Fractals

Computer Graphics

Definition

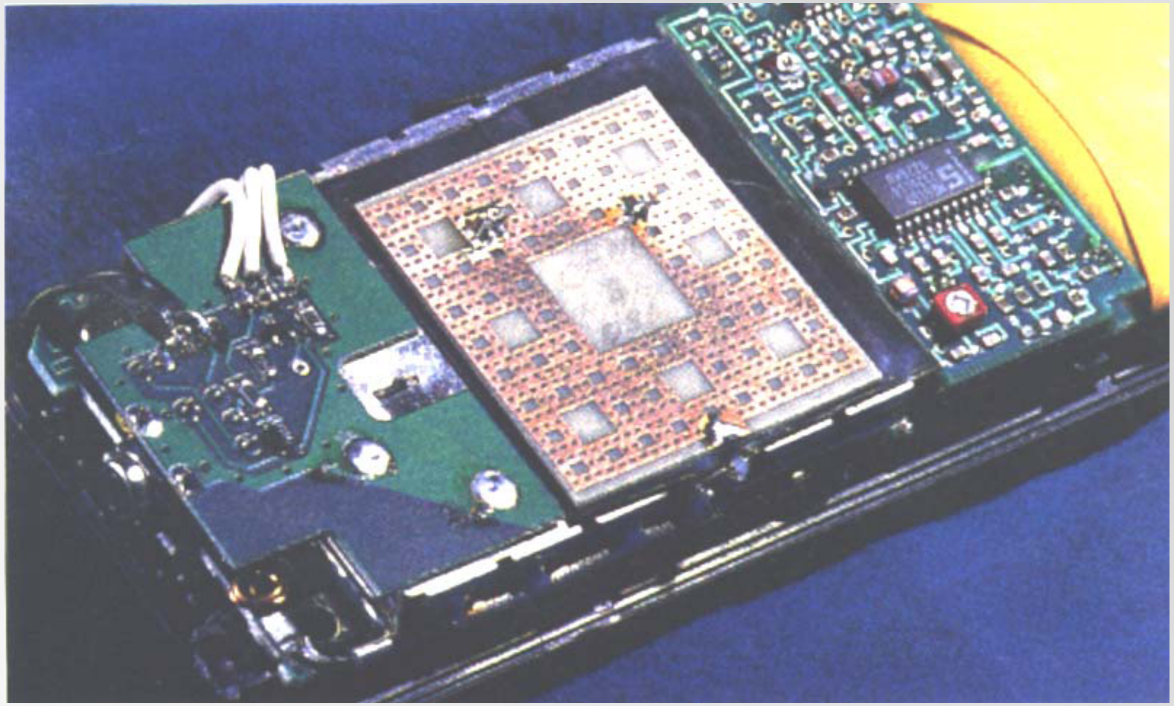
- "**Fractal**" concept coined by Benoit Mandelbrot in 1975
- Relevant researchers
  - Cantor, Lyapunov, Peano, Koch, Sierpinski, Julia , Mandelbrot
- Fractal Objects are characterized by:
  - **Infinite detail** at each point
  - **Self-similarity** among its parts and its whole
- We will always see the same figure regardless of how far we are from it (real fractal)

Characterization

- **Euclidean Geometry**
  - Equations
  - Artificially created Objects
  - Differentiable, locally smooth
  - More detail as we get closer to the object
- **Fractal Geometry**
  - Procedures
  - Natural Objects
  - Non-differentiable, locally bumpy
  - Same detail as we get closer to the object

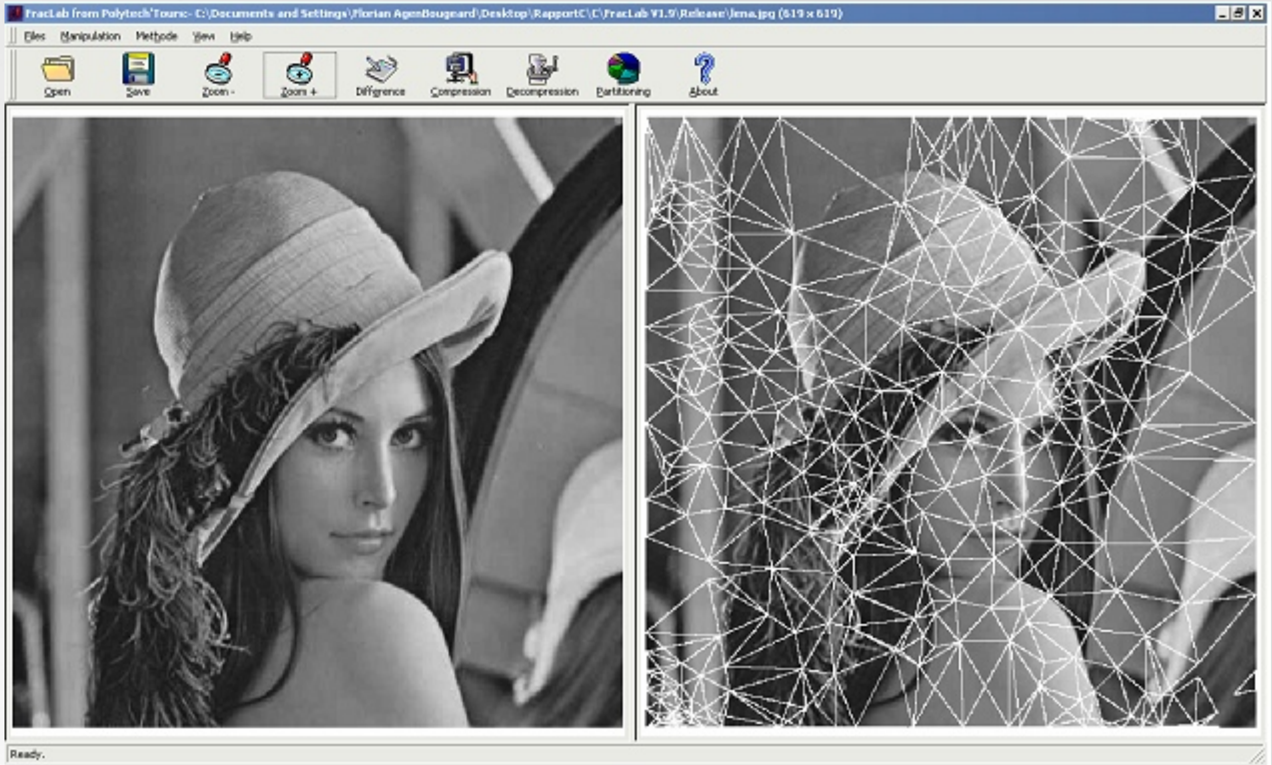
Applications

- **Fractal Antennas** (band width)



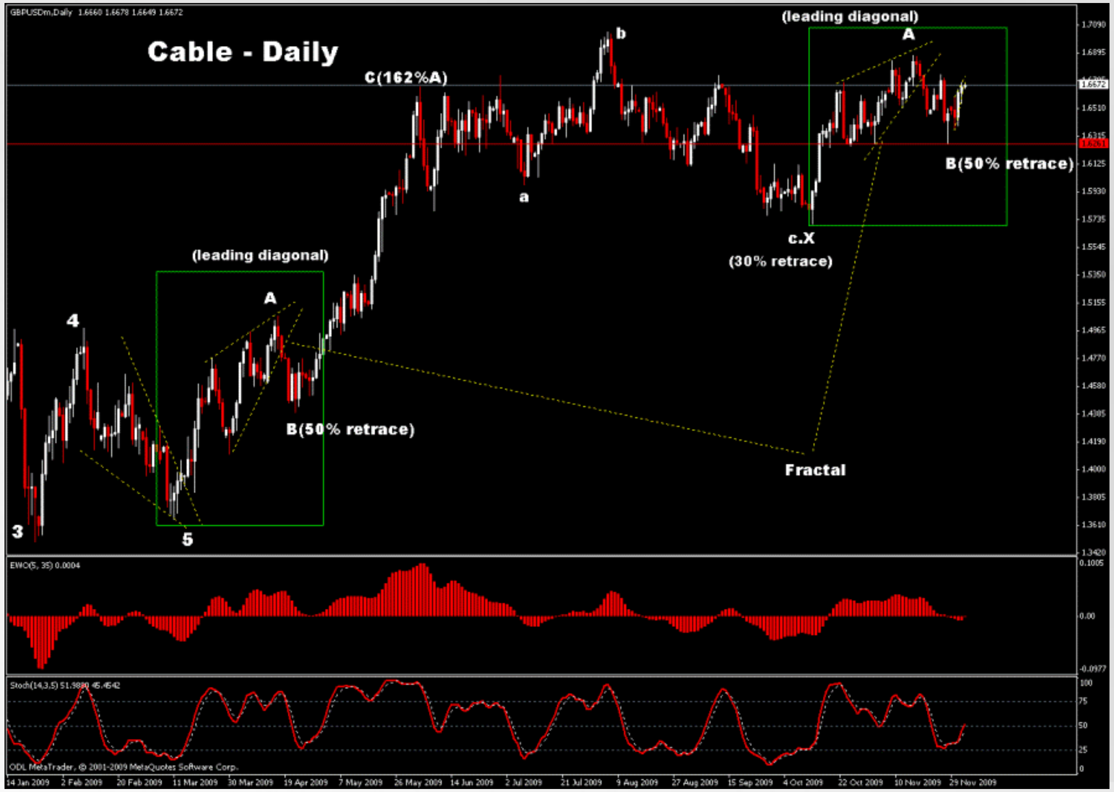
Applications

- **Image Compression**



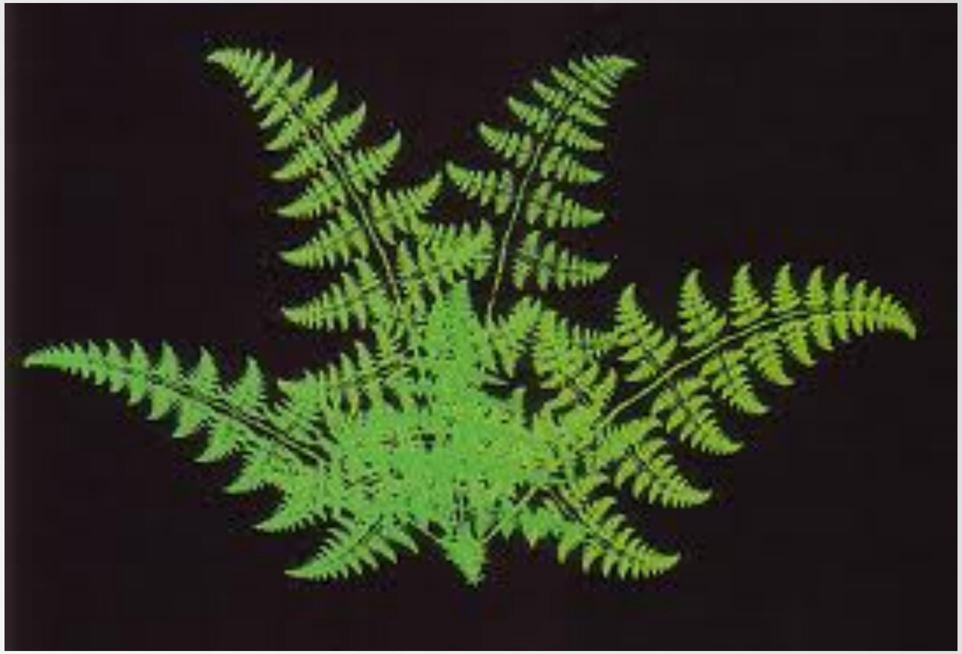
Applications

- **Dynamic Systems**



Applications

- **Natural shapes modeling**

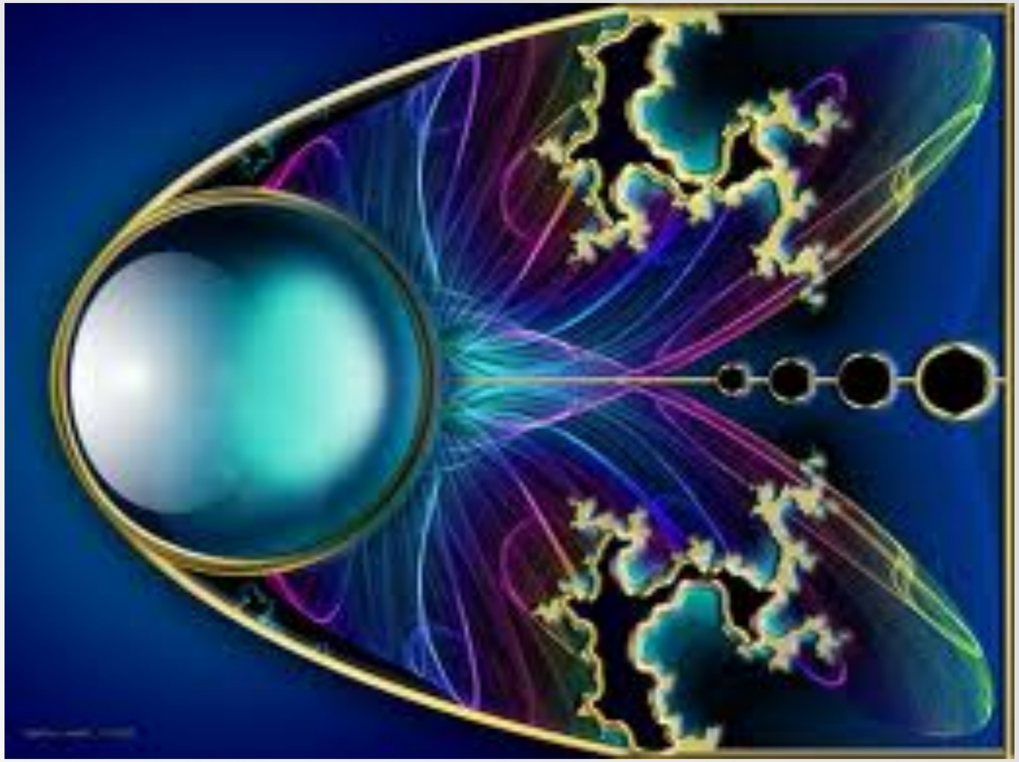


Applications

- **Natural shapes modeling**

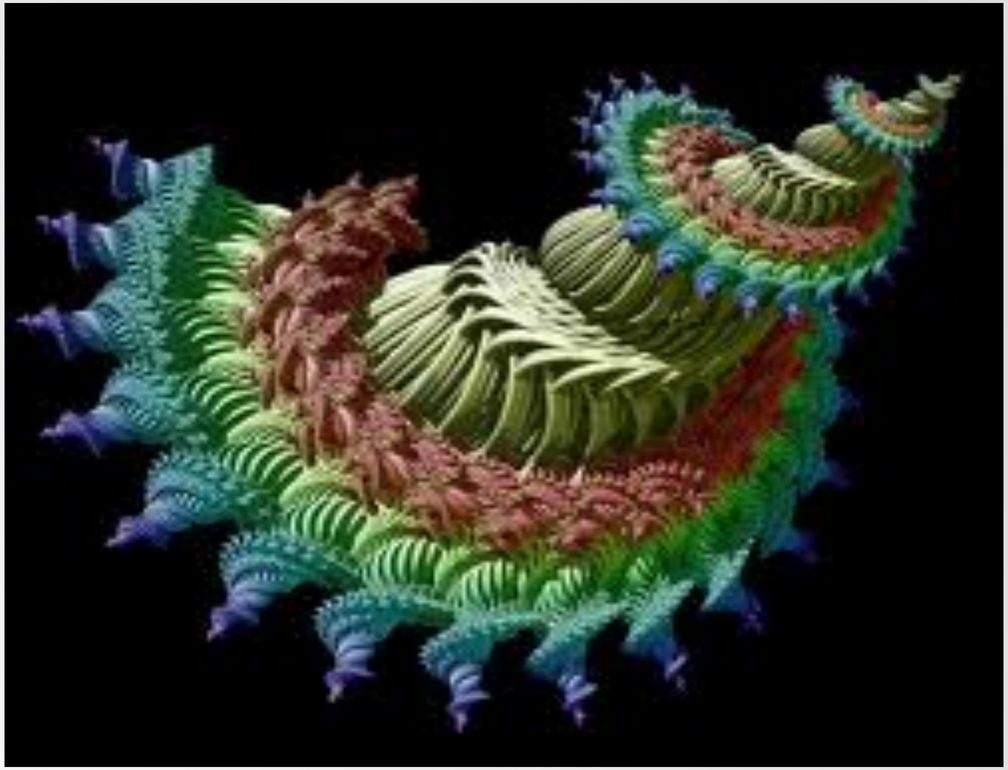
Applications

- **Fractal Art**



Applications

- **Fractal Art**



Generation Procedure

- A transformation function, $F(X)$, is recursively applied to all points within a region of space
  - We start by selecting a initial point P_0
  - Each step $F(X)$ is applied, generating successive levels of detail $P_1=F(P_0)$, $P_2=F(P_1)$, ... $P_{k+1}=F(P_k)$, …
- $F(X)$ can be applied to:
  - Set of points
  - Set of primitives (lines, surfaces, solids, color areas…)

Generation Procedure

- F(X) defined in terms of **transformations**:
  - Geometric (scaling, translation, rotation)
  - Non-linear coordinates (and decision parameters)
- F(X) defined in terms of **procedures**
  - Deterministic
  - Stochastic
- Those variations smaller than a pixel can not be displayed → F(X) is applied a finite number of steps
  - To increase detail, we can always zoom and apply F(X) again

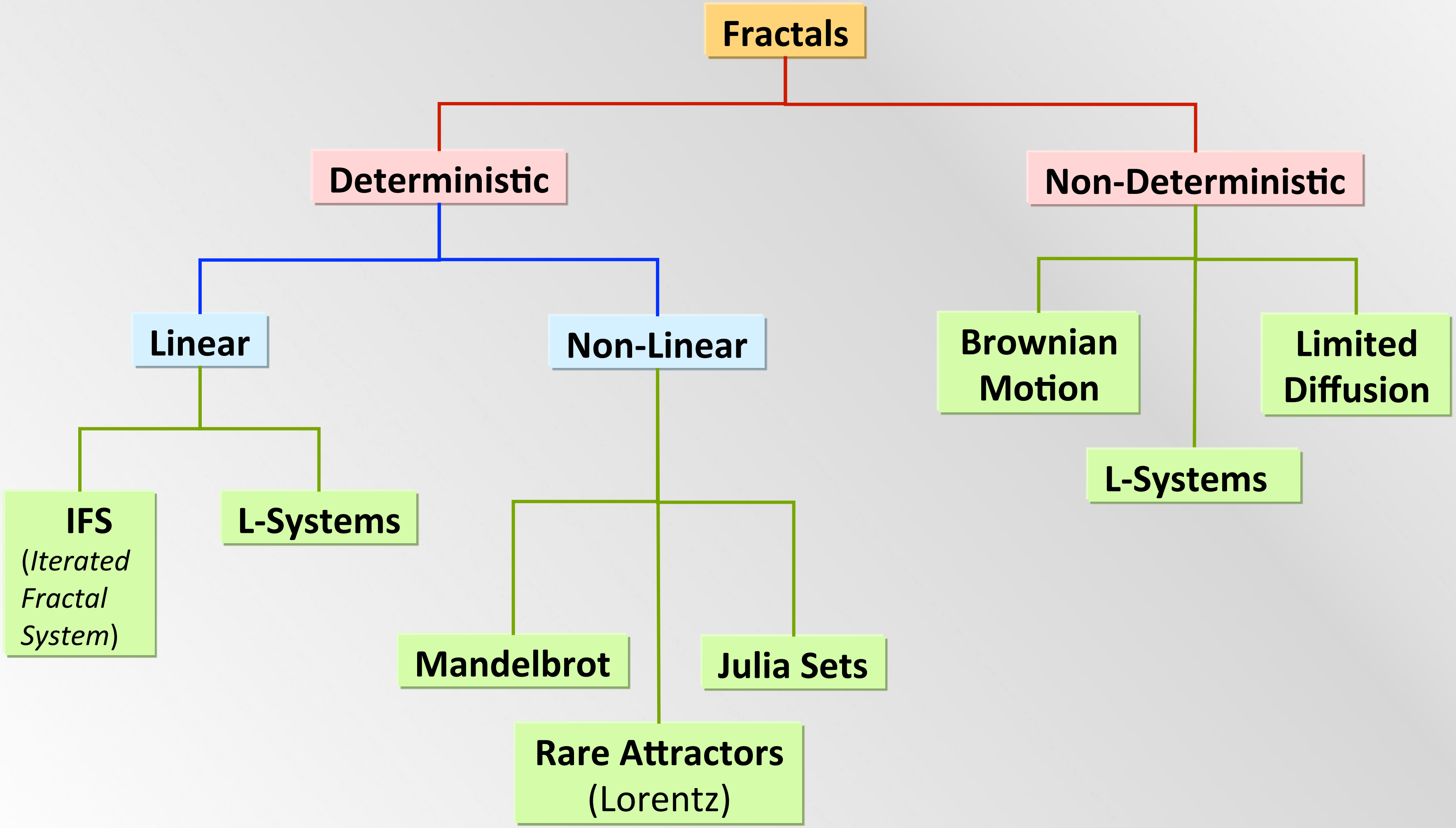
Fractals Classification

- **Self-Similar**
  - Its parts are scaled versions of its whole. A single scale parameter (**S**) is applied to the whole shape
  - **Statistically self-similar** if apply random variations

    → trees, plants…
- **Self-Affine**
  - A different scale factor can be applied to each axis: $\mathbf{S_x}$ **,** $\mathbf{S_y}$ **,** $\mathbf{S_z}$
  - **Statistically self-affine** if apply random variations

    → water, clouds…
- **Invariant**
  - Non-linear transformations
  - Quadratic fractals (Mandelbrot), self-inverse fractals…

Fractals Classification



Fractal Dimension

- Objects have an integer dimension in Euclidean geometry and a **fractional or fractal dimension** in fractal geometry
- **Fractal dimension** is the measure of detail variation in a fractal object
- An object fractal dimension is always greater than its Euclidean (topological) dimension
- Given a fractal dimension, fractal objects can be created through recursive procedures
- We can sometimes calculate an object fractal dimension from its properties through other procedures (difficult task)

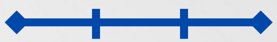
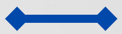
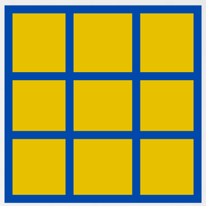
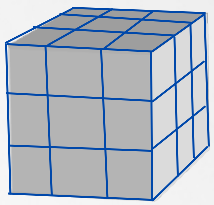
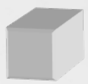
Fractal Dimension

- We calculate a deterministic self-similar fractal dimension (only one scale factor) by using analogies to an Euclidean object subparts
- Given a self-similar fractal, we have:
  - **DE** = Euclidean dimension
  - **D** = fractal dimension
  - **N** = number of similar parts
  - **S** = scale factor

$$\mathbf{NS^{DE} = 1 \quad \rightarrow \quad NS^{D} = 1 \quad \rightarrow \quad N = \left(\frac{1}{S}\right)^{D}}$$

$$\mathbf{\ln(N) = \ln\left(\frac{1}{S}\right)^{D} = D\ln\left(\frac{1}{S}\right) \quad \rightarrow \quad D = \frac{\ln(N)}{\ln\left(\frac{1}{S}\right)}}$$

Fractal Dimension

		N of parts (N)	Scale factor (S)	Dimension (D)
		**3**	**1/3**	1
		6	1/6	
		42	1/42	
		**$9 = 3^2$**	**1/3**	2
		$36 = 6^2$	1/6	
		$1764 = 42^2$	1/42	
		**$27 = 3^3$**	**1/3**	3
		$216 = 6^3$	1/6	
		$74088 = 42^3$	1/42	

Fractal Dimension

- **Dimension of a fractal curve within 2-dim plane** (litorals)
  - Usually $1 < D \leq 2$ (smoother as it gets closer to 1)
    - $D = 2$ when it fills a finite region of the plane
  - $2 < D < 3$ when it is a self intersecting curve able to fill the plane an infinite number of times
- **Dimension of a spatial fractal curve**
  - Usually $1 < D$ but can be $2 < D \leq 3$ without being self intersecting
    - $D = 3$ when it fills a finite space volume
  - $3 < D < 4$ when it is a self intersecting curve able to fill a space volume an infinite number of times

Fractal Dimension

- **Dimension of a fractal surface** (land, clouds, water)
  - Usually $2 < D \leq 3$ (smoother as it gets closer to 2)
    - $D = 3$ when it fills a space volume
  - $3 < D < 4$ when it is a self intersecting surface able to fill the space volume an infinite number of times

- **Dimension of a fractal solid** (water vapor density, temperature differences)
  - Usually $3 < D \leq 4$
  - $4 < D$ for some self-intersecting solids

Deterministic Fractals

- Build a self-similar fractal:
  - **Initiator** – certain geometric shape
  - **Generator** – pattern followed by iteration resulting subparts
- When building a self-similar fractal, we can add a random factor to the iteration process (trees, plants…)
  - Random generator selection
  - Random coordinates translation
- Twisted and bumpy shapes → Rotation and scaling of fractal objects

Deterministic Fractals

- **Cantor Set** (1883) – subset of the interval [0,1] obtained by eliminating, in each step, the central segment for all intervals obtained in the prior iteration

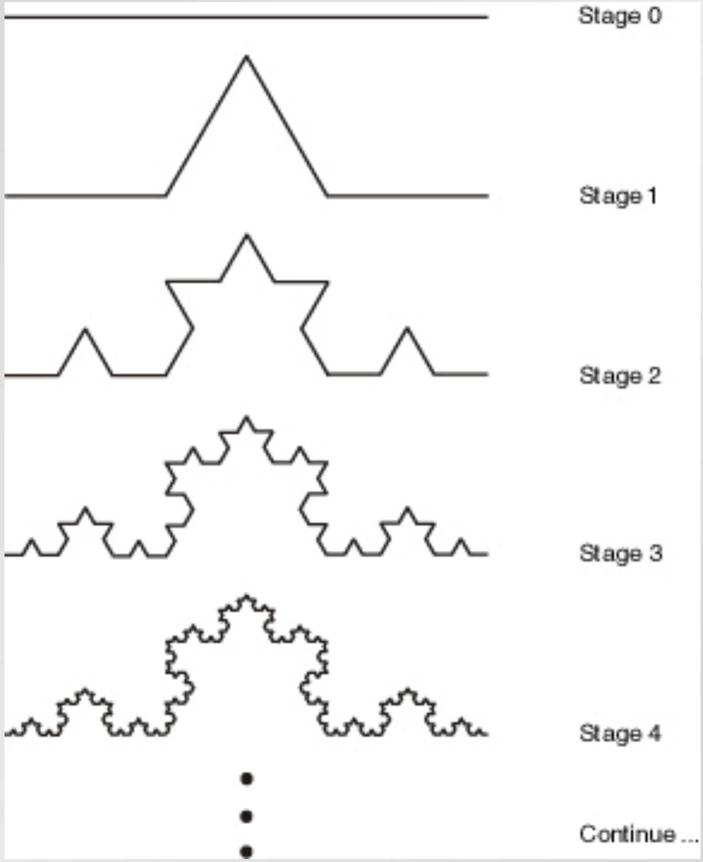
$$D = \frac{\ln 2}{\ln 3} = 0,63$$

Deterministic Fractals

- **Koch curve** (1904)

$$D = \frac{\ln 4}{\ln 3} = 1,26$$

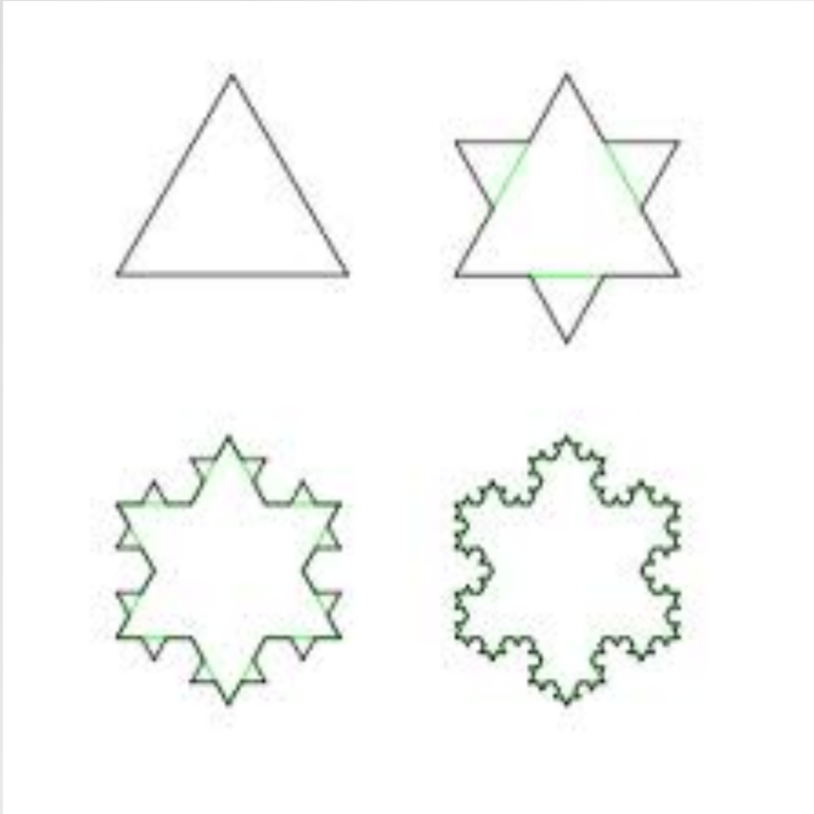
  - Continuous in all points
  - Non-differentiable in any point



Deterministic Fractals

- **Koch snowflake**(1904)

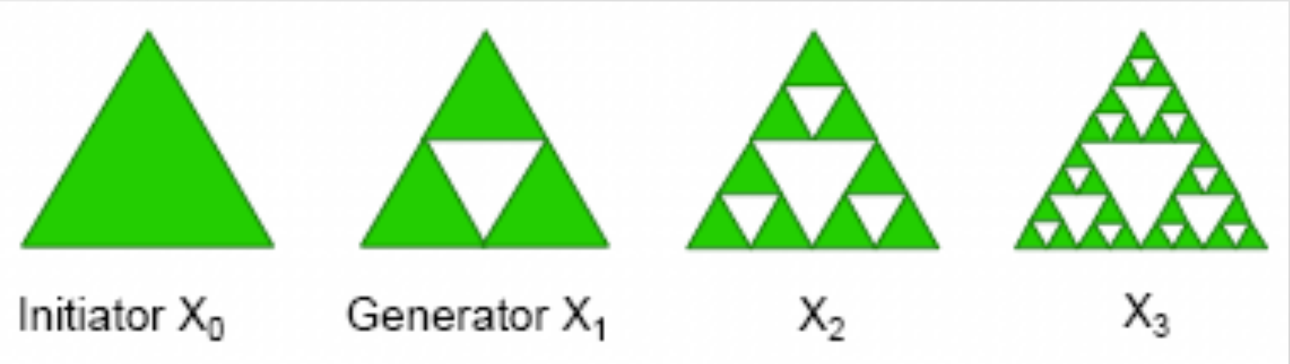
$$D = \frac{\ln 4}{\ln 3} = 1,26$$



Deterministic Fractals

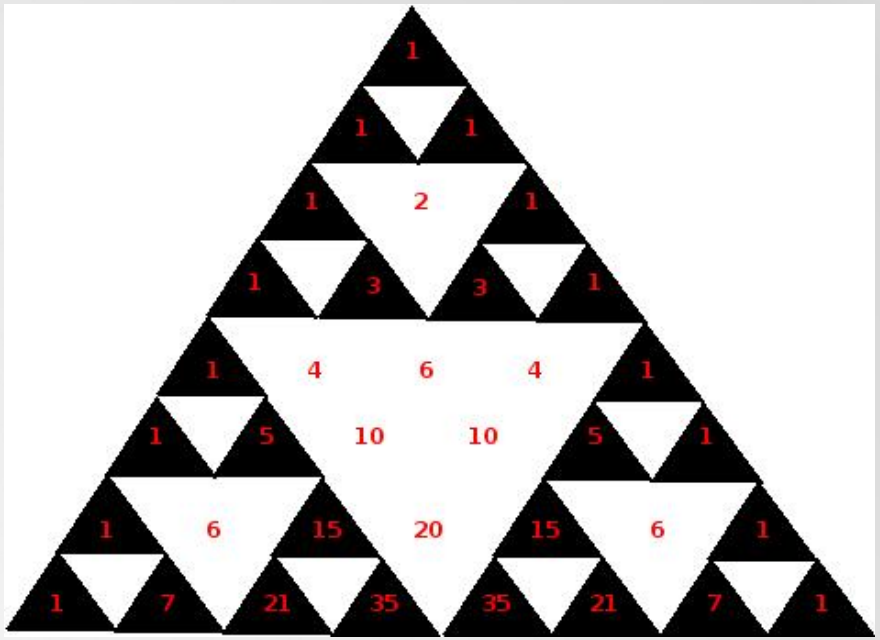
- **Sierpinski Triangle**

$$D = \frac{\ln 3}{\ln 2} = 1,58$$



Deterministic Fractals

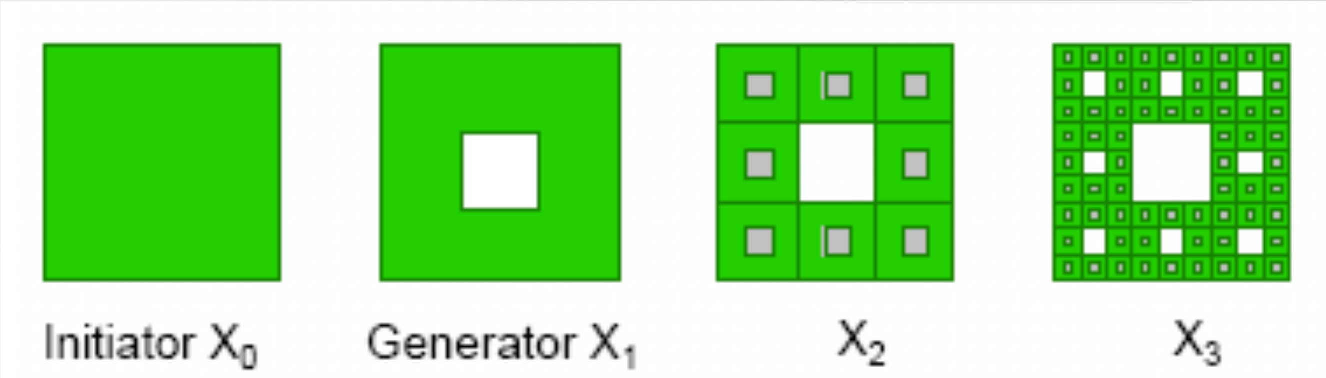
- **Sierpinski Triangle**
  - We place a black triangle on each odd number obtained in the Pascal Triangle.



Deterministic Fractals

- Sierpinsky Carpet

$$\mathbf{D} = \frac{\ln(8)}{\ln(3)} = 1{,}89$$

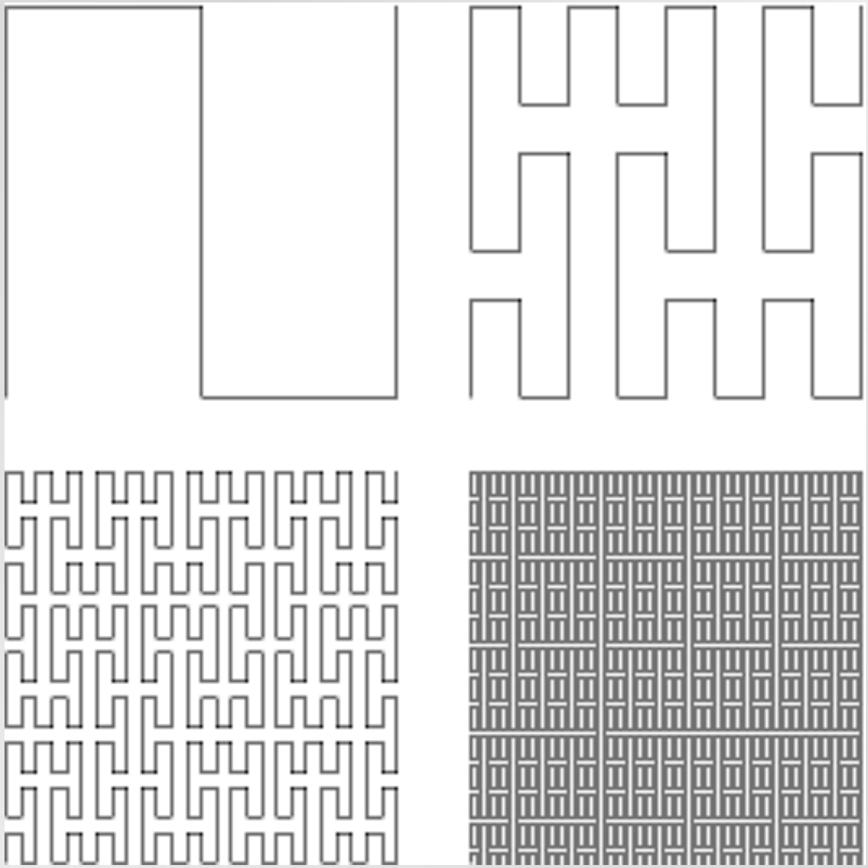


Deterministic Fractals

- **Peano Curve**

$$D = \frac{\ln 9}{\ln 3} = 2$$

  - It fills the whole plane in the limit
  - Similar to **Hilbert curve**

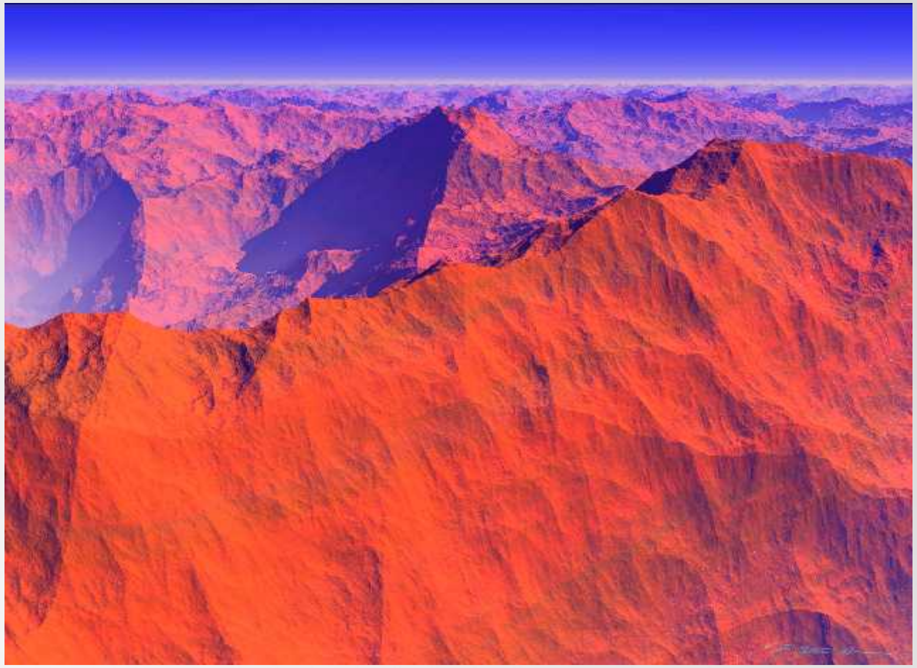


Stochastic Fractals

- **Fractional Brownian Motion** – standard Brownian motion extension, determined by:
  - Random number (associated to certain probability distribution)
  - Time Step
- Iterative process – from the initial position on the XY plane, we generate movement :
  - Random direction
  - Random length
- 2-dim matrix consisting of fractional Brownian motion “jumps” placed on a 2-dim grid (model land, ground…) , or placed on a a sphere (planet)

Stochastic Fractals

- Adjust fractal dimension when calculation Brownian Motion “jumps” to obtain realistic images (mountain)
- Scale jumps → increase/decrease Brownian Motion “jumps”



Grammar Models

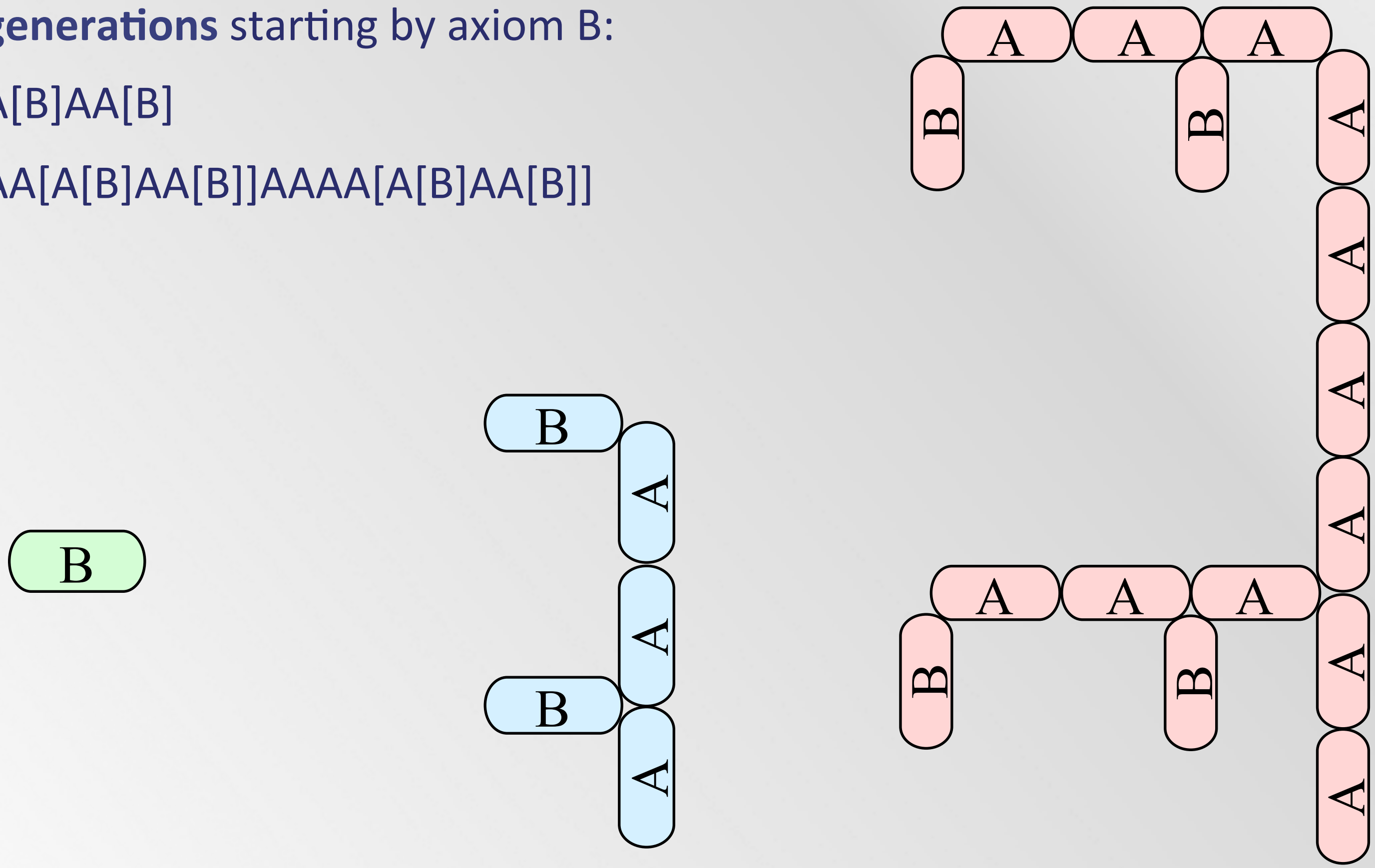
- Smith presents a method for describing the structure of certain plaints, originally developed by Lindenmayer, by using parallel graph grammar languages(L-grammars)
- These languages are described by a grammar consisting of a **set of productions, all applied at once**
- **Self – similarity**: the pattern described in the nth generation word is contained (repeated) in the (n+1)th generation word
- **Specification:**
  - A word in the language represents a sequence of segments in a graph structure
  - Bracketed portions represent portions that branch from the symbol preceding them

Grammar Models

- Grammar with the following **alphabet**
  - **{A, B, [,]}**
- **Production rules**
  - **A → AA**
  - **B → A[B]AA[B]**
- Starting by axiom A, the first generations are:
  1) AA
  2) AAAA , etc.
- Starting by axiom B, the first generations are:
  1) A[B]AA[B]
  2) AA[A[B]AA[B]]AAAA[A[B]AA[B]] etc.

Grammar Models

- **First generations** starting by axiom B:
    1) A[B]AA[B]
    2) AA[A[B]AA[B]]AAAA[A[B]AA[B]]

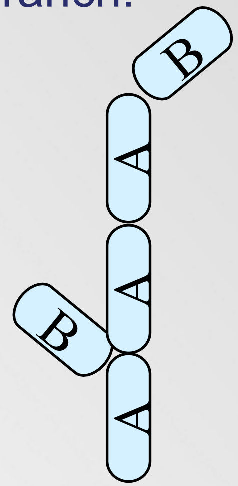


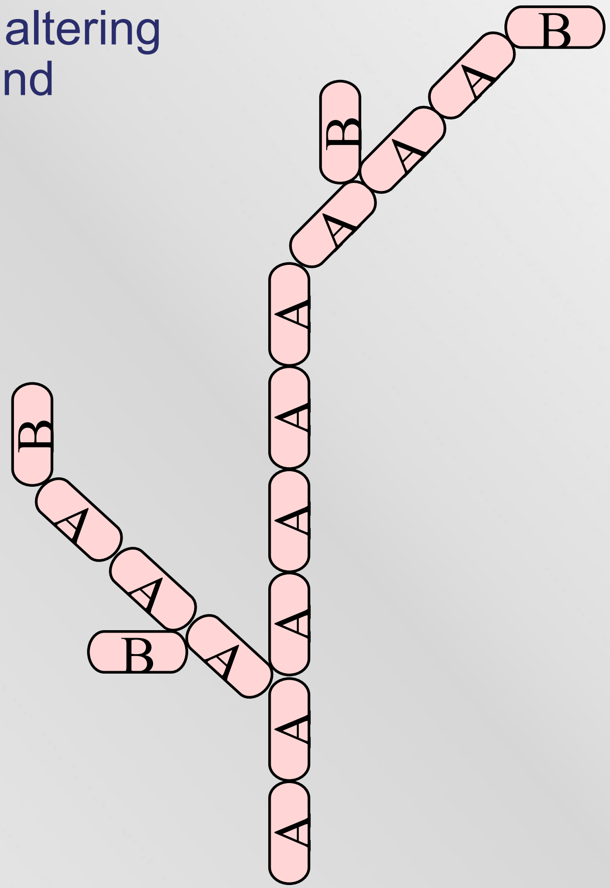
Grammar Models

- By adding parentheses “( )” to the language and altering the second production to be A[B]AA(B), the second generation becomes:

  AA[A[B]AA(B)]AAAA(A[B]AA(B))

- Square brackets denote a left branch and parentheses denote a right branch:





Grammar Models

- The grammar itself has **no inherent geometric content**, so using a grammar-based model requires
  - A **grammar** to generate the word
  - A **geometric interpretation** of the language to generate an object
    - Choose varying branching angles
    - Different thickness for the lines
    - Drawing a “flower” or “leaf” at each terminal node

Grammar Models

- Grammar Enhancements:
  - The grammars have been enriched to allow for keeping track of the **“age” of a letter in a word →** different transformations for old and young letters
  - To accurately represent the actual biology of plants during development, productions of the grammas are **applied probabilistically** rather than deterministically (Reffye´s model)
    - Start from a stem with a bud on its tip, which may undergo different transitions (leaf, bud, internode…)

Grammar Models

- En Fractint:

  Plant02{

  angle 18

  axiom f

  f=F[<4+F]F[<4-F][F]

  }

- < > varía el color
- \+ – varía el ángulo

Quadratic Fractals

- Recursive function, **F(z), recursively applied to points within the complex space** (complex function).

- In the complex plane, a point and its module are represented as:

$$\mathbf{z} = \mathbf{x} + \mathbf{iy} \qquad |\mathbf{Z}| = (\mathbf{x}^2 + \mathbf{y}^2)^{1/2}$$

- We define a function $\mathbf{F}: \mathbf{Z} \rightarrow \mathbf{Z}$ as:

$$\mathbf{z_{k+1}} = \mathbf{F}(\mathbf{z_k}) = \mathbf{z^2}_{\mathbf{k}} + \mathbf{c}$$

- Special family of transformations (λ being complex) is:

$$\mathbf{F}(\mathbf{z_k}) = \boldsymbol{\lambda}\, \mathbf{z^2}_{\mathbf{k}}(\mathbf{1} - \mathbf{z_k})$$

Quadratic Fractals

- When we recursively apply squares to a complex number, it:
  - Tends to infinity if $|z|>1$
  - Tends to zero if $|z|<1$
  - **Stays at $|z|=1$** if $|z|=1$

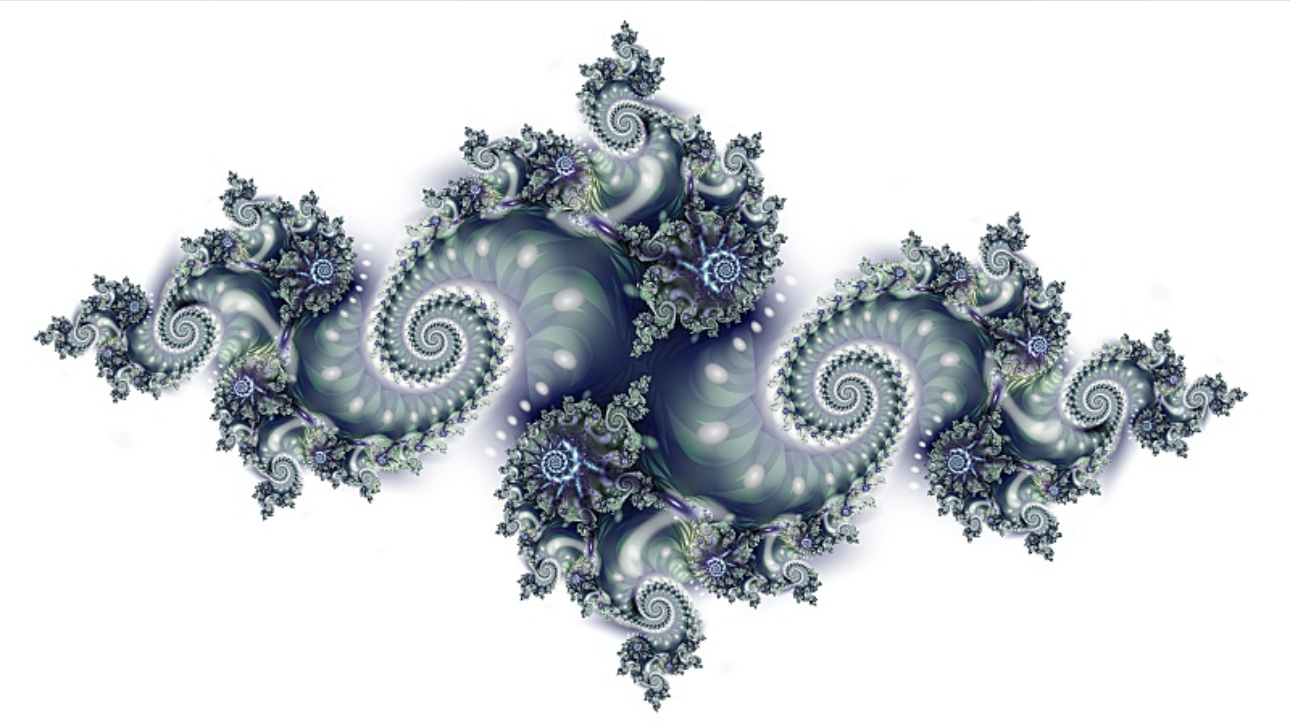
- Depending on the initial point, $\mathbf{z_0 = c}$, the transformation succession:
  - Diverges to infinity if $|z|>1$
  - Converges to an attraction point (*attractor*) if $|z|<1$
  - **Are periodic if** $|z|=1$

Quadratic Fractals – Julia

- **Julia set** – fractal frontier which separates
  - Points diverging to infinity
  - Points converging to an attraction point
- For each value of **c = a + bi**, there is a different Julia set
- To calculate positions in the fractal curve, we could:
  1) Try positions until we find a point for which the succession transformation is periodic (does not converge or diverge)
  2) Calculate the inverse function (when possible)

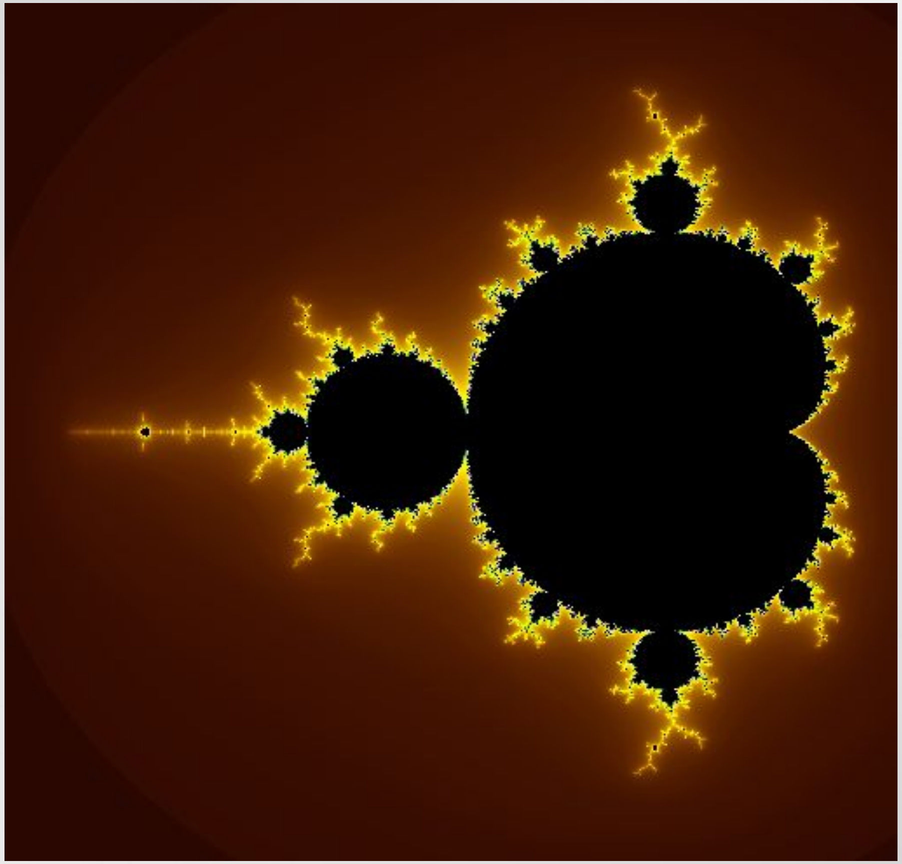
Quadratic Fractals – Julia

- **Julia set example**



Quadratic Fractals – Mandelbrot

- **Mandelbrot set**
  - King of mathematical monsters
  - Contains infinite copies of itself
  - Copies are connected to the main body through “chains” (connected set)
  - It has a fractal dimension of **2**



Quadratic Fractals – Mandelbrot

- **Mandelbrot set** – calculate the Julia set for each value of c and color with a:
  - **Black point** when the Julia set is connected (continuous)
  - **White point** when the Julia set is not connected (non-continuous)
- Our transformation function is:

$$z_{k+1} = F(z_k) = z^2_k + c$$

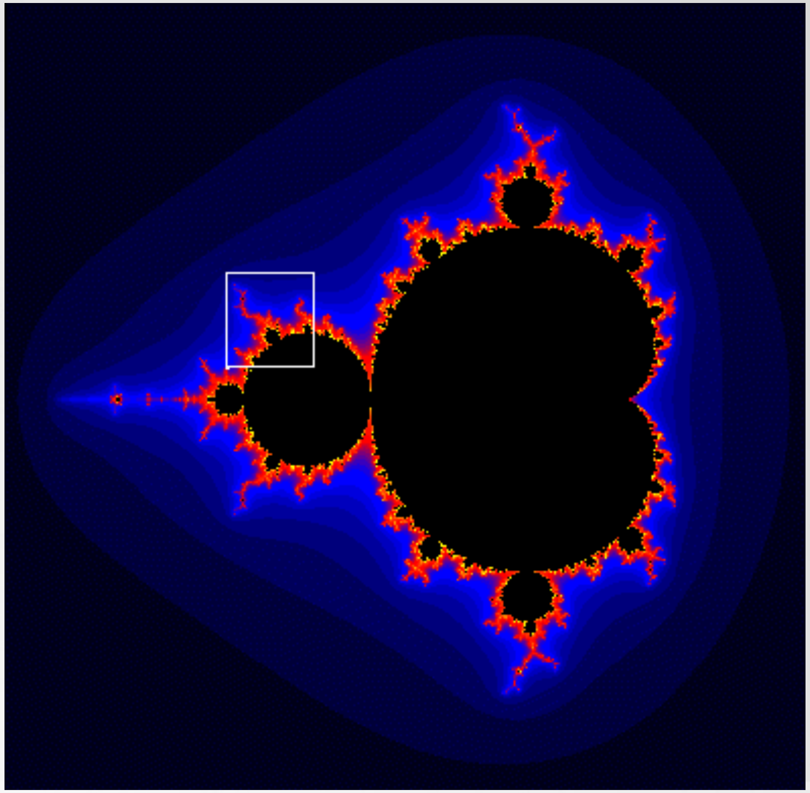
- We select an initial point $\mathbf{z_0 = 0 + 0i}$ and apply the transformation function **for each value of c** (of the form a + bi)
- A point **c belongs to the Mandelbrot set** if only if all points generated by this function have a finite module

Quadratic Fractals – Mandelbrot

- We fix a **maximum radius n** as a limit for the successive positions
- In practice, if $|z_k| > n$, successive values of $|z_k|$ will get bigger every time
- We fix a maximum number of iterations **k**
  - If $|z_k| > n$ before kth iteration (function diverges)
    → color point c white
  - If $|z_k| < n$ before kth iteration (function converges)
    → color point c black (c belongs to the set)
- For a certain value of c, we represent "speed of divergence" with different colors

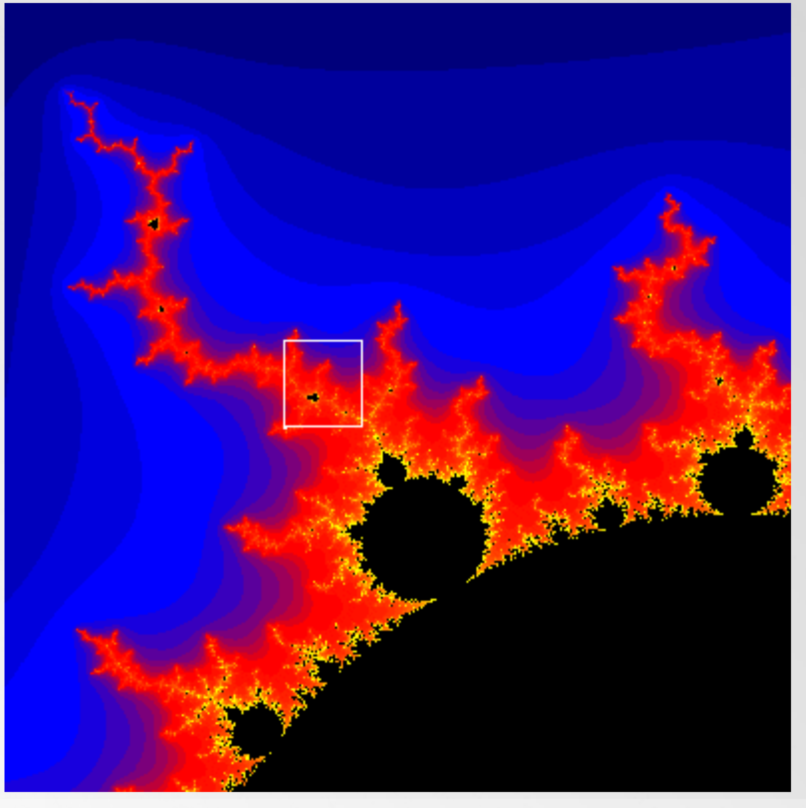
Quadratic Fractals – Mandelbrot

- Example for **n = 16 →** "blue points" need less than 10 iterations to go to infinity!!



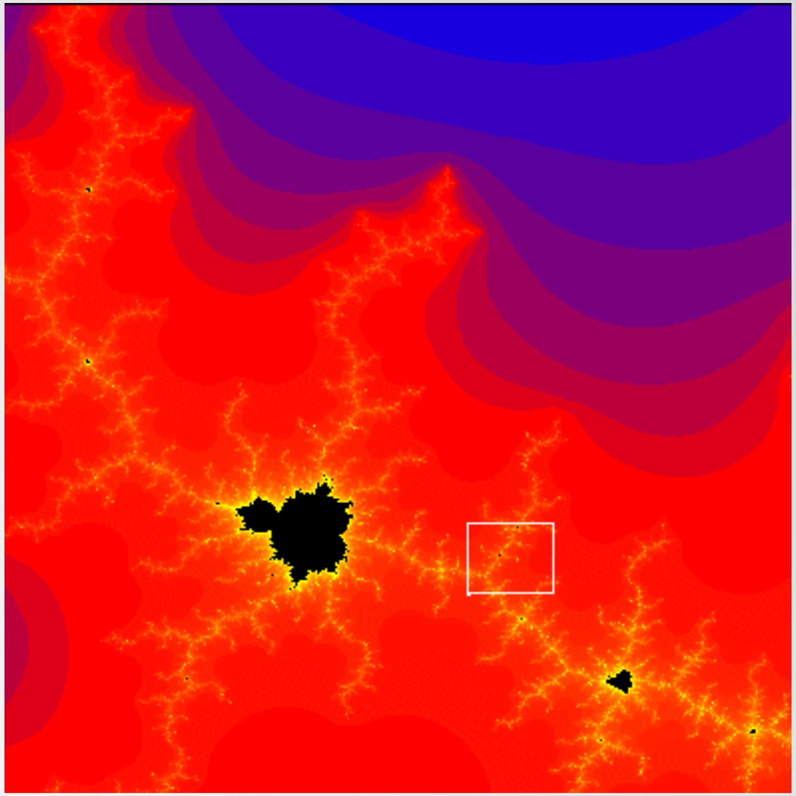
Quadratic Fractals – Mandelbrot

- More intense blue means more iterations needed for divergence



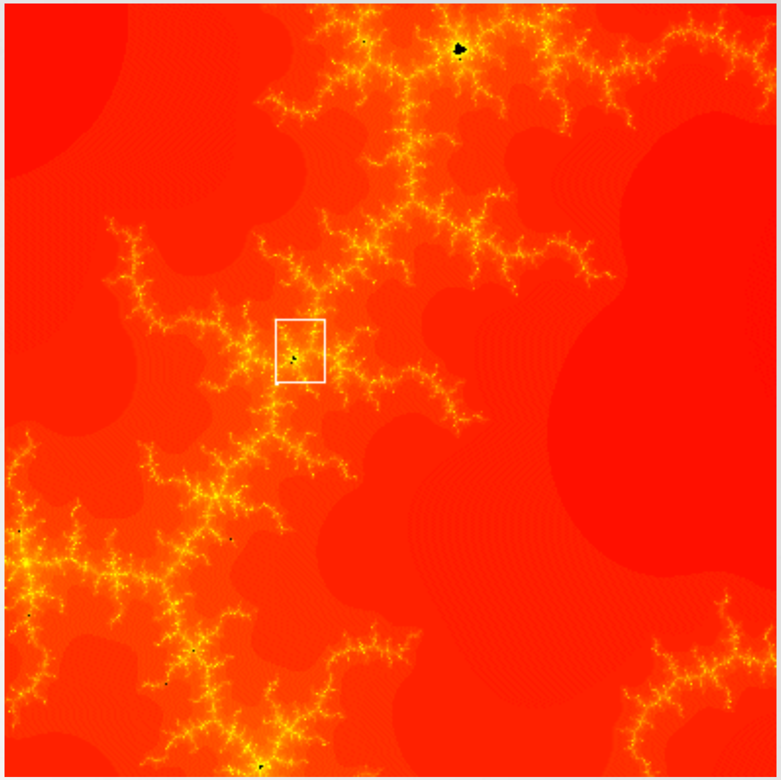
Quadratic Fractals – Mandelbrot

- “Red points” need more than 10 iterations to diverge



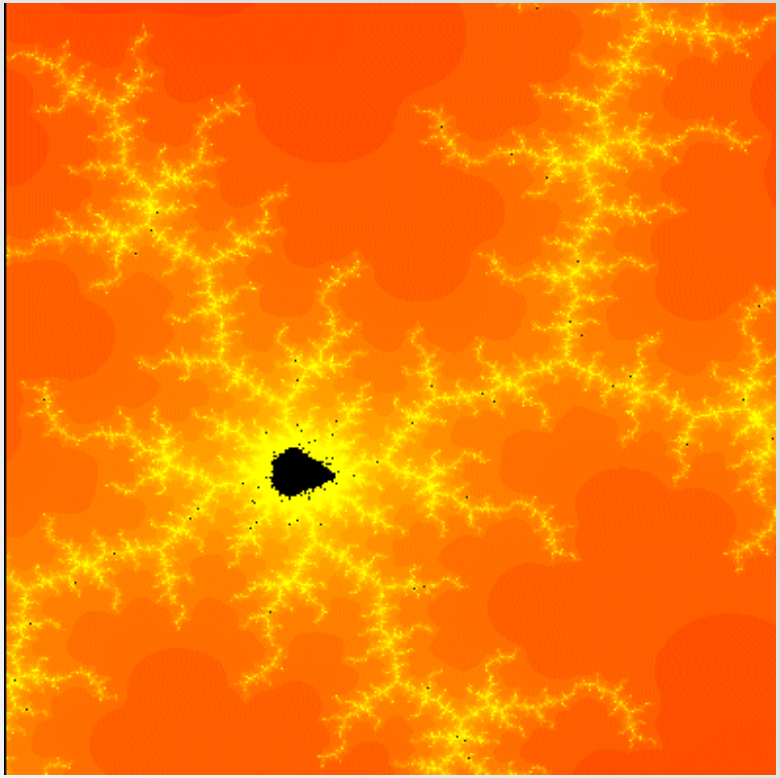
Quadratic Fractals – Mandelbrot

- "Yellow points" need many more than 10 iterations



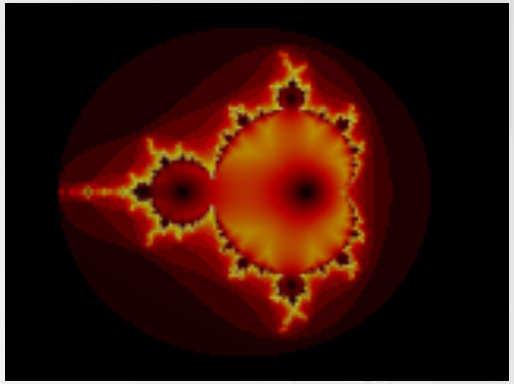
Quadratic Fractals – Mandelbrot

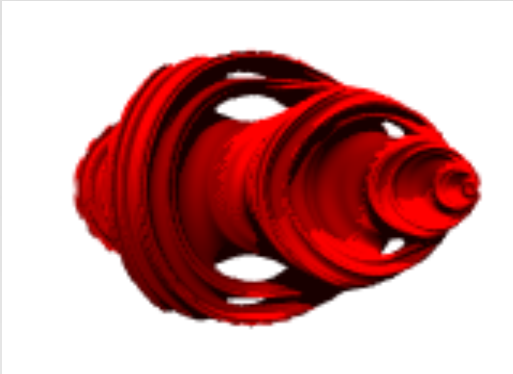
- Mandelbrot set is actually made of the black points within the “yellow frontier”

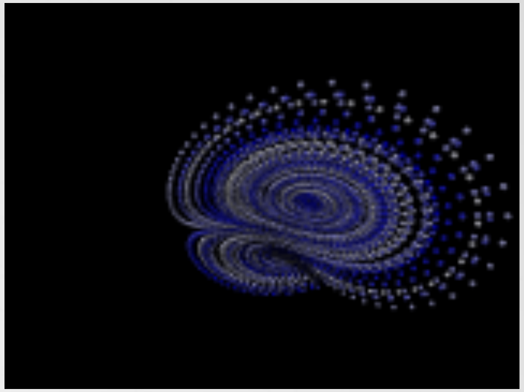


Fractals – PovRay

- Textures → Patterns → **Fractal Patterns**







Fractals – PovRay

- http://paulbourke.net/exhibition/povfrac/final/