A General Framework for Sample-Efficient Function Approximation in RL

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CS @ UCLA

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Joint work with Zixiang Chen*, Chris Junchi Li*, Quanquan Gu, Michael I. Jordan

Outline

01 **Background**
- Preliminaries
- Literature of RL models
- Model-based and Model-free RL
- General Function Approximation

02 **Our ABC Framework**
- Scope and sample complexity of our framework
- Admissible bellman characterization
- Functional eluder dimension
- Decomposable estimation function

03 **MDP Instances**
- Linear mixture MDP
- Low witness rank
- Kernelized nonlinear regulator (KNR)

04 **Algorithm and Main Results**
- Algorithm
- Proof sketch
- Main theorem

05 **Implications for specific MDPs**
- Linear mixture MDPs
- Kernelized nonlinear regulator (KNR)
- Low witness rank

Applications of RL

https://www.deepmind.com/research/highlighted-research/alphago https://openai.com/research/solving-rubiks-cube

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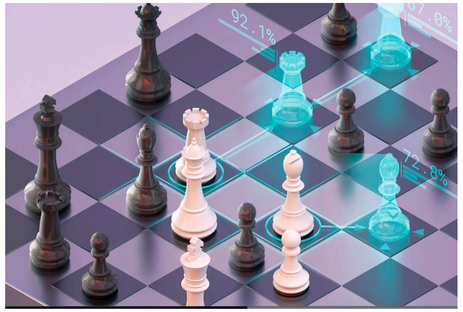

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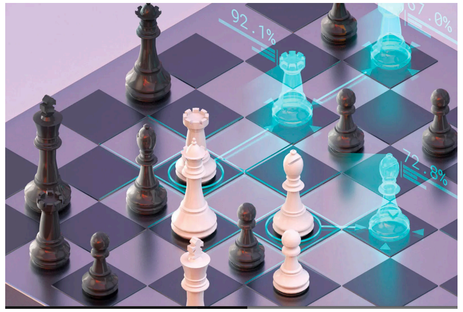

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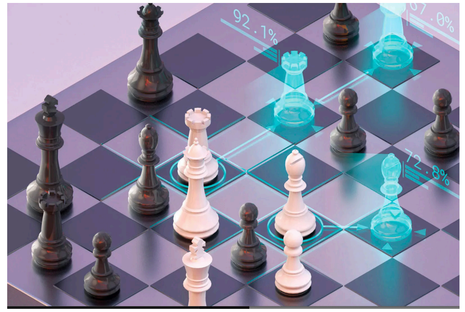

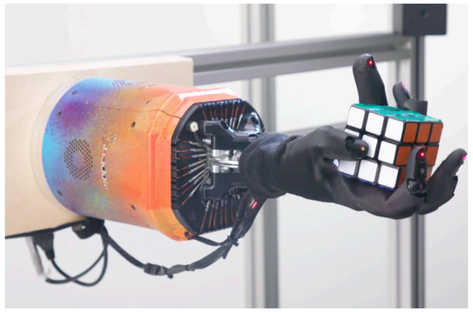

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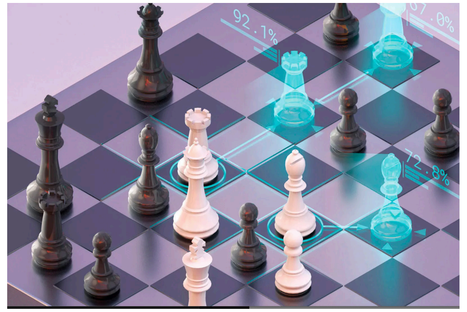


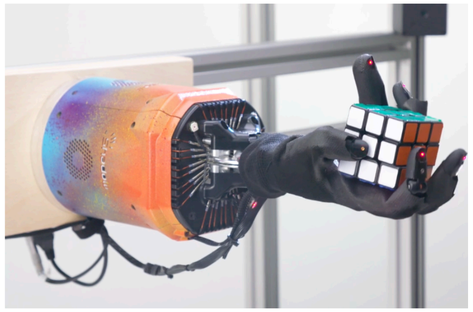


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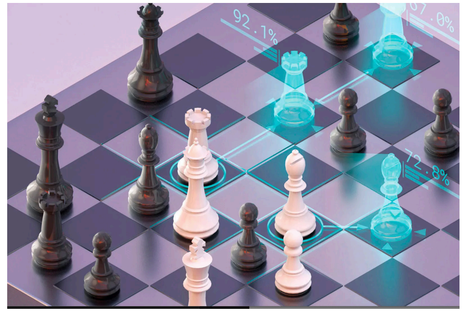


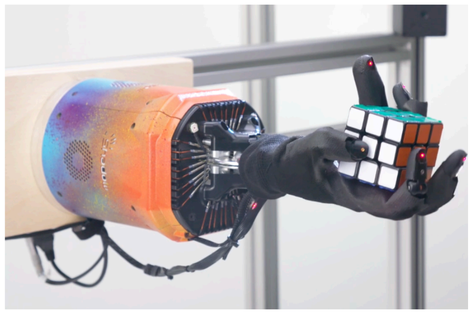


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Background

Markov Decision Processes (MDP)

$\mathcal{S}$: space of feasible states

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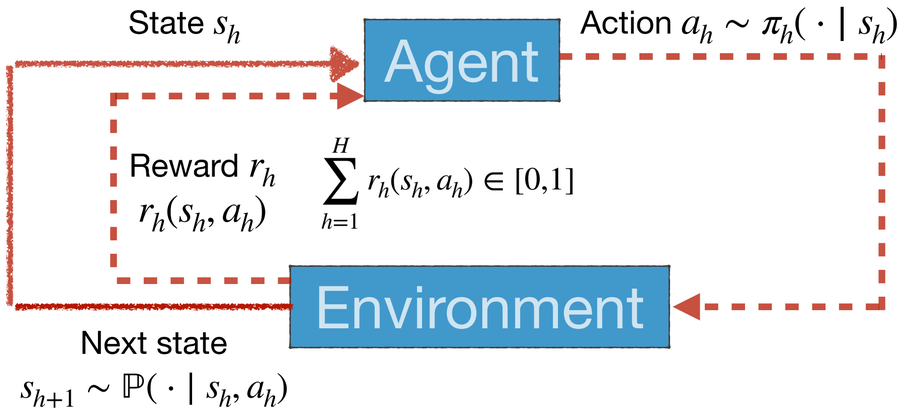
H: the horizon in each episode

$\mathbb{P} := \{\mathbb{P}_h\}_{h\in[H]}$: transition probability

$r_h(s,a) \geq 0$: the reward received



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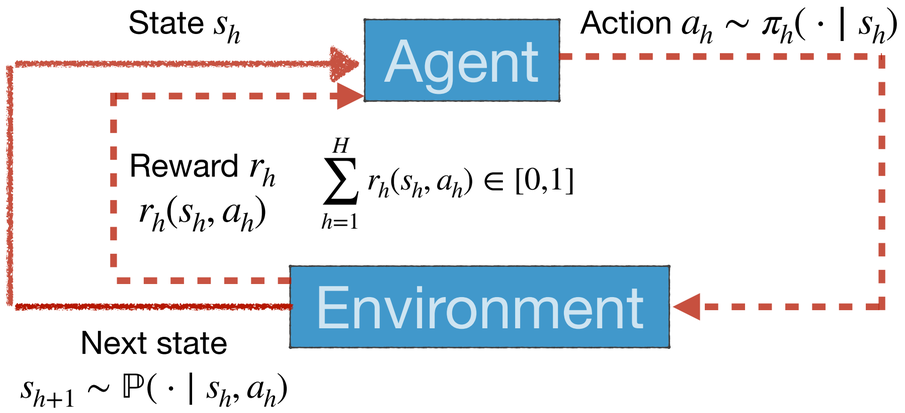

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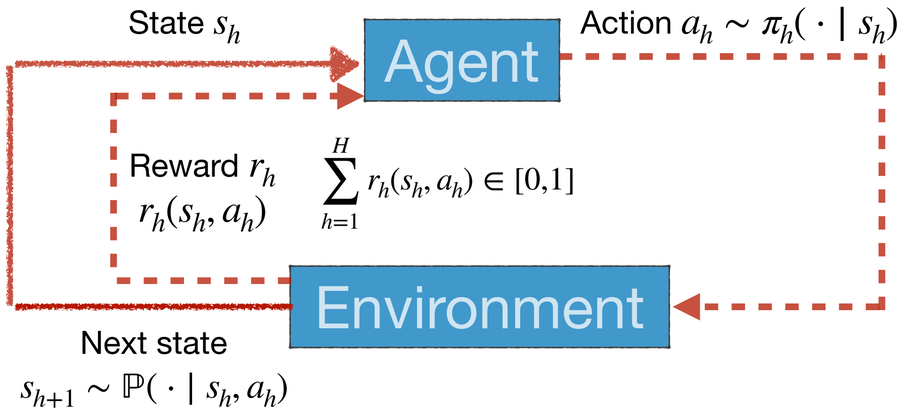

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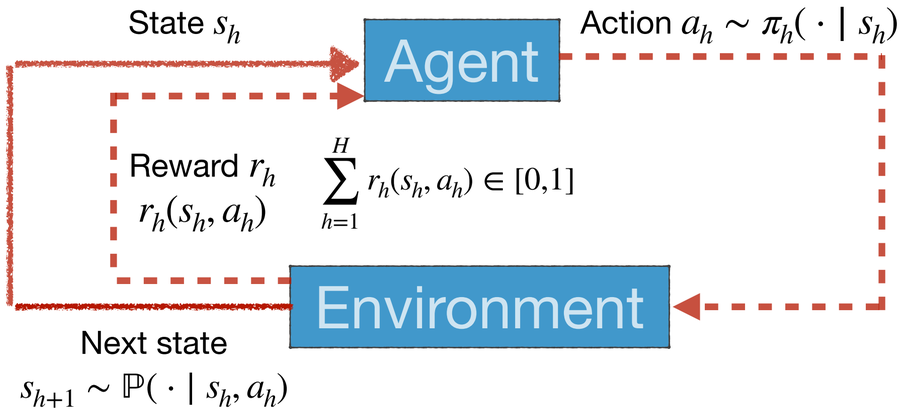

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- $Q_h^\pi(s,a) := \mathbb{E}_\pi \left[\sum_{h'=h}^{H} r_{h'}(s_{h'}, a_{h'}) \middle| s_h = s, a_h = a \right]$, $V_h^\pi(s) = Q_h^\pi(s, \pi(s))$

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- Cumulative regret: $\text{Regret}(T) := \sum_{t=1}^{T} \left[V_1^*(s_1) - V_1^{\pi^t}(s_1) \right]$

Rich Literature of RL Models

- Tabular MDP
- Factored MDP [Kearns & Koller '99]
- Predictive State Rrepresentations [Littman et al. '01]
- State Aggregation [Li '09, Dong et al. '20]
- Block MDP [Jiang et al. '17]
- Linear Quadratic Regulator (LQR) [Dean et al. '19]
- Linear MDP [Jin et al. '20]
- Linear Mixture MDP [Modi et al. '20, Ayoub et al. '20, Zhou et al.'20]
- Bellman Complete [Munos '05, Zanette et al '20]
- Kernelized Nonlinear Regulator [Kakade et al. '20]
- Low Occupancy Complexity [Du et al. '21]
- Linear Q*/V* [Du et al. '21]
- Kernel Reactive POMDPs [Krishnamurthy et al. '16]
- Flambe/Feature Selection [Agarwal et al. '20]
- Generalized Linear Bellman Complete [Wang et al. '19]

Model-based vs Model-free RL

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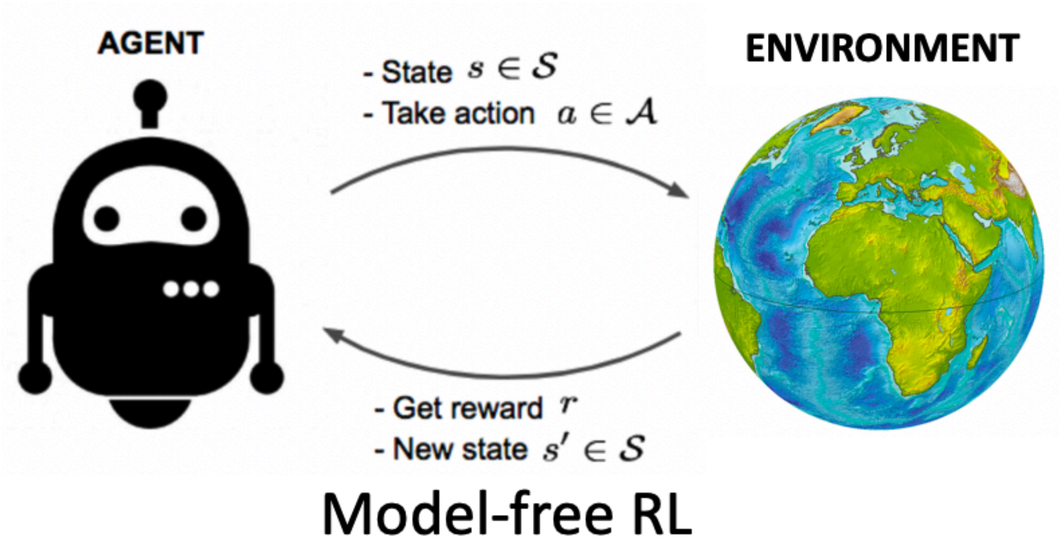
Diagram of model-free reinforcement learning



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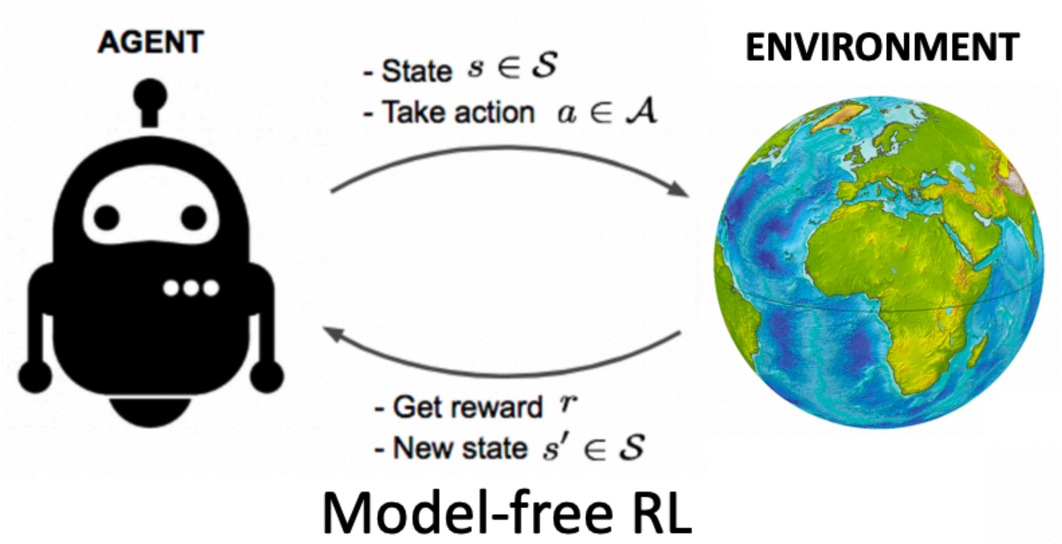
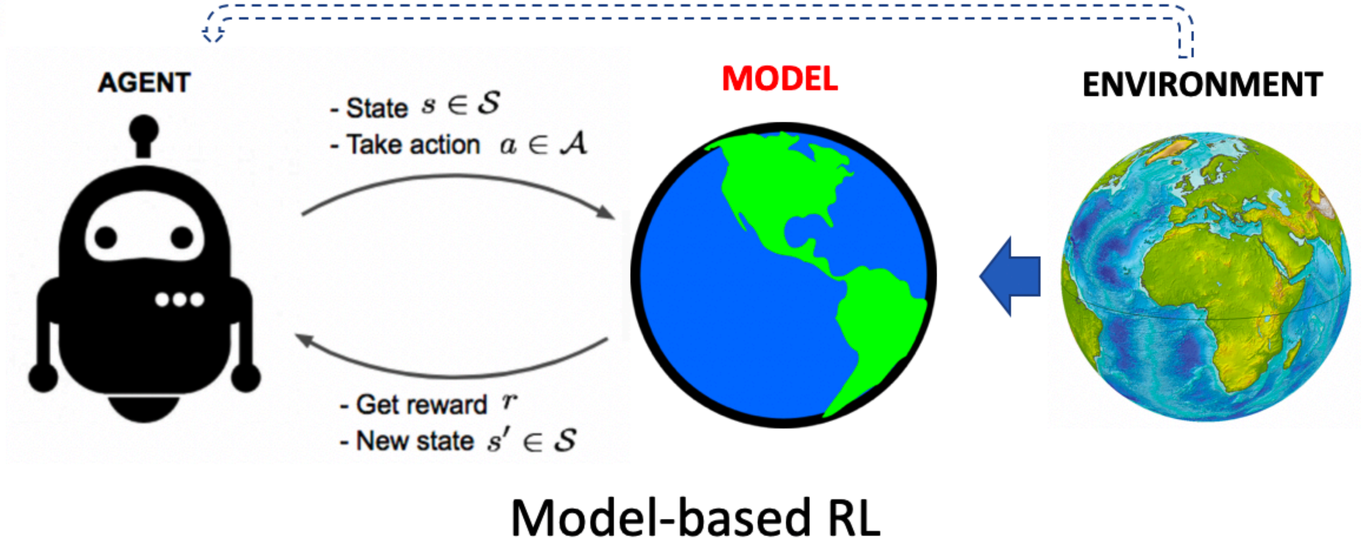


Diagram of model-based reinforcement learning



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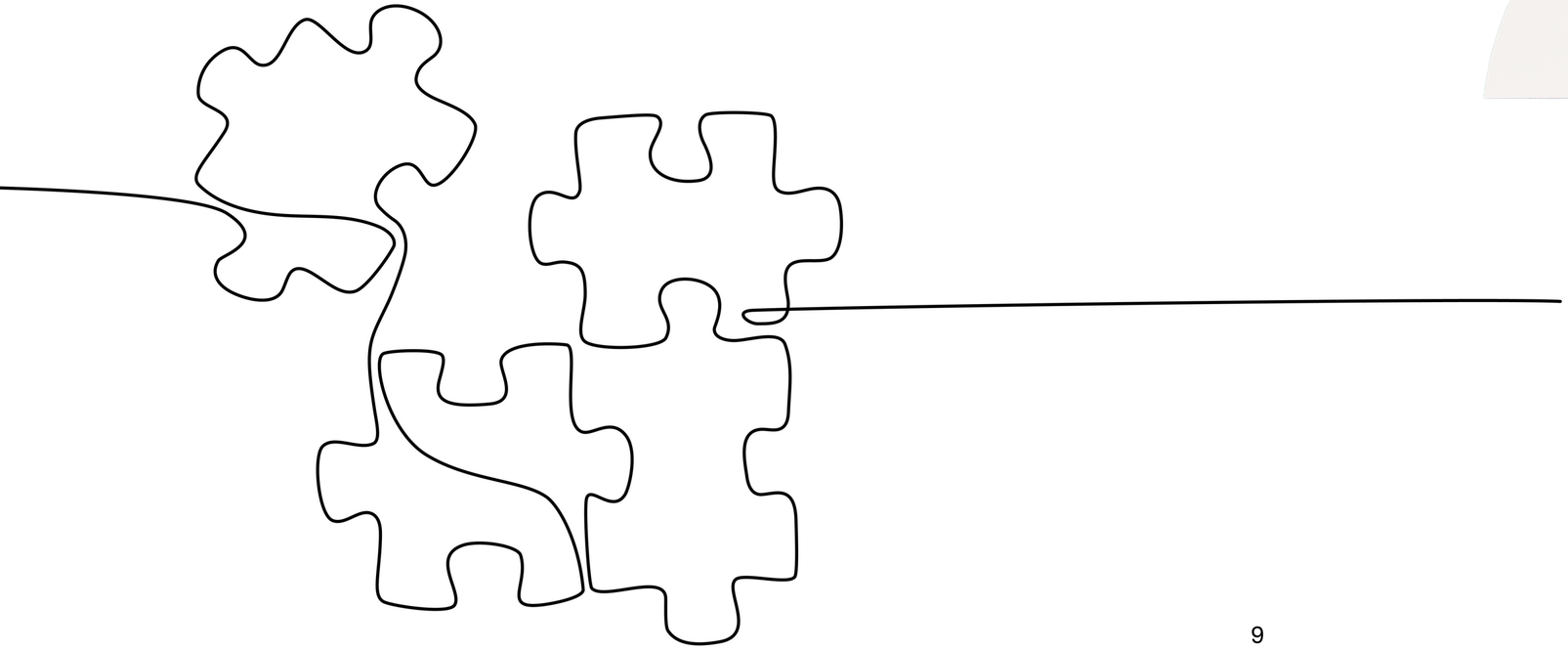
A necessary condition

Is there a unified framework that includes all model-free and model-based RL classes while maintaining sharp sample efficiency



Our ABC Framework

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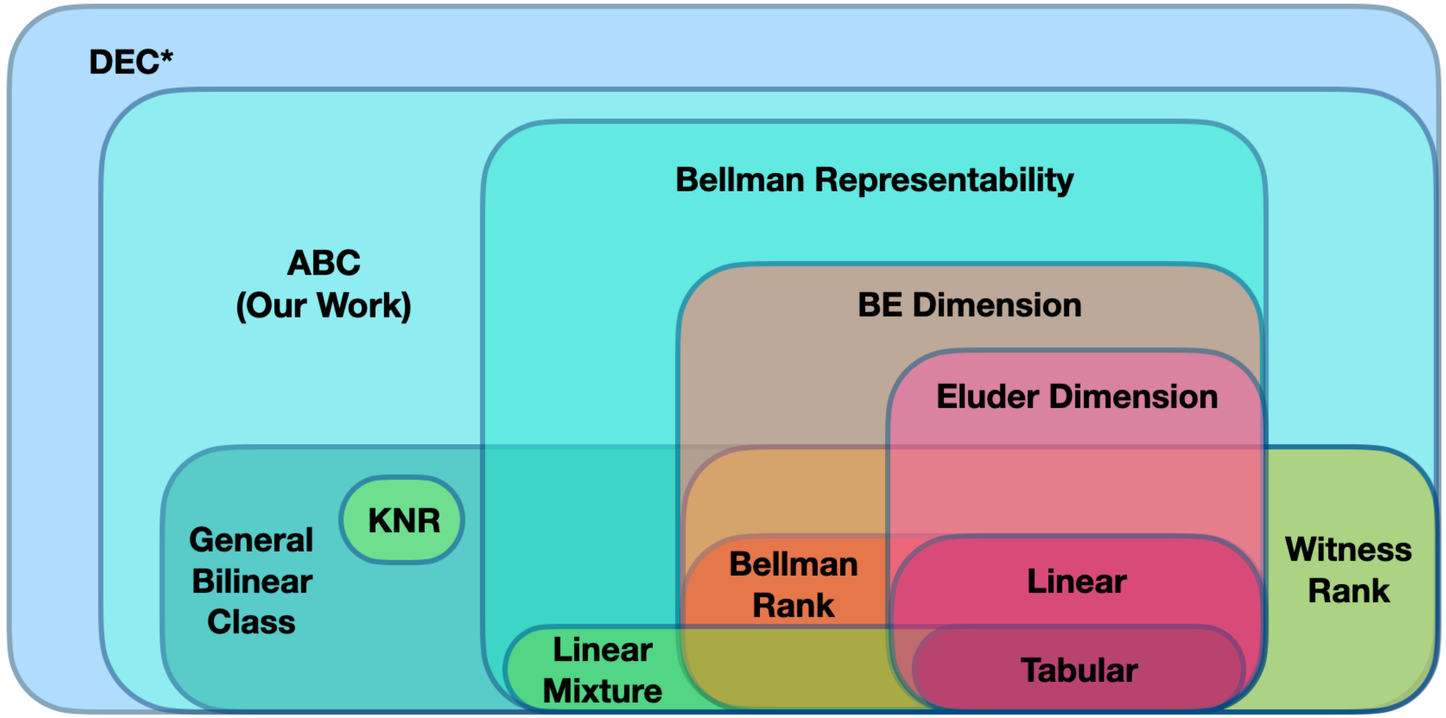

Venn-Diagram Visualization of Prevailing Sample-Efficient RL Classes

Comparison of Sample Complexity

Comparison of sample complexity for different MDP models under different RL frameworks

Comparison of Sample Complexity

	Bilinear Class	Low BE Dimensio	DEC and Bellman Representability	ABC Class (with Low FE
Linear MDPs [Yang & Wang ’19; Jin et				
Linear Mixture MDPs				
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Linear Mixture MDPs	d^3H^4/ε^2	✘	d^3H^3/ε^2	d^2H^2/ε^2
Bellman Rank [Jiang et al. '17]	$d^2H^5\|\mathcal{A}\|/\varepsilon^2$	$dH^2\|\mathcal{A}\|/\varepsilon^2$	$d^2H^3\|\mathcal{A}\|/\varepsilon^2$	$dH^2\|\mathcal{A}\|/\varepsilon^2$
Eluder Dimension [Wang et al. '20]	✘	$\dim_E H^2/\varepsilon^2$	$\dim_E^2 H^3/\varepsilon^2$	$\dim_E H^2/\varepsilon^2$
Witness Rank [Sun et al. '19]				$W_\kappa H^2\|\mathcal{A}\|/\varepsilon^2$
Low Occupancy Complexity	d^3H^4/ε^2	d^2H^2/ε^2	d^3H^3/ε^2	d^2H^2/ε^2
Kernelized Nonlinear Regulator [Kakade et al. '20]				
Linear Q*/V* [Du et al. '21]	d^3H^4/ε^2	d^2H^2/ε^2	d^3H^3/ε^2	d^2H^2/ε^2

Comparison of sample complexity for different MDP models under different RL frameworks

Comparison of Sample Complexity

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Linear Mixture MDPs	d^3H^4/ε^2	✘	d^3H^3/ε^2	d^2H^2/ε^2
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01 **Background**
- Preliminaries
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- Model-based and Model-free RL
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02 **Our ABC Framework**
- Scope and sample complexity of our framework
- Admissible bellman characterization
- Functional eluder dimension
- Decomposable estimation function

03 **MDP Instances**
- Linear mixture MDP
- Low witness rank
- Kernelized nonlinear regulator (KNR)

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$\{Q_f\}, V_{h,f}(s) = \max_a Q_{h,f}(s,a)$

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$Q_{h,f} = Q^*_{h,M_f}, V_{h,f} = V^*_{h,M_f}$

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$\ell_{h,f}(o_h, f_{h+1}, g_h, v) : (\mathcal{S} \times \mathcal{A} \times \mathcal{S}) \times \mathcal{F} \times \mathcal{G} \times \mathcal{V} \to \mathbb{R}^{d_s}$

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Realizability assumption:

$f^* \in \mathcal{F}, Q^*_h(s,a) = Q_{h,f^*}(s,a)$

Generalized Completeness:

$\mathcal{T}_h \mathcal{F}_{h+1} \subseteq \mathcal{G}_h$

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$$\max_{v \in \mathscr{V}} \mathbb{E}_{s_h \sim \pi_g, a_h \sim \pi_{op}} \left\| \mathbb{E}_{s_{h+1}} \left[\ell_{h,g}(o_h, f_{h+1}, f_h, v) \mid s_h, a_h \right] \right\|^2 \geq \left(G_{h,f^*}(f, g) \right)^2$$

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$$\kappa \cdot \left| \mathbb{E}_{s_h, a_h \sim \pi_f} \left[Q_{h,f}(s_h, a_h) - r(s_h, a_h) - V_{h+1,f}(s_{h+1}) \right] \right| \leq \left| G_{h,f^*}(f, f) \right|$$

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When taking

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Further, if $G_{h,f^*}(f, g)$ is $\left\langle W_h(f), X_h(g) \right\rangle$, reduces to low Bellman rank

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- The length of the existing longest sequence $f_1, \ldots, f_n \in \mathscr{F}$ satisfying for some $\epsilon' \geq \epsilon$ and any $2 \leq t \leq n$, there exists $g \in \mathscr{F}$ such that $\sqrt{\sum_{i=1}^{t-1} \big(G(g, f_i)\big)^2} \leq \epsilon'$ while $|G(g, f_t)| > \epsilon'$

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- When $G_h(f, g) := \mathbb{E}_{\pi_{h,g}}(Q_{h,f} - \mathcal{T}_h Q_{f,h+1})$, $\dim_{\mathsf{FE}}(\mathcal{F}, G, \epsilon) = \dim_{\mathsf{BE}}(\mathcal{F}, G, \epsilon)$

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- Global discriminator optimality requires a rich enough $\mathcal{V}$, which is generally satisfied by existing MDP models

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Linear Mixture MDP

Alex Ayoub, Zeyu Jia, Csaba Szepesvari, Mengdi Wang, and Lin Yang. Model-based reinforcement learning with value-targeted regression. In International Conference on Machine Learning, pages 463–474. PMLR, 2020

Dongruo Zhou, Quanquan Gu, and Csaba Szepesvari. Nearly minimax optimal reinforcement learn-ing for linear mixture markov decision processes. In Conference on Learning Theory, pages 4532–4576. PMLR, 2021a

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$$\mathbb{P}_h(s' \mid s, a) = \left\langle \phi(s, a, s'), \theta_h^* \right\rangle, \quad r_h(s, a) = \left\langle \psi(s, a), \theta_h^* \right\rangle$$

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- Linear in feature mapping $\phi : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \to \mathbb{R}^d$ and $\psi : \mathcal{S} \times \mathcal{A} \to \mathbb{R}^d$

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KNR

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KNR

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Properties of ABC

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- Dominating average estimation function
- Bellman Dominance

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- Coupling function G can be nonlinear

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KNR: $\sqrt{\mathbb{E}_{s_h,a_h\sim\pi_g} \left\| (U_{h,f} - U_h^*)\phi(s_h,a_h) \right\|^2}$

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$$\max_{v\in\mathscr{V}} \mathbb{E}_{s_h\sim\pi_g,a_h\sim\pi_{op}} \left\| \mathbb{E}_{s_{h+1}} \left[\ell_{h,g}(o_h, f_{h+1}, f_h, v) \mid s_h, a_h \right] \right\|^2 \geq \left(G_{h,f^*}(f,g) \right)^2$$

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Properties of ABC

- Coupling function G can be nonlinear

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- ℓ can be vector valued

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03 **MDP Instances**
- Linear mixture MDP
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04 **Algorithm and Main Results**
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- Main theorem

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Algorithm (OPERA)

Set $\pi^t := \pi_{f^t}$ where f^t is taken as $\arg\max_{f \in \mathcal{F}} Q_{1,f}(s_1, \pi_f(s_1))$, subject to

$$\max_{v \in \mathcal{V}} \left\{ \sum_{i=1}^{t-1} \left\| \ell_{h,f^i}(o_h^i, f_{h+1}, f_h, v) \right\|^2 - \inf_{g_h \in \mathcal{G}_h} \sum_{i=1}^{t-1} \left\| \ell_{h,f^i}(o_h^i, f_{h+1}, g_h, v) \right\|^2 \right\} \leq \beta$$

Proof Sketch (Optimization based exploration)

$$\text{Regret}(T) := \sum_{t=1}^{T} \left[V_1^*(s_1) - V_1^{\pi^t}(s_1) \right]$$

- Take $\text{argmax}_{f \in \mathcal{F}} Q_{1,f}(s_1, \pi_f(s_1))$ in a confidence interval $\mathcal{C}^t(\beta)$.
- Our choice of β satisfies $f^* \in \mathcal{C}^t$.
- $V_1^*(s_1) - V_1^{\pi^t}(s_1) \leq V_{1,f^t}(s_1) - V_1^{\pi^t}(s_1)$
- $= \sum_{h=1}^{H} \mathbb{E}_{s_h, a_h \sim \pi^t} \left[Q_{h,f^t}(s_h, a_h) - r_h - V_{h+1,f^t}(s_{h+1}) \right]$

$$\sum_{t=1}^{T} \sum_{h=1}^{H} \left| \mathbb{E}_{s_h, a_h \sim \pi^t} \left[Q_{h,f^t}(s_h, a_h) - r_h - V_{h+1,f^t}(s_{h+1}) \right] \right|$$

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$$\sum_{t=1}^{T} V_1^*(s_1) - V_1^{\pi^t}(s_1) \leq \frac{1}{\kappa} \sum_{t=1}^{T} \sum_{h=1}^{H} |\, G_{h,f^*}(f^t, f^t) \,|$$

Proof Sketch

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Martingale concentration argument (Freedman's Inequality):

$$\sum_{i=1}^{t-1} (G_{h,f^*}(f^t, f^i))^2 \leq \mathcal{O}(\beta)$$

holds with probability at least $1 - \delta$, where
$\beta = c(\log(TH\mathcal{N}_{\mathcal{L}}(\rho)/\delta) + T\rho)$

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$$\sum_{i=1}^{t} |\, G(f_i, g_i)\,| \leq \mathcal{O}\left(\sqrt{\dim_{\mathsf{FE}}(\mathcal{F}, G, \omega)\beta t} + C \cdot \min\{t, \dim_{\mathsf{FE}}(\mathcal{F}, G, \omega)\} + t\omega\right)$$

Main Theorem

Theorem (Regret of OPERA)

For an MDP $\mathcal{M}$ with hypothesis classes $\mathcal{F}$, $\mathcal{G}$ under our structural assumption. We choose $\beta = \mathcal{O}(\log(THN_{\mathcal{L}}(1/T)/\delta))$ in OPERA. We have with probability at least $1-\delta$ the regret can be bounded as

$$\mathit{Regret}(T) \leq \mathcal{O}\left(\frac{H}{\kappa}\sqrt{T \cdot \dim_{FE}\left(\mathcal{F}, G, \sqrt{1/T}\right) \cdot \beta}\right).$$

- Dependent on both the functional eluder dimension $\dim_{\mathsf{FE}}$ and $N_{\mathscr{L}}(\sqrt{1/T})$
- For cases with linear structure of dimension d, i.e. Linear MDPs, Linear Mixture MDPs, Low Occupancy Complexity, Linear Q*/V*, the regret bound $\approx \mathcal{O}\left(\frac{dH}{\kappa}\sqrt{T}\right)$

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Linear Mixture MDPs (Algorithm)

Set $\pi^t := \pi_{f^t}$ where f^t is taken as $\arg\max_{f \in \mathcal{F}} Q_{1,f}(s_1, \pi_f(s_1))$, subject to

$$\max_{v \in \mathcal{V}} \left\{ \sum_{i=1}^{t-1} \left\| \ell_{h,f^t}(o_h^i, f_{h+1}, f_h, v) \right\|^2 - \inf_{g_h \in \mathcal{G}_h} \sum_{i=1}^{t-1} \left\| \ell_{h,f^t}(o_h^i, f_{h+1}, g_h, v) \right\|^2 \right\} \leq \beta$$

Set $\pi^t := \pi_{f^t}$ where f^t is taken as $\arg\max_{f \in \mathcal{F}} Q_{1,f}(s_1, \pi_f(s_1))$, subject to

$$\hat{\theta}_{h,t} = \left(\Phi_h^{t-1} \left(\Phi_h^{t-1} \right)^\top \right)^{-1} \Phi_h^{t-1} \left(y_h^{t-1} \right)^\top, \qquad \left\| \theta_{h,t} - \hat{\theta}_{h,t} \right\|_{\Sigma_h^{t-1}}^2 \leq \beta \quad \forall h \in [H]$$

Where $\hat{\theta}_{h,t}$ is the solution to the value-target regression as in UCRL-VTR [Ayoub et al. '20]

- Global optimization version of UCRL-VTR on linear mixture MDPs
- Exhibits a $dH\sqrt{T}$ regret bound and d^2H^2/ϵ^2 sample complexity result
- Compared with the d^3H^4/ϵ^2 sample complexity (the best-known results on general frameworks that subsumes linear mixture MDPs)

KNR

Set $\pi^t := \pi_{f^t}$ where f^t is taken as $\arg\max_{f \in \mathcal{F}} Q_{1,f}(s_1, \pi_f(s_1))$, subject to

$$\max_{v \in \mathcal{V}} \left\{ \sum_{i=1}^{t-1} \left\| \ell_{h,f^t}(o_h^i, f_{h+1}, f_h, v) \right\|^2 - \inf_{g_h \in \mathcal{G}_h} \sum_{i=1}^{t-1} \left\| \ell_{h,f^t}(o_h^i, f_{h+1}, g_h, v) \right\|^2 \right\} \le \beta$$

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$$\sum_{i=1}^{t-1} \left\| U_{h,f}\phi(s_h^i, a_h^i) - s_{h+1}^i \right\|^2 - \inf_{g_h \in \mathcal{G}_h} \sum_{i=1}^{t-1} \left\| U_{h,g}\phi(s_h^i, a_h^i) - s_{h+1}^i \right\|^2 \le \beta$$

- The confidence set is equivalent to $\left\| (U_{h,f} - \hat{U}_{h,f})(\Sigma_h^{t-1})^{1/2} \right\|^2 \le \beta$
- $\Sigma_h^{t-1} := \Phi_h^{t-1}(\Phi_h^{t-1})^\top$ and $\hat{U}_{h,f}$ is the optimal solution to the least square problem$\arg\min_U \sum_{i=1}^{t-1} \|U\phi(s_h^i, a_h^i) - s_{h+1}^i\|^2$
- Inhomogeneous version of LC^3 [Kakade et al. '20] for KNR.
- Yields a regret bound of$\tilde{\mathcal{O}}\left(\sqrt{d_\phi^2 d_s H^4 T}\right)$. Matching the SOTA regret bound.

Low Witness Rank

Set $\pi^t := \pi_{f^t}$ where f^t is taken as $\arg\max_{f \in \mathcal{F}} Q_{1,f}(s_1, \pi_f(s_1))$, subject to

$$\max_{v \in \mathcal{V}} \left\{ \sum_{i=1}^{t-1} \left(\mathbb{E}_{\tilde{s} \sim f_h} v(s_h^i, a_h^i, \tilde{s}) - v(s_h^i, a_h^i, s_{h+1}^i) \right)^2 - \inf_{g_h \in \mathcal{G}_h} \sum_{i=1}^{t-1} \left(\mathbb{E}_{\tilde{s} \sim g_h} v(s_h^i, a_h^i, \tilde{s}) - v(s_h^i, a_h^i, s_{h+1}^i) \right)^2 \right\} \leq \beta$$

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- Outputs an ε-optimal policy π_{out} within $T = \tilde{\mathcal{O}}\left(H^2 |\mathcal{A}| W_\kappa \beta / (\kappa^2 \epsilon^2)\right)$ trajectories.
- Compared with previous best-known sample complexity result of $\widetilde{O}\left(H^3 W_\kappa^2 |\mathcal{A}| \log(T|\mathcal{M}||\mathcal{V}|/\delta)/(\kappa^2\epsilon^2)\right)$, our sample complexity is superior by a factor of dH up to a polylogarithmic prefactor in model parameters

Takeaway

- Unified framework that subsumes nearly all solvable MDP models, including model-based & model-free.
- A new type of estimation function for optimization-based exploration that can be vector.
- Propose Functional eluder dimension, with a sample-efficient algorithm named OPERA

A GENERAL FRAMEWORK FOR SAMPLE-EFFICIENT FUNCTION APPROXIMATION IN REINFORCEMENT LEARNING

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Thank you for Listening!