Unified Spectral Analysis and Accelerated Convergence in Multi-Player Differentiable Games

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Abstract

In recent years, gradient-based optimization methods have become central to machine learning, particularly in complex frameworks such as multi-agent systems, generative adversarial networks (GANs), and reinforcement learning. These settings often involve multi-player games, where each player optimizes its own objective, leading to challenges such as rotational dynamics and slow convergence in standard methods like gradient descent (GD). This paper presents a unified spectral and convergence analysis of three key gradient-based algorithms—Gradient Descent (GD), Extragradient (EG), and Optimistic Gradient (OG)—in the context of differentiable games. By leveraging spectral techniques, we offer a comprehensive analysis that encompasses bilinear, strongly monotone, and intermediate game structures. Our results establish global and local convergence guarantees with tighter rates than previous studies, while introducing a novel geometric framework to extend momentum-based acceleration techniques to games with complex eigenvalue distributions. These findings enhance the theoretical understanding of optimization in multi-agent systems and provide practical insights for improving convergence in adversarial and cooperative games.

Keywords: Multi-player differentiable games, gradient-based optimization, Nash equilibrium, spectral analysis, extragradient method, optimistic gradient, convergence rates

1 Introduction

Optimization algorithms, particularly gradient-based methods, have been crucial to the advancement of machine learning, particularly in areas like neural network training, generative modeling, and reinforcement learning. Traditionally, these methods were designed for single-objective convex optimization. However, as machine learning frameworks evolve, many modern applications involve multi-agent systems where multiple interacting players aim to optimize their own objectives. These problems, often modeled as differentiable games, arise in areas such as generative adversarial networks (GANs), multi-agent reinforcement learning, and actor-critic models.

In multi-player differentiable games, each player's strategy affects the others, making the optimization task more complex than standard single-objective problems. The goal is often to reach a Nash equilibrium, where no player can unilaterally improve their outcome. However, these games introduce unique challenges, particularly in terms of dynamics. Standard optimization algorithms like gradient descent (GD) struggle to converge due to rotational dynamics, leading to oscillations rather than convergence, as seen in bilinear game examples.

To address these issues, alternative algorithms such as the extragradient (EG) and optimistic gradient (OG) methods have been developed. These methods adjust the gradient updates to handle adversarial and cooperative interactions between players, improving convergence behavior. However, despite these advances, the performance of these methods across the full spectrum of game types, from bilinear to strongly monotone, remains inadequately explored. Much of the existing literature focuses on specific cases, leaving gaps in a unified understanding of these algorithms in more general game settings.

This paper seeks to bridge this gap by providing a comprehensive spectral analysis of gradient-based methods in differentiable games. We analyze the convergence properties of GD, EG, and OG, offering both local and global convergence guarantees. Our work introduces a novel geometric framework to examine the spectral shape of smooth games, revealing how the distribution of eigenvalues influences the difficulty of the game and the potential for acceleration. By leveraging matrix iteration theory, we offer insights into how these methods can be accelerated, particularly in games dominated by bilinear structures or with small imaginary perturbations.

Contributions Our key contributions are as follows:

- We provide a unified spectral analysis of gradient-based methods, encompassing bilinear, strongly monotone, and intermediate game structures. This analysis reveals improved convergence rates for the extragradient (EG) and optimistic gradient (OG) methods.
- We introduce a geometric framework for understanding game conditioning via the concept of *spectral shape*, linking the difficulty of a game to the eigenvalue distribution of the Jacobian. This approach enables us to design an optimal algorithm for bilinear games and derive new lower bounds, demonstrating the near-optimality of EG among first-order methods.
- We establish global convergence guarantees for GD, EG, and OG, and propose an accelerated EG method for bilinear games with imaginary perturbations. We also enhance consensus optimization by incorporating momentum, achieving near-accelerated convergence rates.

Organization The remainder of the paper is organized as follows: Section 2 discusses related work on gradient-based methods in differentiable games and their convergence properties. Section 3 presents the theoretical framework for analyzing gradient-based methods using spectral techniques. Section 4 provides a detailed spectral analysis of extragradient and optimistic gradient methods, with a comparison to gradient descent. In Section 5, we offer global convergence results and prove new lower bounds for the considered methods. Finally, Section 6 concludes the paper and discusses potential future research directions.

2 Unified Convergence and Spectral Analysis of Gradient-Based Methods in Differentiable Multi-Player Games

An increasing number of frameworks rely on optimization problems that involve multiple players and objectives. For instance, actor-critic models [PV16], generative adversarial networks (GANs) [GPM$^+$14] and automatic curricula [SKSF18] can be cast as two-player games.

Hence games are a generalization of the standard single-objective framework. The aim of the optimization is to find *Nash equilibria*, that is to say situations where no player can unilaterally decrease their loss. However, new issues that were not present for single-objective problems arise. The presence of rotational dynamics prevent standard algorithms such as the gradient method to converge on simple bilinear examples [Goo16, BRM$^+$18]. Furthermore, stationary points of the gradient dynamics are not necessarily Nash equilibria [ADLH18, MJS19].

	[Tse95]	[GBV+19]	[MOP19]	[ALW19]	This work §2.4
EG	$c\frac{\mu^2}{L^2}$	-	$\frac{\mu}{4L}$	-	$\frac{1}{4}\left(\frac{\mu}{L}+\frac{\gamma^2}{16L^2}\right)$
OG	-	$\frac{\mu}{4L}$	$\frac{\mu}{4L}$	-	$\frac{1}{4}\left(\frac{\mu}{L}+\frac{\gamma^2}{32L^2}\right)$
CO	-	-	-	$\frac{\gamma^2}{4L_H^2}$	$\frac{\mu^2}{2L_H^2}+\frac{\gamma^2}{2L_H^2}$

Table 1. Summary of the global convergence results presented in §2.4 for extragradient (EG), optimistic gradient (OMD) and consensus optimization (CO) methods. If a result shows that the iterates converge as $\mathcal{O}((1-r)^t)$, the quantity r is reported (the larger the better). The letter c indicates that the numerical constant was not reported by the authors. μ is the strong monotonicity of the vector field, γ is a global lower bound on the singular values of ∇v , L is the Lipschitz constant of the vector field and L_H^2 the Lipschitz-smoothness of $\frac{1}{2}\|v\|_2^2$. For instance, for the so-called bilinear example (Ex. 1), we have $\mu = 0$ and $\gamma = \sigma_{\min}(A)$. Note that for this particular example, previous papers developed a specific analysis that breaks when a small regularization is added (see Ex. 3).

Some recent progress has been made by introducing new methods specifically designed with games or variational inequalities in mind. The main example are the optimistic gradient method (OG) introduced by [Rakhlin and Sridharan(2013)] initially for online learning, consensus optimization (CO) which adds a regularization term to the optimization problem and the extragradient method (EG) originally introduced by [Kor76]. Though these news methods and the gradient method (GD) have similar performance in convex optimization, their behaviour seems to differ when applied to games: unlike gradient, they converge on the so-called bilinear example [Tse95, GBV+19, MOP19, ALW19].

However, linear convergence results for EG and OG (a.k.a extrapolation from the past) in particular have only been proven for either strongly monotone variational inequalities problems, which include strongly convex-concave saddle point problems, or in the bilinear setting separately [Tse95, GBV+19, MOP19].

In this section, we study the dynamics of such gradient-based methods and in particular GD, EG and more generally multi-step extrapolations methods for unconstrained games. Our objective is three-fold. First, taking inspiration from the analysis of GD by [GHP+19], we aim at providing a single precise analysis of EG which covers both the bilinear and the strongly monotone settings and their intermediate cases. Second, we are interested in theoretically comparing EG to GD and general multi-step extrapolations through upper and lower bounds on convergence rates. Third, we provide a framework to extend the unifying results of spectral analysis in global guarantees and leverage it to prove tighter convergence rates for OG and CO.

The remainder of this section is organized as follows: In Section 2, we discuss related work on gradient-based methods in differentiable games and their convergence properties. Section 3 presents the theoretical framework for analyzing gradient-based methods using spectral techniques. Section 4 provides a detailed spectral analysis of extragradient and optimistic gradient methods, with a comparison to gradient descent. In Section 5, we offer global convergence results and prove new lower bounds for the considered methods. Finally, Section 6 concludes the paper and discusses potential future research directions.

2.1 Background and motivation

2.1.1 n-player differentiable games

Following [BRM+18], a *n-player differentiable game* can be defined as a family of twice continuously differentiable losses $l_i : \mathbb{R}^d \to \mathbb{R}$ for $i = 1, \ldots, n$. The parameters for player i are $\omega^i \in \mathbb{R}^{d_i}$ and we note $\omega = (\omega^1, \ldots, \omega^n) \in \mathbb{R}^d$ with $d = \sum_{i=1}^n d_i$. Ideally, we are interested in finding an *unconstrained Nash equilibrium* [VNM44]: that is to say a point $\omega^* \in \mathbb{R}^d$ such that

$$\forall i \in \{1, \ldots, n\}, \quad (\omega^i)^* \in \operatorname*{arg\,min}_{\omega^i \in \mathbb{R}^{d_i}} l_i((\omega^{-i})^*, \omega^i)$$

where the vector $(\omega^{-i})^*$ contains all the coordinates of ω^* except the i^{th} one. Moreover, we say that a game is *zero-sum* if $\sum_{i=1}^n l_i = 0$. For instance, following [Mescheder et al.(2017), GHP+19], the standard formulation of GANs from [GPM+14] can be cast as a two-player zero-sum game. The Nash equilibrium corresponds to the desired situation where the generator exactly capture the data distribution, completely confusing a perfect discriminator.

Let us now define the *vector field*

$$v(\omega) = \left(\nabla_{\omega^1} l_1(\omega) \ , \quad \cdots \quad , \ \nabla_{\omega^n} l_n(\omega)\right)$$

associated to a n-player game and its Jacobian:

$$\nabla v(\omega) = \begin{pmatrix} \nabla^2_{\omega^1} l_1(\omega) & \ldots & \nabla_{\omega^n} \nabla_{\omega^1} l_1(\omega) \\ \vdots & & \vdots \\ \nabla_{\omega^1} \nabla_{\omega^n} l_n(\omega) & \ldots & \nabla^2_{\omega^n} l_n(\omega) \end{pmatrix}$$

We say that v is *L-Lipschitz* for some $L \geq 0$ if $\|v(\omega) - v(\omega')\| \leq L\|\omega - \omega'\|$ $\forall \omega, \omega' \in \mathbb{R}^d$, that v is *μ-strongly monotone* for some $\mu \geq 0$, if $\mu\|\omega - \omega'\|^2 \leq (v(\omega) - v(\omega'))^T(\omega - \omega')$ $\forall \omega, \omega' \in \mathbb{R}^d$.

A Nash equilibrium is always a *stationary* point of the gradient dynamics, i.e. a point $\omega \in \mathbb{R}^d$ such that $v(\omega) = 0$. However, as shown by [ADLH18, MJS19, Berard et al.(2019)], in general, being a Nash equilibrium is neither necessary nor sufficient for being a locally stable stationary point, but if v is monotone, these two notions are equivalent. Hence, in this work we focus on finding stationary points. One important class of games is *saddle-point problems*: two-player games with $l_1 = -l_2$. If v is monotone, or equivalently f is convex-concave, stationary points correspond to the solutions of the min-max problem

$$\min_{\omega_1 \in \mathbb{R}^{d_1}} \max_{\omega_2 \in \mathbb{R}^{d_2}} l_1(\omega_1, \omega_2)$$

[GHP+19] and [BRM+18] mentioned two particular classes of games, which can be seen as the two opposite ends of a spectrum. As the definitions vary, we only give the intuition for these two categories. The first one is *adversarial games*, where the Jacobian has eigenvalues with small real parts and large imaginary parts and the cross terms $\nabla_{\omega_i} \nabla_{\omega_j} l_j(\omega)$, for $i \neq j$, are dominant. Ex. 1 gives a prime example of such game that has been heavily studied: a simple bilinear game whose Jacobian is anti-symmetric and so only has imaginary eigenvalues (see Lem. 23 in App. B.1.5):

Example 1 (Bilinear game)**.**

$$\min_{x \in \mathbb{R}^m} \max_{y \in \mathbb{R}^m} x^T A y + b^T x + c^T y$$

with $A \in \mathbb{R}^{m \times m}$ non-singular, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^m$.

If A is non-singular, there is an unique stationary point which is also the unique Nash equilibrium. The gradient method is known not to converge in such game while the proximal point and extragradient methods converge [Roc76, Tse95].

Bilinear games are of particular interest to us as they are seen as models of the convergence problems that arise during the training of GANs. Indeed, [Mescheder et al.(2017)] showed that eigenvalues of the Jacobian of the vector field with small real parts and large imaginary parts could be at the origin of these problems. Bilinear games have pure imaginary eigenvalues and so are limiting models of this situation. Moreover, they can also be seen as a very simple type of WGAN, with the generator and the discriminator being both linear, as explained in [GBV$^+$19, MGN18].

The other category is *cooperative games*, where the Jacobian has eigenvalues with large positive real parts and small imaginary parts and the diagonal terms $\nabla^2_{\omega_i} l_i$ are dominant. Convex minimization problems are the archetype of such games. Our hypotheses, for both the local and the global analyses, encompass these settings.

2.1.2 Methods and convergence analysis

Convergence theory of fixed-point iterations. Seeing optimization algorithms as the repeated application of some operator allows us to deduce their convergence properties from the spectrum of this operator. This point of view was presented by [Pol87, Ber99] and recently used by [ASS16, Mescheder et al.(2017), GHP$^+$19] for instance. The idea is that the iterates of a method $(\omega_t)_t$ are generated by a scheme of the form:

$$\omega_{t+1} = F(\omega_t) \quad \forall t \geq 0$$

where $F : \mathbb{R}^d \to \mathbb{R}^d$ is an operator representing the method. Near a stationary point ω^*, the behavior of the iterates is mainly governed by the properties of $\nabla F(\omega^*)$ as $F(\omega) - \omega^* \approx \nabla F(\omega^*)(\omega - \omega^*)$. This is formalized by the following classical result:

Theorem 1 ([Pol87])**.** *Let $F : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be continuously differentiable and let $\omega^* \in \mathbb{R}^d$ be a fixed point of F. If $\rho(\nabla F(\omega^*)) < 1$, then for ω_0 in a neighborhood of ω^*, the iterates $(\omega_t)_t$ defined by $\omega_{t+1} = F(\omega_t)$ for all $t \geq 0$ converge linearly to ω^* at a rate of $\mathcal{O}((\rho(\nabla F(\omega^*)) + \epsilon)^t)$ for all $\epsilon > 0$.*

This theorem means that to derive a local rate of convergence for a given method, one needs only to focus on the eigenvalues of $\nabla F(\omega^*)$. Note that if the operator F is linear, there exists slightly stronger results such as Thm. 11 in appendix B.1.3.

Gradient method. Following [GHP$^+$19], we define GD as the application of the operator $F_\eta(\omega) := \omega - \eta v(\omega)$, for $\omega \in \mathbb{R}^d$. Thus we have:

$$\omega_{t+1} = F_\eta(\omega_t) = \omega_t - \eta v(\omega_t) \tag{GD}$$

Proximal point. For v monotone [Min62, Roc76], the proximal point operator can be defined as $P_\eta(\omega) = (\mathrm{Id} + \eta v)^{-1}(\omega)$ and therefore can be seen as an implicit scheme: $\omega_{t+1} = \omega_t - \eta v(\omega_{t+1})$.

Extragradient. EG was introduced by [Kor76] in the context of variational inequalities. Its update rule is

$$\omega_{t+1} = \omega_t - \eta v(\omega_t - \eta v(\omega_t)) \tag{EG}$$

It can be seen as an approximation of the implicit update of the proximal point method. Indeed [Nem04] showed a rate of $\mathcal{O}(1/t)$ for extragradient by treating it as a "good enough" approximation of the proximal point method. To see this, fix $\omega \in \mathbb{R}^d$. Then $P_\eta(\omega)$ is the solution of $z = \omega - \eta v(z)$. Equivalently, $P_\eta(\omega)$ is the fixed point of

$$\varphi_{\eta,\omega} : z \longmapsto \omega - \eta v(z) \tag{1}$$

which is a contraction for $\eta > 0$ small enough. From Picard's fixed point theorem, one gets that the proximal point operator $P_\eta(\omega)$ can be obtained as the limit of $\varphi^k_{\eta,\omega}(\omega)$ when k goes to infinity. What [Nem04] showed is that $\varphi^2_{\eta,\omega}(\omega)$, that is to say the extragradient update, is close enough to the result of the fixed point computation to be used in place of the proximal point update without affecting the sublinear convergence speed. Our analysis of multi-step extrapolation methods will encompass all the iterates $\varphi^k_{\eta,\omega}$ and we will show that a similar phenomenon happens for linear convergence rates.

Optimistic gradient. Originally introduced in the online learning literature [Chiang et al.(2012), Rakhlin and Sridharan(2013)] as a two-steps method, [DISZ18] reformulated it with only one step in the unconstrained case:

$$w_{t+1} = w_t - 2\eta v(w_t) + \eta v(w_{t-1}) \tag{OG}$$

Consensus optimization. Introduced by [Mescheder et al.(2017)] in the context of games, consensus optimization is a second-order yet efficient method, as it only uses a Hessian-vector multiplication whose cost is the same as two gradient evaluations [Pearlmutter(1994)]. We define the CO update as:

$$\omega_{t+1} = \omega_t - (\alpha v(\omega_t) + \beta \nabla H(\omega_t)) \tag{CO}$$

where $H(\omega) = \frac{1}{2}\|v(\omega)\|_2^2$ and $\alpha, \beta > 0$ are step sizes.

2.1.3 p-SCLI framework for game optimization

In this section, we present an extension of the framework of [ASS16] to derive lower bounds for game optimization (also see §B.3). The idea of this framework is to see algorithms as the iterated application of an operator. If the vector field is linear, this transformation is linear too and so its behavior when iterated is mainly governed by its spectral radius. This way, showing a lower bound for a class of algorithms is reduced to lower bounding a class of spectral radii.

We consider $\mathcal{V}_d$ the set of linear vector fields $v : \mathbb{R}^d \to \mathbb{R}^d$, i.e., vector fields v whose Jacobian ∇v is a constant $d \times d$ matrix.[1] The class of algorithms we consider is the class of 1-*Stationary Canonical Linear Iterative algorithms* (1-*SCLI)*. Such an algorithm is defined by a mapping $\mathcal{N} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$. The associated update rule can be defined through,

$$F_{\mathcal{N}}(\omega) = w + \mathcal{N}(\nabla v)v(\omega) \quad \forall \omega \in \mathbb{R}^d \tag{2}$$

This form of the update rule is required by the consistency condition of [ASS16] which is necessary for the algorithm to converge to stationary points, as discussed in §B.3. Also note that 1-SCLI are first-order methods that use only the last iterate to compute the next one. Accelerated methods such as accelerated gradient descent [Nes04] or the heavy ball method [Pol64] belong in fact to the class of 2-SCLI, which encompass methods which uses the last two iterates.

[1]With a slight abuse of notation, we also denote by ∇v this matrix.

As announced above, the spectral radius of the operator gives a lower bound on the speed of convergence of the iterates of the method on affine vector fields, which is sufficient to include bilinear games, quadratics and so strongly monotone settings too.

Theorem 2 ([ASS16]). *For all $v \in \mathcal{V}_d$, for almost every[2] initialization point $\omega_0 \in \mathbb{R}^d$, if $(\omega_t)_t$ are the iterates of $F_{\mathcal{N}}$ starting from ω_0,*

$$\|\omega_t - \omega^*\| \geq \Omega(\rho(\nabla F_{\mathcal{N}})^t \|\omega_0 - \omega^*\|)$$

2.2 Revisiting GD for games

In this section, our goal is to illustrate the precision of the spectral bounds and the complexity of the interactions between players in games. We first give a simplified version of the bound on the spectral radius from [GHP+19] and show that their results also imply that this rate is tight.

Theorem 3. *Let ω^* be a stationary point of v and denote by σ^* the spectrum of $\nabla v(\omega^*)$. If the eigenvalues of $\nabla v(\omega^*)$ all have positive real parts, then*

(i). [GHP+19] For $\eta = \min_{\lambda \in \sigma^*} \Re(1/\lambda)$, the spectral radius of F_η can be upper-bounded as

$$\rho(\nabla F_\eta(\omega^*))^2 \leq 1 - \min_{\lambda \in \sigma^*} \Re(1/\lambda) \min_{\lambda \in \sigma^*} \Re(\lambda)$$

(ii). For all $\eta > 0$, the spectral radius of the gradient operator F_η at ω^* is lower bounded by

$$\rho(\nabla F_\eta(\omega^*))^2 \geq 1 - 4 \min_{\lambda \in \sigma^*} \Re(1/\lambda) \min_{\lambda \in \sigma^*} \Re(\lambda)$$

This result is stronger than what we need for a standard lower bound: using Thm. 2, this yields a lower bound on the convergence of the iterates for all games with affine vector fields.

We then consider a saddle-point problem, and under some assumptions presented below, one can interpret the spectral rate of the gradient method mentioned earlier in terms of the standard strong convexity and Lipschitz-smoothness constants. There are several cases, but one of them is of special interest to us as it demonstrates the precision of spectral bounds.

Example 2 (Highly adversarial saddle-point problem)**.** Consider $\min_{x \in \mathbb{R}^m} \max_{y \in \mathbb{R}^m} f(x, y)$ with f twice differentiable such that

(i). f satisfies, with μ_1, μ_2 and μ_{12} non-negative,

$$\mu_1 I \preccurlyeq \nabla_x^2 f \preccurlyeq L_1 I \quad \mu_2 I \preccurlyeq -\nabla_y^2 f \preccurlyeq L_2 I$$
$$\mu_{12}^2 I \preccurlyeq (\nabla_x \nabla_y f)^T (\nabla_x \nabla_y f) \preccurlyeq L_{12}^2 I$$

such that $\mu_{12} > 2 \max(L_1 - \mu_2, L_2 - \mu_1)$.

(ii). There exists a stationary point $\omega^* = (x^*, y^*)$ and at this point, $\nabla_y^2 f(\omega^*)$ and $\nabla_x \nabla_y f(\omega^*)$ commute and $\nabla_x^2 f(\omega^*)$, $\nabla_y^2 f(\omega^*)$ and $(\nabla_x \nabla_y f(\omega^*))^T (\nabla_x \nabla_y f(\omega^*))$ commute.

[2]For any measure absolutely continuous w.r.t. the Lebesgue measure.

Assumption *(i)* corresponds to a highly adversarial setting as the coupling (represented by the cross derivatives) is much bigger than the Hessians of each player. Assumption *(ii)* is a technical assumption needed to compute a precise bound on the spectral radius and holds if, for instance, the objective is separable, i.e. $f(x,y) = \sum_{i=1}^m f_i(x_i, y_i)$. Using these assumptions, we can upper bound the rate of Thm. 3 as follows:

Corollary 1. *Under the assumptions of Thm. 3 and Ex. 2,*

$$\rho(\nabla F_\eta(\omega^*))^2 \leq 1 - \frac{1}{4}\frac{(\mu_1+\mu_2)^2}{L_{12}^2 + L_1 L_2} \tag{3}$$

What is surprising is that, in some regimes, this result induces faster local convergence rates than the existing upper-bound for EG [Tse95]:

$$1 - \frac{\min(\mu_1, \mu_2)}{4L_{max}} \text{ where } L_{max} = \max(L_1, L_2, L_{12}) \tag{4}$$

If, say, μ_2 goes to zero, that is to say the game becomes unbalanced, the rate of EG goes to 1 while the one of (3) stays bounded by a constant which is strictly less than 1. Indeed, the rate of Cor. 1 involves the arithmetic mean of μ_1 and μ_2, which is roughly the maximum of them, while (4) makes only the minimum of the two appear. This adaptivity to the best strong convexity constant is not present in the standard convergence rates of the EG method. We remedy this situation with a new analysis of EG in the following section.

2.3 Spectral analysis of multi-step EG

In this section, we study the local dynamics of EG and, more generally, of extrapolation methods. Define a *k-extrapolation method* (k-EG) by the operator

$$F_{k,\eta} : \omega \mapsto \varphi_{\eta,\omega}^k(\omega) \text{ with } \varphi_{\eta,\omega} : z \mapsto \omega - \eta v(z) \tag{5}$$

We are essentially considering all the iterates of the fixed point computation discussed in §2.1.2. Note that $F_{1,\eta}$ is GD while $F_{2,\eta}$ is EG. We aim at studying the local behavior of these methods at stationary points of the gradient dynamics, so fix ω^* s.t. $v(\omega^*) = 0$ and let $\sigma^* = \operatorname{Sp} \nabla v(\omega^*)$. We compute the spectra of these operators at this point and this immediately yields the spectral radius on the proximal point operator:

Lemma 1. *The spectra of the k-extrapolation operator and the proximal point operator are given by:*

$$\operatorname{Sp} \nabla F_{\eta,k}(\omega^*) = \left\{ \textstyle\sum_{j=0}^k (-\eta\lambda)^j \mid \lambda \in \sigma^* \right\}$$
$$\textit{and} \quad \operatorname{Sp} \nabla P_\eta(\omega^*) = \left\{ (1+\eta\lambda)^{-1} \mid \lambda \in \sigma^* \right\}$$

Hence, for all $\eta > 0$, the spectral radius of the operator of the proximal point method is equal to:

$$\rho(\nabla P_\eta(\omega^*))^2 = 1 - \min_{\lambda\in\sigma^*} \frac{2\eta\Re\lambda + \eta^2|\lambda|^2}{|1+\eta\lambda|^2} \tag{6}$$

Again, this shows that a k-EG is essentially an approximation of proximal point for small step sizes as $(1+\eta\lambda)^{-1} = \sum_{j=0}^k (-\eta\lambda)^j + \mathcal{O}\left(|\eta\lambda|^{k+1}\right)$. This could suggest that increasing the number of extrapolations might yield better methods but we will actually see that $k = 2$ is enough to achieve a similar rate to proximal. We then bound the spectral radius of $\nabla F_{\eta,k}(\omega^*)$:

Theorem 4. *Let $\sigma^* = \mathrm{Sp}\,\nabla v(\omega^*)$. If the eigenvalues of $\nabla v(\omega^*)$ all have non-negative real parts, the spectral radius of the k-extrapolation method for $k \geq 2$ satisfies:*

$$\rho(\nabla F_{\eta,k}(\omega^*))^2 \leq 1 - \min_{\lambda \in \sigma^*} \frac{2\eta \Re\lambda + \frac{7}{16}\eta^2 |\lambda|^2}{|1+\eta\lambda|^2} \tag{7}$$

$\forall \eta \leq \frac{1}{4^{\frac{1}{k-1}}} \frac{1}{\max_{\lambda\in\sigma^*}|\lambda|}$. *For $\eta = (4\max_{\lambda\in\sigma^*}|\lambda|)^{-1}$, this can be simplified as (noting $\rho := \rho(\nabla F_{\eta,k}(\omega^*))$):*

$$\rho^2 \leq 1 - \frac{1}{4}\left(\frac{\min_{\lambda\in\sigma^*}\Re\lambda}{\max_{\lambda\in\sigma^*}|\lambda|} + \frac{1}{16}\frac{\min_{\lambda\in\sigma^*}|\lambda|^2}{\max_{\lambda\in\sigma^*}|\lambda|^2}\right) \tag{8}$$

The zone of convergence of extragradient as provided by this theorem is illustrated in Fig. 1.

The bound of (8) involves two terms: the first term can be seen as the strong monotonicity of the problem, which is predominant in convex minimization problems, while the second shows that even in the absence of it, this method still converges, such as in bilinear games. Furthermore, in situation in between, this bound shows that the extragradient method exploits the biggest of these quantities as they appear as a sum as illustrated by the following simple example.

Example 3 ("In between" example)**.**

$$\min_{x\in\mathbb{R}} \max_{y\in\mathbb{R}} \frac{\epsilon}{2}(x^2 - y^2) + xy \quad \text{for } 1 \geq \epsilon > 0$$

Though for ϵ close to zero, the dynamics will behave as such, this is not a purely bilinear game. The associated vector field is only ϵ-strongly monotone and convergence guarantees relying only on strong monotonicity would give a rate of roughly $1 - \epsilon/4$. However Thm. 4 yields a convergence rate of roughly $1 - 1/64$ for extragradient.

Similarity to the proximal point method. First, note that the bound (7) is surprisingly close to the one of the proximal method (6). However, one can wonder why the proximal point converges with any step size — and so arbitrarily fast — while it is not the case for the k-EG, even as k goes to infinity. The reason for this difference is that for the fixed point iterates to converge to the proximal point operator, one needs $\varphi_{\eta,\omega}$ to be a contraction and so to have η small enough, at least $\eta < (\max_{\lambda\in\sigma^*}|\lambda|)^{-1}$ for local guarantees. This explains the bound on the step size for k-EG .

Comparison with the gradient method. We can now compare this result for EG with the convergence rate of the gradient method Thm. 3 which was shown to be tight. In general $\min_{\lambda\in\sigma^*}\Re(1/\lambda) \leq (\max_{\lambda\in\sigma^*}|\lambda|)^{-1}$ and, for adversarial games, the first term can be arbitrarily smaller than the second one. Hence, in this setting which is of special interest to us, EG has a much faster convergence speed than GD.

Recovery of known rates. If v is μ-strongly monotone and L-Lipschitz, this bound is at least as precise as the standard one $1 - \mu/(4L)$ as μ lower bounds the real part of the eigenvalues of the Jacobian, and L upper bounds their magnitude, as shown in Lem. 10 in §B.2.2. On the other hand, Thm. 4 also recovers the standard rates for the bilinear problem,[3] as shown below:

[3] Note that by exploiting the special structure of the bilinear game and the fact that $k = 2$, one could derive a better constant in the rate. Moreover, our current spectral tools cannot handle the singularity which arises if the two players have a different number of parameters. We provide sharper results to handle this difficulty in appendix B.5.

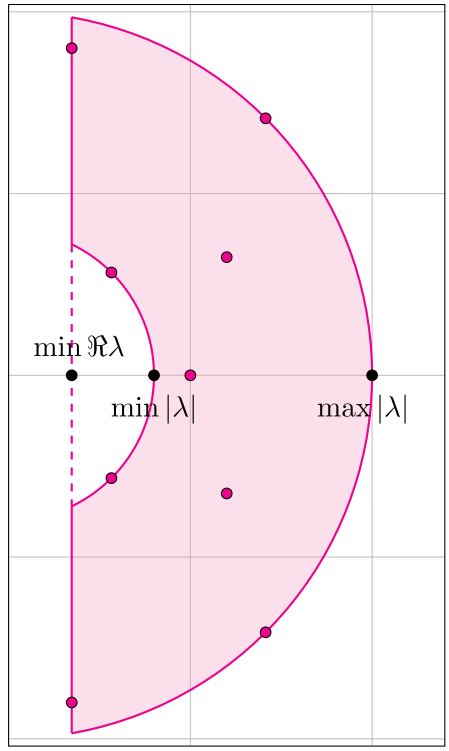

Figure 1. Illustration of the three quantities involved in Thm. 4. The magenta dots are an example of eigenvalues belonging to σ^*. Note that σ^* is always symmetric with respect to the real axis because the Jacobian is a real matrix (and thus non-real eigenvalues are complex conjugates). Note how $\min \Re\lambda$ may be significantly smaller that $\min|\lambda|$.

Corollary 2 (Bilinear game)**.** *Consider Ex. 1. The iterates of the k-extrapolation method with $k \geq 2$ converge globally to ω^* at a linear rate of $\mathcal{O}\Big(\big(1 - \frac{1}{64}\frac{\sigma_{min}(A)^2}{\sigma_{max}(A)^2}\big)^t\Big)$.*

Note that this rate is similar to the one derived by [GHP+19] for alternating gradient descent with negative momentum. This raises the question of whether general acceleration exists for games, as we would expect the quantity playing the role of the condition number in Cor. 2 to appear without the square in the convergence rate of a method using momentum.

Finally it is also worth mentioning that the bound of Thm. 4 also displays the adaptivity discussed in §2.2. Hence, the bound of Thm. 4 can be arbitrarily better than the rate (4) for EG from the literature and also better than the global convergence rate we prove below.

Lower bounds for extrapolation methods. We now show that the rates we proved for EG are tight and optimal by deriving lower bounds of convergence for general extrapolation methods. As described in §2.1.3, a 1-SCLI method is parametrized by a polynomial $\mathcal{N}$. We consider the class of methods where $\mathcal{N}$ is any polynomial of degree at most $k-1$, and we will derive lower bounds for this class. This class is large enough to include all the k'-extrapolation methods for $k' \leq k$ with possibly different step sizes for each extrapolation step (see §B.4 for more examples).

Our main result is that no method of this class can significantly beat the convergence speed of EG of Thm. 4 and Thm. 6. We proceed in two steps: for each of the two terms of these bounds, we provide an example matching it up to a factor. In (i) of the following theorem, we give an example of convex optimization problem which matches the real part, or strong monotonicity, term. Note that this example is already an extension of [ASS16] as the authors only considered constant $\mathcal{N}$. Next, in (ii), we match the other term with a bilinear game example.

Theorem 5. *Let $0 < \mu, \gamma < L$. (i) If $d-2 \geq k \geq 3$, there exists $v \in \mathcal{V}_d$ with a symmetric positive Jacobian whose spectrum is in $[\mu, L]$, such that for any $\mathcal{N}$real polynomial of degree at most $k-1$, $\rho(F_{\mathcal{N}}) \geq 1 - \frac{4k^3}{\pi}\frac{\mu}{L}$.*

(ii) If $d/2 - 2 \geq k/2 \geq 3$ and d is even, there exists $v \in \mathcal{V}_d$ L-Lipschitz with $\min_{\lambda \in \operatorname{Sp} \nabla v} |\lambda| = \sigma_{min}(\nabla v) \geq \gamma$ corresponding to a bilinear game of Example 1 with $m = d/2$, such that, for any $\mathcal{N}$real polynomial of degree at most $k-1$, $\rho(F_{\mathcal{N}}) \geq 1 - \frac{k^3}{2\pi}\frac{\gamma^2}{L^2}$.

First, these lower bounds show that both our convergence analyses of EG are tight, by looking at them for $k=3$ for instance. Then, though these bounds become looser as k grows, they still show that the potential improvements are not significant in terms of conditioning, especially compared to the change of regime between GD and EG . Hence, they still essentially match the convergence speed of EG of Thm. 4 or Thm. 6. Therefore, EG can be considered as optimal among the general class of algorithms which uses at most a fixed number of composed gradient evaluations and only the last iterate. In particular, there is no need to consider algorithms with more extrapolation steps or with different step sizes for each of them as it only yields a constant factor improvement.

2.4 Unified global proofs of convergence

We have shown in the previous section that a spectral analysis of EG yields tight and unified convergence guarantees. We now demonstrate how, combining the strong monotonicity assumption and Tseng's error bound, global convergence guarantees with the same unifying properties might be achieved.

2.4.1 Global Assumptions

[Tse95] proved linear convergence results for EG by using the projection-type error bound [Tse95, Eq. 5] which, in the unconstrained case, i.e. for $v(\omega^*) = 0$, can be written as,

$$\gamma \|\omega - \omega^*\|_2 \leq \|v(\omega)\|_2 \quad \forall \omega \in \mathbb{R}^d. \tag{9}$$

The author then shows that this condition holds for the bilinear game of Example 1 and that it induces a convergence rate of $1 - c\sigma_{min}(A)^2/\sigma_{max}(A)^2$ for some constant $c > 0$. He also shows that this condition is implied by strong monotonicity with $\gamma = \mu$. Our analysis builds on the results from [Tse95] and extends them to cover the whole range of games and recover the optimal rates.

To be able to interpret Tseng's error bound (9), as a property of the Jacobian ∇v, we slightly relax it to,

$$\gamma \|\omega - \omega'\|_2 \leq \|v(\omega) - v(\omega')\|_2, \quad \forall \omega, \omega' \in \mathbb{R}^d \tag{10}$$

This condition can indeed be related to the properties of ∇v as follows:

Lemma 2. *Let v be continuously differentiable and $\gamma > 0$: (10) holds if and only if $\sigma_{min}(\nabla v) \geq \gamma$.*

Hence, γ corresponds to a lower bound on the singular values of ∇v. This can be seen as a weaker "strong monotonicity" as it is implied by strong monotonicity, with $\gamma = \mu$, but it also holds for a square non-singular bilinear example of Example 1 with $\gamma = \sigma_{min}(A)$.

As announced, we will combine this assumption with the strong monotonicity to derive unified global convergence guarantees. Before that, note that this quantities can be related to the spectrum of $\operatorname{Sp} \nabla v(\omega^*)$ as follows – see Lem. 10 in appendix B.2.1,

$$\mu \leq \Re(\lambda), \quad \gamma \leq |\lambda| \leq L \quad \forall \lambda \in \operatorname{Sp} \nabla v(\omega^*) \tag{11}$$

Hence, theses global quantities are less precise than the spectral ones used in Thm. 4, so the following global results will be less precise than the previous ones.

2.4.2 Global analysis EG and OG

We can now state our global convergence result for EG:

Theorem 6. *Let $v : \mathbb{R}^d \to \mathbb{R}^d$ be continuously differentiable and*

(i) μ-strongly monotone for some $\mu \geq 0$,

(ii) L-Lipschitz,

(iii) such that $\sigma_{min}(\nabla v) \geq \gamma$ for some $\gamma > 0$.

Then, for $\eta \leq (4L)^{-1}$, the iterates $(\omega_t)_t$ of (EG) *converge linearly to ω^* as, for all $t \geq 0$,*

$$\|\omega_t - \omega^*\|_2^2 \leq \left(1 - \eta\mu - \frac{7}{16}\eta^2\gamma^2\right)^t \|\omega_0 - \omega^*\|_2^2$$

As for Thm. 4, this result not only recovers both the bilinear and the strongly monotone case, but shows that EG actually gets the best of both world when in between. Furthermore this rate is surprisingly similar to the result of Thm. 4 though less precise, as discussed.

Combining our new proof technique and the analysis provided by [GBV+19], we can derive a similar convergence rate for the optimistic gradient method.

Theorem 7. *Under the same assumptions as in Thm. 6, for $\eta \leq (4L)^{-1}$, the iterates $(\omega_t)_t$ of* (OG) *converge linearly to ω^* as, for all $t \geq 0$,*

$$\|\omega_t - \omega^*\|_2^2 \leq 2\left(1 - \eta\mu - \frac{1}{8}\eta^2\gamma^2\right)^{t+1} \|\omega_0 - \omega^*\|_2^2$$

Interpretation of the condition numbers. As in the previous section, this rate of convergence for EG is similar to the rate of the proximal point method for a small enough step size, as shown by Prop. 9 in §B.2.2. Moreover, the proof of the latter gives insight into the two quantities appearing in the rate of Thm. 6. Indeed, the convergence result for the proximal point method is obtained by bounding the singular values of ∇P_η, and so we compute,[4]

$$(\nabla P_\eta)^T \nabla P_\eta = \left(I_d + \eta\mathcal{H}(\nabla v) + \eta^2 \nabla v \nabla v^T\right)^{-1}$$

where $\mathcal{H}(\nabla v) := \frac{\nabla v + \nabla v^T}{2}$. This explains the quantities L/μ and L^2/γ^2 appear in the convergence rate, as the first corresponds to the condition number of $\mathcal{H}(\nabla v)$ and the second to the condition number of $\nabla v \nabla v^T$. Thus, the proximal point method uses information from both matrices to converge, and so does EG, explaining why it takes advantage of the best conditioning.

[4]We dropped the dependence on ω for compactness.

2.4.3 Global analysis of consensus optimization

In this section, we give a unified proof of CO. A global convergence rate for this method was proven by [ALW19]. However it used a perturbation analysis of HGD. The drawbacks are that it required that the CO update be sufficiently close to the one of HGD and could not take advantage of strong monotonicity. Here, we combine the monotonicity μ with the lower bound on the singular value γ.

As this scheme uses second-order[5] information, we need to replace the Lipschitz hypothesis with one that also controls the variations of the Jacobian of v: we use L_H^2, the Lispchitz smoothness of H. See [ALW19] for how it might be instantiated.

Theorem 8. *Let $v : \mathbb{R}^d \to \mathbb{R}^d$ be continuously differentiable such that*

(i) v is μ- strongly monotone for some $\mu \geq 0$,

(ii) $\sigma_{min}(\nabla v) \geq \gamma$ for some $\gamma > 0$

(iii) H is L_H^2 Lipschitz-smooth.

Then, for $\alpha = (\mu + \sqrt{\mu^2 + 2\gamma^2})/(4L_H^2)$, $\beta = (2L_H^2)^{-1}$ the iterates of CO defined by (CO) satisfy, for all $t \geq 0$,

$$H(\omega_t) \leq \left(1 - \frac{\mu^2}{2L_H^2} - \left(1 + \frac{\mu}{\gamma}\right) \frac{\gamma^2}{2L_H^2}\right)^t H(\omega_0)\,.$$

This result shows that CO has the same unifying properties as EG, though the dependence on μ is worse.

This result also encompasses the rate of HGD [ALW19, Lem. 4.7]. The dependance in μ is on par with the standard rate for the gradient method (see [NS06, Eq. 2.12] for instance). However, this can be improved using a sharper assumption, as discussed in Remark 1 in appendix B.2.3, and so our result is not optimal in this regard.

3 Spectral Analysis and Acceleration in Smooth Games

Recent successes of multi-agent formulations in various areas of deep learning [GPM+14, PV16] have caused a surge of interest in the theoretical understanding of first-order methods for the solution of differentiable multi-player games [PB16, GBV+19, BRM+18, MNG17, MGN18, MJS19]. This exploration hinges on a key question: ***How fast can a first-order method be?*** In convex minimization, [Nes83, Nes04] answered this question with lower bounds for the rate of convergence and an accelerated, momentum-based algorithm matching that optimal lower bound.

The dynamics of numerical methods is often described by a vector field, F, and summarized in the spectrum of its Jacobian. In minimization problems, the eigenvalues of the Jacobian lie on the real line. On strongly convex problems, the *condition number* (the dynamic range of eigenvalues) is at the heart of Nesterov's upper and lower bound results, characterizing the hardness of an minimization problem.

Our understanding of differentiable games is nowhere close to this point. There, the eigenvalues of the Jacobian at the solution are distributed on the complex plane, suggesting a richer, more complex set of dynamics [MNG17, BRM+18]. Some old papers [Kor76, Tse95] and many recent ones [Nem04, CLO14, PB16, MNG17, GBV+19, GHP+19, DISZ18, MOP19, AMLJG19] suggest new methods and provide better upper bounds.

[5] W.r.t. the losses.

All of the above work relies on bounding the magnitude or the real part of the eigenvalues of submatrices of the Jacobian. This coarse-grain approach can be oblivious to the dependence of upper and lower bounds on the exact distribution of eigenvalues on the complex plane. More importantly, the questions of acceleration and optimality have not been answered for smooth games.

In this section, we take a different approach. We use matrix iteration theory to characterize acceleration in smooth games. Our analysis framework revolves around the *spectral shape* of a family of games, defined as the set containing all eigenvalues of the Jacobians of natural gradient dynamics in the family (cf. §3.1.2). This fine-grained analysis framework can captures the dependence of upper and lower bounds on the specific shape of the spectrum. Critically, it allows us to establish acceleration in specific families of smooth games.

The remainder of this section is organized as follows. In Section 2, we review related work on game optimization and matrix iteration theory. Section 3 introduces our spectral shape framework and applies it to bilinear games. In Section 4, we present our main acceleration results for games with small imaginary perturbations. Section 5 extends these results to consensus optimization. Finally, we conclude in Section 6 with a discussion of future research directions and applications to machine learning problems.

3.1 Setting and notation

We consider the problem of finding a stationary point $\omega^* \in \mathbb{R}^d$ of a vector field $F : \mathbb{R}^d \to \mathbb{R}^d$, i.e., $F(\omega^*) = 0.$, the solution of an unconstrained *variational inequality* problem [HP90]. A relevant special case is a n-player convex game, where ω^* corresponds to a Nash equilibrium [VNM44, BRM$^+$18]. Consider n players $i = 1, \dots, n$ who want to minimize their loss $l_i(\omega^{(i)}, \omega^{(-i)})$. The notation $\cdot^{(-i)}$ means all indexes but i. A Nash equilibrium satisfies

$$(\omega^*)^{(i)} \in \operatorname*{arg\,min}_{\omega^{(i)} \in \mathbb{R}^{d_i}} l_i(\omega^{(i)}, (\omega^*)^{(-i)}) \quad \forall i \in \{1, \dots, n\}.$$

In this situation no player can unilaterally reduce its loss. The vector field of the game is

$$F(\omega) = \left[\nabla_{\omega_1} l_1^T(\omega^{(1)}, \omega^{(-1)}), ..., \nabla_{\omega_n} l_n^T(\omega^{(n)}, \omega^{(-n)})\right]^T.$$

3.1.1 First-order methods

To study lower bounds of convergence, we need a class of algorithms. We consider the classic definition[6] of first-order methods from [NY83].

Definition 1. *A* first-order method *generates*

$$\omega_t \in \omega_0 + \textbf{\textit{Span}}\{F(\omega_0), \dots, F(\omega_{t-1})\}, \quad t \geq 1.$$

This class is widely used in large-scale optimization, as it involves only gradient computation. For instance, Nesterov's acceleration belongs to the class of first-order methods. On the contrary, this definition does not cover Adagrad [DHS11], that could conceptually be also considered as first-order. This is due to the diagonal re-scaling, so ω_t can go *outside* the span of gradients. The next proposition gives a way to easily identify first-order methods that fit our definition.

[6] Technically, first-order algorithms are more generally methods that have access only to first-order oracles.

Proposition 1. *[AS16] first-order methods can be written as*

$$\omega_{t+1} = \textstyle\sum_{k=0}^{t} \alpha_k^{(t)} F(\omega_k) + \beta_k^{(t)} \omega_k, \tag{12}$$

where $\sum_{k=0}^{t} \beta_k^{(t)} = 1$*. The method is called oblivious if the coefficients* $\alpha_k^{(t)}$ *and* $\beta_k^{(t)}$ *are known in advance.*

Oblivious methods allow the knowledge of “side information” on the function, like its smoothness constant. Most of first-order methods belong to this class, but it excludes for instance methods with adaptive step-sizes. We show how standard methods fit into this framework.

Gradient method. Consider the gradient method with time-dependant step-size: $\omega_{t+1} = \omega_t - \eta_t F(\omega_t)$. This is a first-order method, where $\alpha_t^{(t)} = -\eta_t$, $\beta_t^{(t)} = 1$ and all the other coefficients set to zero.

Momentum method. The momentum method defines iterates as $\omega_{t+1} = \omega_t - \alpha F(\omega_t) + \beta(\omega_t - \omega_{t-1})$. It fits into the previous framework with $\alpha_t^{(t)} = -\alpha$, $\beta_t^{(t)} = 1 + \beta$, $\beta_{t-1}^{(t)} = -\beta$.

Extragradient method. Though slightly trickier, the extragradient method (EG) is also encompassed by this definition. The iterates of EG are defined by $\omega_{t+1} = \omega_t - \eta F(\omega_t - \eta F(\omega_t))$ where

$$\begin{cases} \beta_t^{(t)} = 0,\ \beta_{t-1}^{(t)} = 1 & \text{if } t \text{ is odd (update)}\,, \\ \beta_t^{(t)} = 1,\ \beta_{t-1}^{(t)} = 0 & \text{if } t \text{ is even (extrapolation)}\,, \end{cases}$$

and $\alpha_t^{(t)} = -\eta$ the step size.

The next (known) lemma shows that when F is linear, first-order methods can be written using *polynomials.*

Lemma 3. *[e.g., [Chi11]] If* $F(\omega) = A\omega + b$*,*

$$\omega_t - \omega^* = p_t(A)(\omega_0 - \omega^*)\,, \tag{13}$$

where ω^* *satisfies* $A\omega^* + b = 0$ *and* p_t *is a real polynomial of degree at most* t *such that* $p_t(0) = 1$*.*

We denote by $\mathcal{P}_t$ the set of real polynomials of degree at most t such that $p_t(0) = 1$. Hence, the convergence of a first-order method can be analyzed through the sequence of polynomials $(p_t)_t$ it defines.

3.1.2 Problem class

In the previous section, when F is the linear function $F = Ax + b$, the iterates ω_t follow the relation (13) involving the polynomial p_t. Since all first-order methods can be written using polynomials (12), they follow

$$\|\omega_t - \omega^*\|_2 = \|p_t(A)(\omega_0 - \omega^*)\|_2\,. \tag{14}$$

This gives the rate of convergence of the method for a specific matrix A. Instead, we consider a larger class of problems. It consists of a set $\mathcal{M}_K$ of matrices A whose eigenvalues belong to a set K on the complex plane,

$$\mathcal{M}_K \coloneqq \{A \in \mathbb{R}^d :\ \mathrm{Sp}(A) \subset K \subset \mathbb{C}_+\}, \tag{15}$$

where $\mathrm{Sp}(A)$ is the set of eigenvalues of A and $\mathbb{C}_+$ is the set of complex numbers with positive real part. Moreover, we assume that $d \geq 2$ to avoid trivial cases.

3.1.3 Geometric intuition

This section is entirely based on the study of the support K of the eigenvalues of the Jacobian of the operator F, denoted by $\mathbf{J}_F(\omega^*)$. Before detailing our theoretical results, we give a high-level explanation of our objectives. This geometric intuition comes from the fact that the standard assumptions made in the literature correspond to particular problem classes $\mathcal{M}_K$.

Smooth and strongly convex minimization. Consider the minimization of a twice-differentiable, L-smooth and μ-strongly convex function f,

$$\mu\mathbf{I} \preceq \nabla^2 f(\omega) \preceq L\mathbf{I} \quad \forall \omega \in \mathbb{R}^d.$$

There is a link between minimization problems and games, since the vector field F becomes the gradient of the objective, and its Jacobian $\mathbf{J}_F(\omega)$ is the Hessian $\nabla^2 f(\omega)$. Thus, the class corresponding to the minimization of smooth, strongly convex functions is

$$\{F : \forall \omega \in \mathbb{R}^d,\ \operatorname{Sp} \mathbf{J}_F(\omega) \subset [\mu, L]\}\,,\ \ 0 < \mu \leq L\}.$$

Bilinear games. Consider the following problem,

$$\min_{x \in \mathbb{R}^d} \max_{y \in \mathbb{R}^d} x^\top A y\,.$$

Its Jacobian $\mathbf{J}_F(\omega)$ is constant and skew-symmetric. It is a standard linear algebra result (see Lem. 23) to show that $\operatorname{Sp} \mathbf{J}_F(\omega) \in \pm[i\sigma_{\min}(A), i\sigma_{\max}(A)]$.

Variational inequalities. The Lipchitz assumption

$$\|F(\omega) - F(\omega')\|_2^2 \leq L\|\omega - \omega'\|_2^2 \tag{16}$$

implies an upper bound on the magnitude of the eigenvalues of $\mathbf{J}_F(\omega^*)$. The strong monotonicity assumption

$$(\omega - \omega')^T(F(\omega) - F(\omega')) \geq \mu\|\omega - \omega'\|_2^2 \tag{17}$$

implies a lower bound on the real part of the eigenvalues of $\mathbf{J}_F(\omega^*)$ (see Lem. 21 in §C.2) which thus belong to

$$K = \{\lambda \in \mathbb{C} :\ 0 < \mu \leq \Re\lambda,\ |\lambda| \leq L\}$$

This set is the intersection between a circle and a half-plane, as shown in Figure 2 (left).

Fine-grained bounds. [Nem04] provides a lower-bound for the class of strongly monotone and Lipschitz operators (see §3.2.2) excluding the possibility of acceleration in that general setting. It motivates the adoption of more refined assumptions on the eigenvalues of $\mathbf{J}_F(\omega^*)$. We consider the class of games where these eigenvalues belong to a specified set K. Since $\mathbf{J}_F(\omega^*)$ is real, its spectrum is symmetric w.r.t. the real axis, so we assume that K is too. For this class of problem, we have a simple method to compute lower and upper convergence bounds using a class of well studied shapes: ellipses.

Proposition 2 (Ellipse method for lower and upper bound (Informal))**.** *Let $K \subset \mathbb{C}_+$ be a compact set, then any ellipse symmetric w.r.t. the real axis that includes (resp. is included in) K provides an upper (resp. lower) convergence bound for the class of problem $\mathcal{M}_K$ using Polyak momentum with a step-size and a momentum depending on the ellipse.*

See appendix C.3.2, Thm. 21 for the precise result on ellipses. The proposition extends to any shape whose optimal algorithm (resp. lower bound) is known. This propositionheavily relies on the fact that, the optimal method for ellipses is Polyak momentum [NV83].

Any first-order method can be seen as a way to transform the set K. In order to illustrate that we consider Lemma 3: since a first-order method update for a linear operator $F = Ax + b$ can be written using a polynomial p, the eigenvalues to consider are not the ones of A but the ones of $p(A)$. Thus, the set of interest is $p(K)$.

As an example, consider EG with momentum. This consists in applying the momentum method to the transformed vector field $\omega \mapsto F(\omega - \eta F(\omega))$. From a spectral point of view, this is equivalent to first transforming the shape K into $\varphi(K)$ with the extragradient mapping $\varphi_\eta : \lambda \mapsto \lambda(1 - \eta\lambda)$, then study the effect of momentum on $\varphi(K)$. This example of transformation is used in §3.3.4.

3.2 Asymptotic convergence factor

We recall known results that compute lower bounds for some classes of games using the *asymptotic convergence factor* [EN83, ENV85, Nev93]. Then, we illustrate them on two particular classes of problems.

3.2.1 Lower bounds for a class of problems

We now show how to lower bound the worst-case rate of convergence of a *specific* method over the class $\mathcal{M}_K$ (15), with the worst possible initialisation ω_0. We start with equation (14), but this time we pick the worst-case over all matrices $A \in \mathcal{M}_K$, i.e.,

$$\max_{A \in \mathcal{M}_K} \|p_t(A)(\omega_0 - \omega^*)\|_2.$$

Now, we can pick an arbitrary bad initialisation ω_0, in particular, the one that corresponds to the largest eigenvalue of $p_t(A)$ in magnitude. This gives

$$\begin{aligned} \exists \omega_0 : \|\omega_t - \omega^*\|_2 &\geq \max_{A \in \mathcal{M}_K} \rho\Big(p_t(A)\Big) \|\omega_0 - \omega^*\|_2 \\ &= \max_{\lambda \in K} |p_t(\lambda)| \|\omega_0 - \omega^*\|_2 \,. \end{aligned} \tag{18}$$

It remains to lower bound $\max_{\lambda \in K} |p_t(\lambda)|$ over *all possible* first-order methods. This is called the *asymptotic convergence factor*, presented in the next section.

3.2.2 Asymptotic convergence factor

Here we recall the definition of the *asymptotic convergence factor* [EN83], which gives a lower bound for the rate of convergence over matrices which belong to the class $\mathcal{M}_k$ (15), for all possible first-order methods. We mainly follow the definition of [Nev93] (see Rmk. 4 in §C.2 for details).

The simplest way to lower bound $\|\omega_t - \omega^*\|_2$ is given by minimizing (18) over all polynomials corresponding to a first-order method. By Lemma 3, this class of polynomials is given by $\mathcal{P}_t$. Thus, for some ω_0,

$$\|\omega_t - \omega^*\| \geq \min_{p_t \in \mathcal{P}_t} \max_{\lambda \in K} |p_t(\lambda)| \cdot \|\omega_0 - \omega^*\|_2.$$

The *asymptotic convergence factor* $\underline{\rho}(K)$ for the class K is given by taking the *minimum average* rate of convergence over t for any t, i.e.,

$$\underline{\rho}(K) = \inf_{t>0} \min_{p_t \in \mathcal{P}_t} \max_{\lambda \in K} \sqrt[t]{|p_t(\lambda)|}\,. \tag{19}$$

This way, by construction, $\underline{\rho}(K)$ gives a lower-bound on the *worst-case* rate of convergence for the class $\mathcal{M}_K$. We formalize this statement in the proposition below.

Proposition 3. *[Nev93] Let $K \subset \mathbb{C}$ be a subset of $\mathbb{C}$ symmetric w.r.t. the real axis, which does not contain 0 and such that $\mathcal{M}_K \neq \emptyset$. Then, any oblivious first-order method (whose coefficients only depend on K) satisfies,*

$$\forall t \geq 0,\, \exists A \in \mathcal{M}_K,\, \exists \omega_0 : \|\omega_t - \omega^*\|_2 \geq \underline{\rho}(K)^t \|\omega_0 - \omega^*\|_2.$$

However, the object $\underline{\rho}(K)$ may be complicated to obtain as it depends on the solution of a minimax problem *over a set* $K \subset \mathbb{C}_+$. If the set is simple enough, we can lower-bound the asymptotic rate of convergence. We start by giving the two extreme cases: when K is a segment on the real line (convex and smooth minimization) or K is a disc (monotone and smooth games).

3.2.3 Extreme cases: real segments and discs

Smooth and strongly convex minimization. In the case where we are interested in lower-bounds, we can consider the restricted class of functions where $J_F(\omega)(= \nabla^2 f(\omega))$ is constant, i.e., independent of ω. This corresponds to quadratic minimization, and our restricted class becomes

$$\mathcal{M}_K \quad \text{where} \;\; K = [\mu, L].$$

For this specific class, where K is a segment in the real line, the solution to the subproblem associated to the *asymptotic rate of convergence* (19), i.e.,

$$\min_{p \in \mathcal{P}_t} \max_{\lambda \in [\mu, L]} |p(\lambda)| \tag{20}$$

is well-known. The optimal polynomial p_t^* is a properly scaled and translated Chebyshev polynomial of the first kind of degree t [GV61, Man77]. The rate of convergence of p_t evolves with t, but asymptotically converges to

$$\underline{\rho}([\mu, L]) = \tfrac{\sqrt{L} - \sqrt{\mu}}{\sqrt{L} + \sqrt{\mu}}\,.$$

This is the lower bound of [Nes04, Thm. 2.1.13], which corresponds to an accelerated linear rate. The condition number L/μ appears as a square root unlike for the rate of the plain gradient descent, which implies a huge (asymptotic) improvement.

In this section, we have seen that when the spectrum is constrained to be on a segment in the real line, one can expect acceleration. The next section shows that this is not the case for the class of discs.

Discs and strongly monotone vector fields Consider a disc with a real positive center

$$K = \{z \in \mathbb{C} : |z - c| \leq r\}, \quad \text{with } 0 < c < r$$

This time again, the shape is simple enough to have an explicit solution for the optimal polynomials

$$p_t^*(\lambda) = \arg\min_{p_t \in \mathcal{P}_t} \max_{\lambda \in K} |p_t(\lambda)|.$$

In this case, the optimal polynomial reads $p_t^*(\omega) = (1 - \omega/c)^t$, and this corresponds to gradient descent with step-size $\eta = 1/c$. Hence, with this specific shape, gradient method is optimal [ENV85,

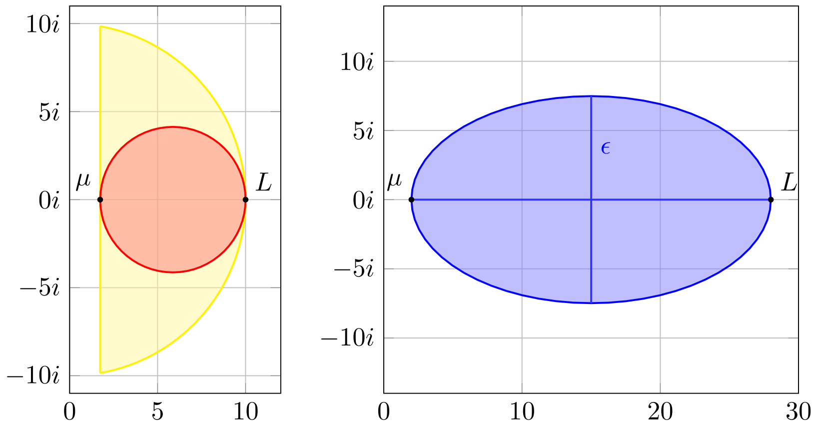

Figure 2. **Left:** Illustration of the proof of Cor. 3. The yellow set correspond to K, the set of strongly monotone problems while the red disc is the disc of center $\frac{1}{2}(\mu + L)$ and radius $\frac{1}{2}(L - \mu)$ which fits inside. **Right:** Illustration of K_ϵ of Prop. 6 with $\epsilon = \sqrt{\mu L}$.

§6.2]; [Nev93, Example 3.8.2]. A direct consequence of this result is a lower bound of convergence for the class of Lipshitz, strongly monotone vector fields, i.e., vector fields F that satisfies (16)-(17). For linear vector fields parameterized by the matrix A as in Lemma 3, this is included in the set

$$\mathcal{M}_K, \; K = \{\lambda \in \mathbb{C} : \; 0 < \mu \leq \Re\lambda, \; |\lambda| \leq L\} \tag{21}$$

This set is the intersection between a circle and a half-plane, as shown in Figure 2 (left). Notice that the disc of center $\frac{\mu+L}{2}$ and radius $\frac{L-\mu}{2}$ actually fits in K, as illustrated by Fig. 2. Since this disc in *included* in K, a lower bound for the disc also gives a lower bound for K, as stated in the following corollary.

Corollary 3. *Let K be defined in* (21)*. Then,*

$$\underline{\rho}(K) > \tfrac{L-\mu}{L+\mu} = 1 - \tfrac{2\mu}{L+\mu}\,.$$

The rate of Cor. 3 is already achieved by first-order methods, without momentum or acceleration, such as EG. Thus, acceleration is *not possible* for the general class of smooth, strongly monotone games.

3.3 Acceleration in games

We present our contributions in this section. The previous section highlights a big contrast between optimization and games. In the former, acceleration is possible, but this does not generalize for the latter. Here, we explore acceleration via a sharp analysis of intermediate cases, like imaginary segments (bilinear games) or thin ellipses (perturbed acceleration), via lower and upper bounds. Since we use spectral arguments, the convergence guarantees of our algorithms are local, but lower bounds remain valid globally.

3.3.1 Local convergence of optimization methods for nonlinear vector fields

Before presenting our result, we recall the classical local convergence theorem from [Pol64]. In this section, we are interested in finding the fixed point ω^* of a vector field V, i.e, $V(\omega^*) = \omega^*$. V here plays the role of an iterative optimization methods and defines iterates according to the fixed-point iteration

$$\omega_{t+1} = V(\omega_t). \tag{22}$$

Analysing the properties of the vector field V is usually challenging, as V can be any nonlinear function. However, under mild assumption, we can simplify the analysis by considering the linearization $V(\omega) \approx V(\omega^*) + \mathbf{J}_V(\omega^*)(\omega - \omega^*)$, where $\mathbf{J}_V(\omega)$ is the Jacobian of V evaluated at ω^*. The next theorem shows we can deduce the rate of convergence of (22) using the spectral radius of $\mathbf{J}_V(\omega^*)$, denoted by $\rho(\mathbf{J}_V(\omega^*))$.

Theorem 9 ([Pol87]). *Let $V : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be continuously differentiable and let ω^* one of its fixed-points. Assume that there exists $\rho^* > 0$ such that,*

$$\rho(\mathbf{J}_V(\omega^*)) \leq \rho^* < 1$$

For ω_0 close to ω^*, (22) converges linearly to ω^* at a rate $\mathcal{O}((\rho^* + \epsilon)^t)$. If V is linear, then $\epsilon = 0$.

Recent works such as [MNG17, GHP+19, DP18] used this connection to study game optimization methods.

Thm. 9 can be applied directly on methods which use only the last iterate, such as gradient or EG. For methods that do not fall into this category, such as momentum, a small adjustment is required, called *system augmentation*.

Consider that $V : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ follows the recursion

$$\omega_{t+1} = V(\omega_t, \omega_{t-1}). \tag{23}$$

Instead we consider its *augmented operator*

$$\begin{bmatrix} \omega_t \\ \omega_{t+1} \end{bmatrix} = V_{\text{augm}}(\omega_t, \omega_{t-1}) = \begin{bmatrix} \omega_t \\ V(\omega_t, \omega_{t-1}) \end{bmatrix},$$

to which we can now apply the previous theorem. This technique is summarized in the following lemma.

Lemma 4. *Let $V : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be continuously differentiable and let ω^* satisfies $V(\omega^*, \omega^*) = \omega^*$. Assume there exists $\rho^* > 0$ such that, $\rho(\mathbf{J}_{V_{augm}}(\omega^*)) \leq \rho^* < 1$. If ω_0 and ω_1 are close to ω^*, then* (22) *converges linearly to ω^* at rate $\big(\rho^* + \epsilon\big)^t$. If V is linear, then $\epsilon = 0$.*

3.3.2 Acceleration for bilinear games

For convex minimization, adding momentum results in an accelerated rate for strongly convex functions we have discuss above. For instance, if $\operatorname{Sp} \nabla F(\omega^*) \subset [\mu, L]$, the Polyak's Heavy-ball method (see the full statement inappendix C.3.1), [Pol64, Thm. 9]

$$\begin{aligned} \omega_{t+1} &= V^{\text{Polyak}}(\omega_t, \omega_{t-1}) \\ &:= \omega_t - \alpha F(\omega_t) + \beta(\omega_t - \omega_{t-1}) \end{aligned} \tag{24}$$

converges (locally) with the accelerated rate

$$\rho(\mathbf{J}_{V^{\text{Polyak}}}(\omega^*, \omega^*)) \leq \tfrac{\sqrt{L} - \sqrt{\mu}}{\sqrt{L} + \sqrt{\mu}}.$$

Another example are bilinear games. Most known methods converge at a rate of $(1 - c\sigma_{min}(A)^2/\sigma_{\max}(A)^2)^t$ for some $c > 0$ [DISZ18, MNG17, GBV+19, GHP+19, LS18, ALW19]. Using results from [ELV89], we show that this rate is suboptimal.

For bilinear games, the eigenvalues of the Jacobian $\mathbf{J}_F$ are purely imaginary (see Lem. 23 inappendix C.3.1), i.e.,

$$K = [i\sigma_{\min}(A), i\sigma_{\max}(A)] \cup [-i\sigma_{\min}(A), -i\sigma_{\max}(A)].$$

A method that follows strictly the vector field F does not converge, as its flow is composed by only concentric circles, thus leading to oscillations. This problem is avoided if we transform the vector field into another one with better properties. For example, the transformation

$$F^{\text{real}}(\omega) = \frac{1}{\eta}(F(\omega - \eta F(\omega)) - F(\omega)) \tag{25}$$

can be seen as a finite-difference approximation of $\nabla\left(\frac{1}{2}\|F\|_2^2\right)$. It is easier to find the equilibrium of V since the eigenvalues of $\mathbf{J}_V(\omega) = -\mathbf{J}_F^2(\omega)$ are located on a real segment. Thus, we can use standard minimization methods like the Polyak Heavy-Ball method.

Proposition 4. *Let F be a vector field such that $\operatorname{Sp}\nabla F(\omega^*) \subset [ia, ib] \cup [-ia, -ib]$, for $0 < a < b$. Setting $\sqrt{\alpha} = \frac{2}{a+b}$, $\sqrt{\beta} = \frac{b-a}{b+a}$, the Polyak Heavy-Ball method (24) on the transformation (25), i.e.,*

$$\omega_{t+1} = \omega_t - \alpha F^{real}(\omega_t) + \beta(\omega_t - \omega_{t-1})\,.$$

converges locally at a linear rate $O\big((1 - \frac{2a}{a+b})^t\big)$.

Using results from [ELV89], we show that this method is optimal. Indeed, for this set, we can compute explicitly $\underline{\rho}(K)$ from (19), the lower bound for the local convergence factor.

Proposition 5. *Let $K = [ia, ib] \cup [-ia, -ib]$ for $0 < a < b$. Then, $\underline{\rho}(K) = \sqrt{\frac{b-a}{b+a}}$.*

Proof. (Sketch). The transformation that we have applied, i.e. $\lambda \mapsto -\lambda^2$, preserves the asymptotic convergence factor $\underline{\rho}$ (up to a square root), as it satisfies the assumptions of [ELV89, Thm. 6]. □

The difference of a square root between the lower bound and the bound on the spectral radius is explained by the fact that the method presented here queries two gradient per iteration and so one of its iterations actually corresponds to two steps of a first-order method as defined in Definition 1.

In this subsection, we showed that when the eigenvalues of the Jacobian are purely real or imaginary, acceleration is possible using momentum on the right vector field. Yet the previous subsection shows it is not the case for general smooth, strongly monotone games. The question of acceleration remains for intermediate shapes, like ellipses. The next subsection shows how to recover an accelerated rate of convergence in this case.

3.3.3 Perturbed acceleration

As we cannot compute $\underline{\rho}$ explicitly for most sets K, we focus on ellipses to answer this question. They have been well studied, and optimal methods are again based on Chebyshev polynomials [Man77].

In this section we study games whose eigenvalues of the Jacobian belong to a thin ellipse. These ellipses correspond to the real segments $[\mu, L]$ perturbed in an elliptic way, see Fig. 2 (right). Mathematically, we have for $0 < \mu < L$ and $\epsilon > 0$, the equation

$$K_\epsilon = \left\{ z \in \mathbb{C} : \left(\frac{\Re z - \frac{\mu+L}{2}}{\frac{L-\mu}{2}} \right)^2 + \left(\frac{\Im z}{\epsilon} \right)^2 \leq 1 \right\}$$

When $\epsilon = 0$ (with the convention that $0/0 = 0$), Polyak momentum achieves the rate of $1 - 2\frac{\sqrt{\mu}}{\sqrt{\mu}+\sqrt{L}}$. However, when $\epsilon = \frac{L-\mu}{2}$, we showed the lower bound of $1 - 2\frac{\mu}{\mu+L}$ in Cor. 3. To check if acceleration still persists for intermediate cases, we study the behaviour of the asymptotic convergence factor (when $L/\mu \to +\infty$) as a function of ϵ. The next proposition uses results from [NV83, ENV85] to show that acceleration is still possible on K_ϵ.

Proposition 6. *Define $\epsilon(\mu, L)$ as $\frac{\epsilon(\mu,L)}{L} = \left(\frac{\mu}{L}\right)^\theta$ with $\theta > 0$ and $a \wedge b = \min(a, b)$. Then, when $\frac{\mu}{L} \to 0$,*

$$\underline{\rho}(K_\epsilon) = \begin{cases} 1 - 2\sqrt{\frac{\mu}{L}} + \mathcal{O}\left(\left(\frac{\mu}{L}\right)^{\theta \wedge 1}\right), & if\ \theta > \frac{1}{2} \\ 1 - 2(\sqrt{2}-1)\sqrt{\frac{\mu}{L}} + \mathcal{O}\left(\frac{\mu}{L}\right), & if\ \theta = \frac{1}{2} \\ 1 - \left(\frac{\mu}{L}\right)^{1-\theta} + \mathcal{O}\left(\left(\frac{\mu}{L}\right)^{1 \wedge (2-3\theta)}\right), & if\ \theta < \frac{1}{2}. \end{cases}$$

Moreover, the momentum method is optimal for K_ϵ. This means there exists $\alpha > 0$ and $\beta > 0$ (function of μ, L and ϵ only) such that if $\operatorname{Sp} \boldsymbol{J}_F(\omega^*) \subset K_\epsilon$*, then, $\rho(\boldsymbol{J}_{V^{Polyak}}(\omega^*, \omega^*)) \leq \underline{\rho}(K_\epsilon)$.*

This shows that the convergence rate interpolates continuously between the accelerated rate and the non-accelerated one. Crucially, for small perturbations, that is to say if the ellipse is thin enough, acceleration persists until $\theta = \frac{1}{2}$ or equivalently $\epsilon \sim \sqrt{\mu L}$. That's why Prop. 6 plays a central role in our forthcoming analyses of accelerated EG and CO.

3.3.4 Accelerating extragradient

We now consider the acceleration of EG using momentum. Its main appealing property is its convergence on bilinear games, unlike the gradient method. On the class of bilinear problems, EG achieves a convergence rate of $(1 - ca^2/b^2)$ for some constant $c > 0$.

In the previous section, we achieved an accelerated rate on bilinear games by applying momentum to the operator $F^{\text{real}}(\omega)$ instead of F, as the Jacobian of F^{real} has real eigenvalues when $\mathbf{J}_F(\omega^*)$ has its spectrum in K. Here we try to apply momentum to the EG operator $F^{\text{e-g}}(\omega)$, defined as

$$F^{\text{e-g}}(\omega) = F(\omega - \eta F(\omega)) \,. \tag{26}$$

Unfortunately, when $\operatorname{Sp} \mathbf{J}_F \subset K$, the spectrum of $F^{\text{e-g}}(\omega^*)$ is never purely real. Using the insight from Prop. 6, we can choose $\eta > 0$ such that we are in the first case of Prop. 6, making acceleration possible.

Proposition 7. *Consider the vector field F, where* $\operatorname{Sp} \boldsymbol{J}_F(\omega^*) \subset [ia, ib] \cup [-ia, -ib]$ *for $0 < a < b$. There exists $\alpha, \beta, \eta > 0$ such that, the operator defined by*

$$\omega_{t+1} = \omega_t - \alpha F(\omega_t - \eta F(\omega_t)) + \beta(\omega_t - \omega_{t-1}) \,,$$

converges locally at a linear rate $O\big(\big(1 - c\frac{a}{b} + M\frac{a^2}{b^2}\big)^t\big)$ where $c = \sqrt{2} - 1$ and M is an absolute constant.

One drawback is that, to achieve fast convergence on bilinear games, one has to tune the two step-sizes α, η of EG precisely and separately. They actually differ by a factor $\frac{b^2}{a^2}$: η is roughly proportional to $\frac{1}{a}$ while α behaves like $\frac{a}{b^2}$ (see Lem. 25 in appendix C.3.4).

3.4 Beyond typical first-order methods

In the previous section, we achieved acceleration with first-order methods for specific problem classes. However, the lower bound from Cor. 3 still prevents us from doing so for the larger problem classes for smooth and strongly monotone games. To bypass this limitation, we can consider going *beyond* first-order methods. In this section, we consider two different approaches. The first one is adaptive acceleration, which is a *non-oblivious* first-order method. The second is consensus optimization, an inversion-free second order method.

3.4.1 Adaptive acceleration

In previous sections, we considered shapes whose optimal polynomial is known. This optimal polynomial lead to an optimal first-order method. However, when the shape is *unknown*, we cannot use better methods than EG with an appropriate stepsize.

Recent work in optimization analysed adaptive algorithms, such as *Anderson Acceleration* [WN11], that are adaptive to the problem constants. They can be seen as an automatic way to find the optimal combination of the previous iterates. Recent works on Anderson Acceleration extended the theory for non-quadratic minimization, by using regularisation [SdB16] (RNA method). The theory has also been extended to "non symmetric operators" [BSd18], and this setting fits perfectly the one of games, as $\mathbf{J}_F(\omega^*)$ is not symmetric.

Anderson acceleration and its extension RNA are similar to quasi-Newton [FS09], but remains first-order methods. Even if they find the optimal first-order method (for linear F), they cannot beat a lower bound similar to Cor. 3, when the number of iterations is smaller than the dimension of the problem. The next section shows how to use *cheap* second-order information to improve the convergence rate.

3.4.2 Momentum consensus optimization

CO [MNG17] iterates as follow:

$$\omega_{t+1} = \omega_t - \alpha\big(F(\omega_t) + \tau \mathbf{J}_F^T(\omega)F(\omega)\big).$$

Albeit being a second-order method, each iteration requires only one Jacobian-vector multiplication. This operation can be computed efficiently by modern machine learning frameworks, with automatic differentiation and back-propagation. For instance, for neural networks, the computation time of this product or the gradient is comparable. Moreover, unlike Newton's method, CO does *not* require a matrix inversion.

Though CO is a second-order method, its analysis can still be reduced to our framework by considering the following transformation of the initial operator $F(\omega)$,

$$F^{\text{cons.}}(\omega) = F(\omega) + \tau\nabla\left(\frac{1}{2}\|F\|^2\right)(\omega)\,. \tag{27}$$

Though the eigenvalues of $\mathbf{J}_{F^{\text{cons.}}}$ are not purely real in general, their imaginary to real part ratio can be controlled by [MNG17, Lem. 9] as,

$$\max_{\lambda\in\operatorname{Sp}\mathbf{J}_{F^{\text{cons.}}}(\omega^*)} \frac{|\Im\lambda|}{|\Re\lambda|} = O\left(\frac{1}{\tau}\right)\,.$$

Therefore, if τ increases, these eigenvalues move closer to the real axis and can be included in a thin ellipse as described by §3.3.3. We then show that, if τ is large enough, this ellipse can be chosen thin enough to fall into the accelerated regime of Prop. 6 and therefore, adding momentum achieves acceleration.

Proposition 8. *Let σ_i be the singular values of $\boldsymbol{J}_F(\omega^*)$. Assume that*

$$\gamma \leq \sigma_i \leq L, \quad \tau = \frac{L}{\gamma^2}.$$

There exists α, β, s.t., momentum applied to $F^{cons.}$,

$$\omega_{t+1} = \omega_t - \alpha F^{cons.}(\omega_t) + \beta(\omega_t - \omega_{t-1})$$

converges locally at a rate $O\Big(\big(1 - c\frac{\gamma}{L} + M\frac{\gamma^2}{L^2}\big)^t\Big)$ where $c = \sqrt{2} - 1$ and M is an absolute constant.

Hence, adding momentum to CO yields an accelerated rate. The assumption on the Jacobian encompasses both strongly monotone and bilinear games. On these two classes of problems, CO is at least as fast as any oblivious first-order method as its rate roughly matches the lower bounds of Prop. 3 and 5.

Note that, choosing τ of this order is what is done by [ALW19] for (non-accelerated) CO. They claim that this point of view – seeing consensus as a perturbation of gradient descent on $\frac{1}{2}\|F\|^2$ – is justified by practice as in the experiments of [MNG17], τ is set to 10.

4 Conclusion

In this paper, we presented a comprehensive spectral analysis of gradient-based methods in multi-player differentiable games, covering a wide range of game types, from bilinear to strongly monotone structures. By leveraging matrix iteration theory, we introduced the concept of spectral shape, linking game difficulty to the geometric distribution of eigenvalues in the Jacobian. This perspective allowed us to propose an optimal algorithm for bilinear games and derive new lower bounds, demonstrating the near-optimality of the extragradient (EG) method among first-order algorithms.

Our analysis unified the behavior of EG, optimistic gradient (OG), and consensus optimization (CO), providing global convergence guarantees and highlighting that these methods leverage different convergence mechanisms depending on the game's structure. Notably, EG outperforms gradient descent (GD) in adversarial settings, while OG and CO achieve similar improvements in specific regimes.

We also explored the potential for acceleration in game dynamics, drawing parallels to Polyak and Nesterov momentum methods in convex optimization. While the possibility of acceleration in adversarial games remains an open question, we proposed momentum-based enhancements to consensus optimization, achieving near-accelerated rates.

Future work will explore extending these findings to stochastic and high-dimensional settings, as well as refining the spectral analysis to handle more intricate eigenvalue distributions, especially in real-world applications like generative adversarial networks (GANs). These directions hold promise for advancing the performance of optimization methods in complex machine learning environments.

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A Related Work

Extragradient was first introduced by [Kor76] in the context of variational inequalities. [Tse95] proves results which induce linear convergence rates for this method in the bilinear and strongly monotone cases. We recover both rates with our analysis. The extragradient method was generalized to arbitrary geometries by [Nem04] as the mirror-prox method. A sublinear rate of $\mathcal{O}(1/t)$ was proven for monotone variational inequalities by treating this method as an approximation of the proximal point method as we will discuss later. More recently, [MLZ$^+$19] proved that, for a broad class of saddle-point problems, its stochastic version converges almost surely to a solution.

Optimistic gradient method is slightly different from EG and can be seen as a kind of extrapolation from the past [GBV$^+$19]. It was initially introduced for online learning [Chiang et al.(2012), Rakhlin and Sridharan(2013)] and subsequently studied in the context of games by [DISZ18], who proved that this method converges on bilinear games. [GBV$^+$19] interpreted GANs as a variational inequality problem and derived OG as a variant of EG which avoids "wasting" a gradient. They prove a linear convergence rate for strongly monotone variational inequality problems. Treating EG and OG as perturbations of the proximal point method, [MOP19] gave new but still separate derivations for the standard linear rates in the bilinear and the strongly convex-concave settings. [LS18] mentioned the potential impact of the interaction between the players, but they only formally show this on bilinear examples: our results show that this conclusion extends to general nonlinear games.

Consensus optimization has been motivated by the use of gradient penalty objectives for the practical training of GANs [Gulrajani et al.(2017), Mescheder et al.(2017)]. It has been analysed by [ALW19] as a perturbation of Hamiltonian gradient descent.

We provide a unified and tighter analysis for these three algorithms leading to faster rates (cf. Tab. 1).

Lower bounds in optimization date back to [NY83] and were popularized by [Nes04]. One issue with these results is that they are either only valid for a finite number of iterations depending on the dimension of the problem or are proven in infinite dimensional spaces. To avoid this issue, [ASS16] introduced a new framework called p-Stationary Canonical Linear Iterative algorithms (p-SCLI). It encompasses methods which, applied on quadratics, compute the next iterate as fixed linear transformation of the p last iterates, for some fixed $p \geq 1$. We build on and extend this framework to derive lower bounds for games for 1-SCLI. Concurrenty, [IAGM19] extended the whole p-SCLI framework to games but excluded extrapolation methods. Note that sublinear lower bounds have been proven for saddle-point problems by [Nem92, Nem04, CLO14, OX18], but they are outside the scope of this paper since we focus on linear convergence bounds.

Matrix iteration theory. There is extensive literature on iterative methods for linear systems, due to their countless applications. An important line of work considers the design of iterative methods through the lens of approximation problems by polynomials on the complex plane. [EN83] then used complex analysis tools to define, for a given compact set, its *asymptotic convergence factor*: it is the optimal asymptotic convergence rate a first-order method can achieve for all linear systems with spectrum in the set. [NV83] bring tools from summability theory to analyze multi-step iterative methods in this framework and provide optimal methods, in particular, the momentum method for ellipses. [ENV85] continued in this direction, summarizing and improving the previous results. Finally [ELV89] study how polynomial transformations of the spectrum help compute the asymptotic convergence factor and the optimal method for a given set, potentially yielding faster convergence.

Acceleration and lower bounds. Lower bounds of convergence are standard in convex optimization [Nes04] but are often non-asymptotic or cast in an infinite-dimensional space. [ASS16, AS16] showed non-asymptotic lower bounds using a framework called p-SCLI close to matrix iteration theory. [IAGM19, AMLJG19] extended this framework to multi-player games, but they consider lower and upper-bounds on the eigenvalues of the Jacobian of the game rather than their distribution in the complex plane. Two main acceleration methods in convex optimization achieve these lower bounds, Polyak's momentum [Pol64] and Nesterov's acceleration [Nes83]. The latter is the only one that has global convergence guarantees for convex functions. Nevertheless, Polyak's momentum still plays a crucial role in the training of large scale machine learning models [SMDH13].

Acceleration for games. Recent work applied acceleration techniques to game optimization. [GHP+19] showed that negative momentum with alternating updates converges on bilinear games, but with the same geometrical rate as EG. [CLO14] provided a version of the mirror-prox method which improves the constant but not its rate. In the context of minimax optimization, [PB16] used Catalyst [LMH15], a generic acceleration method, to improve the convergence of variance-reduced algorithms for min-max problems. In the context of variational inequalities, the standard assumptions on the operator are Lipschitzness and (strong) monotonicity [Tse95, Nes03]. [Nem04] provided a lower bound in $\mathcal{O}(1/t)$ on the convergence rate for smooth monotone games, which suggests that EG is nearly optimal in the strongly monotone case. In our work, we show that acceleration is possible by substituting the smoothness and monotonicity assumptions on the operator into more precise assumptions on the *eigenvalues of its Jacobian*.

B Appendix for Section 2

B.1 Appendix

B.1.1 Notation

We denote by $\mathrm{Sp}(A)$ the spectrum of a matrix A. Its spectral radius is defined by $\rho(A) = \max\{|\lambda| \mid \lambda \in \mathrm{Sp}(A)\}$. We write $\sigma_{min}(A)$ for the smallest singular value of A, and $\sigma_{max}(A)$ for the largest. $\Re$ and $\Im$ denote respectively the real part and the imaginary part of a complex number. We write $A \preccurlyeq B$ for two symmetric real matrices if and only if $B-A$ is positive semi-definite. For a vector $X \in \mathbb{C}^d$, denote its transpose by X^T and its conjugate transpose by X^H. $\|.\|$ denotes an arbitrary norm on $\mathbb{R}^d$ unless specified. We sometimes denote $\min(a,b)$ by $a \wedge b$ and $\max(a,b)$ by $a \vee b$. For $f : \mathbb{R}^d \to \mathbb{R}^d$, we denote by f^k the composition of f with itself k times, i.e. $f^k(\omega) = \underbrace{f \circ f \circ \cdots \circ f}_{k \text{ times}}(\omega)$.

B.1.2 Interpretation of spectral quantities in a two-player zero-sum game

In this appendix section, we are interested in interpreting spectral bounds in terms of the usual strong convexity and Lipschitz continuity constants in a two-player zero-sum game:

$$\min_{x\in\mathbb{R}^m} \max_{y\in\mathbb{R}^p} f(x,y) \tag{28}$$

with f is two times continuously differentiable.

Assume,

$$\mu_1 I_m \preccurlyeq \nabla_x^2 f \preccurlyeq L_1 I_m \tag{29}$$
$$\mu_2 I_p \preccurlyeq -\nabla_y^2 f \preccurlyeq L_2 I_p \tag{30}$$
$$\mu_{12}^2 I_p \preccurlyeq (\nabla_x \nabla_y f)^T (\nabla_x \nabla_y f) \preccurlyeq L_{12}^2 I_p \tag{31}$$

where μ_1, μ_2 and μ_{12} are non-negative constants. Let $\omega^* = (x^*, y^*)$ be a stationary point. To ease the presentation, let,

$$\nabla v(\omega^*) = \begin{pmatrix} \nabla_x^2 f(\omega^*) & (\nabla_x \nabla_y f(\omega^*))^T \\ -(\nabla_x \nabla_y f(\omega^*)) & \nabla_y^2 f(\omega^*) \end{pmatrix} = \begin{pmatrix} S_1 & A \\ -A^T & S_2 \end{pmatrix} \tag{32}$$

Now, more precisely, we are interested in lower bounding $\Re(\lambda)$ and $|\lambda|$ and upper bounding $|\lambda|$ for $\lambda \in \operatorname{Sp} \nabla v(\omega^*)$.

Commutative and square case In this subsection we focus on the square and commutative case as formalized by the following assumptions:

Assumption 1 (Square and commutative case)**.** The following holds:

(i) $p = m = \frac{d}{2}$;

(ii) S_2 and A^T commute;

(iii) S_1, S_2 and AA^T commute.

Assumption 1 holds if, for instance, the objective is separable, i.e. $f(x, y) = \sum_{i=1}^m f_i(x_i, y_i)$. Then, using a well-known linear algebra theorem, Assumption 1 implies that there exists $U \in \mathbb{R}^{d \times d}$ unitary such that $S_1 = U \operatorname{diag}(\alpha_1, \dots, \alpha_m) U^T$, $S_2 = U \operatorname{diag}(\beta_1, \dots, \beta_m) V^T$ and $AA^T = U \operatorname{diag}(\sigma_1^2, \dots, \sigma_d^2) U^T$ where $\alpha_1, \dots, \alpha_m$ are the eigenvalues of S_1, $\beta_1, \dots, \beta_m$ are the eigenvalues of S_2 and $\sigma_1, \dots, \sigma_p$ are the singular values of A. See [Lax07, p. 74] for instance.

Define,

$$\mu = \begin{pmatrix} \mu_1 & \mu_{12} \\ -\mu_{12} & \mu_2 \end{pmatrix}$$

$$L = \begin{pmatrix} L_1 & L_{12} \\ -L_{12} & L_2 \end{pmatrix}$$

Denote by $|\mu|$ and $|L|$ the determinants of theses matrices, and by $\operatorname{Tr} \mu$ and $\operatorname{Tr} L$ their traces.

In this case we get an exact characterization of the spectrum $\nabla v(\omega^*)$, which we denote by $\sigma^* = \operatorname{Sp} \nabla v(\omega^*)$:

Lemma 5. *Under Assumption 1, $\lambda \in sigma^*$ if and only if there exists some $i \leq d$ such that λ is a root of*

$$P_i = X^2 - (\alpha_i + \beta_i) X + \alpha_i \beta_i + \sigma_i^2$$

Proof. We compute the characteristic polynomial of $\nabla v(\omega^*)$ using that S_2 and A^T commute, using the formula for the determinant of a block matrix, which can be found in [Zha05, Section 0.3] for instance.

$$\begin{aligned}\begin{vmatrix} XI - S_1 & -A \\ A^T & XI - S_2 \end{vmatrix} &= |(XI - S_1)(XI - S_2) + AA^T| \\ &= |X^2 I - X(S_1 + S_2) + S_1 S_2 + AA^T| \\ &= \prod_i \left(X^2 - (\alpha_i + \beta_i)X + \alpha_i\beta_i + \sigma_i^2\right)\end{aligned}$$

□

Theorem 10. *Under Assumption 1, we have the following results on the eigenvalues of* $\nabla v(\omega^*)$.

(a) For $i \leq m$*, if* $(\alpha_i - \beta_i)^2 < 4\sigma_i^2$*, the roots of* P_i *satisfy:*

$$\frac{\operatorname{Tr}\mu}{2} \leq \Re(\lambda), \quad \det\mu \leq |\lambda|^2 \leq \det L \quad \forall \lambda \in \mathbb{C} \; s.t. \; P_i(\lambda) = 0 \tag{33}$$

(b) For $i \leq m$*, if* $(\alpha_i - \beta_i)^2 \geq 4\sigma_i^2$*, the roots of* P_i *are real non-negative and satisfy :*

$$\max\left(\mu_1 \wedge \mu_2, \frac{\det\mu}{\operatorname{Tr} L}\right) \leq \lambda \leq L_1 \vee L_2 \quad \forall \lambda \in \mathbb{C} \; s.t. \; P_i(\lambda) = 0 \tag{34}$$

(c) Hence, in general,

$$\mu_1 \wedge \mu_2 \leq \Re\lambda, \quad |\lambda|^2 \leq 2L_{max}^2 \quad \forall \lambda \in \sigma^* \tag{35}$$

where $L_{max} = \max(L_1, L_2, L_{12})$.

Proof. (a) Assume that $(\alpha_i - \beta_i)^2 < 4\sigma_i^2$, i.e. the discriminant of the polynomial P_i of Lem. 5 is negative. Consider λ a root of P_i. Then $\Re\lambda = \frac{\alpha_i + \beta_i}{2}$ and $|\lambda|^2 = \alpha_i\beta_i + \sigma_i^2$. Hence $\Re\lambda \geq \frac{1}{2}\operatorname{Tr}\mu$ and $\det\mu \leq |\lambda|^2 \leq \det L$.

(b) Assume that $(\alpha_i - \beta_i)^2 \geq 4\sigma_i^2$, i.e. the discriminant of the polynomial P_i of Lem. 5 is non-negative. This implies that $\Delta = (\operatorname{Tr} L)^2 - 4\det\mu \geq 0$.

Denote by λ_+ and λ_- the two real roots of P_i. Then

$$\lambda_\pm = \frac{\alpha_i + \beta_i \pm \sqrt{(\alpha_i + \beta_i)^2 - 4(\alpha_i\beta_i + \sigma_i^2)}}{2}$$

Hence

$$\lambda_+ \geq \lambda_- \geq \min_{\max(\operatorname{Tr}\mu,\; 4\det\mu) \leq x \leq \operatorname{Tr} L} \frac{x - \sqrt{x^2 - 4\det\mu}}{2}$$

As $x \mapsto x - \sqrt{x^2 - 4\det\mu}$ is decreasing on its domain, the minimum is reached at $\operatorname{Tr} L$ and is $\frac{\operatorname{Tr} L - \sqrt{\Delta}}{2} \geq 0$. However this lower bound is quite loose when $A = 0$. So note that

$$\lambda_- = \frac{\alpha_i + \beta_i - \sqrt{(\alpha_i + \beta_i)^2 - 4(\alpha_i\beta_i + \sigma_i^2)}}{2} \tag{36}$$

$$
\begin{aligned}
&\geq \frac{\alpha_i + \beta_i - \sqrt{(\alpha_i - \beta_i)^2}}{2} = \alpha_i \wedge \beta_i && (37)\\
&\geq \mu_1 \wedge \mu_2 && (38)\\
& && (39)
\end{aligned}
$$

Similarly,

$$
\lambda_+ \leq \frac{\alpha_i + \beta_i + \sqrt{(\alpha_i - \beta_i)^2}}{2} = \alpha_i \vee \beta_i \leq L_1 \vee L_2 \tag{40}
$$

Finally:

$$
L_1 \vee L_2 \geq \lambda_+ \geq \lambda_- \geq \max\left(\frac{\operatorname{Tr} L - \sqrt{\Delta}}{2}, \mu_1 \wedge \mu_2\right) \tag{41}
$$

Moreover,

$$
\begin{aligned}
\operatorname{Tr} L - \sqrt{\Delta} &= \frac{\left(\operatorname{Tr} L - \sqrt{\Delta}\right)\left(\operatorname{Tr} L + \sqrt{\Delta}\right)}{\operatorname{Tr} L + \sqrt{\Delta}} && (42)\\
&= \frac{4 \det \mu}{\operatorname{Tr} L + \sqrt{\Delta}} && (43)\\
&\geq \frac{2 \det \mu}{\operatorname{Tr} L} && (44)
\end{aligned}
$$

which yields the result.

(c) These assertions are immediate corollaries of the two previous ones.

□

We need the following lemma to be able to interpret Thm. 6 in the context of Example 2, whose assumptions imply Assumption 1.

Lemma 6. *Under Assumption 1, the singular values of $\nabla v(\omega^*)$ can be lower bounded as:*

$$
\mu_{12}(\mu_{12} - \max(L_1 - \mu_2, L_2 - \mu_1)) \leq \sigma_{min}(\nabla v(\omega^*))^2 \tag{45}
$$

In particular, if $\mu_{12} > 2\max(L_1 - \mu_2, L_2 - \mu_1)$, this becomes

$$
\frac{1}{2}\mu_{12}^2 \leq \sigma_{min}(\nabla v(\omega^*))^2 \tag{46}
$$

Proof. To prove this we compute the eigenvalues of $(\nabla v(\omega^*))^T \nabla v(\omega^*)$. We have that,

$$
(\nabla v(\omega^*))^T \nabla v(\omega^*) = \begin{pmatrix} S_1^2 + AA^T & S_1 A - AS_2 \\ A^T S_1 - S_2 A^T & A^T A + S_2^2 \end{pmatrix} \tag{47}
$$

As in the proof of Lem. 5, as Assumption 1 implies that $A^T S_1 - S_2 A^T$ and $A^T A + S_2^2$ commute,

$$
\begin{aligned}
|XI - (\nabla v(\omega^*))^T \nabla v(\omega^*)| &= \left|(XI - S_1^2 - AA^T)(XI - S_2^2 - A^T A) - (S_1 - S_2)^2 AA^T\right| && (48)\\
&= \prod_i \left((XI - \alpha_i^2 - \sigma_i^2)(XI - \beta_i^2 - \sigma_i^2) - (\alpha_i - \beta_i)^2 \sigma_i^2\right) && (49)
\end{aligned}
$$

Let $Q_i(X) = (XI - \alpha_i^2 - \sigma_i^2)(XI - \beta_i^2 - \sigma_i^2) - (\alpha_i - \beta_i)^2\sigma_i^2$. Its discriminant is

$$\Delta_i' = (\alpha_i^2 + \beta_i^2 + 2\sigma_i^2)^2 - 4\left((\alpha_i^2 + \sigma_i^2)(\beta_i^2 + \sigma_i^2) - (\alpha_i - \beta_i)^2)\sigma_i^2\right) \tag{50}$$

$$= (\alpha_i - \beta_i)^2((\alpha_i + \beta_i)^2 + 4\sigma_i^2) \geq 0 \tag{51}$$

Hence the roots of Q_i are:

$$\lambda_{i\pm} = \frac{1}{2}\left(\alpha_i^2 + \beta_i^2 + 2\sigma_i^2 \pm \sqrt{(\alpha_i - \beta_i)^2((\alpha_i + \beta_i)^2 + 4\sigma_i^2)}\right) \tag{52}$$

The smallest is λ_{i-} which can be lower bounded by

$$\lambda_{i-} = \frac{1}{2}\left(\alpha_i^2 + \beta_i^2 + 2\sigma_i^2 - \sqrt{(\alpha_i + \beta_i)^2((\alpha_i + \beta_i)^2 + 4\sigma_i^2)}\right) \tag{53}$$

$$\geq \frac{1}{2}\left(\alpha_i^2 + \beta_i^2 - |\alpha_i^2 - \beta_i^2| + 2\sigma_i(\sigma_i - |\alpha_i - \beta_i|)\right) \tag{54}$$

$$\geq \sigma_i(\sigma_i - |\alpha_i - \beta_i|) \tag{55}$$

$$\geq \mu_{12}(\mu_{12} - \max(L_1 - \mu_2, L_2 - \mu_1)) \tag{56}$$

□

B.1.3 Complement for §2.1

The convergence result of Thm. 9 can be strengthened if the Jacobian is constant as shown below. A proof of this classical result in linear algebra can be found in [ASS16] for instance.

Theorem 11. *Let $F : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a linear operator. If $\rho(\nabla F) < 1$, then for all $\omega_0 \in \mathbb{R}^d$, the iterates $(\omega_t)_t$ defined as above converge linearly to ω^* at a rate of $\mathcal{O}((\rho(\nabla F))^t)$.*

B.1.4 Convergence results of §2.2

Let us restate Thm. 3 for clarity.

Theorem 3. *Let ω^* be a stationary point of v and denote by σ^* the spectrum of $\nabla v(\omega^*)$. If the eigenvalues of $\nabla v(\omega^*)$ all have positive real parts, then*

(i). [GHP+19] For $\eta = \min_{\lambda \in \sigma^*} \Re(1/\lambda)$, the spectral radius of F_η can be upper-bounded as

$$\rho(\nabla F_\eta(\omega^*))^2 \leq 1 - \min_{\lambda\in\sigma^*} \Re(1/\lambda) \min_{\lambda\in\sigma^*} \Re(\lambda)$$

(ii). For all $\eta > 0$, the spectral radius of the gradient operator F_η at ω^* is lower bounded by

$$\rho(\nabla F_\eta(\omega^*))^2 \geq 1 - 4\min_{\lambda\in\sigma^*} \Re(1/\lambda) \min_{\lambda\in\sigma^*} \Re(\lambda)$$

In this subsection, we quickly show how to obtain (i) of Thm. 3 from Theorem 2 of [GHP+19], whose part which interests us now is the following:

Theorem ([GHP+19, part of Theorem 2])**.** *If the eigenvalues of $\nabla v(\omega^*)$ all have positive real parts, then for $\eta = \Re(1/\lambda_1)$ one has*

$$\rho(\nabla F_\eta(\omega^*))^2 \leq 1 - \Re(1/\lambda_1)\delta \tag{57}$$

where $\delta = \min_{1\leq j\leq m} |\lambda_j|^2(2\Re(1/\lambda_j) - \Re(1/\lambda_1))$ and $\operatorname{Sp}\nabla v(\omega^*) = \{\lambda_1, \dots, \lambda_m\}$ sorted such that $0 < \Re(1/\lambda_1) \leq \Re(1/\lambda_2) \leq \dots \leq \Re(1/\lambda_m)$.

Proof of (i) of Thm. 3. By definition of the order on the eigenvalues,

$$\delta = \min_{1\leq j\leq m} |\lambda_j|^2(\Re(1/\lambda_j) + \Re(1/\lambda_j) - \Re(1/\lambda_1)) \tag{58}$$

$$\geq \min_{1\leq j\leq m} |\lambda_j|^2(\Re(1/\lambda_j)) \tag{59}$$

$$= \min_{1\leq j\leq m} \Re(\lambda_j) \tag{60}$$

□

To prove the second part of Thm. 3, we rely on a different part of [GHP+19, Theorem 2] which we recall below:

Theorem ([GHP+19, part of Theorem 2])**.** *The best step-size η^*, that is to say the solution of the optimization problem*

$$\min_{\eta} \rho(\nabla F_\eta(\omega^*))^2 \tag{61}$$

satisfy:

$$\min_{\lambda\in\sigma^*} \Re(1/\lambda) \leq \eta^* \leq 2 \min_{\lambda\in\sigma^*} \Re(1/\lambda) \tag{62}$$

(ii) of Thm. 3 is now immediate.

Proof of (ii) of Thm. 3. By definition of the spectral radius,

$$\rho(\nabla F_{\eta^*}(\omega^*))^2 = \max_{\lambda\in \mathrm{Sp}(\nabla v(\omega^*))} |1-\eta^*\lambda|^2 \tag{63}$$

$$= 1 - \min_{\lambda\in \mathrm{Sp}(\nabla v(\omega^*))} 2\eta^*\Re\lambda - |\eta^*\lambda|^2 \tag{64}$$

$$\geq 1 - \min_{\lambda\in \mathrm{Sp}(\nabla v(\omega^*))} 2\eta^*\Re\lambda \tag{65}$$

$$\geq 1 - 4 \min_{\lambda\in \mathrm{Sp}(\nabla v(\omega^*))} \Re\lambda \min_{\lambda\in\sigma^*} \Re(1/\lambda) \tag{66}$$

□

Corollary 1. *Under the assumptions of Thm. 3 and Ex. 2,*

$$\rho(\nabla F_\eta(\omega^*))^2 \leq 1 - \frac{1}{4}\frac{(\mu_1+\mu_2)^2}{L_{12}^2 + L_1 L_2} \tag{3}$$

Proof. Note that the hypotheses stated in §2.2 correspond to the assumptions of §B.1.2. Moreover, with the notations of this subsection, one has that $4\sigma_i^2 \geq 4\mu_{12}^2$ and $\max(L_1, L_2)^2 \geq (\alpha_i - \beta_i)^2$. Hence the condition $2\mu_{12} \geq \max(L_1, L_2)$ implies that all the eigenvalues of $\nabla v(\omega^*)$ satisfy the case (a) of Thm. 10. Then, using Thm. 3,

$$\rho(\nabla F_\eta(\omega^*))^2 \leq 1 - \min_{\lambda\in\sigma^*} \Re(1/\lambda) \min_{\lambda\in\sigma^*} \Re(\lambda) \tag{67}$$

$$\leq 1 - \left(\frac{\min_{\lambda\in\sigma^*} \Re(\lambda)}{\max_{\lambda\in\sigma^*} |\lambda|}\right)^2 \tag{68}$$

$$\leq 1 - \frac{1}{4}\frac{(\mu_1+\mu_2)^2}{L_{12}^2 + L_1 L_2} \tag{69}$$

□

B.1.5 Spectral analysis of §2.3

We prove Lem. 1.

Lemma 7. *Assuming that the eigenvalues of $\nabla v(\omega^*)$ all have non-negative real parts, the proximal point operator P_η is continuously differentiable in a neighborhood of ω^* . Moreover, the spectra of the k-extrapolation operator and the proximal point operator are given by:*

$$\mathrm{Sp}\, \nabla F_{\eta,k}(\omega^*) = \left\{ \textstyle\sum_{j=0}^{k} (-\eta\lambda)^j \mid \lambda \in \sigma^* \right\} \tag{70}$$

$$\textit{and} \quad \mathrm{Sp}\, \nabla P_\eta(\omega^*) = \left\{ (1+\eta\lambda)^{-1} \mid \lambda \in \sigma^* \right\} \tag{71}$$

Hence, for all $\eta > 0$, the spectral radius of the operator of the proximal point method is equal to:

$$\rho(\nabla P_\eta(\omega^*))^2 = 1 - \min_{\lambda \in \sigma^*} \frac{2\eta \Re\lambda + \eta^2 |\lambda|^2}{|1+\eta\lambda|^2} \tag{72}$$

To prove the result about the k-extrapolation operator, we first show the following lemma, which will be used again later.

Recall that we defined $\varphi_{\eta,\omega} : z \mapsto \omega - \eta v(z)$. We drop the dependence on η in $\varphi_{\eta,\omega}$ for compactness.

Lemma 8. *The Jacobians of $\varphi_\omega^k(z)$ with respect to z and ω can be written as*

$$\nabla_z \varphi_\omega^k(z) = (-\eta)^k \nabla v(\varphi_\omega^{k-1}(z)) \nabla v(\varphi_\omega^{k-2}(z)) \dots \nabla v(\varphi_\omega^0(z)) \tag{73}$$

$$\nabla_\omega \varphi_\omega^k(z) = \sum_{j=0}^{k-1} (-\eta)^j \nabla v(\varphi_\omega^{k-1}(z)) \nabla v(\varphi_\omega^{k-2}(z)) \dots \nabla v(\varphi_\omega^{k-j}(z)) \tag{74}$$

Proof. We prove the result by induction:

- For $k = 1$, $\varphi_\omega(z) = \omega - \eta v(z)$ and the result holds.
- Assume this result holds for $k \geq 0$. Then,

$$\begin{aligned} \nabla_z \varphi_\omega^{k+1}(z) &= \nabla_z \varphi_\omega(\varphi_\omega^k(z)) \nabla_z \varphi_\omega^k(z) && (75)\\ &= -\eta \nabla v(\varphi_\omega^k(z)) (-\eta)^k \nabla v(\varphi_\omega^{k-1}(z)) \dots \nabla v(\varphi_\omega^0(z)) && (76)\\ &= (-\eta)^{k+1} \nabla v(\varphi_\omega^k(z)) \nabla v(\varphi_\omega^{k-1}(z)) \dots \nabla v(\varphi_\omega^0(z)) && (77) \end{aligned}$$

For the derivative with respect to ω, we use the chain rule:

$$\begin{aligned} \nabla_\omega \varphi_\omega^{k+1}(z) &= \nabla_\omega \varphi_\omega(\varphi_\omega^k(z)) + \nabla_z \varphi_\omega(\varphi_\omega^k(z)) \nabla_\omega \varphi_\omega^k(z) && (78)\\ &= I_d - \eta v(\varphi_\omega^k(z)) \sum_{j=0}^{k-1} (-\eta)^j \nabla v(\varphi_\omega^{k-1}(z)) \dots \nabla v(\varphi_\omega^{k-j}(z)) && (79)\\ &= I_d + \sum_{j=0}^{k-1} (-\eta)^{j+1} \nabla v(\varphi_\omega^k(z)) \nabla v(\varphi_\omega^{k-1}(z)) \dots \nabla v(\varphi_\omega^{k-j}(z)) && (80)\\ &= I_d + \sum_{j=1}^{k} (-\eta)^j \nabla v(\varphi_\omega^k(z)) \nabla v(\varphi_\omega^{k-1}(z)) \dots \nabla v(\varphi_\omega^{k+1-j}(z)) && (81)\\ &= \sum_{j=0}^{k} (-\eta)^j \nabla v(\varphi_\omega^k(z)) \nabla v(\varphi_\omega^{k-1}(z)) \dots \nabla v(\varphi_\omega^{k+1-j}(z)) && (82) \end{aligned}$$

□

In the proof of Lem. 1 and later we will use the spectral mapping theorem, which we state below for reference:

Theorem 12 (Spectral Mapping Theorem)**.** *Let $A \in \mathbb{C}^{d\times d}$ be a square matrix, and P be a polynomial. Then,*

$$\operatorname{Sp} P(A) = \{P(\lambda) \mid \lambda \in \operatorname{Sp} A\} \tag{83}$$

See for instance [Lax07, Theorem 4, p. 66] for a proof.

Proof of Lem. 1. First we compute $\nabla F_{\eta,k}(\omega^*)$. As ω^* is a stationary point, it is a fixed point of the extrapolation operators, i.e. $\varphi^j_{\omega^*}(\omega^*) = \omega^*$ for all $j \geq 0$. Then, by the chain rule,

$$\nabla F_{\eta,k}(\omega^*) = \nabla_z \varphi^k_{\omega^*}(\omega^*) + \nabla_\omega \varphi^k_{\omega^*}(\omega^*) \tag{84}$$

$$= (-\eta\nabla v(\omega^*))^k + \sum_{j=0}^{k-1}(-\eta\nabla v(\omega^*))^j \tag{85}$$

$$= \sum_{j=0}^{k}(-\eta\nabla v(\omega^*))^j \tag{86}$$

Hence $\nabla F_{\eta,k}(\omega^*)$ is a polynomial in $\nabla v(\omega^*)$. Using the spectral mapping theorem (Thm. 17), one gets that

$$\operatorname{Sp}\nabla F_{\eta,k}(\omega^*) = \left\{\sum_{j=0}^{k}(-\eta)^j\lambda^j \mid \lambda \in \operatorname{Sp}\nabla v(\omega^*)\right\} \tag{87}$$

For the proximal point operator, first let us prove that it is differentiable in a neighborhood of ω^*. First notice that,

$$\operatorname{Sp}(I_d + \eta\nabla v(\omega^*)) = \{1 + \eta\lambda \mid \lambda \in \operatorname{Sp}\nabla v(\omega^*)\} \tag{88}$$

If the eigenvalues of $\nabla v(\omega^*)$ all have non-negative real parts, this spectrum does not contain zero. Hence $\omega \mapsto \omega + \eta v(\omega)$ is continuously differentiable and has a non-singular differential at ω^*. By the inverse function theorem (see for instance [Rud76]), $\omega \mapsto \omega + \eta v(\omega)$ is invertible in a neighborhood of ω^* and its inverse, which is P_η, is continuously differentiable there. Moreover,

$$\nabla P_\eta(\omega^*) = (I_d + \eta\nabla v(\omega^*))^{-1} \tag{89}$$

Recall that the eigenvalues of a non-singular matrix are exactly the inverses of the eigenvalues of its inverse. Hence,

$$\operatorname{Sp}\nabla P_\eta(\omega^*) = \{\lambda^{-1} \mid \lambda \in \operatorname{Sp}(I_d + \eta\nabla v(\omega^*))\} = \left\{(1+\eta\lambda)^{-1} \mid \lambda \in \operatorname{Sp}\nabla v(\omega^*)\right\} \tag{90}$$

where the last equality follows from the spectral mapping theorem applied to $I_d + \eta\nabla v(\omega^*)$. Now, the bound on the spectral radius of the proximal point operator is immediate. Indeed, its spectral radius is:

$$\rho(\nabla P_\eta(\omega^*))^2 = \max_{\lambda\in\sigma^*}\frac{1}{|1+\eta\lambda|^2} \tag{91}$$

$$= 1 - \min_{\lambda\in\sigma^*}\left(\frac{2\eta\Re\lambda + \eta^2|\lambda|^2}{|1+\eta\lambda|^2}\right) \tag{92}$$

which yields the result. □

Theorem 4. *Let $\sigma^* = \operatorname{Sp} \nabla v(\omega^*)$. If the eigenvalues of $\nabla v(\omega^*)$ all have non-negative real parts, the spectral radius of the k-extrapolation method for $k \geq 2$ satisfies:*

$$\rho(\nabla F_{\eta,k}(\omega^*))^2 \leq 1 - \min_{\lambda \in \sigma^*} \frac{2\eta \Re\lambda + \frac{7}{16}\eta^2 |\lambda|^2}{|1+\eta\lambda|^2} \tag{7}$$

$\forall \eta \leq \frac{1}{4^{\frac{1}{k-1}}} \frac{1}{\max_{\lambda\in\sigma^*}|\lambda|}$. *For $\eta = (4 \max_{\lambda\in\sigma^*}|\lambda|)^{-1}$, this can be simplified as (noting $\rho := \rho(\nabla F_{\eta,k}(\omega^*))$):*

$$\rho^2 \leq 1 - \frac{1}{4}\left(\frac{\min_{\lambda\in\sigma^*}\Re\lambda}{\max_{\lambda\in\sigma^*}|\lambda|} + \frac{1}{16}\frac{\min_{\lambda\in\sigma^*}|\lambda|^2}{\max_{\lambda\in\sigma^*}|\lambda|^2}\right) \tag{8}$$

Proof. Let $L = \max_{\lambda\in\sigma^*}|\lambda|$ and $\eta = \frac{\tau}{L}$ for some $\tau > 0$. For $\lambda \in \sigma^*$,

$$\left|\sum_{j=0}^{k}(-\eta)^j\lambda^j\right|^2 = \frac{|1-(-\eta)^{k+1}\lambda^{k+1}|^2}{|1+\eta\lambda|^2} \tag{93}$$

$$= \frac{1 + 2(-1)^k\eta^{k+1}\Re(\lambda^{k+1}) + \eta^{2(k+1)}|\lambda|^{2(k+1)}}{|1+\eta\lambda|^2} \tag{94}$$

$$= 1 - \frac{2\eta\Re\lambda + \eta^2|\lambda|^2 - 2(-1)^k\eta^{k+1}\Re(\lambda^{k+1}) - \eta^{2(k+1)}|\lambda|^{2(k+1)}}{|1+\eta\lambda|^2} \tag{95}$$

$$= 1 - \frac{2\eta\Re\lambda + \eta^2|\lambda|^2\left(1 - 2(-1)^k\eta^{k-1}\frac{\Re(\lambda^{k+1})}{|\lambda|^2} - \eta^{2(k-1)}|\lambda|^{2(k-1)}\right)}{|1+\eta\lambda|^2} \tag{96}$$

Now we focus on lower bounding the terms in between the parentheses. By definition of η, we have $\eta^{k-1}\frac{|\Re(\lambda^{k+1})|}{|\lambda|^2} \leq \tau^{k-1}$ and $\eta^{2(k-1)}|\lambda|^{2(k-1)} \leq \tau^{2(k-1)}$. Hence

$$1 + 2(-1)^k\eta^{k-1}\frac{\Re(\lambda^{k+1})}{|\lambda|^2} + \eta^{2(k-1)}|\lambda|^{2(k-1)}) \geq 1 - 2\eta^{k-1}\frac{|\Re(\lambda^{k+1})|}{|\lambda|^2} - \eta^{2(k-1)}|\lambda|^{2(k-1)} \tag{97}$$

$$\geq 1 - 2\tau^{k-1} - \tau^{2(k-1)} \tag{98}$$

$$\tag{99}$$

Notice that if $k = 1$, i.e. for the gradient method, we cannot control this quantity. However, for $k \geq 2$, if $\tau \leq (\frac{1}{4})^{\frac{1}{k-1}}$, one gets that

$$1 - 2\tau^{k-1} - \tau^{2(k-1)} \geq 1 - \frac{1}{2} - \frac{1}{16} = \frac{7}{16} \tag{100}$$

which yields the first assertion of the theorem. For the second one, take $\eta = \frac{1}{4L}$, i.e. the maximum step-size authorized for extragradient, and one gets that

$$|1+\eta\lambda|^2 = 1 + 2\eta\Re\lambda + \eta^2|\lambda|^2 \tag{101}$$

$$\leq 1 + 2\frac{1}{4} + \frac{1}{16} = \frac{25}{16} \tag{102}$$

Then,

$$\frac{2\eta\Re\lambda + \frac{7}{16}\eta^2|\lambda|^2}{|1+\eta\lambda|^2} \geq \frac{1}{4}\left(2\frac{16}{25}\frac{\Re\lambda}{L} + \frac{7}{100}\frac{|\lambda|^2}{L^2}\right) \tag{103}$$

$$\geq \frac{1}{4}\left(\frac{\Re\lambda}{L} + \frac{7}{112}\frac{|\lambda|^2}{L^2}\right) \tag{104}$$

$$\geq \frac{1}{4}\left(\frac{\Re\lambda}{L} + \frac{1}{16}\frac{|\lambda|^2}{L^2}\right) \tag{105}$$

which yields the desired result. □

Corollary 2 (Bilinear game)**.** *Consider Ex. 1. The iterates of the k-extrapolation method with $k \geq 2$ converge globally to ω^* at a linear rate of $\mathcal{O}\big(\big(1 - \frac{1}{64}\frac{\sigma_{min}(A)^2}{\sigma_{max}(A)^2}\big)^t\big)$.*

First we need to compute the eigenvalues of ∇v.

Lemma 9. *Let $A \in \mathbb{R}^{m\times m}$ and*

$$M = \begin{pmatrix} 0_m & A \\ -A^T & 0_m \end{pmatrix} \tag{106}$$

Then,

$$\operatorname{Sp} M = \{\pm i\sigma \mid \sigma^2 \in \operatorname{Sp} AA^T\} \tag{107}$$

Proof. Assumption 1 of Appendix B.1.2 holds so we can apply Lem. 5 which yields the result. □

Proof of Cor. 2. The Jacobian is constant here and has following the form:

$$\nabla v = \begin{pmatrix} 0_m & A \\ -A^T & 0_m \end{pmatrix} \tag{108}$$

Applying Lem. 23 yields

$$\operatorname{Sp} \nabla v = \{\pm i\sigma \mid \sigma^2 \in \operatorname{Sp} AA^T\} \tag{109}$$

Hence $\min_{\lambda\in\operatorname{Sp}\nabla v} |\lambda|^2 = \sigma_{min}(A)^2$ and $\max_{\lambda\in\operatorname{Sp}\nabla v} |\lambda|^2 = \sigma_{max}(A)^2$. Using Thm. 4, we have that,

$$\rho(\nabla F_{\eta,k}(\omega^*))^2 \leq \left(1 - \frac{1}{64}\frac{\sigma_{min}(A)^2}{\sigma_{max}(A)^2}\right) \tag{110}$$

Finally, Thm. 11 implies that the iterates of the k-extrapolation converge globally at the desired rate. □

Corollary 4. *Under the assumptions of Cor. 1, the spectral radius of the n-extrapolation method operator is bounded by*

$$\rho(\nabla F_{\eta,k}(\omega^*))^2 \leq 1 - \frac{1}{4}\Big(\frac{1}{2}\frac{\mu_1+\mu_2}{\sqrt{L_{12}^2 + L_1L_2}} + \frac{1}{16}\frac{\mu_{12}^2 + \mu_1\mu_2}{L_{12}^2 + L_1L_2}\Big) \tag{111}$$

Proof. This is a direct consequence of Thm. 4 and Thm. 10, as the latter gives that for any $\lambda \in \operatorname{Sp}\nabla v(\omega^*)$,

$$\frac{\operatorname{Tr}\mu}{2} \leq \Re\lambda, \quad |\mu| \leq |\lambda|^2 \leq |L| \tag{112}$$

as discussed in the proof of Cor. 1. □

B.2 Appendix: Global convergence proofs

In this section, $\|.\|$ denotes the Euclidean norm.

B.2.1 Alternative characterizations and properties of the assumptions

Lemma 2. *Let v be continuously differentiable and $\gamma > 0$: (10) holds if and only if $\sigma_{min}(\nabla v) \geq \gamma$.*

Let us recall (10) here for simplicity:

$$\|\omega - \omega'\| \leq \gamma^{-1} \|v(\omega) - v(\omega')\| \quad \forall \omega, \omega' \in \mathbb{R}^d \tag{10}$$

The proof of this lemma is an immediate consequence of a global inverse theorem from [Had06, Lev20]. Let us recall its statement here:

Theorem 13 ([Had06, Lev20])**.** *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a continuously differentiable map. Assume that, for all $\omega \in \mathbb{R}^d$, ∇f is non-singular and $\sigma_{min}(\nabla f) \geq \gamma > 0$. Then f is a C^1-diffeomorphism, i.e. a one-to-one map whose inverse is also continuously differentiable.*

A proof of this theorem can be found in [Rhe69, Theorem 3.11]. We now proceed to prove the lemma.

Proof of Lem. 2. First we prove the direct implication. By the theorem stated above, v is a bijection from $\mathbb{R}^d$ to $\mathbb{R}^d$, its inverse is continuously differentiable on $\mathbb{R}^d$ and so we have, for all $\omega \in \mathbb{R}^d$:

$$\nabla v^{-1}(v(\omega)) = (\nabla v(\omega))^{-1} . \tag{113}$$

Hence $\|\nabla v^{-1}(v(\omega))\| = (\sigma_{min}(\nabla v(\omega)))^{-1} \leq \gamma^{-1}$.

Consider $\omega, \omega' \in \mathbb{R}^d$ and let $u = v(\omega)$ and $u' = v(\omega')$. Then

$$\|\omega - \omega'\| = \|v^{-1}(u) - v^{-1}(u')\| \tag{114}$$

$$= \left\| \int_0^1 \nabla v^{-1}(tu + (1-t)u')(u - u') \right\| \tag{115}$$

$$\leq \gamma^{-1} \|u - u'\| \tag{116}$$

$$= \gamma^{-1} \|v(\omega) - v(\omega')\| \tag{117}$$

which proves the result.

Conversely, if (10) holds, fix $u \in \mathbb{R}^d$ with $\|u\| = 1$. Taking $\omega' = \omega + tu$ in (10) with $t \neq 0$ and rearranging yields:

$$\gamma \leq \left\| \frac{v(\omega + tu) - v(\omega)}{t} \right\|$$

Taking the limit when t goes to 0 gives that $\gamma \leq \|\nabla v(\omega) u\|$. As it holds for all u such that $\|u\| = 1$ this implies that $\gamma \leq \sigma_{min}(\nabla v)$. □

With the next lemma, we relate the quantities appearing in Thm. 6 to the spectrum of ∇v. Note that the first part of the proof is standard — it can be found in [FP03, Prop. 2.3.2] for instance — and we include it only for completeness.

Lemma 10. *Let $v : \mathbb{R}^d \to \mathbb{R}^d$ be continuously differentiable and*

(i) μ-strongly monotone for some $\mu \geq 0$,

(ii) L-Lispchitz,

(iii) such that $\sigma_{min}(\nabla v) \geq \gamma$ for some $\gamma \geq 0$.

. *Then, for all* $\omega \in \mathbb{R}^d$,

$$\mu\|u\|^2 \leq (\nabla v(\omega)u)^T u \quad \gamma\|u\| \leq \|\nabla v(\omega)u\| \leq L\|u\| \quad \forall u \in \mathbb{R}^d \tag{118}$$

and

$$\mu \leq \Re(\lambda), \quad \gamma \leq |\lambda| \leq L \quad \forall \lambda \in \operatorname{Sp} \nabla v(\omega) \tag{119}$$

Proof. By definition of μ-strong monotonicity, and L-Lispchitz one has that, for any $\omega, \omega' \in \mathbb{R}^d$,

$$\mu\|\omega - \omega'\|^2 \leq (v(\omega) - v(\omega'))^T(\omega - \omega') \tag{120}$$

$$\|v(\omega) - v(\omega')\| \leq L\|\omega - \omega'\| \tag{121}$$

Fix $\omega \in \mathbb{R}^d$, $u \in \mathbb{R}^d$ such that $\|u\| = 1$. Taking $\omega' = \omega + tu$ for $t > 0$ in the previous inequalities and dividing by t yields

$$\mu \leq \frac{1}{t}(v(\omega) - v(\omega + tu))^T u \tag{122}$$

$$\frac{1}{t}\|v(\omega) - v(\omega + tu)\| \leq L \tag{123}$$

Letting t goes to 0 gives

$$\mu \leq (\nabla v(\omega)u)^T u \tag{124}$$

$$\|\nabla v(\omega)u\| \leq L \tag{125}$$

Furthermore, by the properties of the singular values,

$$\|\nabla v(\omega)u\| \geq \gamma \tag{126}$$

Hence, by homogeneity, we have that, for all $u \in \mathbb{R}^d$,

$$\mu\|u\|^2 \leq (\nabla v(\omega)u)^T u \quad \gamma\|u\| \leq \|\nabla v(\omega)u\| \leq L\|u\| \tag{127}$$

Now, take $\lambda \in \operatorname{Sp} \nabla v(\omega)$ an eigenvalue of $\nabla v(\omega)$ and let $Z \in \mathbb{C}^d \setminus \{0\}$ be one of its associated eigenvectors. Note that Z can be written as $Z = X + iY$ with $X, Y \in \mathbb{R}^d$. By definition of Z, we have

$$\nabla v(\omega)Z = \lambda Z \tag{128}$$

Now, taking the real and imaginary part yields:

$$\begin{cases} \nabla v(\omega)X & = \Re(\lambda)X - \Im(\lambda)Y \\ \nabla v(\omega)Y & = \Im(\lambda)X + \Re(\lambda)Y \end{cases} \tag{129}$$

Taking the squared norm and developing the right-hand sides yields

$$\begin{cases} \|\nabla v(\omega)X\|^2 & = \Re(\lambda)^2\|X\|^2 + \Im(\lambda)^2\|Y\|^2 - 2\Re(\lambda)\Im(\lambda)X^T Y \\ \|\nabla v(\omega)Y\|^2 & = \Im(\lambda)^2\|X\|^2 + \Re(\lambda)^2\|Y\|^2 + 2\Re(\lambda)\Im(\lambda)X^T Y \end{cases} \tag{130}$$

Now summing these two equations gives

$$\|\nabla v(\omega)X\|^2 + \|\nabla v(\omega)Y\|^2 = |\lambda|^2(\|X\|^2 + \|Y\|^2) \tag{131}$$

Finally, apply (127) for $u = X$ and $u = Y$:

$$\gamma^2(\|X\|^2 + \|Y\|^2) \leq |\lambda|^2(\|X\|^2 + \|Y\|^2) \leq L^2(\|X\|^2 + \|Y\|^2) \tag{132}$$

As $Z \neq 0$, $\|X\|^2 + \|Y\|^2 > 0$ and this yields $\gamma \leq |\lambda| \leq L$. To get the inequality concerning γ, multiply on the left the first line of (129) by X^T and the second one by Y^T:

$$\begin{cases} X^T(\nabla v(\omega)X) & = \Re(\lambda)\|X\|^2 - \Im(\lambda)X^T Y \\ Y^T(\nabla v(\omega)Y) & = \Im(\lambda)Y^T X + \Re(\lambda)\|Y\|^2 \end{cases} \tag{133}$$

Again, summing these two lines and using (127) yields:

$$\mu(\|X\|^2 + \|Y\|^2) \leq \Re(\lambda)(\|X\|^2 + \|Y\|^2) \tag{134}$$

As $Z \neq 0$, $\|X\|^2 + \|Y\|^2 > 0$ and so $\mu \leq \Re(\lambda)$. $\square$

B.2.2 Proofs of §2.4: extragradient, optimistic and proximal point methods

We now prove a slightly more detailed version of Thm. 6.

Theorem 14. *Let $v : \mathbb{R}^d \to \mathbb{R}^d$ be continuously differentiable and*

(i) μ-strongly monotone for some $\mu \geq 0$,

(ii) L-Lipschitz,

(iii) such that $\sigma_{min}(\nabla v) \geq \gamma$ for some $\gamma > 0$.

Then, for $\eta \leq (4L)^{-1}$, the iterates of the extragradient method $(\omega_t)_t$ converge linearly to ω^* the unique stationary point of v,

$$\|\omega_t - \omega^*\|_2^2 \leq \left(1 - \left(\eta\mu + \frac{7}{16}\eta^2\gamma^2\right)\right)^t \|\omega_0 - \omega^*\|_2^2 \tag{135}$$

For $\eta = (4L)^{-1}$, this can be simplified as: $\|\omega_t - \omega^*\|_2^2 \leq \left(1 - \frac{1}{4}\left(\frac{\mu}{L} + \frac{1}{16}\frac{\gamma^2}{L^2}\right)\right)^t \|\omega_0 - \omega^*\|_2^2$.

The proof is inspired from the ones of [GBV+19, Tse95].

We will use the following well-known identity. It can be found in [GBV+19] for instance but we state it for reference.

Lemma 11. *Let $\omega, \omega', u \in \mathbb{R}^d$. Then*

$$\|\omega + u - \omega'\|^2 = \|\omega - \omega'\|^2 + 2u^T(\omega + u - \omega') - \|u\|^2 \tag{136}$$

Proof.

$$\begin{align} \|\omega + u - \omega'\|^2 &= \|\omega - \omega'\|^2 + 2u^T(\omega - \omega') + \|u\|^2 \tag{137} \\ &= \|\omega - \omega'\|^2 + 2u^T(\omega + u - \omega') - \|u\|^2 \tag{138} \end{align}$$

$\square$

Proof Thm. 14. First note that as $\gamma > 0$, by Thm. 13, v has a stationary point ω^* and it is unique.

Fix any $\omega_0 \in \mathbb{R}^d$, and denote $\omega_1 = \omega_0 - \eta v(\omega_0)$ and $\omega_2 = \omega_0 - \eta v(\omega_1)$. Applying Lem. 11 for $(\omega, \omega', u) = (\omega_0, \omega^*, -\eta v(\omega_1))$ and $(\omega, \omega', u) = (\omega_0, \omega_2, -\eta v(\omega_0))$ yields:

$$\|\omega_2 - \omega^*\|^2 = \|\omega_0 - \omega^*\|^2 - 2\eta v(\omega_1)^T(\omega_2 - \omega^*) - \|\omega_2 - \omega_0\|^2 \tag{139}$$

$$\|\omega_1 - \omega_2\|^2 = \|\omega_0 - \omega_2\|^2 - 2\eta v(\omega_0)^T(\omega_1 - \omega_2) - \|\omega_1 - \omega_0\|^2 \tag{140}$$

Summing these two equations gives:

$$\|\omega_2 - \omega^*\|^2 = \tag{141}$$

$$\|\omega_0 - \omega^*\|^2 - 2\eta v(\omega_1)^T(\omega_2 - \omega^*) - 2\eta v(\omega_0)^T(\omega_1 - \omega_2) - \|\omega_1 - \omega_0\|^2 - \|\omega_1 - \omega_2\|^2 \tag{142}$$

Then, rearranging and using that $v(\omega^*) = 0$ yields that,

$$2\eta v(\omega_1)^T(\omega_2 - \omega^*) + 2\eta v(\omega_0)^T(\omega_1 - \omega_2) \tag{143}$$

$$= 2\eta(v(\omega_1))^T(\omega_1 - \omega^*) + 2\eta(v(\omega_0) - v(\omega_1))^T(\omega_1 - \omega_2) \tag{144}$$

$$= 2\eta(v(\omega_1) - v(\omega^*))^T(\omega_1 - \omega^*) + 2\eta(v(\omega_0) - v(\omega_1))^T(\omega_1 - \omega_2) \tag{145}$$

$$\geq 2\eta\mu\|\omega_1 - \omega^*\|^2 - 2\eta\|v(\omega_0) - v(\omega_1)\|\|\omega_1 - \omega_2\| \tag{146}$$

where the first term is lower bounded using strong monotonicity and the second one using Cauchy-Schwarz's inequality. Using in addition the fact that v is Lipschitz continuous we obtain:

$$2\eta v(\omega_1)^T(\omega_2 - \omega^*) + 2\eta v(\omega_0)^T(\omega_1 - \omega_2) \tag{147}$$

$$\geq 2\eta\mu\|\omega_1 - \omega^*\|^2 - 2\eta L\|\omega_0 - \omega_1\|\|\omega_1 - \omega_2\| \tag{148}$$

$$\geq 2\eta\mu\|\omega_1 - \omega^*\|^2 - (\eta^2 L^2\|\omega_0 - \omega_1\|^2 + \|\omega_1 - \omega_2\|^2) \tag{149}$$

where the last inequality comes from Young's inequality. Using this inequality in (141) yields:

$$\|\omega_2 - \omega^*\|^2 \leq \|\omega_0 - \omega^*\|^2 - 2\eta\mu\|\omega_1 - \omega^*\|^2 + (\eta^2 L^2 - 1)\|\omega_0 - \omega_1\|^2 \tag{150}$$

Now we lower bound $\|\omega_1 - \omega^*\|$ using $\|\omega_0 - \omega^*\|$. Indeed, from Young's inequality we obtain

$$2\|\omega_1 - \omega^*\|^2 \geq \|\omega_0 - \omega^*\|^2 - 2\|\omega_0 - \omega_1\|^2 \tag{151}$$

Hence, we have that,

$$\|\omega_2 - \omega^*\|^2 \leq (1 - \eta\mu)\|\omega_0 - \omega^*\|^2 + (\eta^2 L^2 + 2\eta\mu - 1)\|\omega_0 - \omega_1\|^2 \tag{152}$$

Note that if $\eta \leq \frac{1}{4L}$, as $\mu \leq L$, $\eta^2 L^2 + 2\eta\mu - 1 \leq -\frac{7}{16}$. Therefore, with $c = \frac{7}{16}$,

$$\|\omega_2 - \omega^*\|^2 \leq (1 - \eta\mu)\|\omega_0 - \omega^*\|^2 - c\|\omega_0 - \omega_1\|^2 \tag{153}$$

$$= (1 - \eta\mu)\|\omega_0 - \omega^*\|^2 - c\eta^2\|v(\omega_0)\|^2 \tag{154}$$

Finally, using (iii) and Lem. 2, we obtain:

$$\|\omega_2 - \omega^*\|^2 \leq (1 - \eta\mu - c\eta^2\gamma^2)\|\omega_0 - \omega^*\|^2 \tag{155}$$

which yields the result. □

Proposition 9. *Under the assumptions of Thm. 6, the iterates of the proximal point method method $(\omega_t)_t$ with $\eta > 0$ converge linearly to ω^* the unique stationary point of v,*

$$\|\omega_t - \omega^*\|^2 \leq \left(1 - \frac{2\eta\mu + \eta^2\gamma^2}{1 + 2\eta\mu + \eta^2\gamma^2}\right)^t \|\omega_0 - \omega^*\|^2 \quad \forall t \geq 0 \tag{156}$$

Proof. To proof this convergence result, we upper bound the singular values of the proximal point operator P_η. As v is monotone, by Lem. 10, the eigenvalues of ∇v have all non-negative real parts everywhere. As in the proof of Lem. 1, $\omega \mapsto \omega + \eta v(\omega)$ is continuously differentiable and has a non-singular differential at every $\omega_0 \in \mathbb{R}^d$. By the inverse function theorem, $\omega \mapsto \omega + \eta v(\omega)$ has a continuously differentiable inverse in a neighborhood of ω_0. Its inverse is exactly P_η and it also satisfies

$$\nabla P_\eta(\omega_0) = (I_d + \eta \nabla v(\omega_0))^{-1} \tag{157}$$

The singular values $\nabla P_\eta(\omega_0)$ are the eigenvalues of $(\nabla P_\eta(\omega_0))^T(\nabla P_\eta(\omega_0))$. The latter is equal to:

$$(\nabla P_\eta(\omega_0))^T(\nabla P_\eta(\omega_0)) = \left(I_d + \eta\nabla v(\omega_0) + \eta(\nabla v(\omega_0))^T + \eta^2(\nabla v(\omega_0))^T(\nabla v(\omega_0))\right)^{-1} \tag{158}$$

Now, let $\lambda \in \mathbb{R}$ be an eigenvalue of $(\nabla P_\eta(\omega_0))^T(\nabla P_\eta(\omega_0))$ and let $X \neq 0$ be one of its associated eigenvectors. As $\nabla P_\eta(\omega_0)$ is non-singular, $\lambda \neq 0$ and applying the previous equation yields:

$$\lambda^{-1} X = \left(I_d + \eta\nabla v(\omega_0) + \eta(\nabla v(\omega_0))^T + \eta^2(\nabla v(\omega_0))^T(\nabla v(\omega_0))\right) X \tag{159}$$

Finally, multiply this equation on the left by X^T:

$$\lambda^{-1}\|X\|^2 = \|X\|^2 + \eta X^T(\nabla v(\omega_0) + (\nabla v(\omega_0))^T)X + \eta^2\|\nabla v(\omega_0))X\|^2 \tag{160}$$

Applying the first part of Lem. 10 yields

$$\lambda^{-1}\|X\|^2 \geq (1 + 2\eta\mu + \eta^2\gamma^2)\|X\|^2 \tag{161}$$

Hence, as $X \neq 0$, we have proven that,

$$\sigma_{max}(\nabla v(\omega_0)) \leq (1 + 2\eta\mu + \eta^2\gamma^2)^{-1} \tag{162}$$

This implies that, for all $\omega, \omega' \in \mathbb{R}^d$,

$$\|P_\eta(\omega) - P_\eta(\omega')\|^2 = \left\|\int_0^1 \nabla v(\omega' + t(\omega - \omega'))(\omega - \omega')\right\|^2 \tag{163}$$

$$\leq (1 + 2\eta\mu + \eta^2\gamma^2)^{-1}\|\omega - \omega'\|^2 \tag{164}$$

Hence, as $P_\eta(\omega^*) = \omega^*$, taking $\omega' = \omega^*$ gives the desired global convergence rate. $\square$

Now let us prove the result Thm. 7 regarding Optimistic method.

Theorem 7. *Under the same assumptions as in Thm. 6, for $\eta \leq (4L)^{-1}$, the iterates $(\omega_t)_t$ of* (OG) *converge linearly to ω^* as, for all $t \geq 0$,*

$$\|\omega_t - \omega^*\|_2^2 \leq 2\left(1 - \eta\mu - \frac{1}{8}\eta^2\gamma^2\right)^{t+1} \|\omega_0 - \omega^*\|_2^2$$

Proof. For the beginning of this proof we follow the proof of [GBV$^+$19, Theorem 1] using their notation:

$$\omega'_t = \omega_t - \eta v(\omega'_{t-1}) \tag{165}$$
$$\omega_{t+1} = \omega_t - \eta v(\omega'_t) \tag{166}$$

Note that, with this notation, summing the two upates steps, we recover (OG)

$$\omega'_{t+1} = \omega'_t - 2\eta v(\omega'_t) + \eta v(\omega'_{t-1}) \tag{167}$$

Let us now recall [GBV$^+$19, Equation 88] for a constant step-size $\eta_t = \eta$,

$$\begin{aligned}\|\omega_{t+1} - \omega^*\|_2^2 \leq (1-\eta\mu)\,\|\omega_t - \omega^*\|_2^2 + \eta^2 L^2 (4\eta^2 L^2 \|\omega'_{t-1} - \omega'_{t-2}\|_2^2 - \|\omega'_{t-1} - \omega'_t\|_2^2) \\ - (1 - 2\eta\mu - 4\eta^2 L^2)\|\omega'_t - \omega_t\|_2^2\end{aligned} \tag{168}$$

we refer the reader to the proof of [GBV$^+$19, Theorem 1] for the details on how to get to this equation. Thus with $\eta \leq (4L)^{-1}$, using the update rule $\omega'_t = \omega_t - \eta v(\omega'_{t-1})$, we get,

$$(1 - 2\eta_t\mu - 4\eta_t^2 L^2)\|\omega'_t - \omega_t\|_2^2 \geq \frac{1}{4}\|\omega'_t - \omega_t\|_2^2 = \frac{\eta^2}{4}\|v(\omega'_{t-1})\|_2^2 \geq \frac{\eta^2\gamma^2}{4}\|\omega'_{t-1} - \omega^*\|_2^2 \tag{169}$$

where for the last inequality we used that $\sigma_{\min}(\nabla v) \geq \gamma$ and Lemma 2. Using Young's inequality, the update rule and the Lipchitzness of v, we get that,

$$2\|\omega'_{t-1} - \omega^*\|_2^2 \geq \|\omega_t - \omega^*\|_2^2 - 2\|\omega'_{t-1} - \omega_t\|_2^2 \tag{170}$$
$$= \|\omega_t - \omega^*\|_2^2 - 2\eta^2\|v(\omega'_{t-1}) - v(\omega'_{t-2})\|_2^2 \tag{171}$$
$$\geq \|\omega_t - \omega^*\|_2^2 - 2\eta^2 L^2\|\omega'_{t-1} - \omega'_{t-2}\|_2^2 \tag{172}$$

Thus combining (168), (169) and (172), we get with a constant step-size $\eta \leq (4L)^{-1}$,

$$\|\omega_{t+1} - \omega^*\|_2^2 \leq \left(1 - \eta\mu - \frac{\eta^2\gamma^2}{8}\right)\|\omega_t - \omega^*\|_2^2 + \eta^2 L^2\big((4\eta^2 L^2 + \frac{\eta^2\gamma^2}{4})\|\omega'_{t-1} - \omega'_{t-2}\|_2^2 - \|\omega'_{t-1} - \omega'_t\|_2^2\big)$$

This leads to,

$$\|\omega_{t+1} - \omega^*\|_2^2 + \eta^2 L^2\|\omega'_{t-1} - \omega'_t\|_2^2 \leq \left(1 - \eta\mu - \frac{\eta^2\gamma^2}{8}\right)\|\omega_t - \omega^*\|_2^2 + \eta^2(4L^2 + \frac{\gamma^2}{4})\eta^2 L^2\|\omega'_{t-1} - \omega'_{t-2}\|_2^2 \tag{173}$$

In order to get the theorem statement we need a rate on ω'_t. We first unroll this geometric decrease and notice that

$$\|\omega'_t - \omega^*\|_2^2 \leq 2\|\omega_{t+1} - \omega^*\|_2^2 + 2\|\omega'_t - \omega_{t+1}\|_2^2 \tag{174}$$
$$= 2\|\omega_{t+1} - \omega^*\|_2^2 + 2\eta^2\|v(\omega'_{t-1}) - v(\omega'_t)\|_2^2 \tag{175}$$
$$= 2\|\omega_{t+1} - \omega^*\|_2^2 + 2\eta^2 L^2\|\omega'_{t-1} - \omega'_t\|_2^2 \tag{176}$$

to get (using the fact that $\omega'_0 = \omega'_{-1}$),

$$\|\omega'_t - \omega^*\|_2^2 \leq 2\|\omega_{t+1} - \omega^*\|_2^2 + 2\eta^2 L^2\|\omega'_{t-1} - \omega'_t\|_2^2 \tag{177}$$

$$\leq 2\max\left\{1-\eta\mu-\frac{\eta^2\gamma^2}{8L^2}, 4\eta^2L^2+\frac{\eta^2\gamma^2}{4}\right\}^{t+1}\|\omega_0-\omega^*\|_2^2 \tag{178}$$

With $\eta \leq (4L)^{-1}$ we can use the fact that $\max(\mu,\gamma)\leq L$ to get,

$$1-\eta\mu-\frac{\eta^2\gamma^2}{8} \geq 1-\frac{1}{4}\Big(\frac{\mu}{L}+\frac{\gamma^2}{32L^2}\Big) \geq \frac{3}{4}-\frac{1}{32} \geq \frac{1}{4}+\frac{1}{64} \geq 4\eta^2L^2+\frac{\eta^2\gamma^2}{4} \tag{179}$$

Thus,

$$\max\left\{1-\eta\mu-\frac{\eta^2\gamma^2}{8}, 4\eta^2L^2+\frac{\eta^2\gamma^2}{4}\right\} = 1-\eta\mu-\frac{\eta^2\gamma^2}{8} \qquad \forall \eta\leq(4L)^{-1} \tag{180}$$

leading to the statement of the theorem. Finally, for $\eta=(4L)^{-1}$ that can be simplified into,

$$\begin{aligned}\|\omega_t'-\omega^*\|_2^2 &\leq 2\max\left\{1-\frac{1}{4}\Big(\frac{\mu}{L}+\frac{\gamma^2}{32L^2}\Big), \frac{1}{4}+\frac{\gamma^2}{64L^2}\right\}^{t+1}\|\omega_0-\omega^*\|_2^2 && (181)\\ &= 2\Big(1-\frac{1}{4}\Big(\frac{\mu}{L}+\frac{\gamma^2}{32L^2}\Big)\Big)^{t+1}\|\omega_0-\omega^*\|_2^2 && (182)\end{aligned}$$

□

B.2.3 Proof of §2.4.3: consensus optimization

Let us recall (CO) here,

$$\omega_{t+1} = \omega_t - (\alpha v(\omega_t) + \beta\nabla H(\omega_t)) \tag{CO}$$

where H is the squared norm of v. We prove a more detailed version Thm. 8.

Theorem 15. *Let $v:\mathbb{R}^d\to\mathbb{R}^d$ be continuously differentiable such that*

(i) v is μ- strongly monotone for some $\mu\geq 0$,

(ii) $\sigma_{min}(\nabla v)\geq\gamma$ for some $\gamma>0$

(iii) H is L_H^2 Lipschitz-smooth.

Then, for

$$\alpha^2 \leq \frac{1}{2}\left(\frac{\alpha\mu}{L_H}+\frac{\beta\gamma^2}{2L_H}\right)$$

and $\beta\leq(2L_H)^{-1}$, the iterates of CO defined by (CO) satisfy,

$$H(\omega_t) \leq \left(1-\alpha\mu-\frac{1}{2}\beta\gamma^2\right)H(\omega_0) \tag{183}$$

In particular, for

$$\alpha = \frac{\mu+\sqrt{\mu^2+2\gamma^2}}{4L_H}$$

and $\beta=(2L_H)^{-1}$,

$$H(\omega_t) \leq \left(1-\frac{\mu^2}{2L_H}-\frac{\gamma^2}{2L_H^2}\left(1+\frac{\mu}{\gamma}\right)\right)H(\omega_0) \tag{184}$$

Proof. As H is L_H^2 Lipschitz smooth, we have,

$$H(\omega_{t+1}) - H(\omega_t) \leq \nabla H(\omega_t)^T(\omega_{t+1} - \omega_t) + \frac{L_H^2}{2}\|\omega_{t+1} - \omega_t\|^2$$

Then, replacing $\omega_{t+1} - \omega_t$ by its expression and using Young's inequality,

$$H(\omega_{t+1}) - H(\omega_t) \leq -\alpha\nabla H(\omega_t)^T v(\omega_t) - \beta\|\nabla H(\omega_t)\|^2 + L_H^2\alpha^2\|v(\omega_t)\|^2 + L_H^2\beta^2\|\nabla H(\omega_t)\|^2$$

Note that, crucially, $\nabla H(\omega_t) = \nabla v(\omega_t)^T v(\omega_t)$. Using the first part of Lem. 10 to introduce μ and assuming $\beta \leq (2L_H^2)^{-1}$,

$$H(\omega_{t+1}) - H(\omega_t) \leq -\alpha\mu\|v(\omega_t)\|^2 - \frac{\beta}{2}\|\nabla H(\omega_t)\|^2 + L_H^2\alpha^2\|v(\omega_t)\|^2$$

Finally, using Lem. 10 to introduce γ,

$$\begin{aligned}
H(\omega_{t+1}) - H(\omega_t) &\leq -\alpha\mu\|v(\omega_t)\|^2 - \frac{\beta\gamma^2}{2}\|v(\omega_t)\|^2 + L_H^2\alpha^2\|v(\omega_t)\|^2 \\
&= -2\left(\alpha\mu + \frac{\beta\gamma^2}{2} - L_H^2\alpha^2\right)H(\omega_t)
\end{aligned}$$

Hence, if

$$\alpha^2 \leq \frac{1}{2}\left(\frac{\alpha\mu}{L_H^2} + \frac{\beta\gamma^2}{2L_H^2}\right) \tag{185}$$

then the decrease of H becomes,

$$H(\omega_t) \leq \left(1 - \alpha\mu - \frac{1}{2}\beta\gamma^2\right)H(\omega_0)$$

Now, note that (185) is a second-order polynomial condition on α, so we can compute the biggest α which satisfies this condition. This yields,

$$\begin{aligned}
\alpha &= \frac{\frac{\mu}{2} + \sqrt{\frac{\mu^2}{4} + L_H^2\beta\gamma^2}}{2L_H^2} \\
&= \frac{\mu + \sqrt{\mu^2 + 2\gamma^2}}{4L_H^2}
\end{aligned}$$

where in the second line we defined $\beta = (2L_H^2)^{-1}$. Then the rate becomes,

$$\begin{aligned}
\alpha\mu + \frac{1}{2}\beta\gamma^2 &= \frac{\mu^2}{4L_H^2} + \frac{\mu\sqrt{\mu^2 + 2\gamma^2}}{4L_H^2} + \frac{\gamma^2}{4L_H^2} \\
&\geq \frac{\mu^2}{4L_H^2} + \frac{\mu^2}{4\sqrt{2}L_H^2} + \frac{\mu\gamma}{4L_H^2} + \frac{\gamma^2}{4L_H^2}
\end{aligned}$$

where we use Young's inequality: $\sqrt{2}\sqrt{a+b} \geq \sqrt{a} + \sqrt{b}$. Noting that $\frac{1}{2}(1 + \frac{1}{\sqrt{2}}) \geq 1$ yields the result. $\square$

Remark 1. A common convergence result for the gradient method for variational inequalities problem – see [NS06] for instance – is that the iterates convergence as $O\left(\left(1-\frac{\mu^2}{L^2}\right)^t\right)$ where μ is the monotonicty constant of v and L its Lipschitz constant. However, this rate is not optimal, and also not satisfying as it does not recover the convergence rate of the gradient method for strongly convex optimization. One way to remedy this situation is to use the *co-coercivity* or *inverse strong monotonicity* assumption:

$$\ell(v(\omega)-v(\omega'))^T(\omega-\omega') \geq \|v(\omega)-v(\omega')\|^2 \quad \forall \omega, \omega'$$

This yields a convergence rate of $O\left(\left(1-\frac{\mu}{\ell}\right)^t\right)$ which can be significantly better than the former since ℓ can take all the values of $[L, L^2/\mu]$ [FP03, §12.1.1]. On one hand, if v is the gradient of a convex function, $\ell = L$ and so we recover the standard rate in this case. On the other, one can consider for example the operator $v(w) = Aw$ with $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $a > 0$ and $b \neq 0$ for which $\mu = a, L = \sqrt{a^2+b^2}$ anf $\ell = \mu/L^2$.

B.3 Appendix: The p-SCLI framework for game optimization

The approach we use to prove our lower bounds comes from [ASS16]. Though their whole framework was developed for convex optimization, a careful reading of their proof shows that most of their results carry on to games, at least those in their first three sections. However, we work only in the restricted setting of 1-SCLI and so we actually rely on a very small subset of their results, more exactly two of them.

The first one is Thm. 2 and is crucially used in the derivation of our lower bounds. We state it again for clarity.

Theorem 2 ([ASS16])**.** *For all $v \in \mathcal{V}_d$, for almost every[7] initialization point $\omega_0 \in \mathbb{R}^d$, if $(\omega_t)_t$ are the iterates of $F_\mathcal{N}$ starting from ω_0,*

$$\|\omega_t - \omega^*\| \geq \Omega(\rho(\nabla F_\mathcal{N})^t \|\omega_0 - \omega^*\|)$$

Actually, as $F_\mathcal{N} : \mathbb{R}^d \to \mathbb{R}^d$ is an affine operator and ω^* is one of its fixed point, this theorem is only a reformulation of [ASS16, Lemma 10], which is a standard result in linear algebra. So we actually do not rely on their most advanced results which were proven only for convex optimization problems. For completeness, we state this lemma below and show how to derive Thm. 2 from it.

Lemma 12 ([ASS16, Lemma 10])**.** *Let $A \in \mathbb{R}^{d\times d}$. There exists $c > 0$, $d \geq m \geq 1$ integer and $r \in \mathbb{R}^d$, $r \neq 0$ such that for any $u \in \mathbb{R}^d$ such that $u^T r \neq 0$, for sufficiently large $t \geq 1$ one has:*

$$\|A^t u\| \geq ct^{m-1}\rho(A)^t\|u\| \tag{186}$$

Proof of Thm. 2. $F_\mathcal{N}$ is affine so it can be written as $F_N(\omega) = \nabla F_\mathcal{N}\omega + F_\mathcal{N}(0)$.

Moreover, as $v(\omega^*) = 0$, $F_\mathcal{N}(\omega^*) = \omega^* + \mathcal{N}(\nabla v)v(\omega^*) = \omega^*$. Hence, for all $\omega \in \mathbb{R}^d$,

$$F_\mathcal{N}(\omega) - \omega^* = F_\mathcal{N}(\omega) - F_\mathcal{N}(\omega^*) = \nabla F_\mathcal{N}(\omega - \omega^*) \tag{187}$$

Therefore, for $t \geq 0$,

$$\|\omega_t - w^*\| = \|(\nabla F_\mathcal{N})^t(\omega - \omega^*)\| \tag{188}$$

[7] For any measure absolutely continuous w.r.t. the Lebesgue measure.

Finally, apply the lemma above to $A = \nabla F_{\mathcal{N}}$. The condition $(\omega_0 - \omega^*)^T r \neq 0$ is not satisfied only on an affine subset of dimension 1, which is of measure zero for any measure absolutely continuous w.r.t. the Lebesgue measure. Hence for almost every $\omega_0 \in \mathbb{R}^d$ w.r.t. to such measure, $(\omega_0 - \omega^*)^T r \neq 0$ and so one has, for $t \geq 1$ large enough,

$$\|\omega_t - w^*\| \geq ct^{m-1}\rho(\nabla F_{\mathcal{N}})^t \|\omega_t - \omega^*\| \tag{189}$$

$$\geq c\rho(\nabla F_{\mathcal{N}})^t \|\omega_t - \omega^*\| \tag{190}$$

which is the desired result. □

The other result we use is more anecdotal : it is their consistency condition, which is a necessary condition for an p-SCLI method to converge to a stationary point of the gradient dynamics. Indeed, general 1-SCLI as defined in [ASS16] are given not by one but by two mappings $\mathcal{C}, \mathcal{N} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ and the update rule is

$$F_{\mathcal{N}}(\omega) = \mathcal{C}(\nabla v)w + \mathcal{N}(\nabla v)v(0) \quad \forall \omega \in \mathbb{R}^d \tag{191}$$

However, they show in [ASS16, Thm. 5] that, for a method to converge to a stationary point of v, at least for convex problems, that is to say symmetric positive semi-definite ∇v, $\mathcal{C}$ and $\mathcal{N}$ need to satisfy:

$$I_d - \mathcal{C}(\nabla v) = -\mathcal{N}(\nabla v)\nabla v \tag{192}$$

If $\mathcal{C}$ and $\mathcal{N}$ are polynomials, this equality for all symmetric positive semi-definite ∇v implies the equality on all matrices. Injecting this result in (191) yields the definition of 1-SCLI we used.

B.4 Appendix: Proofs of lower bounds

The class of methods we consider, that is to say the methods whose coefficient mappings $\mathcal{N}$ are any polynomial of degree at most $k-1$, is very general. It includes:

- the k'-extrapolation methods $F_{k',\eta}$ for $k' \leq k$ as defined by (5).
- extrapolation methods with different step sizes for each extrapolation:

$$\omega \longmapsto \varphi_{\eta_1,\omega} \circ \varphi_{\eta_2,\omega} \circ \cdots \circ \varphi_{\eta_k,\omega}(\omega) \tag{193}$$

- cyclic Richardson iterations [OS84]: methods whose update is composed of successive gradient steps with possibly different step sizes for each

$$\omega \longmapsto F_{\eta_1} \circ F_{\eta_2} \circ \cdots \circ F_{\eta_k}(\omega) \tag{194}$$

and any combination of these with at most k composed gradient evaluations.

The lemma below shows how k-extrapolation algorithms fit into the definition of 1-SCLI:

Lemma 13. *For a k-extrapolation method, $\mathcal{N}(\nabla v) = -\eta \sum_{j=0}^{k-1}(-\eta\nabla v)^k$.*

Proof. This result is a direct consequence of Lem. 8. For $\omega \in \mathbb{R}^d$, one gets, by the chain rule,

$$\nabla F_{\eta,k}(\omega) = \nabla_z \varphi_\omega^k(\omega) + \nabla_\omega \varphi_\omega^k(\omega) \tag{195}$$

$$= (-\eta\nabla v)^k + \sum_{j=0}^{k-1}(-\eta\nabla v)^j \tag{196}$$

$$= \sum_{j=0}^{k} (-\eta \nabla v)^j \tag{197}$$

as ∇v is constant. Hence, as expected, $F_{\eta,k}$ is linear so write that, for all $\omega \in \mathbb{R}^d$,

$$F_{\eta,k}(\omega) = \nabla F_{\eta,k} \omega + b \tag{198}$$

If v has a stationary point ω^*, evaluating at ω^* yields

$$\omega^* = \sum_{j=0}^{k} (-\eta \nabla v)^j \omega^* + b \tag{199}$$

Using that $v(\omega^*) = 0$ and so $(\nabla v)\omega^* = -v(0)$, one gets that

$$b = -\eta \sum_{j=1}^{k} (-\eta \nabla v)^{j-1} v(0) \tag{200}$$

and so

$$F_{\eta,k}(\omega) = \omega - \eta \sum_{j=1}^{k} (-\eta \nabla v)^{j-1} v(\omega) \tag{201}$$

which yields the result for affine vector fields with a stationary point. In particular it holds for vector fields such that ∇v is non-singular. As the previous equality is continuous in ∇v, by density of non-singular matrices, the result holds for all affine vector fields.

□

Theorem 5. *Let $0 < \mu, \gamma < L$. (i) If $d - 2 \geq k \geq 3$, there exists $v \in \mathcal{V}_d$ with a symmetric positive Jacobian whose spectrum is in $[\mu, L]$, such that for any $\mathcal{N}$ real polynomial of degree at most $k-1$, $\rho(F_\mathcal{N}) \geq 1 - \frac{4k^3}{\pi} \frac{\mu}{L}$.*

(ii) If $d/2 - 2 \geq k/2 \geq 3$ and d is even, there exists $v \in \mathcal{V}_d$ L-Lipschitz with $\min_{\lambda \in \mathrm{Sp}\, \nabla v} |\lambda| = \sigma_{min}(\nabla v) \geq \gamma$ corresponding to a bilinear game of Example 1 with $m = d/2$, such that, for any $\mathcal{N}$ real polynomial of degree at most $k-1$, $\rho(F_\mathcal{N}) \geq 1 - \frac{k^3}{2\pi} \frac{\gamma^2}{L^2}$.

To ease the presentation of the proof of the theorem, we rely on several lemmas. We first prove (i) and (ii) will follow as a consequence.

In the following, we denote by $\mathbb{R}_{k-1}[X]$ the set of real polynomials of degree at most $k-1$.

Lemma 14. *For, $v \in \mathcal{V}_d$,*

$$\min_{N \in \mathbb{R}_{k-1}[X]} \frac{1}{2} \rho(F_\mathcal{N})^2 = \min_{a_0,\dots,a_{k-1} \in \mathbb{R}} \max_{\lambda \in \mathrm{Sp}\, \nabla v} \frac{1}{2} |1 + \sum_{l=0}^{k-1} a_l \lambda^{l+1}|^2 \tag{202}$$

Proof. Recall the definition of $F_\mathcal{N}$, which is affine by assumption,

$$\forall \omega \in \mathbb{R}^d,\ F_\mathcal{N}(\omega) = w + \mathcal{N}(\nabla v) v(\omega) \tag{203}$$

Then $\nabla F_\mathcal{N} = I_d + \mathcal{N}(\nabla v) \nabla v$. As $\mathcal{N}$is a polynomial, by the spectral mapping theorem (Thm. 17),

$$\mathrm{Sp}\, \nabla F_\mathcal{N} = \{1 + \mathcal{N}(\lambda)\lambda \mid \lambda \in \mathrm{Sp}\, \nabla v\} \tag{204}$$

which yields the result. □

Lemma 15. *Assume that* $\operatorname{Sp} \nabla v = \{\lambda_1, \dots, \lambda_m\} \subset \mathbb{R}$. *Then (202) can be lower bounded by the value of the following problem:*

$$
\begin{aligned}
&\max \sum_{j=1}^{m} \nu_j (\xi_j - \frac{1}{2}\xi_j^2) \\
&s.t. \;\; \nu_j \geq 0, \quad \xi_j \in \mathbb{R}, \; \forall 1 \leq j \leq m \\
&\qquad \sum_{j=1}^{m} \nu_j \xi_j \lambda_j^l = 0, \; \forall 1 \leq l \leq k \\
&\qquad \sum_{j=1}^{m} \nu_j = 1
\end{aligned} \tag{205}
$$

Proof. The right-hand side of (202) can be written as a constrained optimization problem as follows:

$$
\begin{aligned}
&\min_{t, a_0, \dots, a_{k-1}, z_1, \dots, z_m \in \mathbb{R}} t \\
&\text{s.t.} \;\; t \geq \frac{1}{2} z_j^2, \; \forall\, 1 \leq j \leq m \\
&z_j = 1 + \sum_{l=0}^{k-1} a_l \lambda_j^{l+1}, \; \forall\, 1 \leq j \leq m
\end{aligned} \tag{206}
$$

By weak duality, see [BV04] for instance, we can lower bound the value of this problem by the value of its dual. So let us write the Lagrangian of this problem:

$$
\begin{aligned}
&\mathcal{L}(t, a_0, \dots, a_{k-1}, z_1, \dots, z_m, \nu_1, \dots, \nu_m, \chi_1, \dots, \chi_m) \\
&= t + \sum_{j=0}^{m} \nu_j (\frac{1}{2} z_j^2 - t) + \chi_j (1 + \sum_{l=0}^{k-1} a_l \lambda_j^{l+1} - z_j)
\end{aligned} \tag{207}
$$

The Lagrangian is convex and quadratic so its minimum with respect to $t, a_0, \dots, a_{k-1}, z_1, \dots, z_m$ is characterized by the first order condition. Moreover, if there is no solution to the first order condition, its minimum is $-\infty$ (see for instance [BV04, Example 4.5]).

One has that, for any $1 \leq j \leq m$ and $0 \leq l \leq k-1$,

$$
\partial_t \mathcal{L} = 1 - \sum_{j=0}^{m} \nu_j \tag{208}
$$

$$
\partial_{a_l} \mathcal{L} = \sum_{j=0}^{m} \chi_j \lambda_j^{l+1} \tag{209}
$$

$$
\partial_{z_j} \mathcal{L} = \nu_j z_j - \chi_j \tag{210}
$$

Setting these quantities to zero yields the following dual problem:

$$\begin{aligned}
&\max \sum_{j=1,\ \nu_j\neq 0}^{m} \chi_j - \frac{1}{2\nu_j}\chi_j^2 \\
&\text{s.t. } \ \nu_j \geq 0, \quad \chi_j \in \mathbb{R},\ \forall 1 \leq j \leq m \\
&\qquad \sum_{j=1}^{m} \chi_j \lambda_j^l = 0,\ \forall 1 \leq l \leq k \\
&\qquad \nu_j = 0 \implies \chi_j = 0 \\
&\qquad \sum_{j=1}^{m} \nu_j = 1
\end{aligned} \tag{211}$$

Taking $\nu_j \xi_j = \chi_j$ yields the result:

$$\begin{aligned}
&\max \sum_{j=1}^{m} \nu_j (\xi_j - \frac{1}{2}\xi_j^2) \\
&\text{s.t. } \ \nu_j \geq 0, \quad \xi_j \in \mathbb{R},\ \forall 1 \leq j \leq m \\
&\qquad \sum_{j=1}^{m} \nu_j \xi_j \lambda_j^l = 0,\ \forall 1 \leq l \leq k \\
&\qquad \sum_{j=1}^{m} \nu_j = 1
\end{aligned} \tag{212}$$

□

The next lemma concerns Vandermonde matrices and Lagrange polynomials.

Lemma 16. *Let $\lambda_1, \ldots, \lambda_d$ be distinct reals. Denote the Vandermonde matrix by*

$$V(\lambda_1, \ldots, \lambda_d) = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{d-1} \\ 1 & \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{d-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{d-1} \end{pmatrix} \tag{213}$$

Then

$$V(\lambda_1, \ldots, \lambda_d)^{-1} = \begin{pmatrix} L_1^{(0)} & L_2^{(0)} & \ldots & L_d^{(0)} \\ L_1^{(1)} & L_2^{(1)} & \ldots & L_d^{(1)} \\ \vdots & \vdots & & \vdots \\ L_1^{(d-1)} & L_2^{(d-1)} & \ldots & L_d^{(d-1)} \end{pmatrix} \tag{214}$$

where $L_1, L_2, \ldots, L_d$ are the Lagrange interpolation polynomials associated to $\lambda_1, \ldots, \lambda_d$ and $L_j = \sum_{l=0}^{d-1} L_j^{(l)} X^l$ for $1 \leq j \leq d$.

A proof of this result can be found at [Atk89, Theorem 3.1].

The next lemma is the last one before we finally prove the theorem. Recall that in Thm. 5 we assume that $k + 1 \leq d$.

Lemma 17. *Assume that* $\operatorname{Sp}\nabla v = \{\lambda_1, \dots, \lambda_{k+1}\}$ *where* $\lambda_1, \dots, \lambda_{k+1}$ *are distinct non-zero reals. Then the problem of (202) is lower bounded by*

$$\frac{1}{2}\left(\frac{1 - \sum_{j=1}^{k} \frac{\lambda_{k+1}}{\lambda_j} L_j(\lambda_{k+1})}{1 + \sum_{j=1}^{k} |\frac{\lambda_{k+1}}{\lambda_j} L_j(\lambda_{k+1})|}\right)^2 \tag{215}$$

where $L_1, \dots, L_k$ *are the Lagrange interpolation polynomials associated to* $\lambda_1, \dots, \lambda_k$*.*

Proof. To prove this lemma, we start from the result of Lem. 15 and we provide feasible $(\nu_j)_j$ and $(\xi_j)_j$. First, any feasible $(\nu_j)_j$ and $(\xi_j)_j$ must satisfy the k constraints involving the powers of the eigenvalues, which can be rewritten as:

$$V(\lambda_1, \dots, \lambda_k)^T \begin{pmatrix} \nu_1\xi_1\lambda_1 \\ \nu_2\xi_2\lambda_2 \\ \vdots \\ \nu_k\xi_k\lambda_k \end{pmatrix} = -\nu_{k+1}\xi_{k+1} \begin{pmatrix} \lambda_{k+1} \\ \lambda_{k+1}^2 \\ \dots \\ \lambda_{k+1}^k \end{pmatrix} \tag{216}$$

Using the previous lemma yields, for $1 \leq j \leq k$,

$$\nu_j\xi_j = -\nu_{k+1}\xi_{k+1}\frac{1}{\lambda_j}\begin{pmatrix} L_j^{(0)} & L_j^{(1)} & \dots & L_j^{(k-1)} \end{pmatrix}\begin{pmatrix} \lambda_{k+1} \\ \lambda_{k+1}^2 \\ \dots \\ \lambda_{k+1}^k \end{pmatrix} \tag{217}$$

$$= -\nu_{k+1}\xi_{k+1}\frac{\lambda_{k+1}}{\lambda_j}L_j(\lambda_{k+1}) \tag{218}$$

Hence the problem can be rewritten only in terms of the $(\nu_j)_j$ and ξ_{k+1}. Let $c_j = \frac{\lambda_{k+1}}{\lambda_j}L_j(\lambda_{k+1})$. The objective becomes:

$$\sum_{j=1}^{m} \nu_j\left(\xi_j - \frac{1}{2}\xi_j^2\right) = \nu_{k+1}\xi_{k+1}\left(1 - \sum_{j=1}^{k} c_j\right) - \frac{1}{2}\nu_{k+1}\xi_{k+1}^2\left(1 + \sum_{j=0}^{k}\frac{\nu_{k+1}}{\nu_j}c_j^2\right) \tag{219}$$

Choosing $\xi_{k+1} = \frac{1-\sum_{j=1}^{k} c_j}{1+\sum_{j=0}^{k}\frac{\nu_{k+1}}{\nu_j}c_j^2}$ to maximize this quadratic yields:

$$\sum_{j=1}^{m} \nu_j\left(\xi_j - \frac{1}{2}\xi_j^2\right) = \frac{1}{2}\nu_{k+1}\frac{\left(1 - \sum_{j=1}^{k} c_j\right)^2}{1 + \sum_{j=0}^{k}\frac{\nu_{k+1}}{\nu_j}c_j^2} \tag{220}$$

Finally take $\nu_j = \frac{|c_j|}{1+\sum_{j=1}^{k}|c_j|}$ for $j \leq k$ and $\nu_{k+1} = \frac{1}{1+\sum_{j=1}^{k}|c_j|}$ which satisfy the hypotheses of the problem of Lem. 15. With the feasible $(\nu_j)_j$ and $(\xi_j)_j$ defined this way, the value of the objective is

$$\frac{1}{2}\left(\frac{1 - \sum_{j=1}^{k} c_j}{1 + \sum_{j=1}^{k}|c_j|}\right)^2 \tag{221}$$

which is the desired result. □

We finally prove (i) of Thm. 5.

Proof of (i) of Thm. 5. To prove the theorem, we build on the result of Lem. 17. We have to choose $\lambda_1, \ldots, \lambda_{k+1} \in [\mu, L]$ positive distinct such that (215) is big. One could try to distribute the eigenvalues uniformly across the interval but this leads to a lower bound which decreases exponentially in k. To make things a bit better, we use Chebyshev points of the second kind studied by [Sal71]. However we will actually refer to the more recent presentation of [BT04].

For now, assume that k is even and so $k \geq 4$. We will only use that $d - 1 \geq k$ (and not that $d - 2 \geq k$). Define, for $1 \leq j \leq k$, $\lambda_j = \frac{\mu+L}{2} - \frac{L-\mu}{2} \cos \frac{j-1}{k-1}\pi$. Using the barycentric formula of [BT04, Eq. 4.2], the polynomial which interpolates $f_1, \ldots, f_k$ at the points $\lambda_1, \ldots, \lambda_k$ can be written as:

$$P(X) = \frac{\sum_{j=1}^k \frac{w_j}{X-\lambda_j} f_j}{\sum_{j=1}^k \frac{w_j}{X-\lambda_j}} \tag{222}$$

where

$$w_j = \begin{cases} (-1)^{j-1} & \text{if} \quad 2 \leq j \leq k-1 \\ \frac{1}{2}(-1)^{j-1} & \text{if} \quad j \in \{1, k\} \end{cases} \tag{223}$$

Define $Z(X) = \sum_{j=1}^k \frac{w_j}{X-\lambda_j}$.

Now, $\sum_{j=1}^k \frac{\lambda_{k+1}}{\lambda_j} L_j(\lambda_{k+1})$ can be seen as the polynomial interpolating $\frac{\lambda_{k+1}}{\lambda_1}, \ldots, \frac{\lambda_{k+1}}{\lambda_k}$ at the points $\lambda_1, \ldots, \lambda_j$ evaluated at λ_{k+1}. Hence, using the barycentric formula,

$$\sum_{j=1}^k \frac{\lambda_{k+1}}{\lambda_j} L_j(\lambda_{k+1}) = \frac{1}{Z(\lambda_{k+1})} \sum_{j=1}^k \frac{w_j}{\lambda_{k+1} - \lambda_j} \frac{\lambda_{k+1}}{\lambda_j} \tag{224}$$

Similarly, $\sum_{j=1}^k |\frac{\lambda_{k+1}}{\lambda_j} L_j(\lambda_{k+1})|$ can be seen as the polynomial interpolating $|\frac{\lambda_{k+1}}{\lambda_1}| \operatorname{sign}(L_1(\lambda_{k+1})), \ldots, |\frac{\lambda_{k+1}}{\lambda_k}| \operatorname{sign}(L_k(\lambda_{k+1}))$ at the points $\lambda_1, \ldots, \lambda_j$ evaluated at λ_{k+1}. However, from [BT04, Section 3],

$$L_j(\lambda_{k+1}) = \left(\prod_{j=1}^k (\lambda_{k+1} - \lambda_j) \right) \frac{w_j}{\lambda_{k+1} - \lambda_j} \tag{225}$$

and by [BT04, Eq. 4.1],

$$1 = \left(\prod_{j=1}^k (\lambda_{k+1} - \lambda_j) \right) Z(\lambda_{k+1}) \tag{226}$$

Hence

$$\operatorname{sign}(L_j(\lambda_{k+1})) = \operatorname{sign} Z(\lambda_{k+1}) \operatorname{sign} \left(\frac{w_j}{\lambda_{k+1} - \lambda_j} \right) \tag{227}$$

Therefore, using the barycentric formula again,

$$\sum_{j=1}^k \frac{\lambda_{k+1}}{\lambda_j} |L_j(\lambda_{k+1})| = \frac{1}{|Z(\lambda_{k+1})|} \sum_{j=1}^k \left| \frac{w_j}{\lambda_{k+1} - \lambda_j} \right| \frac{\lambda_{k+1}}{\lambda_j} \tag{228}$$

Hence, (215) becomes:

$$\frac{1}{2} \left(\frac{1 - \sum_{j=1}^k \frac{\lambda_{k+1}}{\lambda_j} L_j(\lambda_{k+1})}{1 + \sum_{j=1}^k |\frac{\lambda_{k+1}}{\lambda_j} L_j(\lambda_{k+1})|} \right)^2 \tag{229}$$

$$= \frac{1}{2}\left(\frac{1 - \frac{1}{Z(\lambda_{k+1})}\sum_{j=1}^{k}\frac{w_j}{\lambda_{k+1}-\lambda_j}\frac{\lambda_{k+1}}{\lambda_j}}{1 + \frac{1}{|Z(\lambda_{k+1})|}\sum_{j=1}^{k}\left|\frac{w_j}{\lambda_{k+1}-\lambda_j}\right|\frac{\lambda_{k+1}}{\lambda_j}}\right)^2 \tag{230}$$

$$= \frac{1}{2}\left(1 - \frac{\frac{1}{|Z(\lambda_{k+1})|}\sum_{j=1}^{k}\left|\frac{w_j}{\lambda_{k+1}-\lambda_j}\right|\frac{\lambda_{k+1}}{\lambda_j}\left(1 + \operatorname{sign} Z(\lambda_{k+1})\operatorname{sign}\left(\frac{w_j}{\lambda_{k+1}-\lambda_j}\right)\right)}{1 + \frac{1}{|Z(\lambda_{k+1})|}\sum_{j=1}^{k}\left|\frac{w_j}{\lambda_{k+1}-\lambda_j}\right|\frac{\lambda_{k+1}}{\lambda_j}}\right)^2 . \tag{231}$$

Now take any λ_{k+1} such that $\lambda_1 < \lambda_{k+1} < \lambda_2$. Then, from (226), $\operatorname{sign} Z(\lambda_{k+1}) = (-1)^{k+1} = -1$ as we assume that k is even. By definition of the coefficients w_j, $\operatorname{sign}\frac{w_1}{\lambda_{k+1}-\lambda_1} = +1$. Hence $1 + \operatorname{sign} Z(\lambda_{k+1})\operatorname{sign}\frac{w_1}{\lambda_{k+1}-\lambda_1} = 0$. Similarly, $\operatorname{sign}\frac{w_2}{\lambda_{k+1}-\lambda_2} = +1$ and so $1+\operatorname{sign} Z(\lambda_{k+1})\operatorname{sign}\frac{w_2}{\lambda_{k+1}-\lambda_2} = 0$ too[8].

As the quantity inside the parentheses of (231) is non-negative, we can focus on lower bounding it. Using the considerations on signs we get:

$$\frac{\frac{1}{|Z(\lambda_{k+1})|}\sum_{j=1}^{k}\left|\frac{w_j}{\lambda_{k+1}-\lambda_j}\right|\frac{\lambda_{k+1}}{\lambda_j}\left(1 + \operatorname{sign} Z(\lambda_{k+1})\operatorname{sign}\left(\frac{w_j}{\lambda_{k+1}-\lambda_j}\right)\right)}{1 + \frac{1}{|Z(\lambda_{k+1})|}\sum_{j=1}^{k}\left|\frac{w_j}{\lambda_{k+1}-\lambda_j}\right|\frac{\lambda_{k+1}}{\lambda_j}} \tag{232}$$

$$= \frac{\frac{1}{|Z(\lambda_{k+1})|}\sum_{j=3}^{k}\left|\frac{w_j}{\lambda_{k+1}-\lambda_j}\right|\frac{\lambda_{k+1}}{\lambda_j}\left(1 + \operatorname{sign} Z(\lambda_{k+1})\operatorname{sign}\left(\frac{w_j}{\lambda_{k+1}-\lambda_j}\right)\right)}{1 + \frac{1}{|Z(\lambda_{k+1})|}\sum_{j=1}^{k}\left|\frac{w_j}{\lambda_{k+1}-\lambda_j}\right|\frac{\lambda_{k+1}}{\lambda_j}} \tag{233}$$

$$\leq 2\frac{\sum_{j=3}^{k}\left|\frac{1}{\lambda_{k+1}-\lambda_j}\right|\frac{\lambda_{k+1}}{\lambda_j}}{|Z(\lambda_{k+1})| + \sum_{j=1}^{k}\left|\frac{w_j}{\lambda_{k+1}-\lambda_j}\right|\frac{\lambda_{k+1}}{\lambda_j}} \tag{234}$$

$$\leq 2\frac{\sum_{j=3}^{k}\left|\frac{1}{\lambda_{k+1}-\lambda_j}\right|\frac{\lambda_{k+1}}{\lambda_j}}{\frac{1}{2}\left|\frac{1}{\lambda_{k+1}-\lambda_1}\right|\frac{\lambda_{k+1}}{\lambda_1}} \tag{235}$$

$$\leq 2\frac{(k-2)\left|\frac{1}{\lambda_{k+1}-\lambda_3}\right|\frac{\lambda_{k+1}}{\lambda_3}}{\frac{1}{2}\left|\frac{1}{\lambda_{k+1}-\lambda_1}\right|\frac{\lambda_{k+1}}{\lambda_1}} \tag{236}$$

$$\tag{237}$$

where we used that, for $j \geq 3$, $\left|\frac{1}{\lambda_{k+1}-\lambda_j}\right|\frac{\lambda_{k+1}}{\lambda_j} \leq \left|\frac{1}{\lambda_{k+1}-\lambda_3}\right|\frac{\lambda_{k+1}}{\lambda_3}$ as $\lambda_1 < \lambda_{k+1} < \lambda_2 < \lambda_3 < \cdots < \lambda_k$. Now, recalling that $\lambda_1 = \mu$, and using that $\lambda_1 < \lambda_{k+1} < \lambda_2 < \lambda_3$ for the inequality,

$$2\frac{(k-2)\left|\frac{1}{\lambda_{k+1}-\lambda_3}\right|\frac{\lambda_{k+1}}{\lambda_3}}{\frac{1}{2}\left|\frac{1}{\lambda_{k+1}-\lambda_1}\right|\frac{\lambda_{k+1}}{\lambda_1}} = 4(k-2)\frac{\mu}{\lambda_3}\frac{|\lambda_{k+1}-\lambda_1|}{|\lambda_{k+1}-\lambda_3|} \tag{238}$$

$$\leq 4(k-2)\frac{\mu}{\lambda_3}\frac{|\lambda_2-\lambda_1|}{|\lambda_2-\lambda_3|} \tag{239}$$

$$= 4(k-2)\frac{\mu}{\frac{1}{2}L(1-\cos\frac{2\pi}{k-1}) + \frac{1}{2}\mu(1+\cos\frac{2\pi}{k-1})}\frac{|\cos\frac{\pi}{k-1}-1|}{|\cos\frac{\pi}{k-1}-\cos\frac{2\pi}{k-1}|} \tag{240}$$

$$\leq 8(k-2)\frac{\mu}{L(1-\cos\frac{\pi}{k-1})}\frac{|\cos\frac{\pi}{k-1}-1|}{|\cos\frac{\pi}{k-1}-\cos\frac{2\pi}{k-1}|} \tag{241}$$

[8] We could do without this, but it is free and gives slightly better constants.

$$= 8(k-2)\frac{\mu}{L}\frac{1}{|\cos\frac{\pi}{k-1} - \cos\frac{2\pi}{k-1}|} \tag{242}$$

by definition of the interpolation points. Now, for $k \geq 4$, the sinus is non-negative on $[\frac{\pi}{k-1}, \frac{2\pi}{k-1}]$ and reaches its minimum at $\frac{\pi}{k-1}$. Hence,

$$\left|\cos\frac{\pi}{k-1} - \cos\frac{2\pi}{k-1}\right| = \left|\int_{\pi/(k-1)}^{2\pi/(k-1)} \sin t dt\right| \tag{243}$$

$$= \int_{\pi/(k-1)}^{2\pi/(k-1)} \sin t dt \tag{244}$$

$$\geq \frac{\pi}{k-1}\sin\frac{\pi}{k-1} \tag{245}$$

$$\geq 2\frac{\pi}{(k-1)^2} \tag{246}$$

as $0 \geq \frac{\pi}{k-1} \geq \frac{\pi}{2}$. Putting everything together yields,

$$\frac{1}{2}\left(\frac{1 - \sum_{j=1}^{k}\frac{\lambda_{k+1}}{\lambda_j}L_j(\lambda_{k+1})}{1 + \sum_{j=1}^{k}|\frac{\lambda_{k+1}}{\lambda_j}L_j(\lambda_{k+1})|}\right)^2 \geq \frac{1}{2}\left(1 - \frac{4(k-1)^2(k-2)}{\pi}\frac{\mu}{L}\right)^2 \tag{247}$$

$$\geq \frac{1}{2}\left(1 - \frac{4(k-1)^3)}{\pi}\frac{\mu}{L}\right)^2 \tag{248}$$

which yields the desired result by the definition of the problem of (202).

The lower bound holds for any v such that $\operatorname{Sp}\nabla v = \{\lambda_1, \dots, \lambda_{k+1}\}$. As $\{\lambda_1, \dots, \lambda_{k+1}\} \subset [\mu, L]$, one can choose v of the form $v = \nabla f$ where $f : \mathbb{R}^d \to \mathbb{R}$ is a μ-strongly convex and L-smooth quadratic function with $\operatorname{Sp}\nabla^2 f = \{\lambda_1, \dots, \lambda_{k+1}\}$.

Now, we tackle the case k odd, with $k \geq 3$ and $d-1 \geq k+1$. Note that if $\mathcal{N}$ is a real polynomial of degree at most $k-1$, it is also a polynomial of degree at most $(k+1)-1$. Applying the result above yields that there exists $v \in \mathcal{V}_d$ with the desired properties such that,

$$\rho(F_{\mathcal{N}}) \geq 1 - \frac{k^3}{2\pi}\frac{\gamma^2}{L^2} \tag{249}$$

Hence, (i) holds for any $d-2 \geq k \geq 3$. □

Then, (ii) is essentially a corollary of (i).

Proof of (ii) of Thm. 5. For a square zero-sum two player game, the Jacobian of the vector field can be written as,

$$\nabla v = \begin{pmatrix} 0_m & A \\ -A^T & 0_m \end{pmatrix} \tag{250}$$

where $A \in \mathbb{R}^{m\times m}$. By Lem. 5,

$$\operatorname{Sp}\nabla v = \{i\sqrt{\lambda} \mid \lambda \in \operatorname{Sp} AA^T\} \cup \{-i\sqrt{\lambda} \mid \lambda \in \operatorname{Sp} AA^T\} \tag{251}$$

Using Lem. 14, one gets that:

$$\min_{N\in\mathbb{R}_{k-1}[X]} \frac{1}{2}\rho(F_{\mathcal{N}})^2 = \min_{a_0,\dots,a_{k-1}\in\mathbb{R}} \max_{\lambda\in\operatorname{Sp} AA^T} \frac{1}{2}\max\left(\left|1 + \sum_{l=0}^{k-1} a_l(i\sqrt{\lambda})^{l+1}\right|^2, \left|1 + \sum_{l=0}^{k-1} a_l(-i\sqrt{\lambda})^{l+1}\right|^2\right) \tag{252}$$

$$\geq \min_{a_0,\dots,a_{k-1}\in\mathbb{R}} \max_{\lambda\in\operatorname{Sp} AA^T} \frac{1}{2}\left|1+\sum_{l=0}^{k-1} a_l (i\sqrt{\lambda})^{l+1}\right|^2 \tag{253}$$

$$\geq \min_{a_0,\dots,a_{k-1}\in\mathbb{R}} \max_{\lambda\in\operatorname{Sp} AA^T} \frac{1}{2}\left(\Re\left(1+\sum_{l=0}^{k-1} a_l (i\sqrt{\lambda})^{l+1}\right)\right)^2 \tag{254}$$

$$= \min_{a_0,\dots,a_{k-1}\in\mathbb{R}} \max_{\lambda\in\operatorname{Sp} AA^T} \frac{1}{2}\left(1+\sum_{l=1}^{\lfloor k/2\rfloor} a_{2l-1}(-1)^l\lambda^l\right)^2 \tag{255}$$

$$= \min_{a_0,\dots,a_{\lfloor k/2\rfloor-1}\in\mathbb{R}} \max_{\lambda\in\operatorname{Sp} AA^T} \frac{1}{2}\left(1+\sum_{l=1}^{\lfloor k/2\rfloor} a_{l-1}\lambda^l\right)^2 \tag{256}$$

Using Lem. 14 again,

$$\min_{a_0,\dots,a_{\lfloor k/2\rfloor-1}\in\mathbb{R}} \max_{\lambda\in\operatorname{Sp} AA^T} \frac{1}{2}\left(1+\sum_{l=1}^{\lfloor k/2\rfloor} a_{l-1}\lambda^l\right)^2 = \min_{N\in\mathbb{R}_{\lfloor k/2\rfloor-1}[X]} \frac{1}{2}\rho(\widetilde{F}_{\mathcal{N}})^2 \tag{257}$$

where $\widetilde{F}_{\mathcal{N}}$ is the 1-SCLI operator of $\mathcal{N}$, as defined by (2) applied to the vector field $\omega \mapsto AA^T\omega$. Let $S \in \mathbb{R}^{m\times m}$ be a symmetric positive definite matrix given by (i) of this theorem applied with $(\mu, L) = (\gamma^2, L^2)$ and $\left\lfloor \frac{k}{2}\right\rfloor$ instead of k and so such that $\operatorname{Sp} S \subset [\gamma^2, L^2]$. Now choose $A \in \mathbb{R}^{m\times m}$ such that $A^TA = S$, for instance by taking a square root of S (see [Lax07, Chapter 10]). Then,

$$\min_{N\in\mathbb{R}_{\lfloor k/2\rfloor-1}[X]} \frac{1}{2}\rho(\widetilde{F}_{\mathcal{N}})^2 \geq \frac{1}{2}\left(1-\frac{k^3}{2\pi}\frac{\mu}{L}\right) \tag{258}$$

Moreover, by computing $\nabla v^T\nabla v$ and using that $\operatorname{Sp} AA^T = \operatorname{Sp} A^TA$, one gets that $\min_{\lambda\in\operatorname{Sp}\nabla v}|\lambda| = \sigma_{min}(\nabla v) = \sigma_{min}(A) \geq \gamma$ and $\sigma_{max}(\nabla v) = \sigma_{max}(A) \leq L$. □

Remark 2. Interestingly, the examples we end up using have a spectrum similar to the one of the matrix Nesterov uses in the proofs of his lower bounds in [Nes04]. The choice of the spectrum of the Jacobian of the vector field was indeed the choice of interpolation points. Following [Sal71, BT04] we used points distributed across the interval as a cosinus as it minimizes oscillations near the edge of the interval. Therefore, this links the hardness Nesterov's examples to the well-conditioning of families of interpolation points.

B.5 Appendix: Handling singularity

The following theorem is a way to use spectral techniques to obtain geometric convergence rates even if the Jacobian of the vector field at the stationary point is singular. We only need to ensure that the vector field is locally null along these directions of singularity.

In this subsection, for $A \in \mathbb{R}^{m\times p}$, $\mathcal{K}A = \{x \in \mathbb{R}^p \mid Ax = 0\}$ denotes the kernel (or the nullspace) of A.

The following theorem is actually a combination of the proof of [Nagarajan and Kolter(2017), Thm. A.4], which only proves asymptotic stability in continuous time with no concern for the rate, and the classic Thm. 9.

Theorem 16. *Consider $h : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^m \times \mathbb{R}^p$ twice continuously differentiable vector field and write $h(\theta, \varphi) = (h_\theta(\theta, \varphi), h_\varphi(\theta, \varphi))$. Assume that $(0, 0)$ is a stationary point, i.e. $h(0, 0) = (0, 0)$ and that there exists $\tau > 0$ such that,*

$$\forall \varphi \in \mathbb{R}^p \cap B(0, \tau), \quad h(0, \varphi) = (0, 0) \tag{259}$$

Let $\rho^* = \rho(\nabla_\theta(\mathrm{Id} + h_\theta)(0, 0))$ and define the iterates $(\theta_t, \varphi_t)_t$ by

$$(\theta_{t+1}, \varphi_{t+1}) = (\theta_t, \varphi_t) + h(\theta_t, \varphi_t) \tag{260}$$

Then, if $\rho^* < 1$, for all $\epsilon > 0$, there exists a neighborhood of $(0, 0)$ such that for any initial point in this neighborhood, the distance of the iterates $(\theta_t, \varphi_t)_t$ to a stationary point of h decreases as $\mathcal{O}((\rho^* + \epsilon)^t)$. If v is linear, this is satisfied with the whole space as a neighborhood for all $\epsilon > 0$.

The following proof is inspired from the ones of [Nagarajan and Kolter(2017), Thm. 4] and [GHP+19, Thm. 1].

Proof. Let $\mathbf{J} = \nabla_\theta h(0, 0) \in \mathbb{R}^{(m+p)\times m}$, $\mathbf{J}_\theta = \nabla_\theta h_\theta(0, 0) \in \mathbb{R}^{m\times m}$ and $\mathbf{J}_\varphi = \nabla_\theta h_\varphi(0, 0) \in \mathbb{R}^{p\times m}$. Let $\epsilon > 0$ and suppose $\rho^* + \epsilon < 1$. By [Ber99, Prop. A.15] there exists a norm $\|.\|$ on $\mathbb{R}^m$ such that the induced matrix norm on $\mathbb{R}^{m\times m}$ satisfy:

$$\|\mathrm{Id} + J_\theta\| \leq \rho^* + \frac{\epsilon}{2} \tag{261}$$

On the contrary the norm on $\mathbb{R}^p$ can be chosen arbitrarily.

The extension of these norms to $\mathbb{R}^m \times \mathbb{R}^p$ is chosen such that $\|\theta, \varphi\| = \|\theta\| + \|\varphi\|$ for simplicity (but this is without loss of generality). In this proof, we denote the d-dimensional balls by $B_d(x, r) = \{y \in \mathbb{R}^d \mid \|x - y\| \leq r\}$ with $x \in \mathbb{R}^d$, $r > 0$.

- Let $\mathbf{J} = \nabla_\theta h(0, 0) \in \mathbb{R}^{(m+p)\times m}$.

  We first show that, for all $\eta > 0$ there exists $\tau \geq \delta > 0$ such that,

  $$\forall (\theta, \varphi) \in B_{m+p}((0, 0), \delta) : \ \|h(\theta, \varphi) - \mathbf{J}\theta\| \leq \eta\|\theta\| \tag{262}$$

  The interesting thing here is that we are completely getting rid of the dependence on φ, both in the linearization and in the bound.

  Let $\varphi \in B_p(0, \tau)$. Then, using that $h(0, \varphi) = 0$, the Taylor development of $h(\theta, \varphi)$ w.r.t. to θ yields:

  $$\begin{aligned} h(\theta, \varphi) &= \nabla_\theta h(0, \varphi)\theta + R(\theta, \varphi) && (263) \\ &= J\theta + (\nabla_\theta h(0, \varphi) - \nabla_\theta h(0, 0))\theta + R(\theta, \varphi) && (264) \end{aligned}$$

  We now deal with the last two terms. First the rest $R(\theta, \varphi)$. As v is assumed to be continuously differentiable, there exists $c > 0$ constant (which depends on τ) such that, for any $\theta \in \mathbb{R}^m$:

  $$\forall \varphi \in B_p(0, \delta) : \|R(\theta, \varphi)\| \leq c\|\theta\|^2 \tag{265}$$

  Hence, for $\theta \in B(0, \frac{\eta}{2c})$, we get that:

  $$\forall \varphi \in B_p(0, \delta) : \|R(\theta, \varphi)\| \leq \frac{\eta}{2}\|\theta\| \tag{266}$$

  Concerning the other term, by continuity, $\nabla_\theta h(0, \varphi) - \nabla_\theta h(0, 0)$ goes to zero as φ goes to zero. Hence, there exists $\delta > 0$, $\delta \leq \min(\tau, \frac{\eta}{2c})$ such that for any $\varphi \in B_p(0, \delta)$, $\|(\nabla_\theta h(0, \varphi) - \nabla_\theta h(0, 0))\theta\| \leq \frac{\eta}{2}\|\theta\|$. Combining the two bounds yields the desired result.

- We now apply the previous result with $\eta = \epsilon/2$. We first examine what this means for $(\theta_{t+1}, \varphi_{t+1})$ when (θ_t, φ_t) is in $B_{m+p}((0,0),\delta)$. However, the neighborhood $B_{m+p}((0,0),\delta)$ is not necessarily stable, so we will again restrict it afterwards. See the proof [Nagarajan and Kolter(2017), Thm. 4] for a more detailed discussion on this. Assume for now that $(\theta_t, \varphi_t) \in B_{m+p}((0,0),\delta)$. Then,

$$\begin{aligned}
\|\theta_{t+1}\| &= \|(\mathrm{Id} + \mathbf{J}_\theta)\theta_t + (h_\theta(\theta_t, \varphi_t) - \mathbf{J}_\theta \theta_t)\| && (267)\\
&\leq \|(\mathrm{Id} + \mathbf{J}_\theta)\theta_t\| + \|(h_\theta(\theta_t, \varphi_t) - \mathbf{J}_\theta \theta_t)\| && (268)\\
&\leq (\rho^* + \epsilon)\|\theta_t\| && (269)
\end{aligned}$$

Consider now the other coordinate φ_{t+1}, still under the assumption that $(\theta_t, \varphi_t) \in B_{m+p}((0,0),\delta)$. Then,

$$\begin{aligned}
\|\varphi_{t+1} - \varphi_t\| &= \|\mathbf{J}_\varphi \theta_t + (h_\varphi(\theta_t, \varphi_t) - \mathbf{J}_\varphi \theta_t)\| && (270)\\
&\leq \|\mathbf{J}_\varphi \theta_t\| + \|(h_\varphi(\theta_t, \varphi_t) - \mathbf{J}_\varphi \theta_t)\| && (271)\\
&\leq (\|\mathbf{J}_\varphi\| + \frac{\epsilon}{2})\|\theta_t\| && (272)
\end{aligned}$$

Now let $V = \{(\theta, \varphi) \in \mathbb{R}^m \times \mathbb{R}^p \,|(\theta, \varphi) \in B((0,0),\delta),\ (1 + \frac{\|\mathbf{J}_\varphi\| + \frac{\epsilon}{2}}{1-\rho^*-\epsilon})\|\theta\| + \|\varphi\| < \delta\}$ neighborhood of $(0,0)$. We show, by induction, that if $(\theta_0, \varphi_0) \in V$, then the iterates stay in $B_{m+p}((0,0),\delta)$.

Assume $(\theta_0, \varphi_0) \in V$. By construction, $(\theta_0, \varphi_0) \in B_{m+p}((0,0),\delta)$. Now assume that $(\theta_0, \varphi_0), (\theta_1, \varphi_1), \ldots, (\theta_t, \varphi_t)$ are in $B_{m+p}((0,0),\delta)$ for some $t \geq 0$. By what has been proven above, first, $\|\theta_{t+1}\| \leq (\rho^* + \epsilon)^{t+1}\|\theta_0\| \leq \|\theta_0\|$. Then,

$$\begin{aligned}
\|\varphi_{t+1}\| &\leq \|\varphi_0\| + \sum_{k=0}^{t} \|\varphi_{k+1} - \varphi_k\| && (273)\\
&\leq \|\varphi_0\| + (\|\mathbf{J}_\varphi\| + \frac{\epsilon}{2}) \sum_{k=0}^{t} \|\theta_k\| && (274)\\
&\leq \|\varphi_0\| + (\|\mathbf{J}_\varphi\| + \frac{\epsilon}{2}) \sum_{k=0}^{t} (\rho^* + \epsilon)^k \|\theta_0\| && (275)\\
&\leq \|\varphi_0\| + \frac{\|\mathbf{J}_\varphi\| + \frac{\epsilon}{2}}{1 - \rho^* - \epsilon}\|\theta_0\| && (276)\\
& && (277)
\end{aligned}$$

Hence, putting the two coordinates together,

$$\begin{aligned}
\|(\theta_{t+1}, \varphi_{t+1})\| &= \|\theta_{t+1}\| + \|\varphi_{t+1}\| && (278)\\
&\leq \|\varphi_0\| + \left(1 + \frac{\|\mathbf{J}_\varphi\| + \frac{\epsilon}{2}}{1 - \rho^* - \epsilon}\right)\|\theta_0\| && (279)\\
&< \delta && (280)
\end{aligned}$$

by definition of V. Hence, $(\theta_{t+1}, \varphi_{t+1}) \in B_{m+p}((0,0),\delta)$ which concludes the induction and the proof.

For the linear operator case, note that we can choose $\tau = +\infty$, $c = 0$, $\eta = 0$ and $\delta = +\infty$. Then we have $V = \mathbb{R}^m \times \mathbb{R}^p$. □

By a linear base change, we get the more practical corollary:

Corollary 5. *Let $F : \mathbb{R}^d \to \mathbb{R}^d$ be twice continuously differentiable and $\omega^* \in \mathbb{R}^d$ be a fixed point. Assume that there exists $\delta > 0$ such that for all $\xi \in \mathcal{K}(\nabla F(\omega^*) - I_d) \cap B(0,\delta)$, $\omega^* + \xi$ is still a fixed point and that $\mathcal{K}(\nabla F(\omega^*) - I_d)^2 = \mathcal{K}(\nabla F(\omega^*) - I_d)$. Define*

$$\rho^* = \max\{|\lambda| \,|\, \lambda \in \operatorname{Sp} \nabla F(\omega^*),\, \lambda \neq 1\} \tag{281}$$

and assume $\rho^* < 1$. Consider the iterates $(\omega_t)_t$ built from $\omega_0 \in \mathbb{R}^d$ as:

$$\omega_{t+1} = F(\omega_t) \quad \forall t \geq 0 \tag{282}$$

Then, for all $\epsilon > 0$, for any ω_0 in a neighborhood of ω^*, the distance of the iterates $(\omega_t)^t$ to fixed points of F decreases in $\mathcal{O}((\rho^* + \epsilon)^t)$.

Moreover, if F is linear, we can take this neighborhood to be the whole space and $\epsilon = 0$.

Proof. We consider the spaces $\mathcal{K}(\nabla F(\omega^*) - \lambda I_d)^{m_\lambda}$, $\lambda \in \operatorname{Sp} \nabla F(\omega^*)$ where m_λ denotes the multiplicity of the eigenvalue λ as root of the characteristic polynomial of $\nabla F(\omega^*)$. Then, we have,

$$\mathbb{R}^d = \bigoplus_{\lambda \in \operatorname{Sp} \nabla F(\omega^*)} \mathcal{K}(\nabla F(\omega^*) - \lambda I_d)^{m_\lambda}$$

see [Lax07, Chap. 6] for instance.

Now, using that $\mathcal{K}(\nabla F(\omega^*) - I_d)^2 = \mathcal{K}(\nabla F(\omega^*) - I_d)$, we have that $\mathcal{K}(\nabla F(\omega^*) - I_d)^{m_1} = \mathcal{K}(\nabla F(\omega^*) - I_d)$. Hence, the whole space can be decomposed as $\mathbb{R}^d = \mathcal{K}(\nabla F(\omega^*) - I_d) \oplus E$ where $E = \bigoplus_{\lambda \in \operatorname{Sp} \nabla F(\omega^*) \setminus \{1\}} \mathcal{K}(\nabla F(\omega^*) - \lambda I_d)^{m_\lambda}$. Note that E is stable by $\nabla F(\omega^*)$ and so $\rho(\nabla F(\omega^*)|_E) = \rho^*$ as defined in the statement of the theorem. Denote by $P \in \mathbb{R}^d \times \mathbb{R}^d$ the (invertible) change of basis such that $\mathcal{K}(\nabla F(\omega^*) - I_d)$ is sent on the subspace $\mathbb{R}^m \times \{0\}^p$ and E on the subspace $\{0\}^m \times \mathbb{R}^p$, where m and p are the respective dimensions of $\mathcal{K}(\nabla F(\omega^*) - I_d)$ and E. Then, we apply the previous theorem Thm. 16 with h defined by,

$$(\theta, \varphi) + h(\theta, \varphi) = PF\left(\omega^* + P^{-1}(\theta, \varphi)\right)$$

which concludes the proof. □

Remark 3. In general the condition $\mathcal{K}(\nabla F(\omega^*) - I_d)^2 = \mathcal{K}(\nabla F(\omega^*) - I_d)$ will be equivalent to $\mathcal{K}\nabla v(\omega^*)^2 = \mathcal{K}\nabla v(\omega^*)$ where v is the game vector field. We keep this remark informal but we prove this for extragradient below as an example. Indeed, as seen with $1 - SCLI$ in §2.1.3 with (2), $\nabla F(\omega^*)$ is of the form $\mathrm{Id} + \mathcal{N}(\nabla v(\omega^*))\nabla v(\omega^*)$ where $\mathcal{N}$ is a polynomial. Hence, $(\nabla F(\omega^*) - I_d)^j = \mathcal{N}(\nabla v(\omega^*))^j \nabla v(\omega^*)^j$. Moreover, in practice, $\mathcal{N}(\nabla v(\omega^*))$ will be chosen — e.g. by the choice of the step-size — to be non-singular. Hence, $\mathcal{K}(\nabla F(\omega^*) - I_d)^j = \mathcal{K}\nabla v(\omega^*)^j$ and so $\mathcal{K}(\nabla F(\omega^*) - I_d)^2 = \mathcal{K}(\nabla F(\omega^*) - I_d)$ will be equivalent to $\mathcal{K}\nabla v(\omega^*)^2 = \mathcal{K}\nabla v(\omega^*)$.

We now prove a lemma concerning extragradient which as a first step before apply Cor. 5. We could have proven this result for k-extrapolation methods but we focus on extragradient for simplicity.

Lemma 18. *Let $F_{2,\eta} : \omega \to \omega - \eta v(\omega - \eta v(\omega))$ denote the extragradient operator. Assume that v is L-Lipschitz. Then, if $0 < \eta < \frac{1}{L}$, for ω^* stationary point of v,*

$$\mathcal{K}(\nabla F_{2,\eta}(\omega^*) - I_d) = \mathcal{K}\nabla v(\omega^*)$$

and

$$\mathcal{K}(\nabla F_{2,\eta}(\omega^*) - I_d)^2 = \mathcal{K}(\nabla F_{2,\eta}(\omega^*) - I_d) \iff \mathcal{K}\nabla v(\omega^*)^2 = \mathcal{K}\nabla v(\omega^*)$$

Proof. We have $\nabla F_{2,\eta}(\omega^*) = I_d - \eta\nabla v(\omega^*)(I_d - \eta\nabla v(\omega^*))$ and so $\nabla F_{2,\eta}(\omega^*) - I_d = -\eta\nabla v(\omega^*)(I_d - \eta\nabla v(\omega^*))$. As $\nabla v(\omega^*)$ and $I_d - \eta\nabla v(\omega^*))$ commute, for $j \in \{1, 2\}$,

$$(\nabla F_{2,\eta}(\omega^*) - I_d)^j = (-\eta\,(I_d - \eta\nabla v(\omega^*)))^j\,\nabla v(\omega^*)^j$$

By the choice of η, $\eta(I_d - \eta\nabla v(\omega^*))$ is non-singular and so $\mathcal{K}(\nabla F_{2,\eta}(\omega^*) - I_d)^j = \mathcal{K}\nabla v(\omega^*)^j$ which yields the result. □

The whole framework developed implies in particular that Thm. 4 actually also yields convergence guarantees for extragradient on more general bilinear games than those considered in Example 2.

Example 4 (Bilinear game with potential singularity)**.** A saddle-point problem of the form:

$$\min_{x\in\mathbb{R}^m}\max_{y\in\mathbb{R}^p} x^T Ay + b^T x + c^T y \tag{283}$$

with $A \in \mathbb{R}^{m\times p}$ not null, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^p$.

Corollary 6. *Consider the bilinear game of Example 4. The iterates of extragradient with* $\eta = (4\sigma_{max}(A))$ *converge globally to* ω^* *at a linear rate of* $\mathcal{O}\big((1 - \frac{1}{64}\frac{\widetilde{\sigma}_{min}(A)^2}{\sigma_{max}(A)^2})^t\big)$ *where* $\widetilde{\sigma}_{min}(A)$ *is the smallest non-zero singular value of* A.

Proof. Let ω^* be a stationary point of the associated vector field v. Then, $\nabla v(\omega^*) = \begin{pmatrix} O & A \\ -A^T & 0 \end{pmatrix}$ which is skew-symmetric. Note that if $\eta = (4\sigma_{max}(A))^{-1}$, then $0 < \eta < L$ where L is the Lipschitz constant of v.

We check that $\mathcal{K}\nabla v(\omega^*)^2 = \mathcal{K}\nabla v(\omega^*)$. Let $X \in \mathbb{R}^{m+p}$ such that $\nabla v(\omega^*)^2 X = 0$. As $\nabla v(\omega^*)$ is skew-symmetric, this is equivalent to $\nabla v(\omega^*)^T\nabla v(\omega^*)X = 0$ which implies that $\|\nabla v(\omega^*)\| = 0$ which implies our claim. By Lem. 18, this implies that $\mathcal{K}(\nabla F_{2,\eta}(\omega^*) - I_d)^2 = \mathcal{K}(\nabla F_{2,\eta}(\omega^*) - I_d)$. Moreover, if $\xi \in \mathcal{K}(\nabla F(\omega^*) - I_d)$ then by Lem. 18, $\xi \in \mathcal{K}\nabla v(\omega^*)$ and so $v(\omega^* + \xi) = 0$ too. Hence the hypothesises of Cor. 5 are satisfied. Then, by our choice of η and Lem. 1,

$$\begin{aligned}
\rho^* &= \max\{|\lambda| \,|\, \lambda \in \operatorname{Sp}\nabla F_{2,\eta}(\omega^*),\ \lambda \neq 1\} \\
&= \max\{|1 - \eta\lambda(1 - \eta\lambda)| \,|\, \lambda \in \operatorname{Sp}\nabla v(\omega^*),\ \lambda \neq 0\} \\
&= \max\{|1 - \eta\lambda(1 - \eta\lambda)| \,|\, \lambda = \pm i\sigma,\ \sigma^2 \in \operatorname{Sp} AA^T,\ \sigma \neq 0\}
\end{aligned}$$

by a similar reasoning as Lem. 23 since $\operatorname{Sp} AA^T \setminus \{0\} = \operatorname{Sp} A^T A \setminus \{0\}$.

The result is now a consequence of the proof of Thm. 4. □

C Appendix for Section 3

C.1 Appendix: Linear algebra results

Theorem 17 (Spectral Mapping Theorem)**.** *Let* $A \in \mathbb{C}^{d\times d}$ *and* P *be a polynomial. Then,*

$$\operatorname{Sp} P(A) = \{P(\lambda) \mid \lambda \in \operatorname{Sp} A\} \tag{284}$$

See for instance [Lax07, Theorem 4, p. 66] for a proof.

C.2 Appendix: Proofs of general lemmas

Proposition 1. *[AS16] first-order methods can be written as*

$$\omega_{t+1} = \textstyle\sum_{k=0}^{t} \alpha_k^{(t)} F(\omega_k) + \beta_k^{(t)} \omega_k, \tag{12}$$

where $\sum_{k=0}^{t} \beta_k^{(t)} = 1$. *The method is called oblivious if the coefficients* $\alpha_k^{(t)}$ *and* $\beta_k^{(t)}$ *are known in advance.*

Proof. The fact that any first-order method as defined by definition 1 satisfies such relations is immediate. The converse can be shown by induction. Assume that $(\omega_t)_t$ are generated by the rule of Prop. 1. For $t = 0$, the condition of definition 1 is trivial. Assume that for all $k \leq t$, $w_k \in \omega_0 + \mathbf{Span}\{F(\omega_0), \ldots, F(\omega_{k-1})\}$. Then,

$$\begin{aligned}
\omega_{t+1} &= \sum_{k=0}^{t} \alpha_k^{(t)} F(\omega_k) + \beta_k^{(t)} \omega_k \\
&= \omega_0 + \sum_{k=0}^{t} \alpha_k^{(t)} F(\omega_k) + \beta_k^{(t)} (\omega_k - \omega_0) \\
&\in \omega_0 + \mathbf{Span}\{F(\omega_0), \ldots, F(\omega_t)\}\,.
\end{aligned}$$

□

Lemma 3. *[e.g., [Chi11]] If* $F(\omega) = A\omega + b$,

$$\omega_t - \omega^* = p_t(A)(\omega_0 - \omega^*)\,, \tag{13}$$

where ω^* *satisfies* $A\omega^* + b = 0$ *and* p_t *is a real polynomial of degree at most* t *such that* $p_t(0) = 1$.

Proof. We use Prop. 1 to prove this result by induction. For $t = 0$, the statement holds. Now assume that for all $k \leq t$, $\omega_k - \omega^* = p_k(A)(\omega_0 - \omega^*)$ with $p_{t'}$ a real polynomial of degree at most t' such that $p_{t'}(0) = 1$ (and which depends only on the coefficients of Prop. 1). Note that if $F(\omega^*) = 0$, then as F is linear, we can rewrite F as $F(\omega) = A(\omega - \omega^*)$. Then, by Prop. 1, as $\sum_{k=0}^{t} \beta_k^{(t)} = 1$,

$$\begin{aligned}
\omega_{t+1} - \omega^* &= \sum_{k=0}^{t} \alpha_k^{(t)} F(\omega_k) + \beta_k^{(t)} (\omega_k - \omega^*) \\
&= \sum_{k=0}^{t} \alpha_k^{(t)} A(\omega_k - \omega^*) + \beta_k^{(t)} (\omega_k - \omega^*) \\
&= \sum_{k=0}^{t} \alpha_k^{(t)} A p_k(A)(\omega_0 - \omega^*) + \beta_k^{(t)} p_k(A)(\omega_0 - \omega^*) \\
&= p_{t+1}(A)(\omega_0 - \omega^*)
\end{aligned}$$

where $p_{t+1}(X) = \sum_{k=0}^{t} \alpha_k^{(t)} X p_k(X) + \beta_k^{(t)} p_k(X)$, which is a real polynomial of degree at most $t+1$. Then $p_{t+1}(0) = \sum_{k=0}^{t} \beta_k^{(t)} p_k(0) = 1$, which concludes the proof. □

Lemma 4. *Let* $V : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ *be continuously differentiable and let* ω^* *satisfies* $V(\omega^*, \omega^*) = \omega^*$. *Assume there exists* $\rho^* > 0$ *such that,* $\rho(\mathbf{J}_{V_{augm}}(\omega^*)) \leq \rho^* < 1$. *If* ω_0 *and* ω_1 *are close to* ω^*, *then* (22) *converges linearly to* ω^* *at rate* $\left(\rho^* + \epsilon\right)^t$. *If* V *is linear, then* $\epsilon = 0$.

Proof. This is a direct application of Thm. 9 to $V_{augm} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$. $\square$

Proposition 3. *[Nev93] Let $K \subset \mathbb{C}$ be a subset of $\mathbb{C}$ symmetric w.r.t. the real axis, which does not contain 0 and such that $\mathcal{M}_K \neq \emptyset$. Then, any oblivious first-order method (whose coefficients only depend on K) satisfies,*

$$\forall t \geq 0, \exists A \in \mathcal{M}_K, \exists \omega_0 : \|\omega_t - \omega^*\|_2 \geq \underline{\rho}(K)^t \|\omega_0 - \omega^*\|_2.$$

Note. If we work in $\mathbb{C}^d$ this proposition is immediate. However, as we constrain ourselves to real vectors and matrices, this is slightly more difficult. This is why we need the matrix representation of complex numbers which is described in the following lemma.

Lemma 19. *Define, for $z \in \mathbb{C}$, the real 2×2 matrix $C(z) = \begin{pmatrix} \Re z & -\Im z \\ \Im z & \Re z \end{pmatrix}$. Then,*

(i) The spectrum of $C(z)$ is $\operatorname{Sp} C(z) = \{z, \overline{z}\}$.

(ii) C is $\mathbb{R}$-linear,

$$\forall z, z' \in \mathbb{C},\, a, a' \in \mathbb{R}, \quad C(az + a'z') = aC(z) + a'C(z')$$

(iii) C is a multiplicative group homorphism,

$$\forall z, z' \in \mathbb{C}, \quad C(zz') = C(z)C(z')$$

We now show a small lemma which will be useful to construct matrices in $\mathcal{M}_K$.

Lemma 20. *Let $K \subset \mathbb{C}$ be a subset of $\mathbb{C}$ symmetric w.r.t. the real axis, and such that $\mathcal{M}_K \neq \emptyset$. If $d \geq 3$, then,*

$$\{A \in \mathbb{R}^{d-2} : \operatorname{Sp}(A) \subset K\} \neq \emptyset$$

Proof. We consider two cases, depending on the parity of d.

- Assume that d is odd. We show that this implies that K intersects the real axis. Let M be a matrix in $\mathcal{M}_K$ as it is non-empty by assumption. Then, as the dimension d is odd, M has at least one real eigenvalue, i.e. $\operatorname{Sp} M \cap \mathbb{R} \neq \emptyset$. Hence, $K \cap \mathbb{R} \neq \emptyset$ and let $\nu K \cap \mathbb{R}$ be such an element. Then, the matrix $\operatorname{diag}(\nu, \dots, \nu) \in \mathbb{R}^{(d-2)\times(d-2)}$, which is the square diagonal matrix of size $d-2$ with only ν on its diagonal, belongs to $\{A \in \mathbb{R}^{d-2} : \operatorname{Sp}(A) \subset K\}$ which proves the claim.

- Assume that d is even. As $\mathcal{M}_K \neq \emptyset$, $K \neq \emptyset$ and so take $\lambda \in K$. As K is assumed to be symmetric w.r.t. the real axis, $\overline{\lambda}$ belongs to K too. As d is even, we can then define the matrix $M = \operatorname{diag}(C(\lambda), \dots, C(\lambda)) \in \mathbb{R}^{(d-2)\times(d-2)}$ which a real block-diagonal matrix. Its spectrum is $\operatorname{Sp} M = \operatorname{Sp} C(\lambda) = \{\lambda, \overline{\lambda}\} \subset K$ so it proves the claim.

$\square$

Proof. We write this proof with $\omega^* = 0$ without loss of generality. Consider an oblivious first-order method, given by its sequence of polynomials $p_t \in \mathcal{P}_t$, $t \geq 0$. Fix $t \geq 0$ and take $\lambda \in \arg\max_{z \in K} |p_t(z)|$.

We now build $A \in \mathcal{M}_K$ which has λ as an eigenvalue. First assume that $d \geq 3$. Then, by Lem. 20, there exists $M \in \mathbb{R}^{(d-2)\times(d-2)}$ such that $\operatorname{Sp} M \subset K$. Now construct A as,

$$A = \left(\begin{array}{c|c} C(\lambda) & 0_{2\times(d-2)} \\ \hline 0_{(d-2)\times 2} & M \end{array}\right)$$

If $d = 2$, simply take $A = C(\lambda)$.

As A is block-diagonal, $\operatorname{Sp} A = \operatorname{Sp} C(z) \cup \operatorname{Sp} M = \{\lambda, \overline{\lambda}\} \cup \operatorname{Sp} M$. By definition $\operatorname{Sp} M \subset K$ and, as $\lambda \in K$ and K is symmetric w.r.t. the real axis, $\{\lambda, \overline{\lambda}\} \subset K$ too. Hence $\operatorname{Sp} A \subset K$ and so $A \in \mathcal{M}_K$.

We now look at the iterates of the method applied to the vector field $x \mapsto Ax$ to prove the claim. As $\omega^* = 0$, $\|\omega_t - \omega^*\|_2 = \|\omega_t\|_2 = \|p_t(A)\omega_0\|_2$.

To explicit $p_t(A)$, we need to compute $p_t(C(\lambda))$. But, as p_t is a real polynomial, by Lem. 19, we have $p_t(C(\lambda)) = C(p_t(\lambda))$. Hence,

$$p_t(A) = \left(\begin{array}{c|c} C(p_t(\lambda)) & 0_{2\times(d-2)} \\ \hline 0_{(d-2)\times 2} & p_t(M) \end{array}\right)$$

Now take $\omega_0 = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}^T$. Then $\|p_t(A)\omega_0\|^2 = (\Re(p_t(\lambda)))^2 + (\Im(p_t(\lambda)))^2 = |p_t(\lambda)|^2$ and so $\|p_t(A)\omega_0\| = \max_{z\in K} |p_t(z)|\|\omega_0\| \geq \underline{\rho}(K)^t\|\omega_0\|$, which yields the result.

□

Remark 4 (Definition of the asymptotic convergence factor in the matrix iteration literature)*.* The original definitions of the asymptotic convergence factor for linear systems iterations and in particular the one in [Nev93] (which is called *optimal reduction factor* in their work), are actually different from the one we presented here. Indeed, the authors work with complex numbers all along so they consider methods with potentially non-real coefficients. Hence, they define the asymptotic convergence factor as,

$$\underline{\rho}(K)' = \inf_{t>0} \min_{q_t\in\mathcal{Q}_t} \max_{\lambda\in K} \sqrt[t]{|q_t(\lambda)|} \tag{285}$$

where $\mathcal{Q}_t$ is the set of *complex* polynomials q_t of degree at most t such that $q_t(0) = 1$. However, for infinite K which are symmetric w.r.t. the real axis, these two definitions, the one with the complex polynomials and the one with the real polynomials, coincide, as, for all $t \geq 0$,

$$\min_{q_t\in\mathcal{Q}_t} \max_{\lambda\in K} |q_t(\lambda)| = \min_{p_t\in\mathcal{P}_t} \max_{\lambda\in K} |p_t(\lambda)| \tag{286}$$

This is a consequence of the uniqueness of such minimizers, see [Nev93, Cor. 3.5.4].

The following lemma justifies our choice of spectral problem class for strongly monotone and Lipschitz vector fields.

Lemma 21. *Let $F : \mathbb{R}^d \to \mathbb{R}^d$ be a continuously differentiable vector field μ-strongly monotone and L-Lipschitz. Then, for all $\omega \in \mathbb{R}^d$,*

$$\mu \leq \Re\lambda\,, \quad |\lambda| \leq L\,, \quad \forall \lambda \in \operatorname{Sp} \boldsymbol{J}_F(\omega) \tag{287}$$

Proof. Fix $\omega \in \mathbb{R}^d$. The first step is standard, see [FP03, Prop. 2.3.2] for instance. From the strong monotonicity and the Lipschitz assumptions, for any $\omega' \in \mathbb{R}^d$,

$$(\omega - \omega')^T(F(\omega) - F(\omega')) \geq \mu\|\omega - \omega'\|^2\,, \quad \|F(\omega) - F(\omega')\| \leq L\|\omega - \omega'\|$$

Take $u \in \mathbb{R}^d$. Letting $\omega' = \omega + tu$, dividing by, respectively, t^2 and t, and letting t goes to zero yields,

$$u^T \mathbf{J}_F(\omega) u \geq \mu \|u\|^2 \,, \quad \|\mathbf{J}_F(\omega) u\| \leq L\|u\|$$

From the second inequality, we get that $\|\mathbf{J}_F(\omega)\| \leq L$ and so the magnitudes of the eigenvalues of $\mathbf{J}_F(\omega)$ are bounded by L. From the first one, we get that $\mathcal{H}(\mathbf{J}_F(\omega)) := \frac{\mathbf{J}_F(\omega)+\mathbf{J}_F(\omega)^T}{2} \succeq \mu I_d$. Now, for $\lambda \in \operatorname{Sp} \mathbf{J}_F(\omega)$, and $v \in \mathbb{C}^d$ associated eigenvector with $\|v\| = 1$, then $\mathbf{J}_F(\omega)v = \lambda v$ and so $\lambda = \overline{v}^T \mathbf{J}_F(\omega) v$. In particular $\Re\lambda = \frac{\lambda + \overline{\lambda}}{2} = \overline{v}^T \mathcal{H}\left(\mathbf{J}_F(\omega)\right) v \geq \mu \|v\|^2 = \mu$, which yields the result. $\square$

Lemma 22 ([ENV85, §6.2]; [Nev93, Example 3.8.2])**.** *Let $K = \{z \in \mathbb{C} : |z - c| \leq r\}$ with $c > r > 0$. Then, for all $t \geq 0$, the polynomial*

$$p_t^*(z) = \left(1 - \frac{z}{c}\right)^t$$

is optimal, i.e.,

$$p_t^* \in \arg\min_{p_t \in \mathcal{P}_t} \max_{z \in K} |p_t(z)|$$

and so $\underline{\rho}(K) = \frac{r}{c}$. Moreover, the gradient method with step-size $1/c$ is optimal for K: for any vector field F such that $\operatorname{Sp} \boldsymbol{J}_F(\omega^*) \subset K$, the gradient operator defined by

$$\omega_{t+1} = V_{grad}(\omega_t) = \omega_t - \eta F(\omega_t) \tag{288}$$

satisfy, for $\eta = \frac{1}{c}$,

$$\rho(\boldsymbol{J}_{V_{grad}}(\omega^*)) \leq \frac{r}{c} \tag{289}$$

This result is only briefly discussed in the references above and as consequence of broader theories. For completeness and simplicity we give a simpler proof using Rouché's theorem. We recall a simplified version of this theorem, see [BN10, Thm. 10.10] for a proof.

Theorem 18 (Rouché)**.** *Let f and g be analytic functions, and $D = \{z \in \mathbb{C} \,|\, |z - z_c| < R\}$ for $z_C \in \mathbb{C}$ and $R > 0$. If for all $z \in \partial D$ the boundary of D it holds that $|f(z)| > |g(z)|$, then the number of zeroes of $f - g$ inside D (counted with multiplicity) is the same as the number of zeroes of f inside D.*

Proof of Lem. 22. Let $p_t^*(z) = \left(1 - \frac{z}{c}\right)^t$ which belongs to $\mathcal{P}_t$. For the sake of contradiction assume that p_t^* is not optimal, i.e. there exists $q_t \in \mathcal{P}_t$ different from p_t^* such that

$$\max_{z \in K} |p_t^*(z)| > \max_{z \in K} |q_t(z)|$$

where K was defined in the statement as $K = \{z \in \mathbb{C} : |z - c| \leq r\}$ with $c > r > 0$. Observe that $|p_t^*|$ reaches its maximum $\left(\frac{r}{c}\right)^t$ on K everywhere on the boundary of K,

$$\max_{z \in K} |p_t^*(z)| = \left(\frac{r}{c}\right)^t = |p_t^*(z_b)| \quad \forall z_b \in \partial K$$

$\square$

Hence, for all $z_b \in \partial K$,

$$|q_t(z_b)| \leq \max_{z \in K} |q_t(z)| < \max_{z \in K} |p_t^*(z)| = |p_t^*(z_b)|$$

Therefore, as q_t and p_t^* are polynomials and in particular analytic, we can apply Rouché's theorem with $D = \operatorname{int} K$ and this yields that the number of zeroes of $p_t^* - q_t$ in $\operatorname{int} K$ is the same as the number of zeroes of p_t^* in $\operatorname{int} K$. On the one hand, c, which belongs to the interior of K, is a zero of multiplicity t of p_t^*. On the other hand, as q_t and p_t^* are in $\mathcal{P}_t$, they satisfy $p_t^*(0) = 1 = q_t(0)$ and so $(p_t^* - q_t)(0) = 0$. However, as $c > r$, 0 is not in K. So, as $(p_t^* - q_t)$ is of degree at most t, it can have at most $t-1$ remaining zeroes (counted with multiplicity) in $\operatorname{int} K$. This contradicts the conclusion of Rouché's theorem that $p_t^* - q_t$ must have exaclty t zeroes inside K. Therefore, there exists no such q_t and so $p_t^* \in \arg\min_{p_t \in \mathcal{P}_t} \max_{z \in K} |p_t(z)|$.

Moreover, this implies that $\min_{p_t \in \mathcal{P}_t} \max_{z \in K} |p_t(z)| = \left(\frac{r}{c}\right)^t$ and so that $\underline{\rho}(K) = \frac{r}{c}$.

What is left to check is the bound $\rho(\mathbf{J}_V(\omega^*)) \leq \frac{r}{c}$. Recall that $V_{grad}(\omega) = \omega - \eta F(\omega)$ and so that $\mathbf{J}_{V_{\text{grad}}}(\omega) = I_d - \eta \mathbf{J}_F(\omega)$. By the spectral mapping theorem (Thm. 17),

$$\operatorname{Sp} \mathbf{J}_{V_{\text{grad}}}(\omega) = \{1 - \eta\lambda \mid \lambda \in \operatorname{Sp} \mathbf{J}_F(\omega)\}$$

Letting $\eta = \frac{1}{c}$ and using that $\operatorname{Sp} \mathbf{J}_F(\omega) \subset K$ yields the result.

C.3 Appendix: Acceleration related proofs

C.3.1 Bilinear games

We recall Polyak's theorem.

Theorem 19 ([Pol64, Thm. 9])**.** *Let $0 < \mu < L$. Define Polyak's Heavy-ball method as*

$$\omega_{t+1} = V_{\alpha,\beta}^{Polyak}(\omega_t, \omega_{t-1}) = \omega_t - \alpha F(\omega_t) + \beta(\omega_t - \omega_{t-1}) \tag{290}$$

For $\alpha = \frac{4}{(\sqrt{\mu}+\sqrt{L})^2}$ and $\beta = \left(\frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}\right)^2$ and for any vector field F such that $\operatorname{Sp} \nabla F(\omega^*) \subset [\mu, L]$, then

$$\rho(\nabla V_{\alpha,\beta}^{Polyak}(\omega^*, \omega^*)) \leq \underline{\rho}([\mu, L]) = \frac{\sqrt{L} - \sqrt{\mu}}{\sqrt{L} + \sqrt{\mu}} \tag{291}$$

In this subsection we first prove the following result.

Proposition 4. *Let F be a vector field such that $\operatorname{Sp} \nabla F(\omega^*) \subset [ia, ib] \cup [-ia, -ib]$, for $0 < a < b$. Setting $\sqrt{\alpha} = \frac{2}{a+b}$, $\sqrt{\beta} = \frac{b-a}{b+a}$, the Polyak Heavy-Ball method (24) on the transformation (25), i.e.,*

$$\omega_{t+1} = \omega_t - \alpha F^{real}(\omega_t) + \beta(\omega_t - \omega_{t-1})\,.$$

converges locally at a linear rate $O\big((1 - \frac{2a}{a+b})^t\big)$.

Proof. This proposition follows from Thm. 19. Indeed, the Jacobian of V^{real} at ω^* is,

$$\begin{aligned}
\mathbf{J}_{V^{real}}(\omega^*) &= \frac{1}{\eta}(\mathbf{J}_F(\omega^*)(\operatorname{Id} - \eta \mathbf{J}_F(\omega^*)) - \mathbf{J}_F(\omega^*)) \\
&= -\mathbf{J}_F(\omega^*)^2
\end{aligned}$$

where we used $F(\omega^*) = \omega^*$. Now, we can deduce the spectrum of $\mathbf{J}_{V^{real}}(\omega^*)$ from the one of $\mathbf{J}_F(\omega^*)$ using the spectral mapping theorem Thm. 17,

$$\begin{aligned}\operatorname{Sp}\mathbf{J}_{V^{real}}(\omega^*) &= \{-\lambda^2 \mid \lambda \in \operatorname{Sp}\mathbf{J}_F(\omega^*)\}\\ &\subset \{-\lambda^2 \mid \lambda \in \pm[ia, ib]\}\\ &\subset [a^2, b^2]\,.\end{aligned}$$

We can now apply Polyak's momentum method to V^{real} and we get the desired bound on the spectral radius by Thm. 19, with $\alpha = \frac{4}{(a+b)^2}$ and $\beta = \left(\frac{b-a}{b+a}\right)^2$. □

We now prove the following lemma, in order to use Prop. 4 on bilinear games. Note that as A is square, $\sigma_{min}(A)^2$ and $\sigma_{max}(A)^2$ actually correspond to, respectively, the smallest and the largest eigenvalues of AA^T

Lemma 23. *Consider the bilinear game*

$$\min_{x\in\mathbb{R}^m}\max_{y\in\mathbb{R}^m} x^T A y + b^T x + c^T y \tag{292}$$

Let $F : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ be the associated vector field. Then,

$$\operatorname{Sp}\nabla F(\omega^*) \subset [i\sigma_{min}(A), i\sigma_{max}(A)] \cup [-i\sigma_{min}(A), -i\sigma_{max}(A)] \tag{293}$$

Proof. We have $F(\omega) = \begin{pmatrix} Ay + b \\ -A^T x - c\end{pmatrix}$ and so

$$\nabla F(\omega) = \begin{pmatrix} 0_{m\times m} & A \\ -A^T & 0_{m\times m}\end{pmatrix}$$

We compute the characteristic polynomial of $\nabla F(\omega)$ using the bloc determinant formula, which can be found in [Zha05, Section 0.3], as A^T and I_m commute,

$$\begin{aligned}\det(XI_{2m} - A) &= \begin{vmatrix} XI_m & -A \\ A^T & XI_m\end{vmatrix}\\ &= \det(X^2 I_m + AA^T)\,.\end{aligned}$$

Hence, $\operatorname{Sp}\nabla F(\omega) = \{\pm i\lambda \,|\, \lambda^2 \in \operatorname{Sp} AA^T\}$ which gives the result. □

We now prove the optimality of this method. For this we rely on [ELV89, Thm. 6], that we state below for completeness.

Theorem 20 ([ELV89, Thm. 6])**.** *Let $\Omega \subset \mathbb{C}$ be a compact set such that $0 \notin \Omega$, Ω has no isolated points and $\mathbb{C}\cup\infty\setminus\Omega$ is of finite connectivity. Consider t_n polynomial of degree n such that $t_n(0) = 0$ and define $\widetilde{\Omega} = t_n(\Omega)$. If, $t_n^{-1}(\widetilde{\Omega}) = \Omega$, then we have,*

$$\underline{\rho}(\Omega) = \underline{\rho}(\widetilde{\Omega})^{1/n}$$

Proposition 5. *Let $K = [ia, ib] \cup [-ia, -ib]$ for $0 < a < b$. Then, $\underline{\rho}(K) = \sqrt{\frac{b-a}{b+a}}$.*

Proof. We use Thm. 20 with $\Omega = \pm[ia, ib]$, $t_2(X) = -X^2$ and $\widetilde{\Omega} = [a^2, b^2]$. We get,

$$\underline{\rho}(\underbrace{\pm[ia, ib]}_{=K}) = \underline{\rho}([a^2, b^2])^{1/2} = \sqrt{\frac{b-a}{b+a}}\,.$$

□

C.3.2 Ellipses

Define, for $a, b, c \geq 0$, the ellipse

$$E(a,b,c) = \left\{ \lambda \in \mathbb{C} : \frac{(\Re\lambda - c)^2}{a^2} + \frac{(\Im\lambda)^2}{b^2} \leq 1 \right\} \tag{294}$$

As mentioned earlier, we work with shapes symmetric w.r.t. the real axis and in $\mathbb{C}_+$ (the set of complex number with non-negative real part). So the ellipses we consider have their center on the positive real axis and we we will require below that $0 \notin E(a,b,c)$. Ellipses have been studied in the context of matrix iteration, due to their flexibility and their link to the momentum method. The next theorem can be considered as a summary and reinterpretation of the literature on the subject. The way to obtain it from the literature, and a partial proof, are deferred to the Appendix C.4.

Theorem 21. *Let $a, b \geq 0$, $c > 0$, $(a,b) \neq 0$, such that $0 \notin E(a,b,c)$. Then, if $\rho(a,b,c) < 1$,*

$$\underline{\rho}(E(a,b,c)) = \rho(a,b,c) \tag{295}$$

where

$$\rho(a,b,c) = \begin{cases} \frac{a}{c} & \textit{if } a = b \\ \frac{c - \sqrt{b^2+c^2-a^2}}{a-b} & \textit{otherwise} \end{cases} \tag{296}$$

Assume that F is any vector field $F : \mathbb{R}^d \to \mathbb{R}^d$ that satisfies $\operatorname{Sp} \nabla F(\omega^*) \subset E(a,b,c)$. There exists $\alpha(a,b,c) > 0$, $\beta(a,b,c) \in (-1,1]$, whose signis the same as $a - b$ such that, for the momentum operator $V : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$,

$$V(\omega, \omega') = (\omega - \alpha F(\omega) + \beta(\omega - \omega'), \omega') \tag{297}$$

we have

$$\rho(\boldsymbol{J}_V(\omega^*, \omega^*)) \leq \rho(a,b,c) \tag{298}$$

More exactly the corresponding parameters are given by,

$$\beta(a,b,c) = \begin{cases} 0 & \textit{if } a = b \\ 2c\frac{c-\sqrt{c^2+b^2-a^2}}{a^2-b^2} - 1 & \textit{otherwise} \end{cases} \qquad \alpha(a,b,c) = \frac{1+\beta}{c} = \begin{cases} \frac{1}{c} & \textit{if } a = b \\ 2\frac{c-\sqrt{c^2+b^2-a^2}}{a^2-b^2} & \textit{otherwise} \end{cases} \tag{299}$$

and $\beta(a,b,c)$ can be written $\beta = \chi(a,b,c)(a-b)$ with $\chi(a,b,c) > 0$.

Remark 5 (On the sign of the momentum parameter)*.* As briefly mentioned in the theorem, and detailed in Prop. 14, the optimal momentum parameter $\beta(a,b,c)$ has the same sign as $a - b$, i.e., more exactly, there exists $\chi(a,b,c) > 0$ such that $\beta(a,b,c) = \chi(a,b,c)(a-b)$. Hence the sign of the optimal β has a nice geometric interpretation, which answers some of the questions left open by [GHP+19].

- In the case where $a > b$, or equivalently $\beta > 0$, the ellipse is more elongated in the direction of the real axis. The extreme case is a segment on the real line that corresponds to strongly convex optimization.
- In the case where $a < b$, or equivalently $\beta < 0$, the ellipse is more elongated in the direction of the imaginary axis.
- Finally, when $a = b$ we have a disk instead of an ellipse. For such shape, we have no momentum, which means that gradient descent is optimal as seen in §3.2.3.

C.3.3 Perturbed acceleration

In this subsection, we prove Prop. 6. Note that the constants in the $\mathcal{O}(.)$ are absolute.

Proposition 10. *Define $\epsilon(\mu, L)$ as $\frac{\epsilon(\mu,L)}{L} = \left(\frac{\mu}{L}\right)^\theta$ with $\theta > 0$ and $a \wedge b = \min(a,b)$. Then, when $\frac{\mu}{L} \to 0$,*

$$\underline{\rho}(K_\epsilon) = \begin{cases} 1 - 2\sqrt{\frac{\mu}{L}} + \mathcal{O}\left(\left(\frac{\mu}{L}\right)^{\theta\wedge 1}\right), & if\ \theta > \frac{1}{2} \\ 1 - 2(\sqrt{2}-1)\sqrt{\frac{\mu}{L}} + \mathcal{O}\left(\left(\frac{\mu}{L}\right)\right), & if\ \theta = \frac{1}{2} \\ 1 - \left(\frac{\mu}{L}\right)^{1-\theta} + \mathcal{O}\left(\left(\frac{\mu}{L}\right)^{1\wedge(2-3\theta)}\right), & if\ \theta < \frac{1}{2}. \end{cases}$$

Moreover, the momentum method is optimal for K_ϵ. This means that with $\alpha\left(\frac{L-\mu}{2}, \epsilon, \frac{L+\mu}{2}\right) > 0$ and $\beta\left(\frac{L-\mu}{2}, \epsilon, \frac{L+\mu}{2}\right) > 0$ (as defined in Thm. 21), if $\operatorname{Sp} \boldsymbol{J}_F(\omega^*) \subset K_\epsilon$, then, $\rho(\boldsymbol{J}_{V^{Polyak}}(\omega^*, \omega^*)) \leq \underline{\rho}(K_\epsilon)$.

where K_ϵ is the ellipse defined by,

$$K_\epsilon = \left\{ z \in \mathbb{C} : \left(\frac{\Re z - \frac{\mu+L}{2}}{\frac{L-\mu}{2}}\right)^2 + \left(\frac{\Im z}{\epsilon}\right)^2 \leq 1 \right\} \tag{300}$$

Proof. A direct application of Thm. 21 using K_ϵ gives $\underline{\rho}(K_\epsilon) = \rho(\frac{L-\mu}{2}, \epsilon(\mu, L), \frac{L+\mu}{2})$.
We now study $\rho(\frac{L-\mu}{2}, \epsilon(\mu, L), \frac{L+\mu}{2})$. First note that

$$\rho(a, b, c) = 1 - \frac{a - b - c + \sqrt{b^2 + c^2 - a^2}}{a - b}$$

and so

$$1 - \rho(a, b, c) = \frac{\sqrt{b^2 + c^2 - a^2} + a - b - c}{a - b}.$$

We now replace a, b and c by their expressions (and multiply the denominator and numerator by 2).

$$\begin{aligned} 1 - \underline{\rho}(K_\epsilon) &= \frac{\sqrt{4\epsilon^2 + (L+\mu)^2 - (L-\mu)^2} + (L-\mu) - 2\epsilon - (L+\mu)}{L - \mu - 2\epsilon} \\ &= 2\frac{\sqrt{\epsilon^2 + \mu L} - 2\epsilon - 2\mu}{L - \mu - 2\epsilon} \end{aligned}$$

Define $t = \frac{\mu}{L}$, then $\frac{\epsilon}{L} = t^\theta$. We are now interested in studying the behaviour of $1 - \underline{\rho}(K_\epsilon)$ when t goes to zero.

$$\begin{aligned} 1 - \underline{\rho}(K_\epsilon) &= 2\frac{\sqrt{t^{2\theta} + t} - 2t^\theta - 2t}{1 - t - 2t^\theta} \\ &= 2(\sqrt{t^{2\theta} + t} - 2t^\theta - 2t)(1 + t + 2t^\theta + \mathcal{O}\left(t^{2(\theta\wedge 1)}\right), \\ &= 2\left(\sqrt{t^{2\theta} + t} - 2t^\theta - 2t\right)\left(1 + \mathcal{O}\left(t^{\theta\wedge 1}\right)\right), \end{aligned}$$

where $a \wedge b$ denotes $\min(a, b)$.

- If $\theta = \frac{1}{2}$. This is the smallest θ with which acceleration happens.

$$
\begin{aligned}
1 - \underline{\rho}(K_\epsilon) &= 2\left(\sqrt{2t} - 2\sqrt{t} - 2t\right)\left(1 + \mathcal{O}\left(\sqrt{t}\right)\right) \\
&= 2\left(\sqrt{2} - 1\right)\sqrt{t}\left(1 + \mathcal{O}\left(\sqrt{t}\right)\right)\left(1 + \mathcal{O}\left(\sqrt{t}\right)\right) \\
&= 2\left(\sqrt{2} - 1\right)\sqrt{t} + \mathcal{O}\left(t\right) .
\end{aligned}
$$

- If $\theta > \frac{1}{2}$. This regime is "better" than the previous one, i.e., the perturbation is even smaller so we get a similar asymptotic behavior, up to an improved constant.

$$
\begin{aligned}
1 - \underline{\rho}(K_\epsilon) &= 2\left(\sqrt{t^{2\theta} + t} - 2t^{\theta} - 2t\right)\left(1 + \mathcal{O}\left(t^{\theta\wedge 1}\right)\right) \\
&= 2\sqrt{t}\left(\sqrt{t^{2\theta-1} + 1} - 2t^{\theta-1/2} - 2\sqrt{t}\right)\left(1 + \mathcal{O}\left(t^{\theta\wedge 1}\right)\right) \\
&= 2\sqrt{t}\left(1 + \mathcal{O}\left(t^{2\theta-1}\right) - 2t^{\theta-1/2} - 2\sqrt{t}\right)\left(1 + \mathcal{O}\left(t^{\theta\wedge 1}\right)\right) \\
&= 2\sqrt{t} + \mathcal{O}\left(t^{\theta\wedge 1}\right)
\end{aligned}
$$

- If $\theta < \frac{1}{2}$. In this regime, terms in t^θ are limiting as they are bigger than $\sqrt{t}$, so we do not get the rate in $\sqrt{t}$.

$$
\begin{aligned}
1 - \underline{\rho}(K_\epsilon) &= 2\left(\sqrt{t^{2\theta} + t} - t^{\theta} - t\right)\left(1 + \mathcal{O}\left(t^{\theta}\right)\right) \\
&= 2\left(t^{\theta}\sqrt{1 + t^{1-2\theta}} - t^{\theta} - t\right)\left(1 + \mathcal{O}\left(t^{\theta}\right)\right) \\
&= 2\left(t^{\theta}\left(1 + \frac{1}{2}t^{1-2\theta} + \mathcal{O}\left(t^{2-4\theta}\right)\right) - t^{\theta} - t\right)\left(1 + \mathcal{O}\left(t^{\theta}\right)\right) \\
&= 2t^{1-\theta}\left(\frac{1}{2} + \mathcal{O}\left(t^{1-2\theta}\right) - t^{\theta}\right)\left(1 + \mathcal{O}\left(t^{\theta}\right)\right) \\
&= t^{1-\theta} + \mathcal{O}\left(t^{1\wedge(2-3\theta)}\right)
\end{aligned}
$$

□

Before going further we need to introduce this technical lemma.

Lemma 24. *For any real $m \geq 2$, we have*

$$\forall x \in [0,1] : \quad \frac{(1-x)^2}{(1-\frac{x}{m})^2} + x \leq 1 \tag{301}$$

Proof. Indeed,

$$
\begin{aligned}
(1-x^2) + x\left(1 - \frac{x}{m}\right)^2 - \left(1 - \frac{x}{m}\right)^2 &= 1 - 2x + x^2 + x - 2\frac{x^2}{m} + \frac{x^3}{m^2} - 1 + 2\frac{x}{m} - \frac{x^2}{m^2} && (302) \\
&= \left(\frac{2}{m} - 1\right)x + \left(1 - \frac{2}{m}\right)x^2 + \frac{x^3 - x^2}{m^2} && (303) \\
&= \left(1 - \frac{2}{m}\right)x(x-1) + \frac{x^3 - x^2}{m^2} && (304) \\
& && (305)
\end{aligned}
$$

Then, as $m \geq 2$, $1 - \frac{2}{m} \geq 0$, and so, since $0 \leq x \leq 1$, $\left(1 - \frac{2}{m}\right)x(x-1) \leq 0$. As $0 \leq x \leq 1$, we also have $\frac{x^3 - x^2}{m^2} \leq 0$ which concludes the proof. □

C.3.4 Acceleration of extragradient on bilinear games

We now prove that we can accelerate EG on bilinear games.

Proposition 7. *Consider the vector field F, where $\operatorname{Sp} \boldsymbol{J}_F(\omega^*) \subset [ia, ib] \cup [-ia, -ib]$ for $0 < a < b$. There exists $\alpha, \beta, \eta > 0$ such that, the operator defined by*

$$\omega_{t+1} = \omega_t - \alpha F(\omega_t - \eta F(\omega_t)) + \beta(\omega_t - \omega_{t-1}) \,,$$

converges locally at a linear rate $O\big(\big(1 - c\frac{a}{b} + M\frac{a^2}{b^2}\big)^t\big)$ where $c = \sqrt{2} - 1$ and M is an absolute constant.

This proposition is a consequence of Lem. 4 combinded with the following result.

Proposition 11. *Consider the vector field F, where $\operatorname{Sp} \boldsymbol{J}_F(\omega^*) \subset [ia, ib] \cup [-ia, -ib]$ for $0 < a < b$. There exists $\alpha, \beta, \eta > 0$ such that, the operator defined by*

$$\begin{aligned} \omega_{t+1} &= V^{Polyak+e-g}(\omega_t, \omega_{t-1}), \\ &= \omega_t - \alpha F(\omega_t - \eta F(\omega_t)) + \beta(\omega_t - \omega_{t-1}) \end{aligned}$$

satisfies, with $c = \sqrt{2} - 1$, and absolute constants in the $\mathcal{O}(.)$,

$$\rho(\boldsymbol{J}_{V^{Polyak+e-g}}(\omega^*, \omega^*)) \leq 1 - c\tfrac{a}{b} + O(\tfrac{a^2}{b^2}) \tag{306}$$

More precisely, the parameters are chosen as:

$$\eta = \frac{b}{a\sqrt{2b^2 - \frac{a^2}{2}}} \quad \alpha = \alpha\left(\eta\left(b^2 - \frac{a^2}{2}\right), b, \eta b^2\right) \quad \beta = \beta\left(\eta\left(b^2 - \frac{a^2}{2}\right), b, \eta b^2\right)$$

with where the functions $\alpha(\cdot)$ and $\beta(\cdot)$ are the ones defined in Thm. 21.

Note that this proposition actually requires α and η to be tuned separately and they are actually very different. They actually differ by a factor $\frac{b}{a}$: η is roughly proportional to $\frac{1}{a}$ while α behaves like $\frac{b^2}{a}$.

Proof. Consider $F^{\text{e-g}}(\omega) = F(\omega - \eta F(\omega))$. Then, for ω^* such that $F(\omega^*) = 0$, we have,

$$\mathbf{J}_{F^{\text{e-g}}}(\omega^*) = \mathbf{J}_F(\omega^*) - \eta \mathbf{J}_F(\omega^*)^2$$

Hence, by Thm. 17,

$$\operatorname{Sp} \nabla \mathbf{J}_{F^{\text{e-g}}}(\omega^*) \subset \{z - \eta z^2 \,|\, z \in \pm[ia, ib]\} = \{i\lambda + \eta\lambda^2 \,|\, \lambda \in \pm[a, b]\}$$

As we want to apply Prop. 10, we now look for $\epsilon, \overline{\mu}, \overline{L}$ such that the ellipsis

$$K(\epsilon, \overline{\mu}, \overline{L}) = \left\{ z \in \mathbb{C} : \left(\frac{\Re z - \frac{\overline{\mu} + \overline{L}}{2}}{\frac{\overline{L} - \overline{\mu}}{2}} \right)^2 + \left(\frac{\Im z}{\epsilon} \right)^2 \leq 1 \right\} \tag{307}$$

contains $\operatorname{Sp} \nabla \mathbf{J}_{F^{\text{e-g}}}(\omega^*)$. For this we will choose $\epsilon, \overline{\mu}, \overline{L}$ such that

$$\{i\lambda + \eta\lambda^2 \,|\, \lambda \in \pm[a, b]\} \subset K(\epsilon, \overline{\mu}, \overline{L}) \,,$$

which is equivalent to for all $\lambda \in [a,b]$,

$$\left(\frac{\eta\lambda^2 - \frac{\overline{\mu}+\overline{L}}{2}}{\frac{\overline{L}-\overline{\mu}}{2}}\right)^2 + \left(\frac{\lambda}{\epsilon}\right)^2 \leq 1 \tag{308}$$

Note that the left-hand side is convex in λ^2 so that we only need to check this inequality for the limit values $\lambda = a$ and $\lambda = b$. Hence, now we have reduced the problem to that of looking for $\overline{\mu}$, $\overline{L}$ and ϵ such that $\lambda = a$ and $\lambda = b$ satisfy (308). This is equivalent to look for an ellipse $K(\epsilon, \overline{\mu}, \overline{L})$ that contains $ib + \eta b^2$ and $ia + \eta a^2$.

We now construct this ellipsis explicitly. As we want to apply Prop. 10, we want ϵ as small as possible. So we start with $ib + \eta b^2$ as it is the one with the largest imaginary part, compared to $ia + \eta a^2$. We choose the center of the ellipse – which must lie on the real axis – such that it is placed at the same abscisse as $ib + \eta b^2$, i.e. the same real part. So we define $\frac{\overline{\mu}+\overline{L}}{2} = \eta b^2$. We need another condition to fix $\overline{\mu}$ and $\overline{L}$. To make sure that $ia + \eta a^2$ is also in the ellipsis, we need to choose $\overline{\mu}$ small enough. Define $\overline{\mu} = \frac{\eta a^2}{m}$ with $m > 0$ to be chosen later. This fixes the value of $\overline{L}$ as,

$$\overline{L} = 2\eta b^2 - \overline{\mu} = 2\eta b^2 - \eta\frac{a^2}{m}$$

We choose ϵ so that $ib + \eta b^2$ is in the ellipsis, and as we chose the center to be $\frac{\overline{\mu}+\overline{L}}{2} = \eta b^2$, we define $\epsilon = b$. This way $ib + \eta b^2 \in K(\epsilon, \overline{\mu}, \overline{L})$. We must now check that $ia + \eta a^2 \in K(\epsilon, \overline{\mu}, \overline{L})$. For this we check that $\lambda = a$ satisfies (308),

$$\left(\frac{\eta a^2 - \frac{\overline{\mu}+\overline{L}}{2}}{\frac{\overline{L}-\overline{\mu}}{2}}\right)^2 + \left(\frac{a}{\epsilon}\right)^2 \tag{309}$$

$$= \left(\frac{\eta a^2 - \eta b^2}{\eta b^2 - \eta\frac{a^2}{m}}\right)^2 + \left(\frac{a}{b}\right)^2 \tag{310}$$

$$= \frac{(1-x)^2}{(1-\frac{x}{m})^2} + x\,, \tag{311}$$

$$\tag{312}$$

where $x = \frac{a^2}{b^2} \in [0,1]$. By Lem. 24, for $m = 2$, this quantity is smaller than one and so $ia + \eta a^2 \in K(\epsilon, \overline{\mu}, \overline{L})$. Hence, $K(\epsilon, \overline{\mu}, \overline{L})$ contains $\mathrm{Sp}\, \nabla \mathbf{J}_{F^{\text{e-g}}}(\omega^*)$.

Before we apply Prop. 10, we need to make sure that we are in the accelerated regime, that is to say ϵ is small enough compared to $\overline{\mu}$ and $\overline{L}$. Fortunately, we have not chosen η yet. So we define it so that we reach the accelerated regime,

$$\frac{\epsilon}{\overline{L}} = \sqrt{\frac{\overline{\mu}}{\overline{L}}} \iff \epsilon = \sqrt{\overline{\mu}\overline{L}} \iff \eta = \frac{b}{a\sqrt{2b^2 - \frac{a^2}{m}}}\,.$$

We now apply Prop. 10. As $\frac{\overline{\mu}}{\overline{L}}$ goes to zero, $\underline{\rho}(K(\epsilon, \overline{\mu}, \overline{L})) = 1 - 2(\sqrt{2}-1)\sqrt{\frac{\overline{\mu}}{\overline{L}}} + \mathcal{O}\left(\frac{\overline{\mu}}{\overline{L}}\right)$.

Now note that $\frac{\overline{\mu}}{\overline{L}} = \frac{a^2}{2mb^2 - a^2}$, so if $\frac{a}{b} \to 0$ then $\frac{\mu}{L} \to 0$. Moreover, $\frac{\overline{\mu}}{\overline{L}} = \frac{a^2}{2mb^2} + \mathcal{O}\left(\frac{a^4}{b^4}\right)$. Hence, as we chose $m = 2$,

$$\underline{\rho}(K(\epsilon, \overline{\mu}, \overline{L})) = 1 - (\sqrt{2}-1)\frac{a}{b} + \mathcal{O}\left(\frac{a^2}{b^2}\right)$$

The parameters of the momentum method applied to $F^{\text{e-g}}$ are then chosen according to Prop. 6 so that we reach this rate locally. $\square$

In the previous proposition we showed that EG can be accelerated but that it requires a careful choice of the parameters α, β, η. The following lemma describes the general behavior of these quantities when the condition number worsens.

Lemma 25. *In the context of the previous proposition, Prop. 11, it holds, when $\frac{a}{b} \to 0$,*

$$\begin{aligned}
\eta &= \frac{1}{a}\left(\frac{1}{\sqrt{2}} + \mathcal{O}\left(\frac{a^2}{b^2}\right)\right)\\
\alpha &= \frac{a}{b^2}\left(2\sqrt{2} + \mathcal{O}\left(\frac{a}{b}\right)\right)\\
\beta &= 1 - 2\sqrt{3}\frac{a}{b} + \mathcal{O}\left(\frac{a^2}{b^2}\right).
\end{aligned}$$

Proof. For compactness, denote $t = \frac{a}{b} > 0$. So we study the asymptotic behavior of α and η when t goes to 0. By definition of η we have,

$$\eta = \frac{b}{a\sqrt{2b^2 - \frac{a^2}{2}}} = \frac{1}{\sqrt{2}a\sqrt{1-\frac{t^2}{4}}} = \frac{1}{\sqrt{2}a}\left(1 + \frac{t^2}{8} + \mathcal{O}\left(t^4\right)\right)$$

which gives the first claim as $\frac{t^2}{8} + \mathcal{O}\left(t^4\right) = \mathcal{O}\left(t^2\right)$.

Before moving to the second claim, let us consider some consequences of this asymptotic expansion of η. Indeed, we have,

$$\eta b^2 = \eta\frac{a^2}{t^2} = \frac{a}{\sqrt{2}t^2}\left(1 + \frac{t^2}{8} + \mathcal{O}\left(t^4\right)\right)$$

and, using the expansion above,

$$\eta\left(b^2 - \frac{a^2}{2}\right) = \eta b^2\left(1 - \frac{t^2}{2}\right) = \frac{a}{\sqrt{2}t^2}\left(1 + \frac{t^2}{8} + \mathcal{O}\left(t^4\right)\right)\left(1 - \frac{t^2}{2}\right) = \frac{a}{\sqrt{2}t^2}\left(1 - \frac{3t^2}{8} + \mathcal{O}\left(t^4\right)\right)$$

We can now study the behavior of $\alpha = \alpha\left(\eta\left(b^2 - \frac{a^2}{2}\right), b, \eta b^2\right)$. Recall that $\alpha(., ., .)$ is the function defined in Thm. 21 and that its definition depends on whether its first and second arguments are equal. However, suing the expansion above, the ratio of its first and second arguments is

$$\frac{\eta\left(b^2 - \frac{a^2}{2}\right)}{b} = \frac{1}{\sqrt{2}t}\left(1 - \frac{3t^2}{8} + \mathcal{O}\left(t^4\right)\right),$$

which diverges to infinity as t goes to zero. Hence, when t is close enough to zero, $\eta\left(b^2 - \frac{a^2}{2}\right)$ and b are different and so we have,

$$\alpha = \alpha\left(\eta\left(b^2 - \frac{a^2}{2}\right), b, \eta b^2\right) = 2\frac{\eta b^2 - \sqrt{(\eta b^2)^2 + b^2 - \left(\eta\left(b^2 - \frac{a^2}{2}\right)\right)^2}}{\left(\eta\left(b^2 - \frac{a^2}{2}\right)\right)^2 - b^2}$$

We first consider the term under the square root,

$$
\begin{aligned}
\sqrt{(\eta b^2)^2 + b^2 - \left(\eta\left(b^2 - \frac{a^2}{2}\right)\right)^2} &= \sqrt{\frac{a^2}{2t^4}\left(1 + \frac{t^2}{8} + \mathcal{O}\left(t^4\right)\right)^2 + \frac{a^2}{t^2} - \frac{a^2}{2t^4}\left(1 - \frac{3t^2}{8} + \mathcal{O}\left(t^4\right)\right)^2} \\
&= \sqrt{\frac{a^2}{2t^4}\left(1 + \frac{t^2}{4} + \mathcal{O}\left(t^4\right)\right) + \frac{a^2}{t^2} - \frac{a^2}{2t^4}\left(1 - \frac{3t^2}{4} + \mathcal{O}\left(t^4\right)\right)} \\
&= \sqrt{\frac{3}{2}\frac{a^2}{t^2} + a^2 \times \mathcal{O}\left(1\right)} \\
&= \sqrt{\frac{3}{2}}\frac{a}{t}\sqrt{1 + \mathcal{O}\left(t^2\right)} \\
&= \sqrt{\frac{3}{2}}\frac{a}{t}\left(1 + \mathcal{O}\left(t^2\right)\right)
\end{aligned}
$$

We can now give the expansion of α when $t = \frac{a}{b}$ goes to zero,

$$
\begin{aligned}
\alpha &= 2\frac{\eta b^2 - \sqrt{(\eta b^2)^2 + b^2 - \left(\eta\left(b^2 - \frac{a^2}{2}\right)\right)^2}}{\left(\eta\left(b^2 - \frac{a^2}{2}\right)\right)^2 - b^2} \\
&= 2\frac{\frac{a}{\sqrt{2}t^2}\left(1 + \mathcal{O}\left(t^2\right)\right) - \sqrt{\frac{3}{2}}\frac{a}{t}\left(1 + \mathcal{O}\left(t^2\right)\right)}{\frac{a^2}{2t^4}\left(1 + \mathcal{O}\left(t^2\right)\right) - \frac{a^2}{t^2}} \\
&= 2\frac{\frac{1}{\sqrt{2}}\left(1 + \mathcal{O}\left(t^2\right)\right) - \sqrt{\frac{3}{2}}t\left(1 + \mathcal{O}\left(t^2\right)\right)}{\frac{a}{2t^2}\left(1 + \mathcal{O}\left(t^2\right)\right) - a}
\end{aligned}
$$

by multiplying both the numerator and the denominator by $\frac{t^2}{a}$. Then, if we factorize $\frac{a}{t^2}$ in the denominator, we get,

$$
\begin{aligned}
\alpha &= 2\frac{t^2}{a}\frac{\frac{1}{\sqrt{2}}\left(1 + \mathcal{O}\left(t^2\right)\right) - \sqrt{\frac{3}{2}}t\left(1 + \mathcal{O}\left(t^2\right)\right)}{\frac{1}{2}\left(1 + \mathcal{O}\left(t^2\right)\right) - t^2} \\
&= 2\frac{t^2}{a}\frac{\frac{1}{\sqrt{2}} - \sqrt{\frac{3}{2}}t + \mathcal{O}\left(t^2\right)}{\frac{1}{2} + \mathcal{O}\left(t^2\right)} \\
&= 2\sqrt{2}\frac{t^2}{a}(1 - \sqrt{3}t + \mathcal{O}\left(t^2\right)),
\end{aligned}
$$

which yields the result for α.

Recall that, from the definition of $\alpha(\cdot)$ and $\beta(\cdot)$ in Thm. 21, we have,

$$
\beta = \eta b^2 \alpha - 1,
$$

and so, when $t = \frac{a}{b}$ goes to zero,

$$
\beta = \frac{a}{\sqrt{2}t^2}(1 + \mathcal{O}\left(t^2\right)) \times 2\sqrt{2}\frac{t^2}{a}(1 - \sqrt{3}t + \mathcal{O}\left(t^2\right)) - 1 = 1 - 2\sqrt{3}t + \mathcal{O}\left(t^2\right).
$$

□

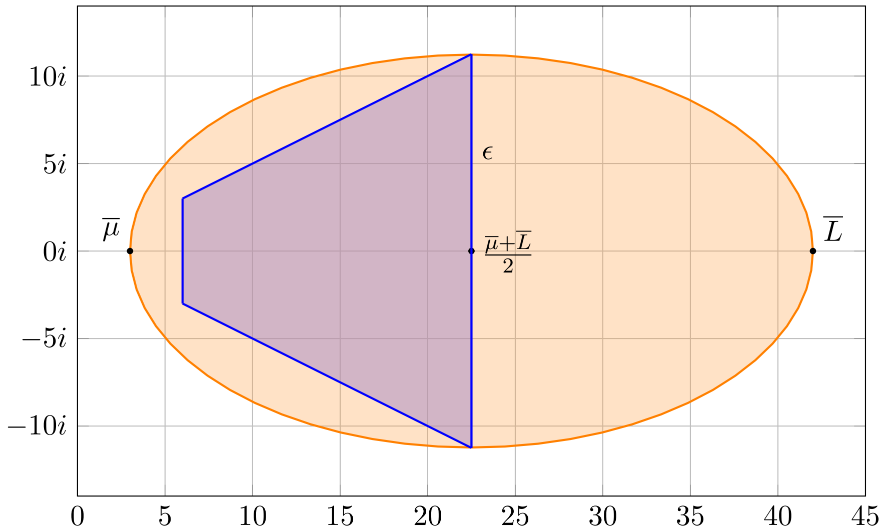

Figure 3. Illustration of the proof of Prop. 12. Lem. 26 guarantees that the spectrum of $\mathbf{J}_{F^{\text{cons.}}}$ is located inside a trapezoid (in blue). We then find a suitable ellipse of the form of Prop. 10 (in orange) which contains it.

C.3.5 Consensus optimization and momentum

The general idea behind the proof of Prop. 12 is illustrated by fig. 3. Using the following lemma, we first prove that the eigenvalues of the consensus optimization operator $F^{cons.}$ are contained in a trapezoid (in blue on the figure). Then, we find a suitable ellipse of the form of Prop. 10 (in orange) such that the trapezoid, thus the spectrum of $\mathbf{J}_{F^{cons.}}$ as shown by Lem. 26, fits inside.

First we need to refine [MNG17, Lem. 9].

Lemma 26. *Let $A \in \mathbb{R}^{d\times d}$ be a square matrix. Let σ_i be the singular values of A. Assume that*

$$\gamma \leq \sigma_i \leq L, \quad \mathcal{H}(A) \coloneqq \tfrac{A+A^T}{2} \succeq \mu I_d$$

with $\gamma > 0$ and $\mu \geq 0$. Then, for $\tau > 0$ such that $\tau\gamma^2 \geq \mu(1+2\tau\mu)$,

$$\max\left\{\frac{|\Im\lambda|}{|\Re\lambda|}\;\middle|\; \lambda \in \operatorname{Sp}(A+\tau A^TA)\right\} \leq \frac{\gamma}{\mu+\tau\gamma^2}$$

Moreover, for $\lambda \in \operatorname{Sp}(A+\tau A^TA)$, we have $\mu+\tau\gamma^2 \leq \Re\lambda \leq L+\tau L^2$.

Proof. In this proof, for M a real matrix, $\mathcal{H}(M) = \frac{M+M^T}{2}$ its Hermitian part and $\mathcal{S}(M = (M - M^T)/2$ its skew-symmetric part.

Let $B = A+\tau A^TA$. Let $\lambda \in \operatorname{Sp} B$ and let $v \in \mathbb{C}^d$ its associated eigenvector with $\|v\| = 1$. Then

$$\Re\lambda = \frac{\lambda+\overline{\lambda}}{2} = \overline{v}^T\mathcal{H}(B)v = \overline{v}^T\mathcal{H}(A)v + \tau\|Av\|^2 \geq \mu\|v\|^2 + \tau\|Av\|^2 = \mu+\tau\|Av\|^2 \tag{313}$$

by the assumption on $\mathcal{H}(A)$. We now deal with the imaginary part,

$$\Im\lambda = \frac{\lambda-\overline{\lambda}}{2i} = \frac{1}{i}\overline{v}^T\mathcal{S}(A)v$$

However, this quantity is hard to bound. Thus, we rewrite it using $\overline{v}^T A \overline{v}$ and $\overline{v}^T \mathcal{H}(A) v$. We have that $\overline{v}^T \mathcal{H}(A) v$ and $\frac{1}{i}\overline{v}^T \mathcal{S}(A) v$ are real and correspond respectively to the real and imaginary parts of $\overline{v}^T A v$. Hence,

$$(\Im\lambda)^2 = \left(\Im \overline{v}^T A v)\right)^2 = |\overline{v}^T A v|^2 - (\Re \overline{v}^T A v)^2 \leq |\overline{v}^T A v|^2$$

Using Cauchy-Schwarz inequality we get,

$$(\Im\lambda)^2 \leq \|Av\|^2$$

Finally,

$$\frac{|\Im\lambda|}{|\Re\lambda|} \leq \frac{\|Av\|}{\mu + \tau\|Av\|^2} = \varphi(\|Av\|)$$

with $\varphi : x \mapsto \frac{x}{\mu+\tau x^2}$. Using the derivative φ', we have that φ is non-decreasing before $x = \sqrt{\frac{\mu}{\tau}}$ and non-increasing after. Note that $\|Av\| \geq \sigma_{min}(A) \geq \gamma$. Hence, if $\tau\gamma^2 \geq \mu$, then $\varphi(\|Av\|) \leq \varphi(\gamma) = \frac{\gamma}{\mu+\tau\gamma^2}$ which concludes the proof of the first part of the lemma.

Now, take $\lambda \in \mathrm{Sp}(A + \tau A^T A)$. Then the inequality $\mu + \tau\gamma^2 \leq \Re\lambda$ comes from (313). The other one is

$$\Re\lambda \leq |\lambda| = \|Bv\| \leq \|Av\| + \tau\|Av\|^2 \leq L + \tau L^2$$

□

We can now proceed to show Prop. 8 by proving the more detailed proposition below.

Proposition 12. *Let σ_i be the singular values and eigenvalues of $J_F(\omega^*)$. Assume that*

$$\gamma \leq \sigma_i \leq L, \quad \tfrac{\boldsymbol{J}_F(\omega^*)+\boldsymbol{J}_F(\omega^*)}{2} \succeq \mu I_d$$

Define $F^{cons.}(\omega) = F(\omega) + \tau\nabla(\frac{1}{2}\|F\|^2)(\omega)$ with $\tau > 0$ and consider the momentum method applied to $F^{cons.}$,

$$\begin{aligned} \omega_{t+1} &= V^{mom+cons.}(\omega_t, \omega_{t-1}) \\ &= \omega_t - \alpha F^{cons.}(\omega_t) + \beta(\omega_t - \omega_{t-1}). \end{aligned}$$

If $\tau\gamma^2 \geq \mu$ and

$$\frac{\gamma}{\mu + \tau\gamma^2} \leq \sqrt{\frac{3}{2}}\sqrt{\frac{\mu+\tau\gamma^2}{L+\tau L^2}} \tag{314}$$

Then, one can choose $\alpha > 0$ and $\beta > 0$ such that,

$$\rho(\boldsymbol{J}_{V^{mom+cons.}}(\omega^*, \omega^*))) \leq 1 - c\sqrt{\frac{\mu+\tau\gamma^2}{L+\tau L^2}} + \mathcal{O}\left(\frac{\mu+\tau\gamma^2}{L+\tau L^2}\right) \tag{315}$$

More precisely, the parameters are given by,

$$\alpha = \alpha\left(L + \tau L^2 - \frac{\mu+\tau\gamma^2}{2}, \frac{1}{2}\sqrt{\mu+\tau\gamma^2}\sqrt{4(L+\tau L^2) - (\mu+\tau\gamma^2)}, L + \tau L^2\right) \tag{316}$$

$$\beta = \beta\left(L + \tau L^2 - \frac{\mu+\tau\gamma^2}{2}, \frac{1}{2}\sqrt{\mu+\tau\gamma^2}\sqrt{4(L+\tau L^2) - (\mu+\tau\gamma^2)}, L + \tau L^2\right) \tag{317}$$

where the functions $\alpha(\cdot)$ and $\beta(\cdot)$ are the ones defined in Thm. 21.

If $\tau = \frac{L}{\gamma^2}$ Then, $\rho(\boldsymbol{J}_{V^{mom+cons.}}(\omega^*, \omega^*)))$ is bounded by

$$\rho(\boldsymbol{J}_{V^{mom+cons.}}(\omega^*, \omega^*))) \le 1 - (\sqrt{2} - 1)\frac{\gamma}{L} + \mathcal{O}\left(\frac{\gamma^2}{L^2}\right)$$

where the constants in the $\mathcal{O}(.)$ are absolute.

Proof. Similarly to the proof of Prop. 7, we want to apply Prop. 10. We now look for $\epsilon, \overline{\mu}, \overline{L}$ such that the ellipsis

$$K(\epsilon, \overline{\mu}, \overline{L}) = \left\{ z \in \mathbb{C} : \left(\frac{\Re z - \frac{\overline{\mu}+\overline{L}}{2}}{\frac{\overline{L}-\overline{\mu}}{2}} \right)^2 + \left(\frac{\Im z}{\epsilon} \right)^2 \le 1 \right\} \tag{318}$$

contains $\operatorname{Sp} \nabla \mathbf{J}_{F^{\text{cons.}}}(\omega^*)$. First we compute $\mathbf{J}_{F^{\text{cons.}}}(\omega^*)$, for F twice differentiable. Note that $F^{\text{cons.}}$ can be written as $F^{\text{cons.}}(\omega) = F(\omega) + \tau \mathbf{J}_F^T(\omega) F(\omega)$, thus

$$\mathbf{J}_{F^{\text{cons.}}}(\omega^*) = \mathbf{J}_F(\omega^*) + \tau \mathbf{J}_F^T(\omega)\mathbf{J}_F(\omega)$$

Note that the derivative of $\mathbf{J}_F$ does not appear as $F(\omega^*) = 0$. From Lem. 26 with $A = \mathbf{J}_F(\omega^*)$, we have a control on

$$q(\tau) \coloneqq \frac{\gamma}{\mu + \tau\gamma^2} \ge \max\left\{ \frac{|\Im\lambda|}{|\Re\lambda|} \,\middle|\, \lambda \in \operatorname{Sp}(\mathbf{J}_{F^{\text{cons.}}}(\omega^*)) \right\} \ge 0$$

Using $q(\tau)$ and the bounds on the real parts of Lem. 26, we have that the spectrum of $\mathbf{J}_{F^{\text{cons.}}}(\omega^*)$ is inside the following shape,

$$\operatorname{Sp} \mathbf{J}_{F^{\text{cons.}}}(\omega^*) \subset S(\tau) \coloneqq \{\lambda \in \mathbb{C} \mid \mu + \tau\gamma^2 \le \Re\lambda \le L + \tau L^2,\ |\Im\lambda| \le q(\tau)\Re\lambda\}$$

We now only seek to include $S(\tau)$ in an ellipse $K(\epsilon, \overline{\mu}, \overline{L})$. First, we show that we can focus on two points, i.e. we prove that if $(1 + iq(\tau))(\mu + \tau\gamma^2)$ and $(1 + iq(\tau))(L + \tau L^2)$ belong to $K(\epsilon, \overline{\mu}, \overline{L})$, then $S(\tau) \subset K(\epsilon, \overline{\mu}, \overline{L})$.

We have that $S(\tau) \cap \{\Re z \ge 0\}$ is a trapezoid, the convex hull of the four points

$$(1 + iq(\tau))(\mu + \tau\gamma^2)\ ; \quad (1 - iq(\tau)(\mu + \tau\gamma^2)\ ; \quad (1 + iq(\tau))(L + \tau L^2)\ ; \quad (1 - iq(\tau))(L + \tau L^2).$$

As $K(\epsilon, \overline{\mu}, \overline{L})$ is convex, we only need to show that these four points belong to the ellipsis. By horizontal symmetry, we can restrict our analysis to the two points $(1 + iq(\tau))(\mu + \tau\gamma^2)$ and $(1 + iq(\tau))(L + \tau L^2)$.

Therefore, we focus on choosing a symmetric ellipse $K(\epsilon, \overline{\mu}, \overline{L})$ to which $(1 + iq(\tau))(\mu + \tau\gamma^2)$ and $(1 + iq(\tau))(L + \tau L^2)$ belong. The construction of this ellipse is similar to the one of the proof of Prop. 7. Since $(1 + iq(\tau))(L + \tau L^2)$ is the farthest point from the real axis, to be able to choose ϵ as small as possible, we put the center of the ellipse at $(L + \tau L^2)$. This way, we force $\frac{\overline{L}+\overline{\mu}}{2} = L + \tau L^2$.

To make sure $(1 + iq(\tau))(\mu + \tau\gamma^2)$ is also in the ellipsis, we need to choose $\overline{\mu}$ small enough. Define $\overline{\mu} = \frac{\mu + \tau\gamma^2}{m}$ with $m \ge 2$. This fixes the value of $\overline{L}$ as,

$$\overline{L} = 2(L + \tau L^2) - \overline{\mu} = 2(L + \tau L^2) - \frac{\mu + \tau\gamma^2}{m}$$

We now take $\epsilon > 0$ such that $(1 + iq(\tau))(L + \tau L^2)$ is in the ellipsis. We thus have the condition $\epsilon \ge q(\tau)(L + \tau L^2)$. The precise value of ϵ will be chosen later.

We must now check that $(1+iq(\tau))(\mu+\tau\gamma^2) \in K(\epsilon, \overline{\mu}, \overline{L})$. For this we check that this point satisfies the equation of the ellipsis,

$$\left(\frac{\mu+\tau\gamma^2-\frac{\overline{\mu}+\overline{L}}{2}}{\frac{\overline{L}-\overline{\mu}}{2}}\right)^2+q(\tau)^2\left(\frac{\mu+\tau\gamma^2}{\epsilon}\right)^2 \tag{319}$$

$$=\left(\frac{\mu+\tau\gamma^2-(L+\tau L^2)}{L+\tau L^2-\frac{\mu+\tau\gamma^2}{m}}\right)^2+q(\tau)^2\left(\frac{\mu+\tau\gamma^2}{\epsilon}\right)^2 \tag{320}$$

$$\leq\left(\frac{\mu+\tau\gamma^2-(L+\tau L^2)}{L+\tau L^2-\frac{\mu+\tau\gamma^2}{m}}\right)^2+\left(\frac{\mu+\tau\gamma^2}{L+\tau L^2}\right)^2 \tag{321}$$

by the choice of ϵ. Now let $x=\frac{\mu+\tau\gamma^2}{L+\tau L^2}\in[0,1]$. We have,

$$\begin{aligned}\left(\frac{\mu+\tau\gamma^2-\frac{\overline{\mu}+\overline{L}}{2}}{\frac{\overline{L}-\overline{\mu}}{2}}\right)^2+q(\tau)^2\left(\frac{\mu+\tau\gamma^2}{\epsilon}\right)^2 &\leq\left(\frac{1-x}{1-\frac{x}{m}}\right)^2+x^2\\ &\leq\left(\frac{1-x}{1-\frac{x}{m}}\right)^2+x\\ &\leq 1\end{aligned}$$

by application of Lem. 24 as $m\geq 2$.

We now fix ϵ. We want to take $\epsilon=\sqrt{\overline{\mu}\overline{L}}$ so we can apply Prop. 10. Thus we need $q(\tau)(L+\tau L^2)\leq\sqrt{\overline{\mu}\overline{L}}$. Substituting for $\overline{\mu}$ and $\overline{L}$, as $\overline{L}=2(L+\tau L^2)-(\mu+\tau\gamma^2)/2\geq\frac{3}{2}(L+\tau L^2)$, this is implied by $q(\tau)\leq\sqrt{\frac{3}{2}}\sqrt{\frac{\mu+\tau\gamma^2}{L+\tau L^2}}$. Now, if $\tau\gamma^2\geq\mu$, we can apply Lem. 26 and obtain the bound $q(\tau)\leq\frac{\mu+\tau\gamma^2}{L+\tau L^2}$. Hence, if

$$\frac{\gamma}{\mu+\tau\gamma^2}\leq\sqrt{\frac{3}{2}}\sqrt{\frac{\mu+\tau\gamma^2}{L+\tau L^2}} \tag{322}$$

we can apply Prop. 10 with $\epsilon=\sqrt{\overline{\mu}\overline{L}}$. Hence, one can choose $\alpha,\beta>0$ such that,

$$\begin{aligned}\rho(\mathbf{J}_{V^{\mathrm{mom+cons.}}}(\omega^*,\omega^*))) &\leq 1-2(\sqrt{2}-1)\sqrt{\frac{\overline{\mu}}{\overline{L}}}+\mathcal{O}\left(\frac{\overline{\mu}}{\overline{L}}\right),\\ &=1-2(\sqrt{2}-1)\sqrt{\frac{\mu+\tau\gamma^2}{2m(L+\tau L^2)-(\mu+\tau\gamma^2)}}+\mathcal{O}\left(\frac{\mu+\tau\gamma^2}{L+\tau L^2}\right),\\ &\leq 1-(\sqrt{2}-1)\sqrt{\frac{\mu+\tau\gamma^2}{L+\tau L^2}}+\mathcal{O}\left(\frac{\mu+\tau\gamma^2}{L+\tau L^2}\right),\end{aligned}$$

as $m=2$. This yields the first part of the proposition. We now need to find an admissible τ.

Assume $\tau L\geq 1$ and so $L\leq\tau L^2$. Then,

$$\begin{aligned}&\frac{\gamma}{\mu+\tau\gamma^2}\leq\sqrt{\frac{3}{2}}\sqrt{\frac{\mu+\tau\gamma^2}{L+\tau L^2}},\\ \Longleftarrow\ &\frac{\gamma}{\mu+\tau\gamma^2}\leq\frac{\sqrt{3}}{2}\sqrt{\frac{\mu+\tau\gamma^2}{\tau L^2}},\end{aligned}$$

$$
\begin{aligned}
&\iff \frac{\gamma}{\mu+\tau\gamma^2} \le \frac{\sqrt{3}}{2L}\sqrt{\frac{\mu+\tau\gamma^2}{\tau}}, \\
&\iff \frac{r}{\mu+\tau\gamma^2} \le \frac{\sqrt{3}}{2L}\sqrt{\frac{\mu+\tau\gamma^2}{\tau\gamma^2}}, \\
&\Longleftarrow \frac{1}{\mu+\tau\gamma^2} \le \frac{\sqrt{3}}{2L}\sqrt{\frac{\mu+\tau\gamma^2}{\mu+\tau\gamma^2}}, \\
&\iff \frac{1}{\mu+\tau\gamma^2} \le \frac{\sqrt{3}}{2L}\,.
\end{aligned}
$$

After rearranging, we get that the last condition is equivalent to,

$$
\tau \ge \frac{\frac{2}{\sqrt{3}}L - \mu}{\gamma^2}
$$

which is implied by $\tau \ge \frac{L}{\gamma^2}$.

Then, if $\tau \ge \frac{L}{\gamma^2}$, $\tau L \ge 1$ and $\tau\gamma^2 \ge \mu$ and so this condition implies $q(\tau)(L+\tau L^2) \le \sqrt{\mu L}$, which is what we wanted.

Then, for $\tau = \frac{L}{\gamma^2}$, we have

$$
\begin{aligned}
\frac{\mu+\tau\gamma^2}{L+\tau L^2} &= \frac{\gamma^2}{L^2}\frac{1+\mu/L}{1+\gamma^2/L^2} \\
&\ge \frac{\gamma^2}{L^2}\frac{1}{1+\gamma^2/L^2} \\
&= \frac{\gamma^2}{L^2} + \mathcal{O}\left(\frac{\gamma^4}{L^4}\right)
\end{aligned}
$$

and also in particular $\mathcal{O}\left(\frac{\mu+\tau\gamma^2}{L+\tau L^2}\right) = \mathcal{O}\left(\frac{\gamma^2}{L^2}\right)$. Hence,

$$
\rho(\mathbf{J}_{V^{\text{mom+cons.}}}(\omega^*,\omega^*))) \le 1 - (\sqrt{2}-1)\frac{\gamma}{L} + \mathcal{O}\left(\frac{\gamma^2}{L^2}\right)\,.
$$

□

Remark 6. Note that, this rate is roughly similar to the one that can be obtained with the standard momentum method applied to minimizing the objective

$$
f(\omega) = \frac{1}{2}\|F\|^2
$$

Indeed, one can check, at a stationary point ω^*, the eigenvalues of of the Hessian of f are in $[\gamma^2, L^2]$ (with the notations of the previous proposition). So applying Thm. 19 would yield a local convergence rate of $\mathcal{O}\left(\left(1-\frac{2\gamma}{L+\gamma}\right)^t\right)$.

One could then wonder what is the advantage of Consensus Optimization over the latter. Actually a plain gradient descent on $\frac{1}{2}\|F\|^2$ does not behave well in practice unlike Consensus Optimization [MNG17] and can be attracted to unstable equilibria in non-monotone landscapes [LBR+19].

Remark 7. Though this is not the focus of this section, similarly to the result of [ALW19] in the non-accelerated case, taking τ slightly higher, such as $\tau = \frac{2L}{\gamma^2}$, guarantees this same accelerated rate even in non-monotone setting. Indeed, all we need is that $\min_{\lambda \in \operatorname{Sp} \mathbf{J}_F(\omega^*)} \Re\lambda + \tau\gamma^2 > 0$, which is always satisfied by $\tau = \frac{2L}{\gamma^2}$ as the eigenvalues of $\mathbf{J}_F(\omega^*)$ are bounded by L.

C.4 Appendix: Ellipses

C.4.1 Main results

We recall the definition of the ellipses which interests us. Define, for $a, b, c \geq 0$, the ellipse:

$$E(a,b,c) = \left\{ \lambda \in \mathbb{C} : \frac{(\Re\lambda - c)^2}{a^2} + \frac{(\Im\lambda)^2}{b^2} \leq 1 \right\} \tag{323}$$

We adopt the convention that $\frac{0}{0} = 0$ so that for $b = 0$ the ellipse $E(a,b,c)$ degenerates into a real segment.

We now need to define objects related to the momentum method, and in particular its ρ-*convergence region.* For $\alpha, \rho \geq 0$, $\beta \in \mathbb{R}$, define

$$S(\alpha,\beta,\rho) = \{\lambda \in \mathbb{C} : \forall z \in \mathbb{C},\ z^2 - (1 - \alpha\lambda + \beta)z + \beta = 0 \implies |z| \leq \rho\} \tag{324}$$

We call it the ρ-convergence region of the momentum method as it corresponds to the maximal regions of the complex plane where the momentum method converges at speed $\mathcal{O}\left(\rho^t\right)$ if the operator has its eigenvalues in this zone. This is formalized by the following lemma,

Lemma 27 ([Sau64, II.7], [Pol64], [GHP+19, Thm. 3])**.** *Denote the momentum operator applied to the vector field F by*

$$V(\omega,\omega') = (\omega - \alpha F(\omega) + \beta(\omega - \omega'), \omega') \tag{325}$$

with $\alpha \geq 0$ step size and $\beta \in \mathbb{R}$ momentum parameter. Then, for any $\rho \geq 0$,

$$\rho(\nabla V(\omega^*,\omega^*)) \leq \rho \tag{326}$$

if and only if $\operatorname{Sp} \nabla F(\omega^*) \subset S(\alpha,\beta,\rho)$*.*

For a proof of this lemma in the context of games, see the proof of Thm. 3 of [GHP+19].

The next is a geometrical characterization of $S(\alpha,\beta,\rho)$: this is an ellipse, which is described in the following lemma.

Lemma 28 ([NV83, Cor. 6])**.** *If $|\beta| > \rho^2$, $S(\alpha,\beta,\rho) = \emptyset$. Otherwise, if $|\beta| \leq \rho^2$ and $\rho > 0$,*

$$S(\alpha,\beta,\rho) = \left\{ \lambda \in \mathbb{C} : \frac{(1 - \alpha\Re\lambda + \beta)^2}{(1+\tau)^2} + \frac{(\alpha\Im\lambda)^2}{(1-\tau)^2} \leq \rho^2 \right\} \tag{327}$$

where $\tau = \frac{\beta}{\rho^2}$.

As indicated, this lemma is a consequence of the results of [NV83], and more exaclty their section §6. However, their notations are significantly different from ours. We give a few elements to help the readers translate their results into our setting. In §6 of [NV83], they study iterative methods of the form,

$$\omega_{t+1} = \mu_0(\operatorname{Id} - F(\omega_t)) + \mu_1\omega_t + \mu_2\omega_{t-1}\,,$$

with $\mu_0 + \mu_1 + \mu_2 = 1$. Developing and using this relation, their iteration rule becomes,

$$\omega_{t+1} = \omega_t - \mu_0 F(\omega_t) + \mu_2(\omega_{t-1} - \omega_t)$$

Identifying with (325), we get that $\alpha = \mu_0$, $\beta = -\mu_2$ and so $\mu_1 = 1 + \beta - \alpha$.

Moreover, what they denote by $S_\eta(p)$, where p is a variable encompassing the parameters μ_0,μ_1 and μ_2, actually corresponds to $1 - S(\alpha, \beta, \rho)$ with α, β linked to μ_0,μ_1, μ_2 as described above and $\eta = \frac{1}{\rho}$. Indeed[9], $S_\eta(p)$ is meant to be compared to the eigenvalues of $I_d - \nabla F(\omega^*)$ instead of $\nabla F(\omega^*)$. Hence, the center and the semiaxes of the ellipse $1 - S(\alpha, \beta, \gamma)$ are given by (6.3) of [NV83] and once translated in our notations yield Lem. 28.

Remark 8. This lemma actually does not require the complex analysis machinery of [NV83]. This can be proven by hand using this remark on second-order equations. Let $0 < \rho \leq 1$ and let z_1, z_2 denote the two (possibly equal) roots of $X^2 + bX + c$. Then,

$$\max(|z_1|, |z_2|) \leq \rho \iff \begin{cases} |c| \leq \rho \\ |b|^2 + |\Delta| \leq 2\left(\rho^2 + \frac{c^2}{\rho^2}\right) \end{cases},$$

where $\Delta = b^2 - 4c$ denote the discriminant of the equation.

Now, we can introduce one of the main results of [NV83]. This is an answer to the natural question: ***what is*** $\underline{\rho}(S(\alpha, \beta, \rho))$ ***? In particular is it equal to*** ρ***? In other words, is momentum optimal w.r.t. to its convergence sets?*** The answer is yes for the momentum method. Note however that this doe snot hold for all stationary methods, this is linked to tricky questions of existence of branch for the roots of some polynomial equations, see [Nev93, §3.7] for a discussion on this.

Proposition 13 ([NV83, Cor. 10])**.** *Assume* $|\beta| \leq \rho^2 < 1$ *and* $\alpha > 0$*, then*

$$\underline{\rho}(S(\alpha, \beta, \rho)) = \rho \tag{328}$$

Hence, momentum is optimal for the sets $S(\alpha, \beta, \rho)$. What is left to show is that the sets $S(\alpha, \beta, \gamma)$ can represent most ellipses $E(a, b, c)$.

Proposition 14. *Let* $a, b \geq 0$*,* $c > 0$*,* $(a, b) \neq (0, 0)$*. There exists* $\alpha > 0$*,* $\rho > 0$*,* $\beta \in (-1, 1]$*, with* $|\beta| \leq \rho$ *such that* $E(a, b, c) = S(\alpha, \beta, \rho)$ *if and only if* $a^2 \leq b^2 + c^2$*. If it is the case,*

(i) The triple (α, β, ρ) *satisfying such conditions is unique.*

(ii) The corresponding β *can be written* $\beta = \chi(a - b)$ *with* $\chi > 0$*.*

(iii) The corresponding ρ *is equal to:*

$$\rho = \begin{cases} \frac{a}{c} & \textit{if } a = b \\ \frac{c - \sqrt{b^2 + c^2 - a^2}}{a - b} & \textit{otherwise.} \end{cases}$$

(iv) The parameters $\alpha > 0$ *and* $\beta \in (-1, 1]$ *are given by,*

$$\beta = \begin{cases} 0 & \textit{if } a = b \\ 2c\frac{c - \sqrt{c^2 + b^2 - a^2}}{a^2 - b^2} - 1 & \textit{otherwise} \end{cases} \qquad \alpha = \frac{1 + \beta}{c} = \begin{cases} \frac{1}{c} & \textit{if } a = b \\ 2\frac{c - \sqrt{c^2 + b^2 - a^2}}{a^2 - b^2} & \textit{otherwise} \end{cases} \tag{329}$$

[9] This is a standard convention in the linear system theory. They consider $\omega = T\omega + c$ instead of $A\omega + b$ as they use splittings of A.

Proof. Recall these two parametrizations of an ellipse,

$$E(a,b,c) = \left\{\lambda \in \mathbb{C} : \frac{(\Re\lambda - c)^2}{a^2} + \frac{(\Im\lambda)^2}{b^2} \leq 1\right\}$$
$$S(\alpha,\beta,\rho) = \left\{\lambda \in \mathbb{C} : \frac{(1-\alpha\Re\lambda+\beta)^2}{(1+\tau)^2} + \frac{(\alpha\Im\lambda)^2}{(1-\tau)^2} \leq \rho^2\right\}$$

where $\tau = \frac{\beta}{\rho^2}$. Note that $(a,b) \neq (0,0)$, $E(a,b,c)$ is not reduced to a point. So if these ellipses are equal, $\rho > 0$, and we also have $\alpha > 0$ These ellipses are characterised by their centers and their semiaxes so they are equal if and only if,

$$\left\{\begin{array}{l} \frac{1+\beta}{\alpha} = c \\ \rho + \frac{\beta}{\rho} = \alpha a \\ \rho - \frac{\beta}{\rho} = \alpha b \end{array}\right. \iff \left\{\begin{array}{l} \frac{1+\beta}{\alpha} = c \\ \rho = \alpha(a+b) \\ \frac{\beta}{\rho} = \alpha(b-a) \end{array}\right. \iff \left\{\begin{array}{l} \frac{1+\beta}{\alpha} = c \\ \rho = \frac{1+\beta}{2}(\widetilde{a}+\widetilde{b}) \\ \frac{\beta}{\rho} = \frac{1+\beta}{2}(\widetilde{a}-\widetilde{b}), \end{array}\right. \tag{330}$$

where $\widetilde{a} = \frac{a}{c}$ and $\widetilde{b} = \frac{b}{c}$. We further let $\widetilde{\beta} = 1+\beta$. Then, the last two equations imply the following equation on β,

$$\beta = \frac{(1+\beta)^2}{4}(\widetilde{a}^2 - \widetilde{b}^2) \iff \widetilde{\beta} - 1 = \frac{\widetilde{\beta}^2}{4}(\widetilde{a}^2 - \widetilde{b}^2) \tag{331}$$

Its discriminant is $\Delta = 1 - (\widetilde{a}^2 - \widetilde{b}^2)$, which is non-negative if and only if $b^2 + c^2 \geq a^2$.

Before solving this equation, we briefly discuss when it degenerates into a degree one equation. Indeed, if $a = b$, and so $\widetilde{a} = \widetilde{b}$, the unique solution of (331) is $\widetilde{\beta} = 1$ and so $\beta = 0$. Moreover $\rho = \frac{\widetilde{a}+\widetilde{b}}{2} = \frac{a}{c}$.

We now assume that $\widetilde{a}^2 - \widetilde{b}^2 \neq 0$. The two solutions of (331) are,

$$\widetilde{\beta}_\pm = 1 + \beta_\pm = 2\frac{1 \pm \sqrt{\Delta}}{\widetilde{a}^2 - \widetilde{b}^2}$$

We distinguish three cases.

- If $\Delta = 0$. There is only one solution $\widetilde{\beta} = 1 + \beta = 2$ to (331) and so $\beta = 1$.
- If $0 < \Delta < 1$ then in particular $\widetilde{a} > \widetilde{b}$. As $0 < \Delta < 1$, we also have $0 < \widetilde{a}^2 - \widetilde{b}^2 < 1$. This implies that $\widetilde{\beta}_+ > 2(1+\sqrt{\Delta}) > 2$ and so $\beta_+ > 1$ which do not satisfy the desired conditions on β. We show that β_- satisfy them instead. As $\Delta < 1$, $\widetilde{\beta}_- > 0$. Moreover, $\sqrt{\Delta} \geq \Delta$ and so $\widetilde{\beta}_- \leq 2\frac{1-\Delta}{\widetilde{a}^2-\widetilde{b}^2} = 2$ and so $\beta_- \in (-1,1]$.
- If $\Delta > 1$ and so $\widetilde{a} < \widetilde{b}$. One has immediately that $\widetilde{\beta}_+ < 0$ which disqualifies β_+. On the contrary as $\Delta > 1$, $\widetilde{\beta}_- > 0$ and $\widetilde{\beta}_- = 2\frac{\sqrt{1+\widetilde{b}^2-\widetilde{a}^2}-1}{\widetilde{b}^2-\widetilde{a}^2} \leq 2\frac{1+\sqrt{\widetilde{b}^2-\widetilde{a}^2}-1}{\widetilde{b}^2-\widetilde{a}^2} = 2$. And so $\beta_- \in (-1,1]$.

Note that the case $\Delta = 1$ is prevented by the assumption $\widetilde{a}^2 - \widetilde{b}^2 \neq 0$.

In each of the three cases above we ended up with,

$$\beta = \beta_- = \widetilde{\beta}_- - 1 = 2\frac{1-\sqrt{1+\widetilde{b}^2-\widetilde{a}^2}}{\widetilde{a}^2-\widetilde{b}^2} - 1 = 2c\frac{c-\sqrt{c^2+b^2-a^2}}{a^2-b^2} - 1 \in (-1,1]$$

Note that the third equation of (330) easily gives that $\beta = \chi(a-b)$ with $\chi > 0$. We now define ρ with the second equation of (330),

$$\rho = \frac{1+\beta}{2}(\widetilde{a} + \widetilde{b})$$

As β satisfy (331), β and ρ also satisfy the third one of (330). α can then be defined by the first equation of (330). Finally note that the fact that $|\beta| \leq \rho^2$ comes from the combination of the second and the third equations of (330). $\square$

Note that if $0 \notin E(a,b,c)$, then $c^2 > a^2$ and so the hypothesis of the proposition above is satisfied. Thm. 21 is now proven by simply combining all the results in this subsection.

C.4.2 Proof of optimality of momentum on its convergence zones

For this proof, we will need a characterization of $\underline{\rho}(K)$ using Green functions. We will follow the presentation of [Nev93].

Definition 2. *The Green function (with pole at ∞) of a non-empty, connected, unbounded open set $\Omega \subset \mathbb{C}$ is the unique function $g : \Omega \to \mathbb{R}$ such that:*

(i) g is harmonic on Ω.

(ii) $g(z) = \log|z| + \mathcal{O}(1)$ as $|z| \to \infty$.

(iii) $g(z) \xrightarrow[z\to\zeta]{} 0$ for every $\zeta \in \partial\Omega$.

For a compact $K \subset \mathbb{C}$, denote by G_∞ the unbounded connected component of $\overline{\mathbb{C}} \setminus K$. $\underline{\rho}(K)$ can then be obtained from the Green function of G_∞. This is not our concern here, but note that the Green function of G_∞ is guaranteed to exist if its boundary is sufficiently nice (see for instance [Wal35, Ran95] for a thorough treatment of this classical question).

The following theorem is a deep result in complex analysis, which links the minimization problem over polynomial which defines $\underline{\rho}$ to the geometric properties of K through its Green function.

Theorem 22 ([Nev93, Prop. 3.4.6, Thm. 3.4.9])**.** *If G_∞ has a Green function g and if $0 \in G_\infty$,*

$$\underline{\rho}(K) = \exp(-g(0)) \tag{332}$$

We will also need the following complex analysis lemma about the Joukowsky map, see [Neh52, Chap. VI] for instance.

Lemma 29. *Let $\psi(z) = z + \frac{1}{z}$. Then $\psi : \overline{\mathbb{C}} \setminus \{z : |z| \leq 1\} \to \overline{\mathbb{C}} \setminus [-1,1]$ is a conformal mapping. Its inverse ϕ is characterized by: for any $z_0 \notin [-1,1]$, $\phi(z_0)$ is the unique solution of*

$$z^2 - 2zz_0 + 1 = 0 \tag{333}$$

outside $\{z : |z| \leq 1\}$. Moreover, $\phi(z) = 2z + \mathcal{O}(1)$ when $z \to \infty$.

First we begin with a simple lemma about the convergence zones of momentum.

Lemma 30. *If $\rho^2 < 1$, $0 \notin S(\alpha, \beta, \rho)$.*

Proof. Consider the equation

$$z^2 - (1+\beta)z + \beta = 0 \tag{334}$$

Its two roots are β and 1 which yields the result. $\square$

As the boundary of the set plays a special role in the definition of the Green function, we need to have a precise characterization of it. This is done through the the next two lemmas.

Lemma 31. *If* $0 < |\beta| \leq \rho^2$, *then*

$$\mathrm{int}(S(\alpha,\beta,\rho)) = \bigcup_{\rho'>0:|\beta|<\rho'<\rho} S(\alpha,\beta,\rho') \tag{335}$$

Proof. The functions $x \mapsto x \pm \frac{\beta}{x}$ are increasing positive on $]\sqrt{|\beta|}, +\infty[$. So their square is also increasing. By Lem. 28,

$$\mathrm{int}\, S(\alpha,\beta,\rho) = \left\{\lambda \in \mathbb{C} : \frac{(1-\alpha\Re\lambda+\beta)^2}{(1+\tau)^2} + \frac{(\alpha\Im\lambda)^2}{(1-\tau)^2} < \rho^2\right\} \tag{336}$$

Define for $x > \sqrt{|\beta|}$ the function $h_\lambda(x) = \frac{(1-\alpha\Re\lambda+\beta)^2}{(x+\frac{\beta}{x})^2} + \frac{(\alpha\Im\lambda)^2}{(x-\frac{\beta}{x})^2}$, which is continuous and non-increasing. We show the result by double inclusion.

- Let $\lambda \in \mathrm{int}\, S(\alpha,\beta,\rho)$. As $\rho > 0$, $h_\lambda(\rho) < 1$ by (336). As $\rho > \sqrt{|\beta|}$, by continuity of h_λ at ρ, there exists $\rho > \rho' > \sqrt{|\beta|}$ such that $h_\lambda(\rho') < 1$. As $\rho' > \sqrt{|\beta|} \geq 0$, this implies that $\lambda \in S(\alpha,\beta,\rho')$.
- Let $\rho' > 0$ such that $|\beta| < \rho' < \rho$ and take $\lambda \in S(\alpha,\beta,\rho')$. By Lem. 28, as $\rho' > 0$, this implies that $h_\lambda(\rho') \leq 1$. Note that if both $\Im\lambda = 0$ and $1-\alpha\Re\lambda+\beta = 0$, $\lambda \in \mathrm{int}\, S(\alpha,\beta,\rho)$ as $\rho > 0$. Otherwise, if at least one of them is non-zero, this means that h_λ is actually decreasing on $]\sqrt{|\beta|}, +\infty[$. Hence, $h_\lambda(\rho) < h_\lambda(\rho') \leq 1$ and so $\lambda \in \mathrm{int}\, S(\alpha,\beta,\rho)$.

$\square$

Lemma 32. *If* $0 < |\beta| \leq \rho^2$,

$$\partial S(\alpha,\beta,\rho) = S(\alpha,\beta,\rho) \cap \{\lambda \in \mathbb{C} : \exists z \in \mathbb{C},\, z^2 - (1-\alpha\lambda+\beta)z + \beta = 0 \text{ and } |z| = \rho\} \tag{337}$$

Proof. This is a direct consequence of Lem. 31 and the definition of $S(\alpha,\beta,\rho)$. $\square$

Lemma 33. *For* $0 < \beta < \rho^2$,

$$\{\lambda \in \mathbb{R} : (1-\alpha\lambda+\beta)^2 \leq 4\beta\} \subset \mathrm{int}(S(\alpha,\beta,\lambda)) \tag{338}$$

For $0 < -\beta < \rho^2$,

$$\{\lambda \in \mathbb{C} : 1-\alpha\Re\lambda+\beta = 0,\, (\alpha\Im\lambda)^2 \leq 4|\beta|\} \subset \mathrm{int}(S(\alpha,\beta,\lambda)) \tag{339}$$

Proof. First assume that $0 < \beta < \rho^2$. Consider $\lambda \in \mathbb{R}$ such that $(1-\alpha\lambda+\beta)^2 \leq 4\beta$. Using the characterization of Lem. 28, we only need to show that $(1-\alpha\lambda+\beta)^2 < \rho^2(1+\tau)^2$ (as $-\rho^2 < \beta \implies \tau \neq -1$). But, from the definition of τ we get

$$\rho^2(1+\tau)^2 - 4\beta = (\rho - \frac{\beta}{\rho})^2 > 0 \tag{340}$$

Hence $4\beta < \rho^2(1+\tau)^2$ and the result follows from the choice of λ.

The proof for the second point is similar.

$\square$

We can now prove the proposition which was the target of this subsection. Note that the following proof does not encompass the case particular $\rho^2 = |\beta|$ in which the ellipse is degenerate. This falls into the case of segments, which is much simpler, see the aforementioned references (or [Nev93] for a didactic explanation).

Proposition 15. *Assume* $0 < |\beta| < \rho^2 < 1$ *and* $\alpha > 0$*, then*

$$\underline{\rho}(S(\alpha, \beta, \rho)) = \rho \tag{341}$$

Proof. We will build the Green function for $\overline{\mathbb{C}} \setminus S(\alpha, \beta, \rho)$ using Lem. 29.

First, we show that if $\lambda \notin \operatorname{int} S(\alpha, \beta, \rho)$, then $\frac{1-\alpha\lambda+\beta}{2\sqrt{\beta}} \notin [-1, 1]$ where $\sqrt{\beta}$ is a square root (with positive real part) of β. Indeed, assume for the sake of contradiction that it is not the case, i.e. there exists $\lambda \notin \operatorname{int} S(\alpha, \beta, \rho)$ such that $\frac{1-\alpha\lambda+\beta}{2\sqrt{\beta}} \in [-1, 1]$. Assume first that $\beta > 0$. This implies that $\Im(1 - \alpha\lambda + \beta) = 0$ and so that $\lambda \in \mathbb{R}$ as $\alpha \neq 0$. Moreover, as $\beta > 0$, λ satisfies $(1 - \alpha\lambda + \beta)^2 \leq 4\beta$. By Lem. 33, $\lambda \in \operatorname{int} S(\alpha, \beta, \rho)$ which is a contradiction. If $\beta < 0$, $\sqrt{\beta} = i\sqrt{|\beta|}$. This implies that $\Re(1 - \eta\lambda + \beta) = 0$. Moreover, λ satisfies $(\Im(1 - \alpha\lambda + \beta))^2 \leq 4|\beta|$. We get a similar contradiction using Lem. 33.

Take $\lambda \notin \operatorname{int} S(\alpha, \beta, \rho)$. Then, as $\frac{1-\alpha\lambda+\beta}{2\sqrt{\beta}} \notin [-1, 1]$, we can consider $\phi\left(\frac{1-\alpha\lambda+\beta}{2\sqrt{\beta}}\right)$. By Lem. 29, $\phi\left(\frac{1-\alpha\lambda+\beta}{2\sqrt{\beta}}\right)$ is the unique solution of modulus (strictly) greater than one of

$$z^2 - 2\frac{1 - \alpha\lambda + \beta}{2\sqrt{\beta}} z + 1 = 0 \tag{342}$$

Hence $\sqrt{\beta}\phi\left(\frac{1-\alpha\lambda+\beta}{2\sqrt{\beta}}\right)$ is the unique solution of modulus (strictly) greater than $\sqrt{|\beta|}$ of

$$\frac{z^2}{\beta} - 2\frac{1 - \alpha\lambda + \beta}{2\beta} z + 1 = 0 \tag{343}$$

$$\iff z^2 - (1 - \eta\lambda + \beta)z + \beta = 0\,. \tag{344}$$

Let $z_1 = \sqrt{\beta}\phi\left(\frac{1-\alpha\lambda+\beta}{2\sqrt{\beta}}\right)$ and let z_2 be the other root of (344). Then $z_1 z_2 = \beta$ and so $|z_1 z_2| = |\beta|$. Hence, as $|z_1| > \sqrt{|\beta|}$, we have $|z_2| < \sqrt{|\beta|}$. Hence $\sqrt{\beta}\phi\left(\frac{1-\alpha\lambda+\beta}{2\sqrt{\beta}}\right)$ is the solution of greatest magnitude of (344). We will see that this quantity is very regular as a function of λ outside $S(\alpha, \beta, \rho)$. Define

$$\chi : \begin{cases} \mathbb{C} \setminus \operatorname{int} S(\alpha, \beta, \rho) \longrightarrow \mathbb{C} \\ \qquad\qquad \lambda \qquad\quad \longmapsto \sqrt{\beta}\phi\left(\dfrac{1 - \alpha\lambda + \beta}{2\sqrt{\beta}}\right) \end{cases} \tag{345}$$

We can now build our Green function using χ. Define,

$$g : \begin{cases} \mathbb{C} \setminus \operatorname{int} S(\alpha, \beta, \rho) \longrightarrow \mathbb{R} \\ \qquad\qquad \lambda \qquad\quad \longmapsto \log \dfrac{|\chi(\lambda)|}{\rho} \end{cases} \tag{346}$$

Note that as ϕ is continuous on its domain of definition χ is continuous too. Moreover, as $\beta \neq 0$ and $\chi(\lambda)$ is a root of (344), $\chi(\lambda) \neq 0$ for $\lambda \notin \operatorname{int} S(\alpha, \beta, \rho)$. Hence g is well-defined and continuous too on $\mathbb{C} \setminus \operatorname{int} S(\alpha, \beta, \rho)$. We now show that g is the Green function of $G_\infty = \mathbb{C} \setminus S(\alpha, \beta, \gamma)$ according to definition 2.

(i) By Lem. 29, ϕ is analytic and so is χ on the open set $\mathbb{C} \setminus S(\alpha, \beta, \rho)$. Moreover, as mentioned above, $\chi(\lambda) \neq 0$ for $\lambda \notin S(\alpha, \beta, \rho)$ Hence, g is harmonic on $\mathbb{C} \setminus S(\alpha, \beta, \rho) = G_\infty$.

(ii) When $\lambda \to \infty$, $\frac{1-\alpha\lambda+\beta}{2\sqrt{\beta}} \to \infty$ too as $\alpha \neq 0$. Hence, by Lem. 29,

$$g(\lambda) = \log \frac{|\chi(\lambda)|}{\rho} \tag{347}$$

$$= \log \left| \phi \left(\frac{1 - \alpha\lambda + \beta}{2\sqrt{\beta}} \right) \right| + \mathcal{O}(1) \tag{348}$$

$$= \log |\lambda + \mathcal{O}(1)| + \mathcal{O}(1) \tag{349}$$

$$= \log |\lambda| + \mathcal{O}(1) \tag{350}$$

(iii) Let $\zeta \in \partial\left(\mathbb{C} \setminus S(\alpha, \beta, \gamma)\right) = \partial S(\alpha, \beta, \gamma)$. Note that χ is defined on $\mathbb{C} \setminus \operatorname{int} S(\alpha, \beta, \rho)$ on so on $\partial S(\alpha, \beta, \gamma)$. Then, by Lem. 32 and the definition of χ, $|\chi(\zeta)| = \rho$. By continuity of g, $g(\lambda) \xrightarrow[\lambda \to \zeta]{} g(\zeta) = 0$. Hence g is the Green function for G_∞ by definition 2. Moreover, by Lem. 30, $0 \in G_\infty$. We can now apply Thm. 22 to get that $\underline{\rho}(S(\alpha, \beta, \rho)) = \exp(-g(0))$. Finally, we compute $g(0)$. Recall that, as $0 \notin S(\alpha, \beta, \rho)$, $\chi(0)$ is the root of greatest magnitude of

$$z^2 - (1 + \beta)z + \beta = 0 \tag{351}$$

The two roots of this equation are β and 1. As $0 < |\beta| < 1$, $\chi(0) = 1$, so $g(0) = \log \frac{1}{\rho}$ and $\underline{\rho}(S(\alpha, \beta, \rho)) = \rho$.

□