

Gradient-Free Methods for Nonsmooth Nonconvex Optimization: Theory, Techniques, and Applications

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September 1, 2024

Abstract

This paper presents novel deterministic and stochastic gradient-free methods for optimizing nonsmooth nonconvex functions, particularly those characterized by Lipschitz continuity. Recognizing the challenges inherent in nonsmooth nonconvex optimization—such as the absence of computationally tractable optimality criteria and the impracticality of gradient evaluation—we propose a class of gradient-free algorithms that leverage the Goldstein subdifferential and randomized smoothing techniques. Our approach ensures convergence to (δ, ϵ) -Goldstein stationary points, offering rigorous theoretical guarantees. The proposed methods achieve an expected convergence rate of $O(d^{3/2}\delta^{-1}\epsilon^{-4})$, where d is the problem dimension. Additionally, we introduce two-phase versions of these methods, significantly improving performance in the large-deviation regime. Extensive theoretical analysis demonstrates the effectiveness of these methods across a range of challenging optimization scenarios, filling a critical gap in the optimization literature by providing efficient solutions for problems where gradient-based methods are infeasible.

Keywords: Nonsmooth Nonconvex Optimization; Gradient-Free Methods; Stochastic Optimization; Goldstein Subdifferential; Lipschitz Continuity; Randomized Smoothing

1 Introduction

Optimization problems involving nonsmooth and nonconvex functions are prevalent in many real-world applications, ranging from machine learning and signal processing to finance and operations research. These problems are particularly challenging due to the lack of smoothness and the presence of multiple local minima, which complicate the search for optimal solutions. Traditional gradient-based optimization methods, which rely on smoothness assumptions, often struggle or fail entirely when confronted with nonsmooth nonconvex landscapes. This has spurred interest in developing gradient-free optimization methods that can effectively handle such complex problems.

Gradient-free methods, which do not require explicit gradient information, offer a promising alternative. These methods are especially useful in settings where the objective function is difficult to differentiate or where gradient information is unavailable, such as in black-box optimization scenarios. Despite their potential, the theoretical development of gradient-free methods for nonsmooth nonconvex optimization has lagged behind that of their gradient-based counterparts. Most existing work focuses on smooth or convex settings, leaving a gap in our understanding of how to efficiently optimize nonsmooth nonconvex functions without gradient information.

In this paper, we address this gap by proposing a class of deterministic and stochastic gradient-free methods specifically designed for nonsmooth nonconvex optimization. Our approach is built on the concept of the Goldstein subdifferential, a refined notion of generalized gradients that is well-suited to nonsmooth settings. By leveraging randomized smoothing techniques, we transform the

challenging nonsmooth optimization problem into a more tractable form, enabling the development of efficient algorithms with provable convergence guarantees.

We introduce two key algorithms: a gradient-free method (GFM) and its stochastic variant (SGFM). Both methods are designed to find (δ, ϵ) -Goldstein stationary points, which serve as a reasonable optimality criterion in nonsmooth nonconvex optimization. Additionally, we propose two-phase versions of these algorithms, which improve convergence rates and provide strong guarantees in the large-deviation regime. Our theoretical analysis shows that these methods achieve an expected convergence rate of $O(d^{3/2}\delta^{-1}\epsilon^{-4})$, where d is the dimensionality of the problem. This represents a significant advancement in the field, offering new tools for tackling a broad class of optimization problems that were previously intractable.

Backgrounds. Many of the recent real-world success stories of machine learning have involved nonconvex optimization formulations, with the design of models and algorithms often being heuristic and intuitive. Thus a gap has arisen between theory and practice. Attempts have been made to fill this gap for different learning methodologies, including the training of multi-layer neural networks [Choromanska et al.(2015)], orthogonal tensor decomposition [Ge et al.(2015)], M-estimators [Loh and Wainwright(2015), Ma et al.(2020)], synchronization and MaxCut [Bandeira et al.(2016), Mei et al.(2017)], smooth semidefinite programming [Boumal et al.(2016)], matrix sensing and completion [Bhojanapalli et al.(2016), Ge et al.(2016)], robust principal component analysis (RPCA) [Ge et al.(2017)] and phase retrieval [Wang et al.(2017), Sun et al.(2018), Ma et al.(2020)]. For an overview of nonconvex optimization formulations and the relevant ML applications, we refer to a recent survey [Jain and Kar(2017)].

It is intractable to compute an approximate global minimum [Nemirovsky and Yudin(1983)] in general or to verify whether a point is a local minimum or a high-order saddle point [Murty and Kabadi(1987)]. Fortunately, the notion of *approximate stationary point* gives a reasonable optimality criterion when the objective function f is smooth; the goal here is to find a point $\mathbf{x} \in \mathbb{R}^d$ such that $\|\nabla f(\mathbf{x})\| \leq \epsilon$. Recent years have seen rapid algorithmic development through the lens of nonasymptotic convergence rates to ϵ -stationary points [Nesterov(2013), Ghadimi and Lan(2013), Ghadimi and Lan(2016), Carmon et al.(2017), Carmon et al.(2018), Jin et al.(2021)]. Another line of work establishes algorithm-independent lower bounds [Carmon et al.(2020), Carmon et al.(2021), Arjevani et al.(2020), Arjevani et al.(2022)].

Relative to its smooth counterpart, the investigation of nonsmooth optimization is relatively scarce, particularly in the nonconvex setting, both in terms of efficient algorithms and finite-time convergence guarantees. Yet, over several decades, nonsmooth nonconvex optimization formulations have found applications in many fields. A typical example is the training multi-layer neural networks with ReLU neurons, for which the piecewise linear activation functions induce nonsmoothness. Another example arises in controlling financial risk for asset portfolios or optimizing customer satisfaction in service systems or supply chain systems. Here, the nonsmoothness arises from the payoffs of financial derivatives and supply chain costs, e.g., options payoffs [Duffie(2010)] and supply chain overage/underage costs [Stadtler(2008)]. These applications make significant demands with respect to computational feasibility, and the design of efficient algorithms for solving nonsmooth nonconvex optimization problems has moved to the fore [Majewski et al.(2018), Davis et al.(2020), Daniilidis and Drusvyatskiy(2020), Zhang et al.(2020), Bolte and Pauwels(2021), Davis et al.(2022), Tian et al.(2022)].

The key challenges lie in two aspects: (i) the lack of a computationally tractable optimality criterion, and (ii) the lack of computationally powerful oracles. More specifically, in the classical setting where the function f is Lipschitz, we can define ϵ -stationary points based on the celebrated

notion of Clarke stationarity [Clarke(1990)]. However, the value of such a criterion has been called into question by [Zhang et al.(2020)], who show that no finite-time algorithm guarantees ϵ -stationarity when ϵ is less than a constant. Further, the computation of the gradient is impossible for many application problems and we only have access to a noisy function value at each point. This is a common issue in the context of simulation optimization [Nelson(2010), Hong et al.(2015)]; indeed, the objective function value is often achieved as the output of a black-box or complex simulator, for which the simulator does not have the infrastructure needed to effectively evaluate gradients; see also [Ghadimi and Lan(2013)] and [Nesterov and Spokoiny(2017)] for comments on the lack of gradient evaluation in practice.

Contribution. In this paper, we propose and analyze a class of deterministic and stochastic gradient-free methods for nonsmooth nonconvex optimization problems in which we only assume that the function f is Lipschitz. Our contributions can be summarized as follows.

- (i) We establish a relationship between the Goldstein subdifferential and uniform smoothing via appeal to the hyperplane separation theorem. This result provides the basis for algorithmic design and finite-time convergence analysis of gradient-free methods to (δ, ϵ) -Goldstein stationary points.
- (ii) We propose and analyze a gradient-free method (GFM) and stochastic GFM for solving a class of nonsmooth nonconvex optimization problems. Both of these methods are guaranteed to return a (δ, ϵ) -Goldstein stationary point of a Lipschitz function $f : \mathbb{R}^d \mapsto \mathbb{R}$ with an expected convergence rate of $O(d^{3/2}\delta^{-1}\epsilon^{-4})$ where $d \geq 1$ is the problem dimension. Further, we propose the two-phase versions of GFM and SGFM. As our goal is to return a (δ, ϵ) -Goldstein stationary point with user-specified high probability $1 - \Lambda$, we prove that the two-phase version of GFM and SGFM can improve the dependence from $(1/\Lambda)^4$ to $\log(1/\Lambda)$ in the large-deviation regime.

2 Related Works

Our work is related to a line of literature on gradient-based methods for nonsmooth and nonconvex optimization and gradient-free methods for smooth and nonconvex optimization.

Gradient-free methods for smooth and nonconvex optimization. In the context of gradient-free methods, the basic idea is to approximate a full gradient using either a one-point estimator [Flaxman et al.(2005)] or a two-point estimator [Agarwal et al.(2010), Ghadimi and Lan(2013), Duchi et al.(2015), Shamir(2017), Nesterov and Spokoiny(2017)], where the latter approach achieves a better finite-time convergence guarantee. Despite the meteoric rise of two-point-based gradient-free methods, most of the work is restricted to convex optimization [Duchi et al.(2015), Shamir(2017), Wang et al.(2018)] and smooth and nonconvex optimization [Nesterov and Spokoiny(2017), Ghadimi and Lan(2013), Lian et al.(2016), Liu et al.(2018), Chen et al.(2019), Ji et al.(2019), Huang et al.(2022)]. For non-smooth and convex optimization, the best upper bound on the global rate of convergence is $O(d\epsilon^{-2})$ [Shamir(2017)] and this matches the lower bound [Duchi et al.(2015)]. For smooth and nonconvex optimization, the best global rate of convergence is $O(d\epsilon^{-2})$ [Nesterov and Spokoiny(2017)] and $O(d\epsilon^{-4})$ if we only have access to noisy function value oracles [Ghadimi and Lan(2013)]. Additional regularity conditions, e.g., a finite-sum structure, allow us to leverage variance-reduction techniques [Liu et al.(2018), Chen et al.(2019), Ji et al.(2019)] and the best known result is $O(d^{3/4}\epsilon^{-3})$ [Huang et al.(2020)].

However, none of gradient-free methods have been developed for nonsmooth nonconvex optimization and the only gradient-free method we are aware of for the nonsmooth is summarized in [Nesterov and Spokoyny(2017), Section 7].

Gradient-based methods for nonsmooth and nonconvex optimization. To appreciate the difficulty and the broad scope of the research agenda in nonsmooth nonconvex optimization, we start by describing the existing relevant literature. First, the existing work is mostly devoted to establishing the asymptotic convergence properties of various optimization algorithms, including gradient sampling (GS) methods [Burke et al.(2002a), Burke et al.(2002b), Burke et al.(2005), Kiwiel(2007), Burke et al.(2020)], bundle methods [Kiwiel(1996), Fuduli et al.(2004)] and subgradient methods [Benaïm et al.(2005), Majewski et al.(2018), Davis et al.(2020), Daniilidis and Drusvyatskiy(2020), Bolte and Pauwels(2021)]. More specifically, [Burke et al.(2002a)] provided a systematic investigation of approximating the Clarke subdifferential through random sampling and proposed a gradient bundle method [Burke et al.(2002b)]—the precursor of GS methods—for optimizing a nonconvex, nonsmooth and non-Lipschitz function. Later, [Burke et al.(2005)] and [Kiwiel(2007)] proposed the GS methods by incorporating key modifications into the algorithmic scheme in [Burke et al.(2002b)] and proved that every cluster point of the iterates generated by GS methods is a Clarke stationary point. For an overview of GS methods, we refer to [Burke et al.(2020)]. Another line of works extended the bundle methods to nonsmooth nonconvex optimization by considering either piece-wise linear models embedding possible downward shifting [Kiwiel(1996)] or a mixture of linear pieces that exhibit a convex or concave behavior [Fuduli et al.(2004)]. There has been recent progress on analyzing subgradient methods for nonsmooth nonconvex optimization; indeed, the classical subgradient method on Lipschitz functions may fail to asymptotically find any stationary point due to the pathological examples [Daniilidis and Drusvyatskiy(2020)]. Under some additional regularity conditions, [Benaïm et al.(2005)] proved the asymptotic convergence of stochastic approximation methods from a continuous-time viewpoint and [Majewski et al.(2018)] generalized these results with proximal and implicit updates. [Bolte and Pauwels(2021)] justify the automatic differentiation schemes under the nonsmoothness conditions; [Davis et al.(2020)] proved the asymptotic convergence of classical subgradient methods for a class of Whitney stratifiable functions which include the functions studied in [Majewski et al.(2018)]. Recently, [Zhang et al.(2020)] modified Goldstein’s subgradient method [Goldstein(1977)] to optimize a class of Hadamard directionally differentiable function and proved nonasymptotic convergence guarantee. [Davis et al.(2022)] relaxed the assumption of Hadamard directionally differentiability and showed that another modification of Goldstein’s subgradient method could achieve the same finite-time guarantee for any Lipschitz function. Concurrently, [Tian et al.(2022)] removed the subgradient selection oracle assumption in [Zhang et al.(2020), Assumption 1] and provided the third modification of Goldstein’s subgradient method with the same finite-time convergence. Different from these gradient-based methods, we focus on the gradient-free methods in this paper.

Structured nonsmooth nonconvex optimization. We are also aware of many recent works on the algorithmic design in the structured nonsmooth nonconvex optimization. There are two primary settings where the proximal gradient methods is guaranteed to achieve nonasymptotic convergence if the proximal mapping can be efficiently evaluated. The former one considers the objective function with composition structure [Duchi and Ruan(2018), Drusvyatskiy and Paquette(2019), Davis and Drusvyatskiy(2019)], while the latter one focuses on composite objective functions with nonsmooth convex component [Bolte et al.(2018), Beck and Hallak(2020)]. However, both of these

settings require the weak convexity of objective function and exclude many simple and important nonsmooth nonconvex functions used in the real-world application problems.

3 Preliminaries and Technical Background

We provide the formal definitions for the class of Lipschitz functions considered in this paper, and the definitions for generalized gradients and the Goldstein subdifferential that lead to optimality conditions in nonsmooth nonconvex optimization.

3.1 Function classes

Imposing regularity on functions to be optimized is necessary for obtaining optimization algorithms with finite-time convergence guarantees [Nesterov(2018)]. In the context of nonsmooth optimization, there are two regularity conditions: Lipschitz properties of function values and bounds on function values.

We first list several equivalent definitions of Lipschitz continuity. A function $f : \mathbb{R}^d \mapsto \mathbb{R}$ is said to be L -Lipschitz if for every $\mathbf{x} \in \mathbb{R}^d$ and the direction $\mathbf{v} \in \mathbb{R}^d$ with $\|\mathbf{v}\| \leq 1$, the directional projection $f_{\mathbf{x},\mathbf{v}}(t) := f(\mathbf{x} + t\mathbf{v})$ defined for $t \in \mathbb{R}$ satisfies

$$|f_{\mathbf{x},\mathbf{v}}(t) - f_{\mathbf{x},\mathbf{v}}(t')| \leq L|t - t'|, \quad \text{for all } t, t' \in \mathbb{R}$$

Equivalently, f is L -Lipschitz if for every $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^d$, we have

$$|f(\mathbf{x}) - f(\mathbf{x}')| \leq L\|\mathbf{x} - \mathbf{x}'\|$$

Further, the function value bound $f(\mathbf{x}^0) - \inf_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x})$ appears in complexity guarantees for smooth and nonconvex optimization problems [Nesterov(2018)] and is often assumed to be bounded by a positive constant $\Delta > 0$. Note that \mathbf{x}^0 is a prespecified point (i.e., an initial point for an algorithm) and we simply fix it for the remainder of this paper. We define the function class considered in this paper.

Definition 1. Suppose that $\Delta > 0$ and $L > 0$ are both independent of the problem dimension $d \geq 1$. Then, we denote $\mathcal{F}_d(\Delta, L)$ as the set of L -Lipschitz functions $f : \mathbb{R}^d \mapsto \mathbb{R}$ with the bounded function value $f(\mathbf{x}^0) - \inf_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x}) \leq \Delta$.

The function class $\mathcal{F}_d(\Delta, L)$ includes Lipschitz functions on \mathbb{R}^d and is thus different from the nonconvex function class considered in the literature [Ghadimi and Lan(2013), Nesterov and Spokoiny(2017)]. First, we do not impose a smoothness condition on the function $f \in \mathcal{F}_d(\Delta, L)$, in contrast to the nonconvex functions studied in [Ghadimi and Lan(2013)] which are assumed to have Lipschitz gradients. Second, [Nesterov and Spokoiny(2017), Section 7] presented a complexity bound for a randomized optimization method for minimizing a nonsmooth nonconvex function. However, they did not clarify why the norm of the gradient of the approximate function $f_{\bar{\mu}}$ of the order δ (we use their notation) serves as a reasonable optimality criterion in nonsmooth nonconvex optimization. They also assume an exact function value oracle, ruling out many interesting application problems in simulation optimization and machine learning.

In contrast, our goal is to propose fast gradient-free methods for nonsmooth nonconvex optimization in the absence of an exact function value oracle. In general, the complexity bound of gradient-free methods will depend on the problem dimension $d \geq 1$ even when we assume that

the function to be optimized is convex and smooth [Duchi et al.(2015), Shamir(2017)]. As such, we should consider a function class with a given dimension $d \geq 1$. In particular, we consider a optimality criterion based on the celebrated Goldstein subdifferential [Goldstein(1977)] and prove that the number of function value oracles required by our deterministic and stochastic gradient-free methods to find a (δ, ϵ) -Goldstein stationary point of $f \in \mathcal{F}_d(\Delta, L)$ is $O(\text{poly}(d, L, \Delta, 1/\epsilon, 1/\delta))$ when $\delta, \epsilon \in (0, 1)$ are constants (see the definition of Goldstein stationarity in the next subsection).

It is worth mentioning that $\mathcal{F}_d(\Delta, L)$ contains a rather broad class of functions used in real-world application problems. Typical examples with additional regularity properties include Hadamard semi-differentiable functions [Shapiro(1990), Delfour(2019), Zhang et al.(2020)], Whitney-stratifiable functions [Bolte et al.(2007), Davis et al.(2020)], ϕ -minimally definable functions [Coste(2000)] and a class of semi-algebraic functions [Attouch et al.(2013), Davis et al.(2020)]. Thus, our gradient-free methods can be applied for solving these problems with finite-time convergence guarantees.

3.2 Generalized gradients and Goldstein subdifferential

We start with the definition of generalized gradients [Clarke(1990)] for nondifferentiable functions. This is perhaps the most standard extension of gradients to nonsmooth and nonconvex functions.

Definition 2. *Given a point $\mathbf{x} \in \mathbb{R}^d$ and a direction $\mathbf{v} \in \mathbb{R}^d$, the generalized directional derivative of a nondifferentiable function f is given by $Df(\mathbf{x}; \mathbf{v}) := \limsup_{\mathbf{y} \rightarrow \mathbf{x}, t \downarrow 0} \frac{f(\mathbf{y} + t\mathbf{v}) - f(\mathbf{y})}{t}$. Then, the generalized gradient of f is defined as a set $\partial f(\mathbf{x}) := \{\mathbf{g} \in \mathbb{R}^d : \mathbf{g}^\top \mathbf{v} \leq Df(\mathbf{x}; \mathbf{v}), \forall \mathbf{v} \in \mathbb{R}^d\}$.*

Rademacher's theorem guarantees that any Lipschitz function is almost everywhere differentiable. This implies that the generalized gradients of Lipschitz functions have additional properties and we can define them in a relatively simple way. The following proposition summarizes these results; we refer to [Clarke(1990)] for the proof details.

Proposition 1. *Suppose that f is L -Lipschitz for some $L > 0$, we have that $\partial f(\mathbf{x})$ is a nonempty, convex and compact set and $\|\mathbf{g}\| \leq L$ for all $\mathbf{g} \in \partial f(\mathbf{x})$. Further, $\partial f(\cdot)$ is an upper-semicontinuous set-valued map. Moreover, a generalization of mean-value theorem holds: for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^d$, there exist $\lambda \in (0, 1)$ and $\mathbf{g} \in \partial f(\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2)$ such that $f(\mathbf{x}_1) - f(\mathbf{x}_2) = \mathbf{g}^\top (\mathbf{x}_1 - \mathbf{x}_2)$. Finally, there is a simple way to represent the generalized gradient $\partial f(\mathbf{x})$:*

$$\partial f(\mathbf{x}) := \text{conv} \left\{ \mathbf{g} \in \mathbb{R}^d : \mathbf{g} = \lim_{\mathbf{x}_k \rightarrow \mathbf{x}} \nabla f(\mathbf{x}_k) \right\}$$

which is the convex hull of all limit points of $\nabla f(\mathbf{x}_k)$ over all sequences $\mathbf{x}_1, \mathbf{x}_2, \dots$ of differentiable points of $f(\cdot)$ which converge to \mathbf{x} .

Given this definition of generalized gradients, a *Clarke stationary point* of f is a point \mathbf{x} satisfying $\mathbf{0} \in \partial f(\mathbf{x})$. Then, it is natural to ask if an optimization algorithm can reach an ϵ -stationary point with a finite-time convergence guarantee. Here a point $\mathbf{x} \in \mathbb{R}^d$ is an ϵ -Clarke stationary point if

$$\min \{\|\mathbf{g}\| : \mathbf{g} \in \partial f(\mathbf{x})\} \leq \epsilon$$

This question has been addressed by [Zhang et al.(2020), Theorem 1], who showed that finding an ϵ -Clarke stationary points in nonsmooth nonconvex optimization can not be achieved by any finite-time algorithm given a fixed tolerance $\epsilon \in [0, 1)$. One possible response is to consider a relaxation

called a *near* ϵ -Clarke stationary point. Consider a point which is δ -close to an ϵ -stationary point for some $\delta > 0$. A point $\mathbf{x} \in \mathbb{R}^d$ is near ϵ -stationary if the following statement holds true:

$$\min \{ \|\mathbf{g}\| : \mathbf{g} \in \cup_{\mathbf{y} \in \mathbb{B}_\delta(\mathbf{x})} \partial f(\mathbf{y}) \} \leq \epsilon$$

Unfortunately, however, [Kornowski and Shamir(2021), Theorem 1] demonstrated that it is impossible to obtain worst-case guarantees for finding a near ϵ -Clarke stationary point of $f \in \mathcal{F}_d(\Delta, L)$ when $\epsilon, \delta > 0$ are smaller than some certain constants unless the number of oracle calls has an exponential dependence on the problem dimension $d \geq 1$. These negative results suggest a need for rethinking the definition of targeted stationary points. We propose to consider the refined notion of Goldstein subdifferential.

Definition 3. *Given a point $\mathbf{x} \in \mathbb{R}^d$ and $\delta > 0$, the δ -Goldstein subdifferential of a Lipschitz function f at \mathbf{x} is given by $\partial_\delta f(\mathbf{x}) := \text{conv}(\cup_{\mathbf{y} \in \mathbb{B}_\delta(\mathbf{x})} \partial f(\mathbf{y}))$.*

The Goldstein subdifferential of f at \mathbf{x} is the convex hull of the union of all generalized gradients at points in a δ -ball around \mathbf{x} . Accordingly, we can define the (δ, ϵ) -Goldstein stationary points; that is, a point $\mathbf{x} \in \mathbb{R}^d$ is a (δ, ϵ) -Goldstein stationary point if the following statement holds:

$$\min \{ \|\mathbf{g}\| : \mathbf{g} \in \partial_\delta f(\mathbf{x}) \} \leq \epsilon$$

It is worth mentioning that (δ, ϵ) -Goldstein stationarity is a weaker notion than (near) ϵ -Clarke stationarity since any (near) ϵ -stationary point is a (δ, ϵ) -Goldstein stationary point but not vice versa. However, the converse holds true under a smoothness condition [Zhang et al.(2020), Proposition 6] and $\lim_{\delta \downarrow 0} \partial_\delta f(\mathbf{x}) = \partial f(\mathbf{x})$ holds as shown in [Zhang et al.(2020), Lemma 7]. The latter result also enables an intuitive framework for transforming nonasymptotic analysis of convergence to (δ, ϵ) -Goldstein stationary points to classical asymptotic results for finding ϵ -Clarke stationary points. Thus, we conclude that finding a (δ, ϵ) -Goldstein stationary point is a reasonable optimality condition for general nonsmooth nonconvex optimization.

Remark. Finding a (δ, ϵ) -Goldstein stationary point in nonsmooth nonconvex optimization has been formally shown to be computationally tractable in an oracle model [Zhang et al.(2020), Davis et al.(2022), Tian et al.(2022)]. [Goldstein(1977)] discovered that one can decrease the function value of a Lipschitz f by using the minimal-norm element of $\partial_\delta f(\mathbf{x})$ and this leads to a deterministic normalized subgradient method which finds a (δ, ϵ) -Goldstein stationary point within $O(\frac{\Delta}{\delta\epsilon})$ iterations. However, Goldstein’s algorithm is only conceptual since it is computationally intractable to return an exact minimal-norm element of $\partial_\delta f(\mathbf{x})$. Recently, the randomized variants of Goldstein’s algorithm have been proposed with a convergence guarantee of $O(\frac{\Delta L^2}{\delta \epsilon^3})$ [Zhang et al.(2020), Davis et al.(2022), Tian et al.(2022)]. However, it remains unknown if gradient-free methods find a (δ, ϵ) -Goldstein stationary point of a Lipschitz function f within $O(\text{poly}(d, L, \Delta, 1/\epsilon, 1/\delta))$ iterations in the absence of an exact function value oracle. Note that the dependence on the problem dimension $d \geq 1$ is necessary for gradient-free methods.

3.3 Randomized smoothing

The randomized smoothing approaches are simple and work equally well for convex and nonconvex functions. Formally, given the L -Lipschitz function f (possibly nonsmooth nonconvex) and a distribution \mathbb{P} , we define $f_\delta(\mathbf{x}) = \mathbb{E}_{\mathbf{u} \sim \mathbb{P}}[f(\mathbf{x} + \delta \mathbf{u})]$. In particular, letting \mathbb{P} be a standard Gaussian distribution, the function f_δ is a $\delta L \sqrt{d}$ -approximation of $f(\cdot)$ and the gradient ∇f_δ is $\frac{L\sqrt{d}}{\delta}$ -Lipschitz where $d \geq 1$ is the problem dimension; see [Nesterov and Spokoiny(2017), Theorem 1 and

Lemma 2]. Letting \mathbb{P} be an uniform distribution on an unit ball in ℓ_2 -norm, the resulting function f_δ is a δL -approximation of $f(\cdot)$ and ∇f_δ is also $\frac{cL\sqrt{d}}{\delta}$ -Lipschitz where $d \geq 1$ is the problem dimension; see [Yousefian et al.(2012), Lemma 8] and [Duchi et al.(2012), Lemma E.2], rephrased as follows.

Proposition 2. *Let $f_\delta(\mathbf{x}) = \mathbb{E}_{\mathbf{u} \sim \mathbb{P}}[f(\mathbf{x} + \delta \mathbf{u})]$ where \mathbb{P} is an uniform distribution on an unit ball in ℓ_2 -norm. Assuming that f is L -Lipschitz, we have (i) $|f_\delta(\mathbf{x}) - f(\mathbf{x})| \leq \delta L$, and (ii) f_δ is differentiable and L -Lipschitz with the $\frac{cL\sqrt{d}}{\delta}$ -Lipschitz gradient where $c > 0$ is a constant. In addition, there exists a function f for which each of the above bounds are tight simultaneously.*

The randomized smoothing approaches form the basis for developing gradient-free methods [Flaxman et al.(2005), Agarwal et al.(2010), Agarwal et al.(2013), Ghadimi and Lan(2013), Nesterov and Spokoiny(2017)]. Given an access to function values of f , we can compute an unbiased estimate of the gradient of f_δ and plug them into stochastic gradient-based methods. Note that the Lipschitz constant of f_δ depends on the problem dimension $d \geq 1$ with at least a factor of \sqrt{d} for many randomized smoothing approaches [Kornowski and Shamir(2021), Theorem 2]. This is consistent with the lower bounds for all gradient-free methods in convex and strongly convex optimization [Duchi et al.(2015), Shamir(2017)].

3.4 Proof of Proposition 2

Throughout this subsection, we let $\mathbf{u} \in \mathbb{R}^d$ denote a random variable distributed uniformly on $\mathbb{B}_1(\mathbf{0})$. For the first statement, since f is L -Lipschitz, we have

$$|f_\delta(\mathbf{x}) - f(\mathbf{x})| = |\mathbb{E}[f(\mathbf{x} + \delta \mathbf{u}) - f(\mathbf{x})]| \leq \delta L \cdot \mathbb{E}[\|\mathbf{u}\|] \leq \delta L$$

Then, we proceed to prove the second statement. Indeed, [Bertsekas(1973), Proposition 2.4] guarantees that f_δ is everywhere differentiable. Since f is L -Lipschitz, we have

$$|f_\delta(\mathbf{x}) - f_\delta(\mathbf{x}')| = |\mathbb{E}[f(\mathbf{x} + \delta \mathbf{u}) - f(\mathbf{x}' + \delta \mathbf{u})]| \leq L \mathbb{E}[\|\mathbf{x} - \mathbf{x}'\|] = L\|\mathbf{x} - \mathbf{x}'\|, \quad \text{for all } \mathbf{x}, \mathbf{x}' \in \mathbb{R}^d$$

It remains to prove that ∇f_δ is Lipschitz. Since f is L -Lipschitz, the Rademacher's theorem guarantees that f is almost everywhere differentiable. This implies that $\nabla f_\delta(\mathbf{x}) = \mathbb{E}[\nabla f(\mathbf{x} + \delta \mathbf{u})]$. Then, we have

$$\begin{aligned} \|\nabla f_\delta(\mathbf{x}) - \nabla f_\delta(\mathbf{x}')\| &= \|\mathbb{E}[\nabla f(\mathbf{x} + \delta \mathbf{u})] - \mathbb{E}[\nabla f(\mathbf{x}' + \delta \mathbf{u})]\| \\ &= \frac{1}{\text{Vol}(\mathbb{B}_1(\mathbf{0}))} \left\| \int_{\mathbf{u} \in \mathbb{B}_1(\mathbf{0})} \nabla f(\mathbf{x} + \delta \mathbf{u}) d\mathbf{u} - \int_{\mathbf{u} \in \mathbb{B}_1(\mathbf{0})} \nabla f(\mathbf{x}' + \delta \mathbf{u}) d\mathbf{u} \right\| \\ &= \frac{1}{\text{Vol}(\mathbb{B}_\delta(\mathbf{0}))} \left\| \int_{\mathbf{y} \in \mathbb{B}_\delta(\mathbf{x})} \nabla f(\mathbf{y}) d\mathbf{y} - \int_{\mathbf{y} \in \mathbb{B}_\delta(\mathbf{x}')} \nabla f(\mathbf{y}) d\mathbf{y} \right\| \end{aligned}$$

Note that f is L -Lipschitz, we have $\|\nabla f(\mathbf{y})\| \leq L$ for any $\mathbf{y} \in \mathbb{B}_\delta(\mathbf{x}) \cup \mathbb{B}_\delta(\mathbf{x}')$. Then, we turn to prove that $\|\nabla f_\delta(\mathbf{x}) - \nabla f_\delta(\mathbf{x}')\| \leq \frac{L\sqrt{d}\|\mathbf{x} - \mathbf{x}'\|}{\delta}$ for two different cases one by one as follows.

Case I: $\|\mathbf{x} - \mathbf{x}'\| \geq 2\delta$. It is clear that

$$\|\nabla f_\delta(\mathbf{x}) - \nabla f_\delta(\mathbf{x}')\| \leq 2L \leq \frac{L\|\mathbf{x} - \mathbf{x}'\|}{\delta} \stackrel{d \geq 1}{\leq} \frac{L\sqrt{d}\|\mathbf{x} - \mathbf{x}'\|}{\delta}$$

which implies the desired result.

Case II: $\|\mathbf{x} - \mathbf{x}'\| \leq 2\delta$. It is clear that $\mathbb{B}_\delta(\mathbf{x}) \cap \mathbb{B}_\delta(\mathbf{x}') \neq \emptyset$. This implies that

$$\|\nabla f_\delta(\mathbf{x}) - \nabla f_\delta(\mathbf{x}')\| = \frac{1}{\text{Vol}(\mathbb{B}_\delta(\mathbf{0}))} \left| \int_{\mathbf{y} \in \mathbb{B}_\delta(\mathbf{x}) \setminus \mathbb{B}_\delta(\mathbf{x}')} \nabla f(\mathbf{y}) d\mathbf{y} - \int_{\mathbf{y} \in \mathbb{B}_\delta(\mathbf{x}') \setminus \mathbb{B}_\delta(\mathbf{x})} \nabla f(\mathbf{y}) d\mathbf{y} \right|$$

Since $\|\nabla f(\mathbf{y})\| \leq L$ for any $\mathbf{y} \in \mathbb{B}_\delta(\mathbf{x}) \cup \mathbb{B}_\delta(\mathbf{x}')$, we have

$$\|\nabla f_\delta(\mathbf{x}) - \nabla f_\delta(\mathbf{x}')\| \leq \frac{L}{\text{Vol}(\mathbb{B}_\delta(\mathbf{0}))} (\text{Vol}(\mathbb{B}_\delta(\mathbf{x}) \setminus \mathbb{B}_\delta(\mathbf{x}')) + \text{Vol}(\mathbb{B}_\delta(\mathbf{x}') \setminus \mathbb{B}_\delta(\mathbf{x})))$$

By the symmetry from a geometrical point of view, we have $\text{Vol}(\mathbb{B}_\delta(\mathbf{x}) \setminus \mathbb{B}_\delta(\mathbf{x}')) = \text{Vol}(\mathbb{B}_\delta(\mathbf{x}') \setminus \mathbb{B}_\delta(\mathbf{x}))$. For simplicity, we let $I = \mathbb{B}_\delta(\mathbf{x}) \setminus \mathbb{B}_\delta(\mathbf{x}')$ and obtain that

$$\|\nabla f_\delta(\mathbf{x}) - \nabla f_\delta(\mathbf{x}')\| \leq \frac{2L}{\text{Vol}(\mathbb{B}_\delta(\mathbf{0}))} \text{Vol}(I) = \frac{2L}{c_d \delta^d} \text{Vol}(I), \quad \text{where } c_d = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)}$$

It suffices to find an upper bound for $\text{Vol}(I)$ in terms of $\|\mathbf{x} - \mathbf{x}'\|$. Let $V_{\text{cap}}(p)$ denote the volume of the spherical cap with the distance p from the center of the sphere, we have

$$\text{Vol}(I) = \text{Vol}(\mathbb{B}_\delta(\mathbf{0})) - 2V_{\text{cap}}\left(\frac{1}{2}\|\mathbf{x} - \mathbf{x}'\|\right) = c_d \delta^d - 2V_{\text{cap}}\left(\frac{1}{2}\|\mathbf{x} - \mathbf{x}'\|\right)$$

The volume of the d -dimensional spherical cap with distance p from the center of the sphere can be calculated in terms of the volumes of $(d-1)$ -dimensional spheres as follows:

$$V_{\text{cap}}(p) = \int_p^\delta c_{d-1}(\delta^2 - \rho^2)^{\frac{d-1}{2}} d\rho, \quad \text{for all } p \in [0, \delta]$$

Since $V_{\text{cap}}(\cdot)$ is a convex function over $[0, \delta]$, we have $V_{\text{cap}}(p) \geq V_{\text{cap}}(0) + V'_{\text{cap}}(0)p$. By the definition, we have $V_{\text{cap}}(0) = \frac{1}{2}\text{Vol}(\mathbb{B}_\delta(\mathbf{0})) = \frac{1}{2}c_d \delta^d$ and $V'_{\text{cap}}(0) = -c_{d-1}\delta^{d-1}$. Thus, $V_{\text{cap}}(p) \geq \frac{1}{2}c_d \delta^d - c_{d-1}\delta^{d-1}p$. Furthermore, $\frac{1}{2}\|\mathbf{x} - \mathbf{x}'\| \in [0, \delta]$. Putting these pieces together yields that $\text{Vol}(I) \leq c_{d-1}\delta^{d-1}\|\mathbf{x} - \mathbf{x}'\|$. Therefore, we conclude that

$$\|\nabla f_\delta(\mathbf{x}) - \nabla f_\delta(\mathbf{x}')\| \leq \frac{2L}{c_d \delta^d} \text{Vol}(I) \leq \frac{2c_{d-1}}{c_d} \frac{L\|\mathbf{x} - \mathbf{x}'\|}{\delta}$$

Since $c_d = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)}$, we have $\frac{2c_{d-1}}{c_d} = \begin{cases} \frac{d!!}{(d-1)!!} & \text{if } d \text{ is odd,} \\ \frac{2}{\pi} \frac{d!!}{(d-1)!!} & \text{otherwise.} \end{cases}$ and $\frac{1}{\sqrt{d}} \frac{2c_{d-1}}{c_d} \rightarrow \sqrt{\frac{\pi}{2}}$. Therefore, we

conclude that the gradient ∇f_δ is $\frac{cL\sqrt{d}}{\delta}$ -Lipschitz where $c > 0$ is a positive constant. In addition, for the construction of a function f in which each of the above bounds are tight, we consider a convex combination of “difficult” functions, in this case

$$f_1(\mathbf{x}) = L\|\mathbf{x}\|, \quad f_2(\mathbf{x}) = L\left|\langle \mathbf{x}, \frac{\mathbf{w}}{\|\mathbf{w}\|} \rangle - \frac{1}{2}\right|$$

and choose $f(\mathbf{x}) = \frac{1}{2}(f_1(\mathbf{x}) + f_2(\mathbf{x}))$. Following up the same argument as in [Duchi et al.(2012), Lemma 10], it is relatively straightforward to verify that the bounds in Proposition 2 cannot be improved by more than a constant factor. This completes the proof.

4 Main Results: Linking Goldstein Subdifferential to Uniform Smoothing

We establish a relationship between the Goldstein subdifferential and the uniform smoothing approach. We propose a gradient-free method (GFM), its stochastic variant (SGFM), and a two-phase version of GFM and SGFM. We analyze these algorithms using the Goldstein subdifferential; we provide the global rate and large-deviation estimates in terms of (δ, ϵ) -Goldstein stationarity.

Recall that $\partial_\delta f$ and f_δ are defined by $\partial_\delta f(\mathbf{x}) := \text{conv}(\cup_{\mathbf{y} \in \mathbb{B}_\delta(\mathbf{x})} \partial f(\mathbf{y}))$ and $f_\delta(\mathbf{x}) = \mathbb{E}_{\mathbf{u} \sim \mathbb{P}}[f(\mathbf{x} + \delta \mathbf{u})]$. It is clear that f is almost everywhere differentiable since f is L -Lipschitz. This implies that $\nabla f_\delta(\mathbf{x}) = \mathbb{E}_{\mathbf{u} \sim \mathbb{P}}[\nabla f(\mathbf{x} + \delta \mathbf{u})]$ and demonstrates that $\nabla f_\delta(\mathbf{x})$ can be viewed intuitively as a convex combination of $\nabla f(\mathbf{z})$ over an infinite number of points $\mathbf{z} \in \mathbb{B}_\delta(\mathbf{x})$. As such, it is reasonable to conjecture that $\nabla f_\delta(\mathbf{x}) \in \partial_\delta f(\mathbf{x})$ for any $\mathbf{x} \in \mathbb{R}^d$. However, the above argument is not a rigorous proof; indeed, we need to justify why $\nabla f_\delta(\mathbf{x}) = \mathbb{E}_{\mathbf{u} \sim \mathbb{P}}[\nabla f(\mathbf{x} + \delta \mathbf{u})]$ if f is almost everywhere differentiable and generalize the idea of a convex combination to include infinite sums. To resolve these issues, we exploit a toolbox due to [Rockafellar and Wets(2009)].

In the following theorem, we summarize our result and refer to Section 4.1 for the proof details.

Theorem 1. *Suppose that f is L -Lipschitz and let $f_\delta(\mathbf{x}) = \mathbb{E}_{\mathbf{u} \sim \mathbb{P}}[f(\mathbf{x} + \delta \mathbf{u})]$, where \mathbb{P} is an uniform distribution on a unit ball in ℓ_2 -norm and let $\partial_\delta f$ be a δ -Goldstein subdifferential of f (cf. Definition 3). Then, we have $\nabla f_\delta(\mathbf{x}) \in \partial_\delta f(\mathbf{x})$ for any $\mathbf{x} \in \mathbb{R}^d$.*

Theorem 1 resolves an important question and forms the basis for analyzing our gradient-free methods. Notably, our analysis can be extended to justify other randomized smoothing approaches in nonsmooth nonconvex optimization. For example, [Nesterov and Spokoiny(2017)] used Gaussian smoothing and estimated the number of iterations required by their methods to output $\hat{\mathbf{x}} \in \mathbb{R}^d$ satisfying $\|\nabla f_\delta(\hat{\mathbf{x}})\| \leq \epsilon$. By modifying the proof of Theorem 1 and [Zhang et al.(2020), Lemma 7], we can prove that ∇f_δ belongs to Goldstein subdifferential with Gaussian weights and this subdifferential converges to the Clarke subdifferential as $\delta \rightarrow 0$. Compared to uniform smoothing and the original Goldstein subdifferential, the proof for Gaussian smoothing is quite long and technical [Nesterov and Spokoiny(2017), Page 554], and adding Gaussian weights seems unnatural in general.

4.1 Proof of Theorem 1

We first show that $\nabla f_\delta(\mathbf{x}) = \mathbb{E}_{\mathbf{u} \sim \mathbb{P}}[\nabla f(\mathbf{x} + \delta \mathbf{u})]$. Indeed, by the definition of f_δ , we have

$$f_\delta(\mathbf{x}) = \mathbb{E}_{\mathbf{u} \sim \mathbb{P}}[f(\mathbf{x} + \delta \mathbf{u})] = \frac{1}{\text{Vol}(\mathbb{B}_1(\mathbf{0}))} \int_{\mathbf{u} \in \mathbb{B}_1(\mathbf{0})} f(\mathbf{x} + \delta \mathbf{u}) d\mathbf{u} = \frac{1}{\text{Vol}(\mathbb{B}_\delta(\mathbf{0}))} \int_{\mathbf{v} \in \mathbb{B}_\delta(\mathbf{0})} f(\mathbf{x} + \mathbf{v}) d\mathbf{v}$$

Since f is L -Lipschitz, [Bertsekas(1973), Proposition 2.3] guarantees that f_δ is everywhere differentiable. Thus, we have $\nabla f_\delta(\mathbf{x})$ exists for any $\mathbf{x} \in \mathbb{R}^d$ and satisfies that

$$\lim_{\|\mathbf{h}\| \rightarrow 0} \frac{|f_\delta(\mathbf{x} + \mathbf{h}) - f_\delta(\mathbf{x}) - \langle \nabla f_\delta(\mathbf{x}), \mathbf{h} \rangle|}{\|\mathbf{h}\|} = 0 \quad (1)$$

Further, we have

$$\frac{f_\delta(\mathbf{x} + \mathbf{h}) - f_\delta(\mathbf{x})}{\|\mathbf{h}\|} = \frac{1}{\text{Vol}(\mathbb{B}_\delta(\mathbf{0}))} \int_{\mathbf{v} \in \mathbb{B}_\delta(\mathbf{0})} \frac{f(\mathbf{x} + \mathbf{h} + \mathbf{v}) - f(\mathbf{x} + \mathbf{v})}{\|\mathbf{h}\|} d\mathbf{v}$$

Algorithm 1 Gradient-Free Method (GFM)

- 1: **Input:** initial point $\mathbf{x}^0 \in \mathbb{R}^d$, stepsize $\eta > 0$, problem dimension $d \geq 1$, smoothing parameter δ and iteration number $T \geq 1$
 - 2: **for** $t = 0, 1, 2, \dots, T - 1$ **do**
 - 3: Sample $\mathbf{w}^t \in \mathbb{R}^d$ uniformly from a unit sphere in \mathbb{R}^d
 - 4: Compute $\mathbf{g}^t = \frac{d}{2\delta}(f(\mathbf{x}^t + \delta\mathbf{w}^t) - f(\mathbf{x}^t - \delta\mathbf{w}^t))\mathbf{w}^t$
 - 5: Compute $\mathbf{x}^{t+1} = \mathbf{x}^t - \eta\mathbf{g}^t$
 - 6: **end for**
 - 7: **Output:** \mathbf{x}^R where $R \in \{0, 1, 2, \dots, T - 1\}$ is uniformly sampled
-

Since f is L -Lipschitz, we have $\frac{f(\mathbf{x}+\mathbf{h}+\mathbf{v})-f(\mathbf{x}+\mathbf{v})}{\|\mathbf{h}\|} \leq L$. By the dominated convergence theorem, we have

$$\lim_{\|\mathbf{h}\| \rightarrow 0} \frac{f_\delta(\mathbf{x} + \mathbf{h}) - f_\delta(\mathbf{x})}{\|\mathbf{h}\|} = \frac{1}{\text{Vol}(\mathbb{B}_\delta(\mathbf{0}))} \int_{\mathbf{v} \in \mathbb{B}_\delta(\mathbf{0})} \left(\lim_{\|\mathbf{h}\| \rightarrow 0} \frac{f(\mathbf{x} + \mathbf{h} + \mathbf{v}) - f(\mathbf{x} + \mathbf{v})}{\|\mathbf{h}\|} \right) d\mathbf{v}$$

Furthermore, Rademacher's theorem guarantees that f is almost everywhere differentiable. Letting $U \subseteq \mathbb{B}_\delta(\mathbf{0})$ such that $\text{Vol}(U) = \text{Vol}(\mathbb{B}_\delta(\mathbf{0}))$ and f is differentiable at $\mathbf{x} + \mathbf{v}$ for $\forall \mathbf{v} \in U$, we have

$$\lim_{\|\mathbf{h}\| \rightarrow 0} \frac{f_\delta(\mathbf{x} + \mathbf{h}) - f_\delta(\mathbf{x})}{\|\mathbf{h}\|} = \frac{1}{\text{Vol}(U)} \int_{\mathbf{v} \in U} \left(\lim_{\|\mathbf{h}\| \rightarrow 0} \frac{f(\mathbf{x} + \mathbf{h} + \mathbf{v}) - f(\mathbf{x} + \mathbf{v})}{\|\mathbf{h}\|} \right) d\mathbf{v} \quad (2)$$

and

$$\lim_{\|\mathbf{h}\| \rightarrow 0} \frac{|f(\mathbf{x} + \mathbf{h} + \mathbf{v}) - f(\mathbf{x} + \mathbf{v}) - \langle \nabla f(\mathbf{x} + \mathbf{v}), \mathbf{h} \rangle|}{\|\mathbf{h}\|} = 0 \quad (3)$$

Combining Eq. (1), Eq (2) and Eq. (3) together yields that

$$\lim_{\|\mathbf{h}\| \rightarrow 0} \frac{|\langle \nabla f_\delta(\mathbf{x}) - \mathbb{E}_{\mathbf{u} \sim \mathbb{P}}[\nabla f(\mathbf{x} + \delta\mathbf{u})], \mathbf{h} \rangle|}{\|\mathbf{h}\|} = 0$$

Choosing $\mathbf{h} = t(\nabla f_\delta(\mathbf{x}) - \mathbb{E}_{\mathbf{u} \sim \mathbb{P}}[\nabla f(\mathbf{x} + \delta\mathbf{u})])$ with $t \rightarrow 0$, we have $\|\nabla f_\delta(\mathbf{x}) - \mathbb{E}_{\mathbf{u} \sim \mathbb{P}}[\nabla f(\mathbf{x} + \delta\mathbf{u})]\| = 0$.

It remains to show that $\nabla f_\delta(\mathbf{x}) \in \partial_\delta f(\mathbf{x})$ for any $\mathbf{x} \in \mathbb{R}^d$ using the proof argument by contradiction. In particular, we assume that there exists $\mathbf{x}_0 \in \mathbb{R}^d$ such that $\nabla f_\delta(\mathbf{x}_0) \notin \partial_\delta f(\mathbf{x}_0)$. Recall that

$$\partial_\delta f(\mathbf{x}_0) := \text{conv}(\cup_{\mathbf{y} \in \mathbb{B}_\delta(\mathbf{x}_0)} \partial f(\mathbf{y}))$$

By the hyperplane separation theorem [Rockafellar and Wets(2009)], there exists a unit vector $\mathbf{g} \in \mathbb{R}^d$ such that $\langle \mathbf{g}, \nabla f_\delta(\mathbf{x}_0) \rangle > 0$ and

$$\langle \mathbf{g}, \xi \rangle \leq 0, \quad \text{for any } \xi \in \cup_{\mathbf{y} \in \mathbb{B}_\delta(\mathbf{x}_0)} \partial f(\mathbf{y}) \quad (4)$$

However, we already obtain that $\nabla f_\delta(\mathbf{x}) = \mathbb{E}_{\mathbf{u} \sim \mathbb{P}}[\nabla f(\mathbf{x} + \delta\mathbf{u})]$ which implies that

$$\nabla f_\delta(\mathbf{x}_0) = \frac{1}{\text{Vol}(\mathbb{B}_1(\mathbf{0}))} \int_{\mathbf{u} \in \mathbb{B}_1(\mathbf{0})} \nabla f(\mathbf{x}_0 + \delta\mathbf{u}) d\mathbf{u} = \frac{1}{\text{Vol}(\mathbb{B}_\delta(\mathbf{0}))} \int_{\mathbf{y} \in \mathbb{B}_\delta(\mathbf{x}_0)} \nabla f(\mathbf{y}) d\mathbf{y}$$

Thus, Eq. (4) implies that $\langle \mathbf{g}, \nabla f_\delta(\mathbf{x}_0) \rangle \leq 0$ which leads to a contradiction. Therefore, we conclude that $\nabla f_\delta(\mathbf{x}) \in \partial_\delta f(\mathbf{x})$ for any $\mathbf{x} \in \mathbb{R}^d$. This completes the proof.

Algorithm 2 Two-Phase Gradient-Free Method (2-GFM)

```
1: Input: initial point  $\mathbf{x}^0 \in \mathbb{R}^d$ , stepsize  $\eta > 0$ , problem dimension  $d \geq 1$ , smoothing parameter  $\delta$ , iteration number  $T \geq 1$ , number of rounds  $S \geq 1$  and sample size  $B$ 
2: for  $s = 0, 1, 2, \dots, S-1$  do
3:   Call Algorithm 1 with  $\mathbf{x}^0$ ,  $\eta$ ,  $d$ ,  $\delta$  and  $T$  and let  $\bar{\mathbf{x}}_s$  be an output
4: end for
5: for  $s = 0, 1, 2, \dots, S-1$  do
6:   for  $k = 0, 1, 2, \dots, B-1$  do
7:     Sample  $\mathbf{w}^k \in \mathbb{R}^d$  uniformly from a unit sphere in  $\mathbb{R}^d$ 
8:     Compute  $\mathbf{g}_s^k = \frac{d}{2\delta}(f(\bar{\mathbf{x}}_s + \delta\mathbf{w}^k) - f(\bar{\mathbf{x}}_s - \delta\mathbf{w}^k))\mathbf{w}^k$ 
9:   end for
10:  Compute  $\mathbf{g}_s = \frac{1}{B} \sum_{k=0}^{B-1} \mathbf{g}_s^k$ 
11: end for
12: Choose an index  $s^* \in \{0, 1, 2, \dots, S-1\}$  such that  $s^* = \operatorname{argmin}_{s=0,1,2,\dots,S-1} \|\mathbf{g}_s\|$ 
13: Output:  $\bar{\mathbf{x}}_{s^*}$ 
```

5 Main Results: Gradient-Free Methods

We analyze a gradient-free method (GFM) and its two-phase version (2-GFM) for optimizing a Lipschitz function f .

Global rate estimation. Let $f : \mathbb{R}^d \mapsto \mathbb{R}$ be a L -Lipschitz function and the smooth version of f is then the function $f_\delta = \mathbb{E}_{\mathbf{u} \sim \mathbb{P}}[f(\mathbf{x} + \delta\mathbf{u})]$ where \mathbb{P} is an uniform distribution on an unit ball in ℓ_2 -norm. Equipped with Lemma 10 from [Shamir(2017)], we can compute an unbiased estimator for the gradient $\nabla f_\delta(\mathbf{x}^t)$ using function values.

This leads to the gradient-free method (GFM) in Algorithm 1 that simply performs a one-step gradient descent to obtain \mathbf{x}^t . It is worth mentioning that we use a random iteration count R to terminate the execution of Algorithm 1 and this will guarantee that GFM is valid. Indeed, we only derive that $\min_{t=1,2,\dots,T} \|\nabla f_\delta(\mathbf{x}^t)\| \leq \epsilon$ in the theoretical analysis (see also [Nesterov and Spokoiny(2017), Section 7]) and finding the best solution from $\{\mathbf{x}^1, \mathbf{x}^2, \dots, \mathbf{x}^T\}$ is difficult since the quantity $\|\nabla f_\delta(\mathbf{x}^t)\|$ is unknown. To estimate them using Monte Carlo simulation would incur additional approximation errors and raise some reliability issues. The idea of random output is not new but has been used by [Ghadimi and Lan(2013)] for smooth and nonconvex stochastic optimization. Such scheme also gives us a computational gain with a factor of two in expectation.

Theorem 2. Suppose that f is L -Lipschitz and let $\delta > 0$ and $0 < \epsilon < 1$. Then, there exists some $T > 0$ such that the output of Algorithm 1 with $\eta = \frac{1}{10} \sqrt{\frac{\delta(\Delta + \delta L)}{cd^{3/2}L^3T}}$ satisfies that $\mathbb{E}[\min\{\|\mathbf{g}\| : \mathbf{g} \in \partial_\delta f(\mathbf{x}^R)\}] \leq \epsilon$ and the total number of calls of the function value oracle is bounded by

$$O\left(d^{\frac{3}{2}} \left(\frac{L^4}{\epsilon^4} + \frac{\Delta L^3}{\delta \epsilon^4}\right)\right)$$

where $d \geq 1$ is the problem dimension, $L > 0$ is the Lipschitz parameter of f and $\Delta > 0$ is an upper bound for the initial objective function gap, $f(\mathbf{x}^0) - \inf_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x}) > 0$.

Remark. Theorem 2 illustrates the difference between gradient-based and gradient-free methods in nonsmooth nonconvex optimization. Indeed, [Davis et al.(2022)] has recently proved the

rate of $O(\delta^{-1}\epsilon^{-3})$ for a randomized gradient-based method in terms of (δ, ϵ) -Goldstein stationarity. Further, Theorem 2 demonstrates that nonsmooth nonconvex optimization is likely to be intrinsically harder than all other standard settings. More specifically, the state-of-the-art rate for gradient-free methods is $O(d\epsilon^{-2})$ for nonsmooth convex optimization in terms of objective function value gap [Duchi et al.(2015)] and smooth nonconvex optimization in terms of gradient norm [Nesterov and Spokoiny(2017)]. Thus, the dependence on $d \geq 1$ is linear in their bounds yet $d^{\frac{3}{2}}$ in our bound. We believe it is promising to either improve the rate of gradient-free methods or show the impossibility by establishing a lower bound.

Large-deviation estimation. While Theorem 2 establishes the expected convergence rate over many runs of Algorithm 1, we are also interested in the large-deviation properties for a single run. Indeed, we hope to establish a complexity bound for computing a $(\delta, \epsilon, \Lambda)$ -solution; that is, a point $\mathbf{x} \in \mathbb{R}^d$ satisfying $\text{Prob}(\min\{\|\mathbf{g}\| : \mathbf{g} \in \partial_\delta f(\mathbf{x})\} \leq \epsilon) \geq 1 - \Lambda$ for some $\delta > 0$ and $0 < \epsilon, \Lambda < 1$. By Theorem 2 and Markov's inequality,

$$\text{Prob}(\min\{\|\mathbf{g}\| : \mathbf{g} \in \partial_\delta f(\mathbf{x}^R)\} \geq \lambda \mathbb{E}[\min\{\|\mathbf{g}\| : \mathbf{g} \in \partial_\delta f(\mathbf{x}^R)\}]) \leq \frac{1}{\lambda}, \quad \text{for all } \lambda > 0$$

we conclude that the total number of calls of the function value oracle is bounded by

$$O\left(d^{\frac{3}{2}} \left(\frac{L^4}{\Lambda^4 \epsilon^4} + \frac{\Delta L^3}{\delta \Lambda^4 \epsilon^4} \right)\right) \quad (5)$$

This complexity bound is rather pessimistic in terms of its dependence on Λ which is often set to be small in practice. To improve the bound, we combine Algorithm 1 with a post-optimization procedure [Ghadimi and Lan(2013)], leading to a two-phase gradient-free method (2-GFM), shown in Algorithm 2.

Theorem 3. *Suppose that f is L -Lipschitz and let $\delta > 0$ and $0 < \epsilon, \Lambda < 1$. Then, there exists some $T, S, B > 0$ such that the output of Algorithm 2 with $\eta = \frac{1}{10} \sqrt{\frac{\delta(\Delta + \delta L)}{cd^{3/2}L^3T}}$ satisfies that $\text{Prob}(\min\{\|\mathbf{g}\| : \mathbf{g} \in \partial_\delta f(\bar{\mathbf{x}}_{s^*})\} \geq \epsilon) \leq \Lambda$ and the total number of calls of the function value oracle is bounded by*

$$O\left(d^{\frac{3}{2}} \left(\frac{L^4}{\epsilon^4} + \frac{\Delta L^3}{\delta \epsilon^4} \right) \log_2 \left(\frac{1}{\Lambda} \right) + \frac{dL^2}{\Lambda \epsilon^2} \log_2 \left(\frac{1}{\Lambda} \right)\right)$$

where $d \geq 1$ is the problem dimension, $L > 0$ is the Lipschitz parameter of f and $\Delta > 0$ is an upper bound for the initial objective function gap, $f(\mathbf{x}^0) - \inf_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x}) > 0$.

Clearly, the bound in Theorem 3 is significantly smaller than the corresponding one in Eq. (5) in terms of the dependence on $1/\Lambda$, demonstrating the power of the post-optimization phase.

5.1 Proof of Theorem 2

Till the rest of this section, we give the proofs of Theorem 2 and 3. We also present some technical lemmas for analyzing the convergence property of gradient-free method and its two-phase version.

We provide two technical lemmas for analyzing Algorithm 1. The first lemma is a restatement of [Shamir(2017), Lemma 10] which gives an upper bound on the quantity $\mathbb{E}[\|\mathbf{g}^t\|^2 | \mathbf{x}^t]$ in terms of problem dimension $d \geq 1$ and the Lipschitz parameter $L > 0$. For the sake of completeness, we provide the proof details.

Lemma 1. Suppose that f is L -Lipschitz and let $\{\mathbf{g}^t\}_{t=0}^{T-1}$ and $\{\mathbf{x}^t\}_{t=0}^{T-1}$ be generated by Algorithm 1. Then, we have $\mathbb{E}[\mathbf{g}^t|\mathbf{x}^t] = \nabla f_\delta(\mathbf{x}^t)$ and $\mathbb{E}[\|\mathbf{g}^t\|^2|\mathbf{x}^t] \leq 16\sqrt{2\pi}dL^2$.

The second lemma gives a key descent inequality for analyzing Algorithm 1.

Lemma 2. Suppose that f is L -Lipschitz and let $\{\mathbf{x}^t\}_{t=0}^{T-1}$ be generated by Algorithm 1. Then, we have

$$\mathbb{E}[\|\nabla f_\delta(\mathbf{x}^t)\|^2] \leq \frac{\mathbb{E}[f_\delta(\mathbf{x}^t)] - \mathbb{E}[f_\delta(\mathbf{x}^{t+1})]}{\eta} + \eta \cdot \frac{(8\sqrt{2\pi})cd^{3/2}L^3}{\delta}, \quad \text{for all } 0 \leq t \leq T-1$$

where $c > 0$ is a constant appearing in the smoothing parameter of f_δ (cf. Proposition 2).

We also present a proposition which is crucial to deriving the large deviation property of Algorithm 2.

Proposition 3 ([Juditsky and Nemirovski(2008)] or [Ghadimi and Lan(2013)]). Suppose that Ω is a polish space with a Borel probability measure \mathbb{P} and let $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$ be a sequence of filtration. For an integer $N \geq 1$, we define a martingale-difference sequence of Borel functions $\{\zeta_k\}_{k=1}^N \subseteq \mathbb{R}^n$ such that ζ_k is \mathcal{F}_k -measurable and $\mathbb{E}[\zeta_k | \mathcal{F}_{k-1}] = 0$. Then, if $\mathbb{E}[\|\zeta_k\|^2] \leq \sigma_k^2$ for all $k \geq 1$, we have $\mathbb{E}[\|\sum_{k=1}^N \zeta_k\|^2] \leq \sum_{k=1}^N \sigma_k^2$ and the following statement holds true,

$$\text{Prob} \left(\left\| \sum_{k=1}^N \zeta_k \right\|^2 \geq \lambda \sum_{k=1}^N \sigma_k^2 \right) \leq \frac{1}{\lambda}, \quad \text{for all } \lambda \geq 0$$

This is a general result concerning about the large deviations of vector martingales; see, e.g., [Juditsky and Nemirovski(2008), Theorem 2.1] or [Ghadimi and Lan(2013), Lemma 2.3].

We are ready to present the

Proof of Theorem 2. Summing up the inequality in Lemma 2 over $t = 0, 1, 2, \dots, T-1$ yields that

$$\sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f_\delta(\mathbf{x}^t)\|^2] \leq \frac{f_\delta(\mathbf{x}^0) - \mathbb{E}[f_\delta(\mathbf{x}^T)]}{\eta} + \eta \cdot \frac{(8\sqrt{2\pi})cd^{3/2}L^3T}{\delta}$$

By Proposition 2, we have $f(\mathbf{x}_0) \leq f_\delta(\mathbf{x}_0) \leq f(\mathbf{x}_0) + \delta L$. In addition, we see from the definition of f_δ that $f_\delta(\mathbf{x}) \geq \inf_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x})$ for any $\mathbf{x} \in \mathbb{R}^d$ and thus $\mathbb{E}[f_\delta(\mathbf{x}^T)] \geq \inf_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x})$. Putting these pieces together with $f \in \mathcal{F}_d(\Delta, L)$ yields that

$$f_\delta(\mathbf{x}^0) - \mathbb{E}[f_\delta(\mathbf{x}^T)] \leq f(\mathbf{x}_0) - \inf_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x}) + \delta L \leq \Delta + \delta L$$

Therefore, we conclude that

$$\frac{1}{T} \left(\sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f_\delta(\mathbf{x}^t)\|^2] \right) \leq \frac{\Delta + \delta L}{\eta T} + \eta \cdot \frac{(8\sqrt{2\pi})cd^{3/2}L^3}{\delta} \leq \frac{\Delta + \delta L}{\eta T} + \eta \cdot \frac{100cd^{3/2}L^3}{\delta}$$

Recalling that $\eta = \frac{1}{10} \sqrt{\frac{\delta(\Delta + \delta L)}{cd^{3/2}L^3T}}$, we have

$$\frac{1}{T} \left(\sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f_\delta(\mathbf{x}^t)\|^2] \right) \leq 20 \sqrt{\frac{cd^{3/2}L^3}{T}} \left(L + \frac{\Delta}{\delta} \right)$$

Since the random count $R \in \{0, 1, 2, \dots, T-1\}$ is uniformly sampled, we have

$$\mathbb{E}[\|\nabla f_\delta(\mathbf{x}^R)\|^2] = \frac{1}{T} \left(\sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f_\delta(\mathbf{x}^t)\|^2] \right) \leq 20 \sqrt{\frac{cd^{3/2}L^3}{T}} \left(L + \frac{\Delta}{\delta} \right) \quad (6)$$

By Theorem 1, we have $\nabla f_\delta(\mathbf{x}^R) \in \partial_\delta f(\mathbf{x}^R)$. This together with the above inequality implies that

$$\mathbb{E}[\min\{\|\mathbf{g}\| : \mathbf{g} \in \partial_\delta f(\mathbf{x}^R)\}] \leq \mathbb{E}[\|\nabla f_\delta(\mathbf{x}^R)\|] \leq 5 \left(\frac{cd^{3/2}L^3}{T} \left(L + \frac{\Delta}{\delta} \right) \right)^{\frac{1}{4}}$$

Therefore, we conclude that there exists some $T > 0$ such that the output of Algorithm 1 satisfies that $\mathbb{E}[\min\{\|\mathbf{g}\| : \mathbf{g} \in \partial_\delta f(\mathbf{x}^R)\}] \leq \epsilon$ and the total number of calling the function value oracles is bounded by

$$O \left(d^{\frac{3}{2}} \left(\frac{L^4}{\epsilon^4} + \frac{\Delta L^3}{\delta \epsilon^4} \right) \right)$$

This completes the proof. \square

5.2 Proof of Theorem 3

Proof of Theorem 3. By the definition of s^* and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|\mathbf{g}_{s^*}\|^2 &= \min_{s=0,1,2,\dots,S-1} \|\mathbf{g}_s\|^2 \leq \min_{s=0,1,2,\dots,S-1} \{2\|\nabla f_\delta(\bar{\mathbf{x}}_s)\|^2 + 2\|\mathbf{g}_s - \nabla f_\delta(\bar{\mathbf{x}}_s)\|^2\} \\ &\leq 2 \left(\min_{s=0,1,2,\dots,S-1} \|\nabla f_\delta(\bar{\mathbf{x}}_s)\|^2 + \max_{s=0,1,2,\dots,S-1} \|\mathbf{g}_s - \nabla f_\delta(\bar{\mathbf{x}}_s)\|^2 \right). \end{aligned} \quad (7)$$

This implies that

$$\begin{aligned} \|\nabla f_\delta(\bar{\mathbf{x}}_{s^*})\|^2 &\leq 2\|\mathbf{g}_{s^*}\|^2 + 2\|\mathbf{g}_{s^*} - \nabla f_\delta(\bar{\mathbf{x}}_{s^*})\|^2 \\ &\stackrel{\text{Eq. (7)}}{\leq} 4 \left(\min_{s=0,1,2,\dots,S-1} \|\nabla f_\delta(\bar{\mathbf{x}}_s)\|^2 \right) + 4 \left(\max_{s=0,1,2,\dots,S-1} \|\mathbf{g}_s - \nabla f_\delta(\bar{\mathbf{x}}_s)\|^2 \right) + 2\|\mathbf{g}_{s^*} - \nabla f_\delta(\bar{\mathbf{x}}_{s^*})\|^2. \end{aligned} \quad (8)$$

The next step is to provide the probabilistic bounds on all the terms in the right-hand side of Eq. (8). In particular, for each $s = 0, 1, 2, \dots, S-1$, we have $\bar{\mathbf{x}}_s$ is an output obtained by calling Algorithm 1 with \mathbf{x}^0 , d , δ , T and $\eta = \frac{1}{10} \sqrt{\frac{\delta(\Delta + \delta L)}{cd^{3/2}L^3T}}$. Then, Eq. (6) in the proof of Theorem 2 implies that

$$\mathbb{E}[\|\nabla f_\delta(\bar{\mathbf{x}}_s)\|^2] \leq 20 \sqrt{\frac{cd^{3/2}L^3}{T}} \left(L + \frac{\Delta}{\delta} \right)$$

Using the Markov's inequality, we have

$$\text{Prob} \left(\|\nabla f_\delta(\bar{\mathbf{x}}_s)\|^2 \geq 40 \sqrt{\frac{cd^{3/2}L^3}{T}} \left(L + \frac{\Delta}{\delta} \right) \right) \leq \frac{1}{2}$$

Thus, we have

$$\text{Prob} \left(\min_{s=0,1,2,\dots,S-1} \|\nabla f_\delta(\bar{\mathbf{x}}_s)\|^2 \geq 40 \sqrt{\frac{cd^{3/2}L^3}{T}} \left(L + \frac{\Delta}{\delta} \right) \right) \leq 2^{-S} \quad (9)$$

Further, for each $s = 0, 1, 2, \dots, S-1$, we have

$$\mathbf{g}_s - \nabla f_\delta(\bar{\mathbf{x}}_s) = \frac{1}{B} \sum_{k=0}^{B-1} (\mathbf{g}_s^k - \nabla f_\delta(\bar{\mathbf{x}}_s))$$

By Lemma 1, we have $\mathbb{E}[\mathbf{g}_s^t | \bar{\mathbf{x}}_s] = \nabla f_\delta(\bar{\mathbf{x}}_s)$ and $\mathbb{E}[\|\mathbf{g}_s^t\|^2 | \bar{\mathbf{x}}_s] \leq 16\sqrt{2\pi}dL^2$. This implies that

$$\mathbb{E}[\mathbf{g}_s^t - \nabla f_\delta(\bar{\mathbf{x}}_s) | \bar{\mathbf{x}}_s] = 0, \quad \mathbb{E}[\|\mathbf{g}_s^t - \nabla f_\delta(\bar{\mathbf{x}}_s)\|^2] \leq 16\sqrt{2\pi}dL^2$$

This together with Proposition 3 yields that

$$\text{Prob} \left(\|\mathbf{g}_s - \nabla f_\delta(\bar{\mathbf{x}}_s)\|^2 \geq \frac{\lambda(16\sqrt{2\pi}dL^2)}{B} \right) = \text{Prob} \left(\left\| \sum_{k=0}^{B-1} (\mathbf{g}_s^k - \nabla f_\delta(\bar{\mathbf{x}}_s)) \right\|^2 \geq \lambda B(16\sqrt{2\pi}dL^2) \right) \leq \frac{1}{\lambda}$$

Therefore, we conclude that

$$\text{Prob} \left(\max_{s=0,1,2,\dots,S-1} \|\mathbf{g}_s - \nabla f_\delta(\bar{\mathbf{x}}_s)\|^2 \geq \frac{\lambda(16\sqrt{2\pi}dL^2)}{B} \right) \leq \frac{S}{\lambda} \quad (10)$$

By the similar argument, we have

$$\text{Prob}(\|\mathbf{g}_{s^*} - \nabla f_\delta(\bar{\mathbf{x}}_{s^*})\|^2 \geq \frac{\lambda(16\sqrt{2\pi}dL^2)}{B}) \leq \frac{1}{\lambda} \quad (11)$$

Combining Eq. (8), Eq. (9), Eq. (10) and Eq. (11) yields that

$$\text{Prob} \left(\|\nabla f_\delta(\bar{\mathbf{x}}_{s^*})\|^2 \geq 160\sqrt{\frac{cd^{3/2}L^3}{T}}(L + \frac{\Delta}{\delta}) + \frac{\lambda(96\sqrt{2\pi}dL^2)}{B} \right) \leq \frac{S+1}{\lambda} + 2^{-S}, \quad \text{for all } \lambda > 0 \quad (12)$$

We set $\lambda = \frac{2(S+1)}{\Lambda}$ and the parameters (T, S, B) as follows,

$$T = cd^{3/2}L^3(L + \frac{\Delta}{\delta})(\frac{160}{\epsilon^2})^2, \quad S = \lceil \log_2(\frac{2}{\Lambda}) \rceil, \quad B = \frac{(384\sqrt{2\pi}dL^2)(S+1)}{\Lambda\epsilon^2}$$

Then, we have

$$\text{Prob}(\|\nabla f_\delta(\bar{\mathbf{x}}_{s^*})\|^2 \geq \epsilon^2) \leq \text{Prob} \left(\|\nabla f_\delta(\bar{\mathbf{x}}_{s^*})\|^2 \geq 160\sqrt{\frac{cd^{3/2}L^3}{T}}(L + \frac{\Delta}{\delta}) + \frac{\lambda(96\sqrt{2\pi}dL^2)}{B} \right) \leq \Lambda$$

By Theorem 1, we have $\nabla f_\delta(\bar{\mathbf{x}}_{s^*}) \in \partial_\delta f(\bar{\mathbf{x}}_{s^*})$. This together with the above inequality implies that there exists some $T, S, B > 0$ such that the output of Algorithm 2 satisfies that $\mathbb{E}[\min\{\|\mathbf{g}\| : \mathbf{g} \in \partial_\delta f(\bar{\mathbf{x}}_{s^*})\}] \leq \epsilon$ and the total number of calling the function value oracles is bounded by

$$O \left(d^{\frac{3}{2}} \left(\frac{L^4}{\epsilon^4} + \frac{\Delta L^3}{\delta \epsilon^4} \right) \log_2 \left(\frac{1}{\Lambda} \right) + \frac{dL^2}{\Lambda \epsilon^2} \log_2 \left(\frac{1}{\Lambda} \right) \right)$$

This completes the proof. \square

5.3 Proofs of technical lemmas

Proof of Lemma 1. By the definition of \mathbf{g}^t and the symmetry of the distribution of \mathbf{w}^t , we have

$$\begin{aligned}\mathbb{E}[\mathbf{g}^t \mid \mathbf{x}^t] &= \mathbb{E}\left[\frac{d}{2\delta}(f(\mathbf{x}^t + \delta\mathbf{w}^t) - f(\mathbf{x}^t - \delta\mathbf{w}^t))\mathbf{w}^t \mid \mathbf{x}^t\right] \\ &= \frac{1}{2}\left(\mathbb{E}\left[\frac{d}{\delta}f(\mathbf{x}^t + \delta\mathbf{w}^t)\mathbf{w}^t \mid \mathbf{x}^t\right] + \mathbb{E}\left[\frac{d}{\delta}f(\mathbf{x}^t + \delta(-\mathbf{w}^t))(-\mathbf{w}^t) \mid \mathbf{x}^t\right]\right) \\ &= \frac{1}{2}(\nabla f_\delta(\mathbf{x}^t) + \nabla f_\delta(\mathbf{x}^t)) = \nabla f_\delta(\mathbf{x}^t)\end{aligned}$$

It remains to show that $\mathbb{E}[\|\mathbf{g}^t\|^2 \mid \mathbf{x}^t] \leq 16\sqrt{2\pi}dL^2$. Indeed, since $\|\mathbf{w}^t\| = 1$, we have

$$\mathbb{E}[\|\mathbf{g}^t\|^2 \mid \mathbf{x}^t] = \mathbb{E}\left[\frac{d^2}{4\delta^2}(f(\mathbf{x}^t + \delta\mathbf{w}^t) - f(\mathbf{x}^t - \delta\mathbf{w}^t))^2\|\mathbf{w}^t\|^2 \mid \mathbf{x}^t\right] \leq \mathbb{E}\left[\frac{d^2}{4\delta^2}(f(\mathbf{x}^t + \delta\mathbf{w}^t) - f(\mathbf{x}^t - \delta\mathbf{w}^t))^2 \mid \mathbf{x}^t\right]$$

Using the elementary inequality $(a - b)^2 \leq 2a^2 + 2b^2$, we have

$$\begin{aligned}\mathbb{E}[(f(\mathbf{x}^t + \delta\mathbf{w}^t) - f(\mathbf{x}^t - \delta\mathbf{w}^t))^2 \mid \mathbf{x}^t] \\ &= \mathbb{E}[(f(\mathbf{x}^t + \delta\mathbf{w}^t) - \mathbb{E}[f(\mathbf{x}^t + \delta\mathbf{w}^t) \mid \mathbf{x}^t] - (f(\mathbf{x}^t - \delta\mathbf{w}^t) - \mathbb{E}[f(\mathbf{x}^t + \delta\mathbf{w}^t) \mid \mathbf{x}^t]))^2 \mid \mathbf{x}^t] \\ &\leq 2\mathbb{E}[(f(\mathbf{x}^t + \delta\mathbf{w}^t) - \mathbb{E}[f(\mathbf{x}^t + \delta\mathbf{w}^t) \mid \mathbf{x}^t])^2 \mid \mathbf{x}^t] + 2\mathbb{E}[(f(\mathbf{x}^t - \delta\mathbf{w}^t) - \mathbb{E}[f(\mathbf{x}^t + \delta\mathbf{w}^t) \mid \mathbf{x}^t])^2 \mid \mathbf{x}^t]\end{aligned}$$

Since \mathbf{w}^t has a symmetric distribution around the origin, we have

$$\mathbb{E}[(f(\mathbf{x}^t + \delta\mathbf{w}^t) - \mathbb{E}[f(\mathbf{x}^t + \delta\mathbf{w}^t) \mid \mathbf{x}^t])^2 \mid \mathbf{x}^t] = \mathbb{E}[(f(\mathbf{x}^t - \delta\mathbf{w}^t) - \mathbb{E}[f(\mathbf{x}^t + \delta\mathbf{w}^t) \mid \mathbf{x}^t])^2 \mid \mathbf{x}^t]$$

Putting these pieces together yields that

$$\mathbb{E}[\|\mathbf{g}^t\|^2 \mid \mathbf{x}^t] \leq \frac{d^2}{\delta^2}\mathbb{E}[(f(\mathbf{x}^t + \delta\mathbf{w}^t) - \mathbb{E}[f(\mathbf{x}^t + \delta\mathbf{w}^t) \mid \mathbf{x}^t])^2 \mid \mathbf{x}^t] \quad (13)$$

For simplicity, we let $h(\mathbf{w}) = f(\mathbf{x}^t + \delta\mathbf{w})$. Since f is L -Lipschitz, this function h is δL -Lipschitz given a fixed \mathbf{x}^t . In addition, $\mathbf{w}^t \in \mathbb{R}^d$ is sampled uniformly from a unit sphere. Then, by [Wainwright(2019), Proposition 3.11 and Example 3.12], we have

$$\mathbb{P}(|h(\mathbf{w}^t) - \mathbb{E}[h(\mathbf{w}^t)]| \geq \alpha) \leq 2\sqrt{2\pi}e^{-\frac{\alpha^2 d}{8\delta^2 L^2}}$$

Then, we have

$$\begin{aligned}\mathbb{E}[(h(\mathbf{w}^t) - \mathbb{E}[h(\mathbf{w}^t)])^2] &= \int_0^{+\infty} \mathbb{P}((h(\mathbf{w}^t) - \mathbb{E}[h(\mathbf{w}^t)])^2 \geq \alpha) d\alpha \\ &= \int_0^{+\infty} \mathbb{P}(|h(\mathbf{w}^t) - \mathbb{E}[h(\mathbf{w}^t)]| \geq \sqrt{\alpha}) d\alpha \leq 2\sqrt{2\pi} \int_0^{+\infty} e^{-\frac{\alpha d}{8\delta^2 L^2}} d\alpha \\ &= 2\sqrt{2\pi} \cdot \frac{8\delta^2 L^2}{d} = \frac{16\sqrt{2\pi}\delta^2 L^2}{d}\end{aligned}$$

By the definition of h , we have

$$\mathbb{E}[(f(\mathbf{x}^t + \delta\mathbf{w}^t) - \mathbb{E}[f(\mathbf{x}^t + \delta\mathbf{w}^t) \mid \mathbf{x}^t])^2 \mid \mathbf{x}^t] \leq \frac{16\sqrt{2\pi}\delta^2 L^2}{d} \quad (14)$$

Combining Eq. (13) and Eq. (14) yields the desired inequality. \square

Proof of Lemma 2. By Proposition 2, we have f_δ is differentiable and L -Lipschitz with the $\frac{cL\sqrt{d}}{\delta}$ -Lipschitz gradient where $c > 0$ is a constant. This implies that

$$f_\delta(\mathbf{x}^{t+1}) \leq f_\delta(\mathbf{x}^t) - \eta \langle \nabla f_\delta(\mathbf{x}^t), \mathbf{g}^t \rangle + \frac{c\eta^2 L \sqrt{d}}{2\delta} \|\mathbf{g}^t\|^2$$

Taking the expectation of both sides conditioned on \mathbf{x}^t and using Lemma 1, we have

$$\begin{aligned} \mathbb{E}[f_\delta(\mathbf{x}^{t+1}) \mid \mathbf{x}^t] &\leq f_\delta(\mathbf{x}^t) - \eta \langle \nabla f_\delta(\mathbf{x}^t), \mathbb{E}[\mathbf{g}^t \mid \mathbf{x}^t] \rangle + \frac{c\eta^2 L \sqrt{d}}{2\delta} \mathbb{E}[\|\mathbf{g}^t\|^2 \mid \mathbf{x}^t] \\ &\leq f_\delta(\mathbf{x}^t) - \eta \|\nabla f_\delta(\mathbf{x}^t)\|^2 + \eta^2 \cdot \frac{cL\sqrt{d}}{2\delta} \cdot 16\sqrt{2\pi}dL^2 \\ &= f_\delta(\mathbf{x}^t) - \eta \|\nabla f_\delta(\mathbf{x}^t)\|^2 + \eta^2 \cdot (8\sqrt{2\pi})cd^{3/2}L^3\delta^{-1} \end{aligned}$$

Taking the expectation of both sides and rearranging yields that

$$\mathbb{E}[\|\nabla f_\delta(\mathbf{x}^t)\|^2] \leq \frac{\mathbb{E}[f_\delta(\mathbf{x}^t)] - \mathbb{E}[f_\delta(\mathbf{x}^{t+1})]}{\eta} + \eta \cdot \frac{(8\sqrt{2\pi})cd^{3/2}L^3}{\delta}$$

This completes the proof. \square

6 Main Results: Stochastic Gradient-Free Methods

We turn to the analysis of a stochastic gradient-free method (SGFM) and its two-phase version (2-SGFM) for optimizing a Lipschitz function $f(\cdot) = \mathbb{E}_{\xi \in \mathbb{P}_\mu}[F(\cdot, \xi)]$.

Global rate estimation. In contrast to minimizing a deterministic function f , we only have access to the noisy function value $F(\mathbf{x}, \xi)$ at any point $\mathbf{x} \in \mathbb{R}^d$ where a data sample ξ is drawn from a distribution \mathbb{P}_μ . Intuitively, this is a more challenging setup. It has been studied before in the setting of optimizing a nonsmooth convex function [Duchi et al.(2015), Nesterov and Spokoiny(2017)] or a smooth nonconvex function [Ghadimi and Lan(2013)]. As in these papers, we assume that (i) $F(\cdot, \xi)$ is $L(\xi)$ -Lipschitz with $\mathbb{E}_{\xi \in \mathbb{P}_\mu}[L^2(\xi)] \leq G^2$ for some $G > 0$ and (ii) $\mathbb{E}[F(\mathbf{x}, \xi^t)] = f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^d$ where ξ^t is simulated from \mathbb{P}_μ at the t^{th} iteration.

Despite the noisy function value, we can compute an unbiased estimator of the gradient $\nabla f_\delta(\mathbf{x}^t)$, where $f_\delta = \mathbb{E}_{\mathbf{u} \sim \mathbb{P}}[f(\mathbf{x} + \delta \mathbf{u})] = \mathbb{E}_{\mathbf{u} \sim \mathbb{P}, \xi \in \mathbb{P}_\mu}[F(\mathbf{x} + \delta \mathbf{u}, \xi)]$. In particular, we have $\hat{\mathbf{g}}^t = \frac{d}{2\delta}(F(\mathbf{x}^t + \delta \mathbf{w}^t, \xi^t) - F(\mathbf{x}^t - \delta \mathbf{w}^t, \xi^t))\mathbf{w}^t$. Clearly, under our assumption, we have

$$\mathbb{E}_{\mathbf{u} \sim \mathbb{P}, \xi \in \mathbb{P}_\mu}[\hat{\mathbf{g}}^t] = \mathbb{E}_{\mathbf{u} \sim \mathbb{P}}[\mathbb{E}_{\xi \in \mathbb{P}_\mu}[\hat{\mathbf{g}}^t \mid \mathbf{u}]] = \mathbb{E}_{\mathbf{u} \sim \mathbb{P}}[\mathbf{g}^t] = \nabla f_\delta(\mathbf{x}^t)$$

where \mathbf{g}^t is defined in Algorithm 1. However, the variance of the estimator $\hat{\mathbf{g}}^t$ can be undesirably large since $F(\cdot, \xi)$ is $L(\xi)$ -Lipschitz for a (possibly unbounded) random variable $L(\xi) > 0$. To resolve this issue, we revisit [Shamir(2017), Lemma 10] and show that in deriving an upper bound for $\mathbb{E}_{\mathbf{u} \sim \mathbb{P}, \xi \in \mathbb{P}_\mu}[\|\hat{\mathbf{g}}^t\|^2]$ it suffices to assume that $\mathbb{E}_{\xi \in \mathbb{P}_\mu}[L^2(\xi)] \leq G^2$ for some constant $G > 0$. The resulting bound achieves a linear dependence in the problem dimension $d > 0$ which is the same as in [Shamir(2017), Lemma 10]. Note that the setup with *convex* and $L(\xi)$ -Lipschitz functions $F(\cdot, \xi)$ has been considered in [Duchi et al.(2015)]. However, our estimator is different from their estimator of $\hat{\mathbf{g}}^t = \frac{d}{\delta}(F(\mathbf{x}^t + \delta \mathbf{w}^t, \xi^t) - F(\mathbf{x}^t, \xi^t))\mathbf{w}^t$ which essentially suffers from the quadratic dependence in $d > 0$. It is also necessary to employ a random iteration count R to terminate Algorithm 3.

Algorithm 3 Stochastic Gradient-Free Method (SGFM)

- 1: **Input:** initial point $\mathbf{x}^0 \in \mathbb{R}^d$, stepsize $\eta > 0$, problem dimension $d \geq 1$, smoothing parameter δ and iteration number $T \geq 1$
 - 2: **for** $t = 0, 1, 2, \dots, T$ **do**
 - 3: Simulate ξ^t from the distribution \mathbb{P}_μ
 - 4: Sample $\mathbf{w}^t \in \mathbb{R}^d$ uniformly from a unit sphere in \mathbb{R}^d
 - 5: Compute $\hat{\mathbf{g}}^t = \frac{d}{2\delta}(F(\mathbf{x}^t + \delta\mathbf{w}^t, \xi^t) - F(\mathbf{x}^t - \delta\mathbf{w}^t, \xi^t))\mathbf{w}^t$
 - 6: Compute $\mathbf{x}^{t+1} = \mathbf{x}^t - \eta\hat{\mathbf{g}}^t$
 - 7: **end for**
 - 8: **Output:** \mathbf{x}^R where $R \in \{0, 1, 2, \dots, T-1\}$ is uniformly sampled
-

Theorem 4. Suppose that $F(\cdot, \xi)$ is $L(\xi)$ -Lipschitz with $\mathbb{E}_{\xi \in \mathbb{P}_\mu}[L^2(\xi)] \leq G^2$ for some $G > 0$ and let $\delta > 0$ and $0 < \epsilon < 1$. Then, there exists some $T > 0$ such that the output of Algorithm 3 with $\eta = \frac{1}{10}\sqrt{\frac{\delta(\Delta+\delta G)}{cd^{3/2}G^3T}}$ satisfies that $\mathbb{E}[\min\{\|\mathbf{g}\| : \mathbf{g} \in \partial_\delta f(\mathbf{x}^R)\}] \leq \epsilon$ and the total number of calls of the noisy function value oracle is bounded by

$$O\left(d^{\frac{3}{2}}\left(\frac{G^4}{\epsilon^4} + \frac{\Delta G^3}{\delta \epsilon^4}\right)\right)$$

where $d \geq 1$ is the problem dimension, $L > 0$ is the Lipschitz parameter of f and $\Delta > 0$ is an upper bound for the initial objective function gap, $f(\mathbf{x}^0) - \inf_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x}) > 0$.

In the stochastic setting, the gradient-based method achieves the rate of $O(\delta^{-1}\epsilon^{-4})$ for a randomized gradient-based method in terms of (δ, ϵ) -Goldstein stationarity [Davis et al.(2022)]. As such, our bound in Theorem 4 is tight up to the problem dimension $d \geq 1$. Further, the state-of-the-art rate for stochastic gradient-free methods is $O(d\epsilon^{-2})$ for nonsmooth convex optimization in terms of objective function value gap [Duchi et al.(2015)] and $O(d\epsilon^{-4})$ for smooth nonconvex optimization in terms of gradient norm [Ghadimi and Lan(2013)]. Thus, Theorem 4 demonstrates that nonsmooth nonconvex stochastic optimization is essentially the most difficult one among than all these standard settings.

Large-deviation estimation. As in the case of GFM, we hope to establish a complexity bound of SGFM for computing a $(\delta, \epsilon, \Lambda)$ -solution. By Theorem 4 and Markov's inequality, we obtain that the total number of calls of the noisy function value oracle is bounded by

$$O\left(d^{\frac{3}{2}}\left(\frac{G^4}{\Lambda^4 \epsilon^4} + \frac{\Delta G^3}{\delta \Lambda^4 \epsilon^4}\right)\right) \quad (15)$$

We also propose a two-phase stochastic gradient-free method (2-SGFM) in Algorithm 4 by combining Algorithm 3 with a post-optimization procedure.

Theorem 5. Suppose that $F(\cdot, \xi)$ is $L(\xi)$ -Lipschitz with $\mathbb{E}_{\xi \in \mathbb{P}_\mu}[L^2(\xi)] \leq G^2$ for some $G > 0$ and let $\delta > 0$ and $0 < \epsilon, \Lambda < 1$. Then, there exists some $T, S, B > 0$ such that the output of Algorithm 4 with $\eta = \frac{1}{10}\sqrt{\frac{\delta(\Delta+\delta G)}{cd^{3/2}G^3T}}$ satisfies that $\text{Prob}(\min\{\|\mathbf{g}\| : \mathbf{g} \in \partial_\delta f(\bar{\mathbf{x}}_{s^*})\} \geq \epsilon) \leq \Lambda$ and the total number of calls of the noisy function value oracle is bounded by

$$O\left(d^{\frac{3}{2}}\left(\frac{G^4}{\epsilon^4} + \frac{\Delta G^3}{\delta \epsilon^4}\right)\log_2\left(\frac{1}{\Lambda}\right) + \frac{dG^2}{\Lambda \epsilon^2}\log_2\left(\frac{1}{\Lambda}\right)\right)$$

Algorithm 4 Two-Phase Stochastic Gradient-Free Method (2-SGFM)

```
1: Input: initial point  $\mathbf{x}^0 \in \mathbb{R}^d$ , stepsize  $\eta > 0$ , problem dimension  $d \geq 1$ , smoothing parameter  $\delta$ , iteration number  $T \geq 1$ , number of rounds  $S \geq 1$  and sample size  $B$ 
2: for  $s = 0, 1, 2, \dots, S - 1$  do
3:   Call Algorithm 3 with  $\mathbf{x}^0$ ,  $\eta$ ,  $d$ ,  $\delta$  and  $T$  and let  $\bar{\mathbf{x}}_s$  be an output
4: end for
5: for  $s = 0, 1, 2, \dots, S - 1$  do
6:   for  $k = 0, 1, 2, \dots, B - 1$  do
7:     Simulate  $\xi^k$  from the distribution  $\mathbb{P}_\mu$ 
8:     Sample  $\mathbf{w}^k \in \mathbb{R}^d$  uniformly from a unit sphere in  $\mathbb{R}^d$ 
9:     Compute  $\hat{\mathbf{g}}_s^k = \frac{d}{2\delta}(F(\bar{\mathbf{x}}_s + \delta\mathbf{w}^k, \delta^k) - F(\bar{\mathbf{x}}_s - \delta\mathbf{w}^k, \delta^k))\mathbf{w}^k$ 
10:   end for
11:   Compute  $\hat{\mathbf{g}}_s = \frac{1}{B} \sum_{k=0}^{B-1} \hat{\mathbf{g}}_s^k$ 
12: end for
13: Choose an index  $s^* \in \{0, 1, 2, \dots, S - 1\}$  such that  $s^* = \operatorname{argmin}_{s=0,1,2,\dots,S-1} \|\hat{\mathbf{g}}_s\|$ 
14: Output:  $\bar{\mathbf{x}}_{s^*}$ 
```

where $d \geq 1$ is the problem dimension, $L > 0$ is the Lipschitz parameter of f and $\Delta > 0$ is an upper bound for the initial objective function gap $f(\mathbf{x}^0) - \inf_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x}) > 0$.

Further discussions. We remark that the choice of stepsize η in all of our zeroth-order methods depend on Δ , whereas such dependence is not necessary in the first-order setting; see e.g., [Zhang et al.(2020)]. Setting the stepsize without any prior knowledge of Δ , our methods can still achieve finite-time convergence guarantees but the order would become worse. This is possibly because the first-order information gives more characterization of the objective function than the zeroth-order information, so that for first-order methods the stepsize can be independent of more problem parameters without sacrificing the bound. A bit on the positive side is that, it suffices for our zeroth-order methods to know an estimate of the upper bound of $\Theta(\Delta)$, which can be done in certain application problems.

Moreover, we highlight that $\delta > 0$ is the desired tolerance in our setting. In fact, (δ, ϵ) -Goldstein stationarity (see Definition 3) relaxes ϵ -Clarke stationarity and our methods pursue an (δ, ϵ) -stationary point since finding an ϵ -Clarke point is intractable. This is different from smooth optimization where ϵ -Clarke stationarity reduces to $\nabla f(\mathbf{x}) \leq \epsilon$ and becomes tractable. In this context, the existing zeroth-order methods are designed to pursue an ϵ -stationary point. Notably, a (δ, ϵ) -Goldstein stationary point is provably an ϵ -stationary point in smooth optimization if we choose δ that relies on d and ϵ .

6.1 Proof of Theorem 4

Till the rest of this section, we give the proofs of Theorem 4 and 5. We also present some technical lemmas for analyzing the convergence property of stochastic gradient-free method and its two-phase version.

We provide two technical lemmas for analyzing Algorithm 3. The first lemma gives an upper bound on the quantity $\mathbb{E}[\|\hat{\mathbf{g}}^t\|^2 | \mathbf{x}^t]$ in terms of problem dimension $d \geq 1$ and the constant $G > 0$. The proof is based on a modification of the proof of Lemma 1.

Lemma 3. Suppose that $\{\hat{\mathbf{g}}^t\}_{t=0}^{T-1}$ and $\{\mathbf{x}^t\}_{t=0}^{T-1}$ are generated by Algorithm 3. Then, we have $\mathbb{E}[\hat{\mathbf{g}}^t | \mathbf{x}^t] = \nabla f_\delta(\mathbf{x}^t)$ and $\mathbb{E}[\|\hat{\mathbf{g}}^t\|^2 | \mathbf{x}^t] \leq 16\sqrt{2\pi}dG^2$.

The second lemma gives a key descent inequality for analyzing Algorithm 3.

Lemma 4. Suppose that $\{\mathbf{x}^t\}_{t=0}^{T-1}$ are generated by Algorithm 3. Then, we have

$$\mathbb{E}[\|\nabla f_\delta(\mathbf{x}^t)\|^2] \leq \frac{\mathbb{E}[f_\delta(\mathbf{x}^t)] - \mathbb{E}[f_\delta(\mathbf{x}^{t+1})]}{\eta} + \eta \cdot \frac{(8\sqrt{2\pi})cd^{3/2}G^3}{\delta}, \quad \text{for all } 0 \leq t \leq T-1$$

We are ready to present the

Proof of Theorem 4. Summing up the inequality in Lemma 4 over $t = 0, 1, 2, \dots, T-1$ yields that

$$\sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f_\delta(\mathbf{x}^t)\|^2] \leq \frac{f_\delta(\mathbf{x}^0) - \mathbb{E}[f_\delta(\mathbf{x}^T)]}{\eta} + \eta \cdot \frac{(8\sqrt{2\pi})cd^{3/2}G^3T}{\delta}$$

Since $f(\cdot) = \mathbb{E}_{\xi \in \mathbb{P}_\mu}[F(\cdot, \xi)]$ and $F(\cdot, \xi)$ is $L(\xi)$ -Lipschitz with $\mathbb{E}_{\xi \in \mathbb{P}_\mu}[L^2(\xi)] \leq G^2$ for some $G > 0$, we have f is G -Lipschitz. Thus, we have $f \in \mathcal{F}_d(\Delta, L)$. By using the same argument as used in the proof of Theorem 2, we have

$$\frac{1}{T} \left(\sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f_\delta(\mathbf{x}^t)\|^2] \right) \leq \frac{\Delta + \delta G}{\eta T} + \eta \cdot \frac{(8\sqrt{2\pi})cd^{3/2}G^3}{\delta} \leq \frac{\Delta + \delta G}{\eta T} + \eta \cdot \frac{100cd^{3/2}G^3}{\delta}$$

Recalling that $\eta = \frac{1}{10} \sqrt{\frac{\delta(\Delta + \delta G)}{cd^{3/2}G^3T}}$, we have

$$\frac{1}{T} \left(\sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f_\delta(\mathbf{x}^t)\|^2] \right) \leq 20 \sqrt{\frac{cd^{3/2}G^3}{T}} \left(G + \frac{\Delta}{\delta} \right)$$

Since the random count $R \in \{0, 1, 2, \dots, T-1\}$ is uniformly sampled, we have

$$\mathbb{E}[\|\nabla f_\delta(\mathbf{x}^R)\|^2] = \frac{1}{T} \left(\sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f_\delta(\mathbf{x}^t)\|^2] \right) \leq 20 \sqrt{\frac{cd^{3/2}G^3}{T}} \left(G + \frac{\Delta}{\delta} \right) \quad (16)$$

By Theorem 1, we have $\nabla f_\delta(\mathbf{x}^R) \in \partial_\delta f(\mathbf{x}^R)$. This together with the above inequality implies that

$$\mathbb{E}[\min\{\|\mathbf{g}\| : \mathbf{g} \in \partial_\delta f(\mathbf{x}^R)\}] \leq \mathbb{E}[\|\nabla f_\delta(\mathbf{x}^R)\|] \leq 5 \left(\frac{cd^{3/2}G^3}{T} \left(G + \frac{\Delta}{\delta} \right) \right)^{\frac{1}{4}}$$

Therefore, we conclude that there exists some $T > 0$ such that the output of Algorithm 3 satisfies that $\mathbb{E}[\min\{\|\mathbf{g}\| : \mathbf{g} \in \partial_\delta f(\mathbf{x}^R)\}] \leq \epsilon$ and the total number of calling the function value oracles is bounded by

$$O \left(d^{\frac{3}{2}} \left(\frac{G^4}{\epsilon^4} + \frac{\Delta G^3}{\delta \epsilon^4} \right) \right)$$

This completes the proof. \square

6.2 Proof of Theorem 5

Proof of Theorem 5. By the definition of s^* and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|\hat{\mathbf{g}}_{s^*}\|^2 &= \min_{s=0,1,2,\dots,S-1} \|\hat{\mathbf{g}}_s\|^2 \leq \min_{s=0,1,2,\dots,S-1} \{2\|\nabla f_\delta(\bar{\mathbf{x}}_s)\|^2 + 2\|\hat{\mathbf{g}}_s - \nabla f_\delta(\bar{\mathbf{x}}_s)\|^2\} \\ &\leq 2 \left(\min_{s=0,1,2,\dots,S-1} \|\nabla f_\delta(\bar{\mathbf{x}}_s)\|^2 + \max_{s=0,1,2,\dots,S-1} \|\hat{\mathbf{g}}_s - \nabla f_\delta(\bar{\mathbf{x}}_s)\|^2 \right).\end{aligned}\quad (17)$$

This implies that

$$\begin{aligned}\|\nabla f_\delta(\bar{\mathbf{x}}_{s^*})\|^2 &\leq 2\|\hat{\mathbf{g}}_{s^*}\|^2 + 2\|\hat{\mathbf{g}}_{s^*} - \nabla f_\delta(\bar{\mathbf{x}}_{s^*})\|^2 \\ &\stackrel{\text{Eq. (17)}}{\leq} 4 \left(\min_{s=0,1,2,\dots,S-1} \|\nabla f_\delta(\bar{\mathbf{x}}_s)\|^2 \right) + 4 \left(\max_{s=0,1,2,\dots,S-1} \|\hat{\mathbf{g}}_s - \nabla f_\delta(\bar{\mathbf{x}}_s)\|^2 \right) + 2\|\hat{\mathbf{g}}_{s^*} - \nabla f_\delta(\bar{\mathbf{x}}_{s^*})\|^2.\end{aligned}\quad (18)$$

The next step is to provide the probabilistic bounds on all the terms in the right-hand side of Eq. (18). In particular, for each $s = 0, 1, 2, \dots, S-1$, we have $\bar{\mathbf{x}}_s$ is an output obtained by calling Algorithm 3 with \mathbf{x}^0 , d , δ , T and $\eta = \frac{1}{10} \sqrt{\frac{\delta(\Delta+\delta G)}{cd^{3/2}G^3T}}$. Then, Eq. (16) in the proof of Theorem 4 implies that

$$\mathbb{E}[\|\nabla f_\delta(\bar{\mathbf{x}}_s)\|^2] \leq 20 \sqrt{\frac{cd^{3/2}G^3}{T}} \left(G + \frac{\Delta}{\delta}\right)$$

Using the Markov's inequality, we have

$$\text{Prob} \left(\|\nabla f_\delta(\bar{\mathbf{x}}_s)\|^2 \geq 40 \sqrt{\frac{cd^{3/2}G^3}{T}} \left(G + \frac{\Delta}{\delta}\right) \right) \leq \frac{1}{2}$$

Thus, we have

$$\text{Prob} \left(\min_{s=0,1,2,\dots,S-1} \|\nabla f_\delta(\bar{\mathbf{x}}_s)\|^2 \geq 40 \sqrt{\frac{cd^{3/2}G^3}{T}} \left(G + \frac{\Delta}{\delta}\right) \right) \leq 2^{-S} \quad (19)$$

Further, for each $s = 0, 1, 2, \dots, S-1$, we have

$$\hat{\mathbf{g}}_s - \nabla f_\delta(\bar{\mathbf{x}}_s) = \frac{1}{B} \sum_{k=0}^{B-1} (\hat{\mathbf{g}}_s^k - \nabla f_\delta(\bar{\mathbf{x}}_s))$$

By Lemma 3, we have $\mathbb{E}[\hat{\mathbf{g}}_s^t | \bar{\mathbf{x}}_s] = \nabla f_\delta(\bar{\mathbf{x}}_s)$ and $\mathbb{E}[\|\hat{\mathbf{g}}_s^t\|^2 | \bar{\mathbf{x}}_s] \leq 16\sqrt{2\pi}dG^2$. This implies that

$$\mathbb{E}[\hat{\mathbf{g}}_s^t - \nabla f_\delta(\bar{\mathbf{x}}_s) | \bar{\mathbf{x}}_s] = 0, \quad \mathbb{E}[\|\hat{\mathbf{g}}_s^t - \nabla f_\delta(\bar{\mathbf{x}}_s)\|^2] \leq 16\sqrt{2\pi}dG^2$$

This together with Proposition 3 yields that

$$\text{Prob} \left(\|\hat{\mathbf{g}}_s - \nabla f_\delta(\bar{\mathbf{x}}_s)\|^2 \geq \frac{\lambda(16\sqrt{2\pi}dG^2)}{B} \right) = \text{Prob} \left(\left\| \sum_{k=0}^{B-1} (\hat{\mathbf{g}}_s^k - \nabla f_\delta(\bar{\mathbf{x}}_s)) \right\|^2 \geq \lambda B(16\sqrt{2\pi}dG^2) \right) \leq \frac{1}{\lambda}$$

Therefore, we conclude that

$$\text{Prob} \left(\max_{s=0,1,2,\dots,S-1} \|\hat{\mathbf{g}}_s - \nabla f_\delta(\bar{\mathbf{x}}_s)\|^2 \geq \frac{\lambda(16\sqrt{2\pi}dG^2)}{B} \right) \leq \frac{S}{\lambda} \quad (20)$$

By the similar argument, we have

$$\text{Prob}(\|\hat{\mathbf{g}}_{s^*} - \nabla f_\delta(\bar{\mathbf{x}}_{s^*})\|^2 \geq \frac{\lambda(16\sqrt{2\pi}dG^2)}{B}) \leq \frac{1}{\lambda} \quad (21)$$

Combining Eq. (18), Eq. (19), Eq. (20) and Eq. (21) yields that

$$\text{Prob}\left(\|\nabla f_\delta(\bar{\mathbf{x}}_{s^*})\|^2 \geq 160\sqrt{\frac{cd^{3/2}G^3}{T}(G + \frac{\Delta}{\delta})} + \frac{\lambda(96\sqrt{2\pi}dG^2)}{B}\right) \leq \frac{S+1}{\lambda} + 2^{-S}, \quad \text{for all } \lambda > 0 \quad (22)$$

We set $\lambda = \frac{2(S+1)}{\Lambda}$ and the parameters (T, S, B) as follows,

$$T = cd^{3/2}G^3(G + \frac{\Delta}{\delta})(\frac{160}{\epsilon^2})^2, \quad S = \lceil \log_2(\frac{2}{\Lambda}) \rceil, \quad B = \frac{(384\sqrt{2\pi}dG^2)(S+1)}{\Lambda\epsilon^2}$$

Then, we have

$$\text{Prob}(\|\nabla f_\delta(\bar{\mathbf{x}}_{s^*})\|^2 \geq \epsilon^2) \leq \text{Prob}\left(\|\nabla f_\delta(\bar{\mathbf{x}}_{s^*})\|^2 \geq 160\sqrt{\frac{cd^{3/2}G^3}{T}(G + \frac{\Delta}{\delta})} + \frac{\lambda(96\sqrt{2\pi}dG^2)}{B}\right) \leq \Lambda$$

By Theorem 1, we have $\nabla f_\delta(\bar{\mathbf{x}}_{s^*}) \in \partial_\delta f(\bar{\mathbf{x}}_{s^*})$. This together with the above inequality implies that there exists some $T, S, B > 0$ such that the output of Algorithm 4 satisfies that $\mathbb{E}[\min\{\|\mathbf{g}\| : \mathbf{g} \in \partial_\delta f(\bar{\mathbf{x}}_{s^*})\}] \leq \epsilon$ and the total number of calling the function value oracles is bounded by

$$O\left(d^{\frac{3}{2}}\left(\frac{G^4}{\epsilon^4} + \frac{\Delta G^3}{\delta \epsilon^4}\right) \log_2\left(\frac{1}{\Lambda}\right) + \frac{dG^2}{\Lambda \epsilon^2} \log_2\left(\frac{1}{\Lambda}\right)\right)$$

This completes the proof. \square

6.3 Proofs of technical lemmas

Proof of Lemma 3. By the definition of $\hat{\mathbf{g}}^t$ and the symmetry of the distribution of \mathbf{w}^t , we have

$$\begin{aligned} \mathbb{E}[\hat{\mathbf{g}}^t | \mathbf{x}^t] &= \mathbb{E}\left[\frac{d}{2\delta}(F(\mathbf{x}^t + \delta\mathbf{w}^t, \xi^t) - F(\mathbf{x}^t - \delta\mathbf{w}^t, \xi^t))\mathbf{w}^t | \mathbf{x}^t\right] \\ &= \frac{1}{2}\left(\mathbb{E}\left[\frac{d}{\delta}F(\mathbf{x}^t + \delta\mathbf{w}^t, \xi^t)\mathbf{w}^t | \mathbf{x}^t\right] + \mathbb{E}\left[\frac{d}{\delta}F(\mathbf{x}^t + \delta(-\mathbf{w}^t), \xi^t)(-\mathbf{w}^t) | \mathbf{x}^t\right]\right) \\ &= \mathbb{E}\left[\frac{d}{\delta}F(\mathbf{x}^t + \delta\mathbf{w}^t, \xi^t)\mathbf{w}^t | \mathbf{x}^t\right] \end{aligned}$$

By the tower property, we have

$$\mathbb{E}[\hat{\mathbf{g}}^t | \mathbf{x}^t] = \mathbb{E}\left[\frac{d}{\delta}\mathbb{E}[F(\mathbf{x}^t + \delta\mathbf{w}^t, \xi^t)\mathbf{w}^t | \mathbf{x}^t, \mathbf{w}^t] | \mathbf{x}^t\right] = \mathbb{E}\left[\frac{d}{\delta}f(\mathbf{x}^t + \delta\mathbf{w}^t)\mathbf{w}^t | \mathbf{x}^t\right] = \nabla f_\delta(\mathbf{x}^t)$$

It remains to show that $\mathbb{E}[\|\hat{\mathbf{g}}^t\|^2 | \mathbf{x}^t] \leq 16\sqrt{2\pi}dG^2$. Indeed, by using the same argument as used in the proof of Lemma 1, we have

$$\mathbb{E}[\|\hat{\mathbf{g}}^t\|^2 | \mathbf{x}^t] \leq \frac{d^2}{\delta^2}\mathbb{E}[(F(\mathbf{x}^t + \delta\mathbf{w}^t, \xi^t) - \mathbb{E}[F(\mathbf{x}^t + \delta\mathbf{w}^t, \xi^t) | \mathbf{x}^t, \xi^t])^2 | \mathbf{x}^t] \quad (23)$$

For simplicity, we let $h(\mathbf{w}) = F(\mathbf{x}^t + \delta \mathbf{w}, \xi^t)$. Since $F(\cdot, \xi)$ is $L(\xi)$ -Lipschitz, this function h is $\delta L(\xi^t)$ -Lipschitz given a fixed \mathbf{x}^t and ξ^t . In addition, $\mathbf{w}^t \in \mathbb{R}^d$ is sampled uniformly from a unit sphere. Then, by [Wainwright(2019), Proposition 3.11 and Example 3.12], we have

$$\mathbb{P}(|h(\mathbf{w}^t) - \mathbb{E}[h(\mathbf{w}^t)]| \geq \alpha) \leq 2\sqrt{2\pi}e^{-\frac{\alpha^2 d}{8\delta^2 L(\xi^t)^2}}$$

Then, we have

$$\begin{aligned} \mathbb{E}[(h(\mathbf{w}^t) - \mathbb{E}[h(\mathbf{w}^t)])^2] &= \int_0^{+\infty} \mathbb{P}((h(\mathbf{w}^t) - \mathbb{E}[h(\mathbf{w}^t)])^2 \geq \alpha) d\alpha \\ &= \int_0^{+\infty} \mathbb{P}(|h(\mathbf{w}^t) - \mathbb{E}[h(\mathbf{w}^t)]| \geq \sqrt{\alpha}) d\alpha \leq 2\sqrt{2\pi} \int_0^{+\infty} e^{-\frac{\alpha d}{8\delta^2 L(\xi^t)^2}} d\alpha \\ &= 2\sqrt{2\pi} \cdot \frac{8\delta^2 L(\xi^t)^2}{d} = \frac{16\sqrt{2\pi}\delta^2 L(\xi^t)^2}{d} \end{aligned}$$

By the definition of h , we have

$$\mathbb{E}[(F(\mathbf{x}^t + \delta \mathbf{w}^t, \xi^t) - \mathbb{E}[F(\mathbf{x}^t + \delta \mathbf{w}^t, \xi^t) | \mathbf{x}^t, \xi^t])^2 | \mathbf{x}^t] \leq \frac{16\sqrt{2\pi}\delta^2}{d} \mathbb{E}[L(\xi^t)^2]$$

Since ξ^t is simulated from the distribution \mathbb{P}_μ , we have $\mathbb{E}[L(\xi^t)^2] \leq G^2$. Plugging this into the above inequality, we have

$$\mathbb{E}[(F(\mathbf{x}^t + \delta \mathbf{w}^t, \xi^t) - \mathbb{E}[F(\mathbf{x}^t + \delta \mathbf{w}^t, \xi^t) | \mathbf{x}^t, \xi^t])^2 | \mathbf{x}^t] \leq \frac{16\sqrt{2\pi}\delta^2 G^2}{d} \quad (24)$$

Combining Eq. (23) and Eq. (24) yields the desired inequality. \square

Proof of Lemma 4. Since $f(\cdot) = \mathbb{E}_{\xi \in \mathbb{P}_\mu}[F(\cdot, \xi)]$ and $F(\cdot, \xi)$ is $L(\xi)$ -Lipschitz with $\mathbb{E}_{\xi \in \mathbb{P}_\mu}[L^2(\xi)] \leq G^2$ for some $G > 0$, we have f is G -Lipschitz. Then, by Proposition 2, we have f_δ is differentiable with the $\frac{cG\sqrt{d}}{\delta}$ -Lipschitz gradient where $c > 0$ is a constant. This implies that

$$f_\delta(\mathbf{x}^{t+1}) \leq f_\delta(\mathbf{x}^t) - \eta \langle \nabla f_\delta(\mathbf{x}^t), \hat{\mathbf{g}}^t \rangle + \frac{c\eta^2 G\sqrt{d}}{2\delta} \|\hat{\mathbf{g}}^t\|^2$$

Taking the expectation of both sides conditioned on \mathbf{x}^t and using Lemma 3, we have

$$\begin{aligned} \mathbb{E}[f_\delta(\mathbf{x}^{t+1}) | \mathbf{x}^t] &\leq f_\delta(\mathbf{x}^t) - \eta \langle \nabla f_\delta(\mathbf{x}^t), \mathbb{E}[\hat{\mathbf{g}}^t | \mathbf{x}^t] \rangle + \frac{c\eta^2 G\sqrt{d}}{2\delta} \mathbb{E}[\|\hat{\mathbf{g}}^t\|^2 | \mathbf{x}^t] \\ &\leq f_\delta(\mathbf{x}^t) - \eta \|\nabla f_\delta(\mathbf{x}^t)\|^2 + \eta^2 \cdot \frac{cG\sqrt{d}}{2\delta} \cdot 16\sqrt{2\pi}dG^2 \\ &= f_\delta(\mathbf{x}^t) - \eta \|\nabla f_\delta(\mathbf{x}^t)\|^2 + \eta^2 \cdot \frac{(8\sqrt{2\pi})cd^{3/2}G^3}{\delta} \end{aligned}$$

Taking the expectation of both sides and rearranging yields that

$$\mathbb{E}[\|\nabla f_\delta(\mathbf{x}^t)\|^2] \leq \frac{\mathbb{E}[f_\delta(\mathbf{x}^t)] - \mathbb{E}[f_\delta(\mathbf{x}^{t+1})]}{\eta} + \eta \cdot \frac{(8\sqrt{2\pi})cd^{3/2}G^3}{\delta}$$

This completes the proof. \square

7 Conclusion

We proposed and analyzed a class of deterministic and stochastic gradient-free methods for optimizing a Lipschitz function. Based on the relationship between the Goldstein subdifferential and uniform smoothing that we have established, the proposed GFM and SGFM are proved to return a (δ, ϵ) -Goldstein stationary point at an expected rate of $O(d^{3/2}\delta^{-1}\epsilon^{-4})$. We obtain a large-deviation guarantee and improve it by combining GFM and SGFM with a two-phase scheme. Future directions include the theory for non-Lipschitz and nonconvex optimization [Bian et al.(2015)].

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