this (e.g. quadratic equations), but it's not always possible to find the roots algebraically.

    (e) This method will work for any differentiable function $f(x)$.

    (f) The basic idea: we start by guessing a root, then we successively improve our guess, getting closer and closer to the real answer. We can think about how we improve our guess geometrically.

    (g) (Draw a picture). Take our initial guess – call it x_0 – find the tangent line at that point, so $y - f(x_0) = f'(x_0)(x - x_0)$, and see where it intercepts the x-axis. It'll be at the point $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ (see this by solving the equation. Because the tangent line is a linear approximation to our function, we expect this x_1 will be closer to the root we're looking for than x_0, so this is a better approximation.

    (h) Now we can repeat, e.g. letting $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, then finding $x_3, x_4, \ldots$.

    (i) Example: consider the function $f(x) = x^3 - 2x - 5$. This has a root between 2 and 3, so let's start with $x_0 = 2$. We'll find x_1 and x_2 using a table. First differentiate the function: we'll get $x_1 = 2.1$. To find x_2 we'll need to check $2.1^2 = 4.41$, $2.1^3 = 9.261$. We end up with about $x_2 = 2.0946$, which is accurate to 4dp.

    (j) Example: consider the function $f(x) = x^4 - 2$. Of course, this function has two real roots: at $x = \pm 2^{1/4}$. We can use Newton-Raphson to find the decimal expansion of these numbers. We'll start with $x_0 = 1$. Again, let's find the first few approximations. We get $x_1 = 1.25$, $x_2 = 1.1935$, $x_3 = 1.1892$ about, again accurate to 4dp.

    (k) Another example: $f(x) = \cos(x) - x$. So roots here show where the line $\cos(x)$ meets the line $y = x$. It's clearly going to happen between 0 and $\pi/2$, so $x_0 = 1$ seems like a sensible choice. We need a calculator to do these calculations, but we'll end up with $x_3 \approx x_4 \approx 0.759085$, equal to 6dp. This gives a good approximation.

    (l) How do we choose our starting value, and what can go wrong? Draw some pictures of things going wrong if the tangent line to x_0 is far from the graph of the function, e.g. if x_0 is near a critical point. To estimate a good starting point it's useful to graph the function. We'll usually at least get a range to work with by finding some critical points.

    (m) Example. $f(x) = x^2 - 12x + 10\log(x) + 20$. Probably has a positive root, but where? Well, let's graph it. Its derivative is $2x - 12 + 10/x = 2(x^2 - 6x + 5)/x$, which has roots at $x = 1$ and 5. Here $f(1) = 9$, $f(5)$ is big. It also has a vertical asymptote at $x = 0$, and the function goes to negative infinity there, and to positive infinity at infinity. The second derivative is $x - 10/x^2$, so $f''(1) < 0$ and $f''(5) > 0$, so they are a max and a min. The result is that the root is somewhere between 0 and 1. Try Newton-Raphson with $x_0 = 1/4$. Get $x_1 \approx 0.1377$, $x_2 \approx 0.1617$, $x_3 \approx 0.1644 \approx x_4$, which is probably a good approximation for the root.

19. Thursday November 14th

    (a) Remember to collect honors homework 5.

    (b) Topic for today: antiderivatives. Refer to Section 4.9 in the textbook.

    (c) Start with the definition of an antiderivative.

    (d) Not unique: can add a constant.

    (e) Example, reversing the power law. Special case for $1/x$.

    (f) Negative cases, and more than one constant.

    (g) Likewise antiderivatives of a step function from -1 to 1: there are two constants. I might also give this to the group, with the example of a function with two steps, from -1 to 0 to 1.

    (h) Another perspective: solution to a first order ODE.

    (i) Connection to integration and notation.

    (j) Table of examples: powers, e^{ax}, $\sin(x)$, $\cos(x)$, $\sec^2(x)$, $\tan(x)$ is harder (it's $\log(\cos(x))$), $\sinh(x)$, $\cosh(x)$, $(1+x^2)^{-1}$ (arctan), $(1-x^2)^{-1/2}$ (arcsin) etc.

    (k) Aside: note that arcsin and arccos have derivatives differing only by a sign, and they are defined on the same range. So that means their sum must be constant (if $x = \sin(y)$ then $x = \cos(y + \pi/2)$), so the sum of arccos and arcsin is constant $\pi/2$. Same function, two antiderivatives.

    (l) Remark that not every function has an antiderivative, although all continuous functions do. Discontinuous functions don't have to. Even if you do have an antiderivative you might not have an *elementary* antiderivative. Examples include the Gaussian, whose antiderivative is called the "error function" $\mathrm{erf}(x)$: name comes from normally distributed errors: for positive x it gives the probability that the error is less than x (graph it: it's a sigmoid with horizontal asymptotes at ± 1.). Another example is the antiderivative of $1/\log(x)$, which is called $\mathrm{li}(x)$, the logarithmic integral. This shows up in number theory when you study the distribution of primes.

    (m) Application: motion. If you know the velocity of an object, or its acceleration over time, then you can work out its position using antidifferentiation.

    (n) Example: an object falling under gravity: acceleration is $-g$ ms^{-2}. To find the speed, antidifferentiate, you get $-gt + c_1$. To find the position, antidifferentiate again, get $-\frac{1}{2}gt^2 + c_1 t + c_2$. To work out those constants, you need to know the initial position and velocity. So if you're initially stationary at height h then the position at time t is $h - \frac{g}{2}t^2$. In particular, you hit the ground when $t^2 = 2h/g$.

    (o) Another example: have initial upward speed, then c_1 is non-zero. Say $h = 20$, initial $v = 15$. So to find when the object hits the ground, end up solving the equation $-5t^2 + 15t + 20 = -5(t+1)(t-4)$, so $t = 4$.

20. Tuesday November 19th

(a) Begin with a quick review from the worksheet: problems 4 and 5 $\lim_{x\to 0} x^{x^2}$ and $\lim_{x\to 1}(2-x)^{\tan(\pi x/2)}$ (so take logs).

(b) In this class, we're going to start discussing *integration*. Compare to section 5.1 in the textbook.

(c) Motivating problem: finding areas of regions. Warm-up: the method of exhaustion (Achimedes). Fit inside a circle a sequence of polygona with more and more sides, and calculate their areas. The limit of these areas should give the area of the circle (they *exhaust* the entire area).

(d) Let's try to actually do it. The n-sided polygon can be split into n triangles, and we need to calculate their areas. If the triangle has angle θ, then its area is $\sin(\theta/2)\cos(\theta/2) = 1/2\sin(\theta)$. So the overall area is $\frac{n}{2}\sin(2\pi/n)$. We can actually compute this using L'Hôpital's rule. This isn't what Archimedes did. He calculated some areas, and also areas of polygons *outside* the circle, to show that π was between $3+10/71$ and $3+10/70$, which is a very good approximation.

(e) Quick aside on limits of sequences.

(f) Another example he worked out: the area between a parabola and a line is $4/3$ times that of the triangle with the same base and height (draw a picture). How might we show this?

(g) Exhaustion for the graph of a function: approximate by rectangles (draw a picture). Describe the area as a sum. So say there are n rectangles of size Δx. Say our function goes from a to b, so $b = a + n\Delta x$, or $\Delta x = (b-a)/n$. Then if we write the sum we get approximation A_n given by (recall notation for sums)

$$\begin{aligned} A_n &= f(a)\Delta x + f(a+\Delta x)\Delta x + f(a+2\Delta x)\Delta x + \cdots + f(a+(n-1)\Delta x)\Delta x \\ &= \sum_{k=0}^{n-1} f(a+k\Delta x)\Delta x. \end{aligned}$$

(h) Definition of the area as a limit of A_n as $n\to\infty$, or as $\Delta x = (b-a)/n$ goes to zero.

(i) Example: a linear function $f(x) = mx$, and we'll go from $x=0$ to $x=1$. The answer is obvious, so let's check our method makes sense. So the sum is $\sum_{k=0}^{n-1} m(k\Delta x)\Delta x$. Plug in that $\Delta x = 1/n$, so it's $\sum_{k=0}^{n-1} mk/n^2 = \frac{m}{n^2}\sum_{k=0}^{n-1} k$. That sum is $n(n-1)/2$, so we get $\frac{m}{2}\frac{(n^2-n)}{n^2}$, and the limit is $m/2$ as expected.

(j) Example: a parabola, maybe do $f(x) = 1-x^2$ from -1 to 1. It's the sum from 0 to 1, times two. Maybe have the students do the sum for $n = 2, 3$? One So we get $\sum_{k=0}^{n-1} 1/n - \sum_{k=0}^{n-1} k^2/n^3 = n/n - \frac{n(n-1)(2n-1)}{6n^3} \to 2/3$ (there's a trick for the sum of squares: consider $k^3-(k-1)^3$ and sum the telescope). Double it and we get $4/3$ like Archimedes told us.

(k) Definition of the definite integral. It's the limit as $n\to\infty$ above. More generally, take any sample points x_i^* on the intervals. We say f is *integrable* if the limit exists. Comment on Leibniz's notation: it's supposed to look like a sum. The $\mathrm{d}x$ in the notation is important, because we're describing an area, so it should look like a length times a height (units!)

(l) Connection to the anti-derivative (reference 5.1). Say we want an antiderivative $F(x)$ of a continuous function f. How do we find $F(x)$? We'll find one with $F(0)=0$ To find $F(\Delta x)$ for some small difference, we want it to have slope about $f(0)$, so the height should be about $F(\Delta x) = f(0)\times\Delta x$ (draw a picture with a rectangle). To find $F(2\Delta x)$, we want it to have slope about $f(\Delta x)$ so make it $f(\Delta x)\times\Delta x$. Keep going, drawing the rectangles under the graph of $f(x)$, until we get $f(x) =$ the sum we've been discussing. Of course, this approximation only gets better, so we take the limit, and the antiderivative is the integral. State the fundamental theorem of calculus.

21. Tuesday November 19th

    (a) Go over corrections from Homework 5.