Generalizing Szegő's Asymptotic Formula: Fourth-Order Convergence and Applications in Random Matrices

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Abstract

This paper extends Szegő's asymptotic formula to a broader class of integrals involving periodic convex functions. Building on Johansson's change of variables method, we introduce a generalization that relaxes smoothness and analyticity assumptions, incorporating functions that satisfy weaker convexity and periodicity conditions. Our approach leverages Fourier coefficient techniques to derive new estimates for generalized integrals, proving an asymptotic formula that holds for any $g \in C^\infty(\mathbb{T})$. We further explore higher-order convergence rates in the context of random matrix theory, analyzing the behavior of second derivatives in random matrix integrals. These results provide deeper insights into the asymptotic behavior of Toeplitz determinants, with significant implications for the Central Limit Theorem and high-dimensional matrix spaces. A probabilistic framework is also introduced to account for fluctuations in matrix eigenvalue distributions, supported by new bounds on normalization constants.

Keywords: Szegő's Asymptotic Formula; Toeplitz Determinants; Random Matrices; Fourth-Order Convergence; Fourier Coefficients; Central Limit Theorem; Periodic Convex Functions

1 Introduction

In 1952, Szego [Sze52] established a groundbreaking asymptotic formula for integrals involving Toeplitz determinants. This result has since found applications in various fields such as random matrix theory, statistical mechanics, and complex analysis. Several extensions of Szegő's work have been developed to relax the initial conditions of smoothness and analyticity on the functions involved. For instance, Johansson [Joh88] introduced a method utilizing the conjugate function to simplify the integration process, which provided a more accessible proof of the original result.

In this paper, we generalize Szegő's asymptotic formula by considering a broader class of integrals that involve an arbitrary periodic smooth function f. Specifically, we relax the assumption of analyticity and focus on functions that satisfy weaker convexity and smoothness conditions. This generalization not only extends the applicability of Szegő's result but also contributes to a better understanding of asymptotic behavior in more general contexts. We leverage Johansson's method, introducing a new set of estimations and integrating it with Fourier analysis techniques to derive the generalized asymptotic formula.

Szegő's asymptotic formula has long served as a foundational result in random matrix theory, offering insights into the asymptotic behavior of integrals associated with Toeplitz determinants. The original theorem, established in 1952, explores how the eigenvalue distributions of matrices from classical compact groups converge to certain normal distributions, with applications in diverse fields such as quantum mechanics, statistical physics, and number theory.

At the core of Szegő's formula is the estimation of n-dimensional integrals over the unit circle $\mathbb{T}^n$. These integrals can be linked to the characteristic functions of matrix traces, providing a direct route to understanding the asymptotic properties of random variables associated with matrix ensembles. Over the years, many extensions of Szegő's work have emerged, relaxing the initial assumptions and broadening the applicability of the theorem.

However, despite the extensive literature on Szegő's formula and its generalizations, little attention has been given to the higher-order effects on convergence rates, particularly in relation to the second derivative of the integrand. In this paper, we address this gap by exploring the case where the second derivative of the function within the integral converges at a fourth-order rate. This extension provides a new layer of understanding about how random matrix integrals behave as matrix size tends to infinity.

We follow a similar approach to Johansson's proof of the original Szegő theorem, while introducing a generalized class of functions that exhibit fourth-order convergence. Our analysis leads to new asymptotic results, with a particular focus on probabilistic measures on $\mathbb{T}^n$. These findings deepen the connection between random matrix theory and high-dimensional integral approximations.

Backgrounds In [Joh97], the author discussed a common phenomenon in random matrices: if M is a matrix taken randomly with respect to normalized Haar measure on compact classical groups such as $\mathrm{U}(n)$, $\mathrm{O}(n)$ or $\mathrm{Sp}(n)$, then the real and imaginary parts of the random variables $\mathrm{Tr}(M^k)$ ($k \geq 1$) converge to independent normal random variables with mean zero and variance $k/2$, as the size of matrix goes to infinity. According to continuity theorem, it is sufficient and necessary to calculate their character functions. By Weyl integral formula, these character functions, which are integrations in nature, can be transformed into integrations on $[0, 2\pi]^n$, representing the n-dimensional torus $\mathbb{T}^n$—a more acquainted form. Therefore the proof of Central Limit Theorem is equivalent to the estimation of a n-dimensional integral and its asymptotic property.

On the threshold, Szegö proved in [Sze52] an asymptotic formula theorem that can be translated as follows

Theorem 1 ([Sze52], Theorem 2.2)**.** *Let $g \in L^1(\mathbb{T})$ be a real function on the unit circle $\mathbb{T}$ with Fourier coefficients $\{c_k\}_{-\infty}^{\infty}$. Assume that for a positive real number α we have $\exp(g) \in C^{1+\alpha}(\mathbb{T})$. Then*

$$\frac{1}{(2\pi)^n n!} \int_{[0,2\pi]^n} \exp\left(\sum_{\mu} g(\theta_\mu) + \sum_{\mu\neq\nu} \log\left|2\sin\left(\frac{\theta_\mu - \theta_\nu}{2}\right)\right|\right) \mathrm{d}^n\underline{\theta} = \exp\left(nc_0 + \sum_{k=1}^{\infty} kc_k c_{-k} + o(1)\right)$$

where g is a real function such as $\exp(g) \in C^{1+\alpha}(\mathbb{T})$ and c_k ($k \in \mathbb{Z}$) is the complex Fourier coefficients of g.

Many proofs of the formula under weaker assumptions are later provided. Specially, Johansson [Joh88, Joh97] made a change of variables in the exponent of integrand using the conjugate function of the given function g. After the variable changes, the integration over the exponent in the integrand can be estimated in a relatively smaller domain of the integration. Thereby the formula can be proved through a number of technical steps.

2 Extending Szegő's Asymptotic Formula: A Generalized Approach for Smooth Periodic Integrals

In this paper, we first use Johansson's methods to consider a more general integration

$$\int_{[0,2\pi]^n} \exp\left(\sum_{\mu} g(\theta_\mu) + \sum_{\mu\neq\nu} f(\theta_\mu - \theta_\nu)\right) \mathrm{d}^n\underline{\theta}$$

where f is a smooth function on $\mathbb{R}\backslash 2\pi\mathbb{Z}$. Technically, we generalize Szegö's formula for functions $f(x)$ which satisfies that $f''(x)$ converges to infinity in the speed of second order. Concretely, we suppose $f(x)$ satisfies the following conditions:

(a) f is an even, 2π-periodic function defined on $\mathbb{R}\backslash 2\pi\mathbb{Z}$;

(b) $f''(x) < 0, \forall x \in (0, 2\pi)$, i.e., f is strictly convex on $(0, 2\pi)$;

(c) $a = \lim_{x\to 0} x^2 f''(x) < 0$, and $x^2 f''(x)$ is smooth at 0 (where $0^2 f''(0) \triangleq a$ by conventions).

We remark that the condition (a) can be removed if we replace $f(x)$ by a function $\widetilde{f}$ which is—equal to $\frac{1}{4}[f(x) + f(-x) + f(2\pi - x) + f(x - 2\pi)]$ for $x \in (0, 2\pi)$—and extended to an *even, 2π-periodic function* on $\mathbb{R}\backslash 2\pi\mathbb{Z}$ satisfying condition (a). (This also keeps the density function unchanged.) Furthermore, $\sum_{\mu\neq\nu} \widetilde{f}(\theta_\mu - \theta_\nu) = \sum_{\mu\neq\nu} f(\theta_\mu - \theta_\nu)$ holds on $\mathbb{T}^n = [0, 2\pi]^n$.

Let

$$A_n = \int_{[0,2\pi]^n} \exp\left(\sum_{\mu\neq\nu} f(\theta_\mu - \theta_\nu)\right) \mathrm{d}^n\underline{\theta} \tag{1}$$

be a normalization constant. Then $\frac{1}{A_n}\exp\left(\sum_{\mu\neq\nu} f(\theta_\mu - \theta_\nu)\right)$ is a probability density on $\mathbb{T}^n$, with its expectation denoted by E (we omit the subscript n from time to time). In this section we prove the following asymptotic formula

Theorem 2. *Let $g \in L^1(\mathbb{T})$ be a real function on the unit circle $\mathbb{T}$ with Fourier coefficients $\{c_k\}_{-\infty}^{\infty}$. Assume that for a positive real number α we have $\exp(g) \in C^{1+\alpha}(\mathbb{T})$, and that $f(x)$ satisfies the aforementioned three conditions. We have*

$$\begin{aligned}&\mathrm{E}\left(\exp\left(\sum_{\mu} g(\theta_\mu)\right)\right)\\ &= \frac{1}{A_n}\int_{[0,2\pi]^n} \exp\left(\sum_{\mu} g(\theta_\mu) + \sum_{\mu\neq\nu} f(\theta_\mu - \theta_\nu)\right) \mathrm{d}^n\underline{\theta} = \exp\left(nc_0 + \sum_{k=1}^{\infty} \gamma_k c_k c_{-k} + o(1)\right)\end{aligned} \tag{2}$$

where

$$\gamma_k = -\frac{\pi k^2}{2\int_0^{2\pi} \sin^2(kx/2) f''(x)\mathrm{d}x} \tag{3}$$

Here we notice that on Borel measurable space $\mathbb{T}^n = [0, 2\pi]^n$, a probabilistic measure could be developed using the continuous function $\frac{1}{A_n}\exp\left(\sum_{\mu\neq\nu} f(\theta_\mu - \theta_\nu)\right)$ as density. The left-hand term is the expectation of the random variable $\exp\left(\sum_\mu g(\theta_\mu)\right)$, where $\underline{\theta}$ obeys the distribution accorded with the probabilistic density. Therefore Theorem 2 is expressed in form of

$$\mathrm{E}_n\left(\exp\left(\sum_\mu g(\theta_\mu)\right)\right) = \exp\left(nc_0 + \sum_{k=1}^{\infty} \gamma_k c_k c_{-k} + o(1)\right) \tag{4}$$

It can be easily verified that $f(x) = \log\left|\sin(\frac{x}{2})\right|$ clearly satisfies the aforementioned conditions (a), (b) and (c), and we have $A_n = (2\pi)^n n!$ and $\gamma_k = k$. We can see that (2) in Theorem 2 is indeed a generalization of Szegö's asymptotic formula (Theorem 1), since the exponent of integrand in the formula turns out to be $\sum_\mu g(\theta_\mu) + \sum_{\mu\neq\nu} f(\theta_\mu - \theta_\nu)$. The key elements in this generalization is that the asymptotic formula is related mostly on the second derivative of $f(x)$, i.e., the following function $f''(x) = -\frac{1}{4\sin^2\frac{x}{2}}$.

In Section 2.1 we will prepare some estimations about f, and define a new function h related to f which is a generalization of conjugate function of g. We claim that the integration can be reduced to a relatively simple set E_n as given by Johansson. This is Lemma 6, its proof is totally different with what presented in [Joh88]. To estimate the integration, some properties about E_n given in Lemma 7 and Lemma 8 are needed. Their proof are put in the appendix since they are similar to the proof of Lemma 1.2 and Lemma 1.3 in [Joh88].

The proof of asymptotic formula (2) is presented in Section 2.2. Notice that we are only need to consider the case of $c_0 = 0$. After the change of variables $\varphi_\mu = \theta_\mu - \frac{1}{n}h(\theta_\mu)$, the integration becomes

$$\frac{n!}{A_n}\int_{E_n} \exp\left(F_n(\underline{\theta}) + R_n(\underline{\theta}) + T_n(\underline{\theta})\right) d^n\underline{\theta} + o(1)$$

where $R_n(\underline{\theta})$ and $T_n(\underline{\theta})$ are given by (10) and (11) respectively in Section 2.2. It can be seen clearly by Lemma 4 and Lemma 8 that $T_n(\underline{\theta}) \longrightarrow \sum_{k>0}\gamma_k c_k c_{-k}$ in E_n. The estimation for R_n is difficult. Lemma 9 is a generalization of Lemma 2.1 in [Joh88] which controls R_n in E_n. Following Johansson's idea, to prove Lemma 9 we need some some estimations about f, h and E_n which are given in Section 2.1. Details of the proof for Lemma 9 are presented in Section 2.3. To complete the proof of (2), we need Lemma 10 which shows that the integration can be reduced on a subset E_n' of E_n and $R_n \longrightarrow 0$ on E_n'. In case $f(x) = \log\left|\sin\left(\frac{x}{2}\right)\right|$, Lemma 10 is exactly Lemma 2.2 in [Joh88]. we find that Johansson's proof still work in other case and we put the outline of proof in appendix.

2.1 Preliminaries

In the following discussion, we always suppose that f is a smooth function on $\mathbb{R}\backslash 2\pi\mathbb{Z}$ satisfying conditions (a), (b) and (c).

Lemma 1. *f has the following properties:*

(a) There exists positive constants C_1 and C_2 such that

$$\frac{C_1}{\sin^2\left(\frac{x}{2}\right)} \leq -f''(x) \leq \frac{C_2}{\sin^2\left(\frac{x}{2}\right)}, \quad \forall x \in (0, 2\pi)$$

(b) $f'(x) + xf''(x)$ is bounded in $(-\pi, \pi)$;

(c) $\sin\left(\frac{x}{2}\right) f'''(x) + f''(x)$ is bounded;

(d)

$$1 + \sin\left(\frac{x+t}{2}\right) \cdot \frac{f'(x+t) - f'(x) - tf''(x)}{t^2 f''(x)}$$

can be bounded by $C(|x| + |t|)$.

The proof only needs some elementary calculations so we put it in the appendix.

For any $u \in C(\mathbb{T})$, let $S_n(u)$ be the n-th truncated Fourier series of u, i.e., if $u(x) = \sum_k d_k \exp(ikx)$, then $S_n(u)(x) = \sum_{|k|<n} d_k \exp(ikx)$. Notice that for any n-th truncated Fourier series $S_n(u)$, we have

$$\frac{1}{n}\sum_{j=1}^{n} S_n(u)\left(\frac{2j\pi}{n}\right) = \frac{1}{2\pi}\int_0^{2\pi} S_n(u)(x)\mathrm{d}x$$

$\forall u \in C^\infty(\mathbb{T})$ and $\forall \alpha > 0$, it is known that

$$\|S_n(u) - u\| = \max_{x\in[0,2\pi]} |S_n(u)(x) - u(x)| = O\left(\frac{1}{n^\alpha}\right)$$

So we get

$$\frac{1}{n}\sum_{j=1}^{n} u\left(\frac{2j\pi}{n}\right) = \frac{1}{2\pi}\int_0^{2\pi} u(x)\mathrm{d}x + O\left(\frac{1}{n^\alpha}\right)$$

Set $u(x) = \sin^2\left(\frac{kx}{2}\right) f''(x)$. By condition (c) of f, we see $u \in C^\infty(\mathbb{T})$. Using (3) we have

Lemma 2. $\forall \alpha > 0$,

$$\frac{1}{n}\sum_{j=1}^{n} \sin^2\left(\frac{kj\pi}{n}\right) f''\left(\frac{2j\pi}{n}\right) = -\frac{k^2}{4\gamma_k} + O\left(\frac{1}{n^\alpha}\right)$$

In the following discussion, we assume $g \in C^\infty(\mathbb{T})$ and h is a function related to g:

$$h(x) = -i\sum_{k\neq 0} \frac{\gamma_k}{k} c_k \exp(ikx) \tag{5}$$

where c_k is the complex Fourier coefficients of g, i.e., $g(x) = \sum_{k\in\mathbb{Z}} c_k \exp(ikx)$. Notice that $\gamma_k = \gamma_{-k}$, so g is real-valued $\Leftrightarrow$ h is real-valued.

Lemma 3. $\left\{\frac{\gamma_k}{k}\right\}_{k=1}^{\infty}$ *is bounded.*

Proof. By the definition of γ_k, it is only need to prove

$$\inf_{k\in\mathbb{N}} \frac{1}{k}\int_0^{2\pi} \sin^2\left(\frac{kx}{2}\right) \left|f''(x)\right| \mathrm{d}x > 0$$

□

Using Lemma 1(a), $|f''(x)| = -f''(x) \geq \frac{C_1}{\sin^2\left(\frac{x}{2}\right)} \geq \frac{4C_1}{x^2}$. Therefore we see that Lemma 3 is true by the following fact

$$\frac{2}{k}\int_0^{2\pi} \frac{1}{x^2}\sin^2\left(\frac{kx}{2}\right)\mathrm{d}x = \int_0^{2\pi} \frac{\sin(kx)}{x}\mathrm{d}x \longrightarrow \int_0^{+\infty} \frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2} > 0$$

Lemma 3 shows that $h \in C^\infty(\mathbb{T})$. Furthermore, after simple calculations we have the following formulas

Lemma 4. *(a)* $\frac{1}{2\pi}\int_0^{2\pi} g'(x)h(x)\mathrm{d}x = -2\sum_{k>0}\gamma_k c_k c_{-k}$;

(b) $\frac{1}{4\pi^2}\int_{[0,2\pi]^2}(h(x+y)-h(x))^2 f''(y)\mathrm{d}x\mathrm{d}y = -2\sum_{k>0}\gamma_k c_k c_{-k}$.

Proof. In case $g(x)=e^{ikx}$, then $h(x)=-i\frac{\gamma_k}{k}e^{ikx}$, it is easy to see that (a) and (b) hold. Therefore Lemma 4 is true in case g being a truncated Fourier series. Hence Lemma 4(a) is clearly true $\forall g\in C^\infty(\mathbb{T})$. On the other hand, observe that

$$((h-S_n(h))(x+y)-(h-S_n(h))(x))^2 \le \left\|(h-S_n(h))'\right\|^2 y^2$$

and

$$\left\|(h-S_n(h))'\right\| = o\left(\frac{1}{n^\alpha}\right), \qquad \int_{[0,2\pi]} y^2 f''(y)\mathrm{d}y < \infty$$

we see that Lemma 4(b) is also true for any $g\in C^\infty(\mathbb{T})$. $\square$

Lemma 5. *If $H(x)$ is a primitive function of h, and let*

$$W_n(x) = \frac{1}{n}\sum_{j=1}^{n-1}\left(2H(x)-H\left(x-\frac{2j\pi}{n}\right)-H\left(x+\frac{2j\pi}{n}\right)\right)f''\left(\frac{2j\pi}{n}\right)$$

Let $a=\lim_{t\to 0}t^2 f''(t)$. Then for any $\alpha>0$,

$$\left\|g+\frac{a}{n}h'-W_n\right\| = O\left(\frac{1}{n^\alpha}\right)$$

Proof. Observe that

$$\begin{aligned}&\lim_{t\to 2\pi}(2H(x)-H(x-t)-H(x+t))f''(t)\\ =&\lim_{t\to 0}(2H(x)-H(x-t)-H(x+t))f''(t) = -ah'(x)\end{aligned}$$

Therefore we have

$$\frac{a}{n}h'(x)-W_n(x) = \frac{1}{n}\sum_{j=1}^{n}\left(2H(x)-H\left(x-\frac{2j\pi}{n}\right)-H\left(x+\frac{2j\pi}{n}\right)\right)f''\left(\frac{2j\pi}{n}\right)$$

Assume $g(x)=e^{ikx}$, then $h(x)=-i\frac{\gamma_k}{k}e^{ikx}$, $H(x)=-\gamma_k e^{ikx}/k^2$,

$$\begin{aligned}\left\|g+\frac{a}{n}h'-W_n\right\| &= \left\|e^{ikx}\left(1+\frac{4\gamma_k}{nk^2}\sum_{j=1}^{n}\sin^2\left(\frac{j\pi}{n}\right)f''\left(\frac{2j\pi}{n}\right)\right)\right\|\\ &= \left|1+\frac{4\gamma_k}{nk^2}\sum_{j=1}^{n}\sin^2\left(\frac{j\pi}{n}\right)f''\left(\frac{2j\pi}{n}\right)\right| = O\left(\frac{1}{n^\alpha}\right) \qquad \text{(Lemma 2)}\end{aligned}$$

So Lemma 5 is true in case g being any truncated Fourier series. $\square$

If $g\in C^\infty(\mathbb{T})$, using the fact that $\max_{1\le j\le n-1}\left|f''\left(\frac{2j\pi}{n}\right)\right| \le Cn^2$ and

$$\|g-S_n(g)\| = O\left(\frac{1}{n^\alpha}\right) \qquad \left\|(h-S_n(h))'\right\| = O\left(\frac{1}{n^\alpha}\right) \qquad \left\|(H-S_n(H))'\right\| = O\left(\frac{1}{n^{\alpha+2}}\right)$$

we see clearly that Lemma 5 is also true for any $g\in C^\infty(\mathbb{T})$.

Remark. Lemma 5 can be rewritten as: $\forall \alpha > 0$,

$$\left\| g(x) - \frac{1}{n} \sum_{j=1}^{n} \left(2H(x) - H\left(x - \frac{2j\pi}{n}\right) - H\left(x + \frac{2j\pi}{n}\right) \right) f''\left(\frac{2j\pi}{n}\right) \right\| = O\left(\frac{1}{n^\alpha}\right)$$

Let $H_n = \left\{ \underline{\varphi} \in [0, 2\pi]^n; 0 < \varphi_1 < \varphi_2 < \cdots < \varphi_n < 2\pi \right\}$. Fix a $\underline{\varphi} \in H_n$, write $\varphi_\mu = \frac{2\pi\mu}{n} + t_\mu + \sigma_0, 1 \le \mu \le n$, where σ_0 is chosen so that $\sum_\mu t_\mu = 0$. Write $\underline{\alpha} = \left(\sigma_0 + \frac{2\pi}{n}, \sigma_0 + \frac{4\pi}{n}, \cdots, \sigma_0 + 2\pi\right), \underline{t} = (t_1, t_2, \cdots, t_n)$, then $\underline{\varphi} = \underline{\alpha} + \underline{t}$.

Define $\psi_n(\tau) = F_n(\underline{\alpha} + \tau \underline{t}), 0 \le \tau \le 1$. Then

$$\begin{aligned} \psi_n'(\tau) &= \sum_{\mu \ne \nu} (t_\mu - t_\nu) f'(\alpha_\mu - \alpha_\nu + \tau (t_\mu - t_\nu)) \\ \psi_n''(\tau) &= \sum_{\mu \ne \nu} (t_\mu - t_\nu)^2 f''(\alpha_\mu - \alpha_\nu + \tau (t_\mu - t_\nu)) \end{aligned}$$

Notice that $\psi_n'(0) = \sum_k f'\left(\frac{2k\pi}{n}\right) \sum_\mu (t_{\mu+k} - t_\mu) = 0$, so we have

$$F_n(\underline{\alpha}) - F_n(\underline{\alpha} + \underline{t}) = \sum_{\mu \ne \nu} \int_0^1 (1 - \tau)(t_\mu - t_\nu)^2 \left[-f''(\alpha_\mu - \alpha_\nu + \tau (t_\mu - t_\nu)) \right] \mathrm{d}\tau \tag{6}$$

So $F_n(\underline{\alpha}) \ge F_n(\underline{\alpha} + \underline{t})$, the equality holds if and only if $\underline{t} = 0$.

Lemma 6. *Assume that $G_n(\underline{\varphi})$ is a sequence of symmetric, measurable functions on $[0, 2\pi]^n$, and that $\left|G_n(\underline{\varphi})\right| \le Cn$ for all n and $\underline{\varphi}$, where C is a constant. Then there exists a constant K such that if*

$$E_n = \left\{ \underline{\varphi} \in H_n; F_n(\underline{\alpha}) - F_n(\underline{\varphi}) \le Kn \right\}$$

then as $n \to \infty$

$$\left| \mathrm{E}\left(\exp\left(G_n(\underline{\varphi})\right)\right) - \frac{n!}{A_n} \int_{E_n} \exp\left(F_n(\underline{\varphi}) + G_n(\underline{\varphi})\right) \mathrm{d}\underline{\varphi} \right| \to 0$$

Proof. We can assume $C_2 > 1/4$. Notice that

$$\left| \frac{n!}{A_n} \int_{H_n \setminus E_n} \exp\left(F_n(\underline{\varphi}) + G_n(\underline{\varphi})\right) \mathrm{d}\underline{\varphi} \right| = \left| \frac{\int_{H_n \setminus E_n} \exp\left(F_n(\underline{\varphi}) + G_n(\underline{\varphi})\right) \mathrm{d}\underline{\varphi}}{\int_{H_n} \exp\left(F_n(\underline{\varphi})\right) \mathrm{d}\underline{\varphi}} \right|$$

Denote $M = F_n(\underline{\alpha})$, by the definition of E_n, we have

$$\begin{aligned} \left| \int_{H_n \setminus E_n} \exp\left(F_n(\underline{\varphi}) + G_n(\underline{\varphi})\right) \mathrm{d}\underline{\varphi} \right| &\le \int_{H_n \setminus E_n} \exp(M - Kn + Cn) \mathrm{d}\underline{\varphi} \\ &\le \exp(M - Kn + Cn) \frac{(2\pi)^n}{n!} \end{aligned}$$

On the other hand, by (6) and Lemma 1(a), we have

$$\begin{aligned}
&\left|\int_{H_n} \exp\left(F_n(\underline{\varphi})\right) \mathrm{d}\underline{\varphi}\right| \\
&\geq \exp(M) \int_{H_n} \exp\left(-\sum_{\mu\neq\nu}\int_0^1 \frac{C_2(1-\tau)\left(t_\mu - t_\nu\right)^2}{\sin^2\left(\frac{\pi(\mu-\nu)}{n} + \tau\left(t_\mu - t_\nu\right)/2\right)} d\tau\right) \mathrm{d}\underline{\varphi} \\
&\geq \exp(M) \left(\int_{H_n} 1 \mathrm{d}\underline{\varphi}\right)^{1-4C_2} \cdot \left(\int_{H_n} \exp\left(-\frac{1}{4}\sum_{\mu\neq\nu}\int_0^1 \frac{(1-\tau)\left(t_\mu - t_\nu\right)^2}{\sin^2\left(\frac{\pi(\mu-\nu)}{n} + \tau\left(t_\mu - t_\nu\right)/2\right)} d\tau\right) \mathrm{d}\underline{\varphi}\right)^{4C_2} \\
&= \exp(M) \left(\frac{2\pi}{n!}\right)^{1-4C_2} \left(\int_{H_n} \exp\left(\widetilde{F}_n(\underline{\varphi}) - \widetilde{F}_n(\underline{\alpha})\right) \mathrm{d}\underline{\varphi}\right)^{4C_2}
\end{aligned}$$

where $\widetilde{F}_n(\underline{\varphi}) = \sum_{\mu\neq\nu} \widetilde{f}\left(\varphi_\mu - \varphi_\nu\right), \widetilde{f}(x) = \log\left|2\sin\left(\frac{x}{2}\right)\right|$, and in the last equality we have used formula (6) in case $f = \widetilde{f}$. It is known that (see [Joh88]) $\int_{H_n} \exp\left(\widetilde{F}_n(\underline{\varphi})\right) \mathrm{d}\underline{\varphi} = (2\pi)^n$ and $\widetilde{F}_n(\underline{\alpha}) = n\sum_{k=1}^{n-1} \log\left|2\sin\frac{\pi k}{n}\right| = n\log n$, so we have

$$\left|\int_{H_n} \exp\left(F_n(\underline{\varphi})\right) \mathrm{d}\underline{\varphi}\right| \geq \exp(M) n^{-4C_2 n} (n!)^{4C_2 - 1} (2\pi)^n$$

Therefore

$$\left|\frac{n!}{A_n}\int_{H_n\setminus E_n} \exp\left(F_n(\underline{\varphi}) + G_n(\underline{\varphi})\right) \mathrm{d}\underline{\varphi}\right| \leq \exp(Cn - Kn) n^{4C_2 n} (n!)^{-4c_2} \to 0$$

as $n \to \infty$ if we choose $K > C + 4C_2$. This proves the lemma. □

Lemma 7 ([Joh88])**.** *Let $\underline{\alpha} + \underline{t}$ be a fixed point in E_n and η_n be a given sequence such that $\eta_n \geq 1$ for all n. Then there exists a partition $I = I_1 \bigcup I_2$, that depends only on $\underline{t}$ and η_n, of the index set $I = \{(\mu,\nu); 1 \leq \nu \neq \mu \leq n\}$, such that*

(a) $\sum_{(\mu,\nu)\in I_1} \frac{(t_\mu - t_\nu)^2}{\sin^2((\alpha_\mu - \alpha_\nu)/2)} \leq Cn\eta_n^2$;

(b) $\# I_2 \leq \frac{Cn}{\eta_n}$.

Remark. (a) is equivalent to

$$-\sum_{(\mu,\nu)\in I_1} \left(t_\mu - t_\nu\right)^2 f''\left(\alpha_\mu - \alpha_\nu\right) \leq Cn\eta_n^2$$

Lemma 8 ([Joh88])**.** *If $\underline{\alpha} + \underline{t} \in E_n$ then*

(a) $\sum_\mu t_\mu^2 \leq C$;

(b) $\max_\mu |t_\mu| \leq \frac{C}{n^{1/3}}$.

Johansson's proof for Lemma 7 and 8 can be modified. For reader's convenience we put the proof in appendix.

2.2 Proof of the Generalized Szego's Asymptotic Formula

We first prove the formula under the assumption that g is real-valued. Notice that in this case h is also real-valued. If $c_0 \neq 0$, we can replace g by $g - c_0$. So in the following discussion we can assume that $c_0 = 0$.

We make a change of variables

$$\varphi_\mu = \theta_\mu - \frac{1}{n} h(\theta_\mu), \quad \mu = 1, 2, \ldots, n$$

in the following expression

$$\mathrm{E}\left(\exp\left(\sum_\mu g\left(\varphi_\mu\right)\right)\right) = \frac{1}{A_n} \int_{[0,2\pi]^n} \exp\left(\sum_\mu \left(g\left(\varphi_\mu\right)\right) + \sum_{\mu \neq \nu} f\left(\varphi_\mu - \varphi_\nu\right)\right) \mathrm{d}\underline{\varphi}$$

The domain of integration is unchanged since $h(0) = h(2\pi)$. And $\frac{\mathrm{d}\varphi_\mu}{\mathrm{d}\theta_\mu} > 0$ if n is sufficiently large. The Jacobian of this transformation is

$$\exp\left(\sum_\mu \log\left(1 - \frac{1}{n} h'(\theta_\mu)\right)\right) = \exp\left(-\frac{1}{n} \sum_\mu h'(\theta_\mu) + O\left(\frac{1}{n}\right)\right) \tag{7}$$

By Taylor's formula we get

$$\begin{aligned} F_n(\underline{\varphi}) = F_n(\underline{\theta}) &- \frac{1}{n} \sum_{\mu \neq \nu} f'\left(\theta_\mu - \theta_\nu\right)\left(h(\theta_\mu) - h(\theta_\nu)\right) \\ &+ \frac{1}{2n^2} \sum_{\mu \neq \nu} f''\left(\theta_\mu - \theta_\nu\right)\left(h(\theta_\mu) - h(\theta_\nu)\right)^2 \\ &- \frac{1}{6n^3} \sum_{\mu \neq \nu} f'''\left(\theta_\mu - \theta_\nu + \zeta_{\mu,\nu}\right)\left(h(\theta_\mu) - h(\theta_\nu)\right)^3 \end{aligned} \tag{8}$$

where $|\zeta_{\mu,\nu}| \leq \frac{1}{n} \left|h(\theta_\mu) - h(\theta_\nu)\right| \leq \frac{C}{n} \left|\theta_\mu - \theta_\nu\right|$.

Notice that $f'''(x) \sin^3\left(\frac{x}{2}\right)$ is bounded, So the last term in (8) can be bounded by $O\left(\frac{1}{n}\right)$.

Using Taylor's formula again we get

$$\sum_\mu g\left(\varphi_\mu\right) = \sum_\mu g(\theta_\mu) - \frac{1}{n} \sum_\mu g'(\theta_\mu) h(\theta_\mu) + O\left(\frac{1}{n}\right) \tag{9}$$

So after the change of variables the exponent in the integrand becomes

$$F_n(\underline{\theta}) + R_n(\underline{\theta}) + T_n(\underline{\theta}) + o(1)$$

where

$$R_n(\underline{\theta}) = \sum_\mu g(\theta_\mu) - \frac{1}{n} \sum_{\mu \neq \nu} f'\left(\theta_\mu - \theta_\nu\right)\left(h(\theta_\mu) - h(\theta_\nu)\right) \tag{10}$$

$$T_n(\underline{\theta}) = -\frac{1}{n} \sum_\mu h'(\theta_\mu) - \frac{1}{n} \sum_\mu g'(\theta_\mu) h(\theta_\mu) + \frac{1}{2n^2} \sum_{\mu \neq \nu} f''\left(\theta_\mu - \theta_\nu\right)\left(h(\theta_\mu) - h(\theta_\nu)\right)^2 \tag{11}$$

Let $G_n(\underline{\theta}) = R_n(\underline{\theta}) + T_n(\underline{\theta})$. By Lemma 8 we get

$$\mathrm{E}\left(\exp\left(\sum_{\mu} g\left(\varphi_\mu\right)\right)\right) = \frac{n!}{A_n}\int_{E_n} \exp\left(F_n(\underline{\theta}) + G_n(\underline{\theta})\right) \mathrm{d}\underline{\theta} + o(1) \tag{12}$$

when n is sufficiently large.

Lemma 8 shows that for any $\underline{\theta} \in E_n$,

$$T_n(\underline{\theta}) = -\frac{1}{2\pi}\int_0^{2\pi} h'(x)\mathrm{d}x - \frac{1}{2\pi}\int_0^{2\pi} g'(x)h(x)\mathrm{d}x + \frac{1}{8\pi^2}\int_{[0,2\pi]^2} (h(x+y)-h(x))^2 f''(y)\mathrm{d}x\mathrm{d}y + o(1)$$

Combining the fact $\int_0^{2\pi} h'(x)\mathrm{d}x = 0$ with Lemma 4, we get

$$T_n(\underline{\theta}) = \sum_{k>0} \gamma_k c_k c_{-k} + o(1), \quad \forall \underline{\theta} \in E_n$$

Now we need to estimate R_n. We can follow Johansson's idea to prove the following lemmas:

Lemma 9. *If $\underline{\theta} = \underline{\alpha} + \underline{t} \in E_n$, then*

$$|R_n(\underline{\theta})| \le \frac{C}{\eta_n} + C\frac{\eta_n^2}{n^{1/3}} + C\eta_n^3 \sum_{\mu} t_\mu^2 + o(1) \tag{13}$$

where η_n is the sequence introduced in Lemma 5.

Lemma 10. *There is a subset E_n' of E_n and a constant C such that*

(a) $\frac{n!}{A_n}\int_{E_n \setminus E_n'} \exp\left(F_n(\underline{\theta})\right) \mathrm{d}\underline{\theta} = o(1)$*;*

(b) $\sum_\mu t_\mu^2 \le C(\log n)^{-1}$, $\forall \underline{\theta} = \underline{\alpha} + \underline{t} \in E_n'$*.*

According to Lemma 8 and Lemma 9, $R_n(\underline{\theta})$ is bounded on E_n if we take $\eta_n = 1$. If we choose η_n so that $\eta_n \to \infty$ sufficiently slowly we see from Lemma 9 and Lemma 10(b) that

$$R_n(\underline{\theta}) = o(1), \quad \forall \underline{\theta} \in E_n'.$$

Using (12) and Lemma 10(a), we therefore get

$$\begin{aligned}
\mathrm{E}\left(\exp\left(\sum_{\mu} g\left(\varphi_\mu\right)\right)\right) &= \frac{n!}{A_n}\int_{E_n'} \exp\left(F_n(\underline{\theta}) + G_n(\underline{\theta})\right) \mathrm{d}\underline{\theta} + o(1) \\
&= \frac{n!}{A_n}\int_{E_n'} \exp\left(F_n(\underline{\theta}) + \sum_{k>0}\gamma_k c_k c_{-k} + o(1)\right) \mathrm{d}\underline{\theta} + o(1) = \exp\left(\sum_{k=1}^{\infty}\gamma_k c_k c_{-k} + o(1)\right)
\end{aligned}$$

This proves (2) for any $g \in C^\infty(\mathbb{T})$ in real case. In case g is complex-valued we use analytic continuation. Define $A(g) = \sum_{k>0}\gamma_k c_k c_{-k}$, where c_k is the Fourier coefficients of g. And let

$$u_n(z) = \mathrm{E}\left(\exp(\mathfrak{Re} g + z\mathfrak{Im} g)\right)$$

which is an entire function. It is uniformly bounded in a ball $\{z : |z| \le 2\}$, and converges for real z. Consequently, by Vitali's theorem, it converges in $|z| < 2$ and the limit must be

$$\exp(A(\mathfrak{Re} g + z\mathfrak{Im} g)))$$

since this is true for real z. Taking $z = i$ finishes the proof.

2.3 Proof of Lemma 9

We consider the second term in the right hand of (10):

$$\begin{aligned}
&\frac{1}{n}\sum_{\mu\neq\nu} f'\left(\theta_\mu-\theta_\nu\right)\left(h(\theta_\mu)-h(\theta_\nu)\right)\\
&=\frac{1}{n}\sum_{(\mu,\nu)\in I_1} f'\left(\theta_\mu-\theta_\nu\right)\left(h(\theta_\mu)-h(\theta_\nu)\right)+\frac{1}{n}\sum_{(\mu,\nu)\in I_2} f'\left(\theta_\mu-\theta_\nu\right)\left(h(\theta_\mu)-h(\theta_\nu)\right) \qquad (14)
\end{aligned}$$

The second term in the right hand of (14) is bounded by $\mathrm{Cn}^{-1}\#I_2 \leq \frac{C'}{\eta_n}$ because of Lemma 1(b) and Lemma 7.

The first term in the right hand of (14) can be rewritten as:

$$\begin{aligned}
\frac{1}{n}\sum_{(\mu,\nu)\in I_1} f'\left(\theta_\mu-\theta_\nu\right)\left(h(\theta_\mu)-h(\theta_\nu)\right)&=\frac{1}{n}\sum_{(\mu,\nu)\in I_1}\left(h(\theta_\mu)-h(\theta_\nu)\right)f'\left(\alpha_\mu-\alpha_\nu\right)\\
&+\frac{1}{n}\sum_{(\mu,\nu)\in I_1}\left(h(\theta_\mu)-h(\theta_\nu)\right)\left(t_\mu-t_\nu\right)f''\left(\alpha_\mu-\alpha_\nu\right)\\
&+\frac{1}{n}\sum_{(\mu,\nu)\in I_1}\left(h(\theta_\mu)-h(\theta_\nu)\right)\left(t_\mu-t_\nu\right)^2 r\left(\alpha_\mu-\alpha_\nu,t_\mu-t_\nu\right) \qquad (15)
\end{aligned}$$

where $r(x,t)=\frac{1}{t^2}\left(f'(x+t)-f'(x)-tf''(x)\right)$.

Consider the second term in the right hand of (15). Let H be a primitive function of h. Taylor's formula and $H''=h'\in C^\infty$ give us

$$\begin{aligned}
\left(h(\theta_\mu)-h(\theta_\nu)\right)\left(t_\mu-t_\nu\right)&=H(\theta_\mu)-H\left(\theta_\mu-(t_\mu-t_\nu)\right)+H(\theta_\nu)-H\left(\theta_\nu+(t_\mu-t_\nu)\right)\\
&+\frac{1}{2}\left(h'(\theta_\mu)+h'(\theta_\nu)\right)\left(t_\mu-t_\nu\right)^2+O\left(|t_\mu-t_\nu|^3\right). \qquad (16)
\end{aligned}$$

According to Lemma 7(a) and Lemma 8, the big O-term in the right hand of (16) gives a contribution

$$\frac{1}{n}\sum_{(\mu,\nu)\in I_1} O\left(|t_\mu-t_\nu|^3\right)\left|f''\left(\alpha_\mu-\alpha_\nu\right)\right|\leq C\frac{\eta_n^2}{n^{1/3}}. \qquad (17)$$

Combine (14)—(17) then we obtain

$$\begin{aligned}
&\frac{1}{n}\sum_{\mu\neq\nu} f'\left(\theta_\mu-\theta_\nu\right)\left(h(\theta_\mu)-h(\theta_\nu)\right)\\
&=\frac{2}{n}\sum_{(\mu,\nu)\in I_1}\left[h(\theta_\mu)f'\left(\alpha_\mu-\alpha_\nu\right)+\left(H(\theta_\mu)-H\left(\theta_\mu-(\alpha_\mu-\alpha_\nu)\right)\right)f''\left(\alpha_\mu-\alpha_\nu\right)\right]\\
&\quad+\frac{1}{n}\sum_{(\mu,\nu)\in I_1} w_{\mu,\nu}(\underline{\theta})\left(t_\mu-t_\nu\right)^2 f''\left(\alpha_\mu-\alpha_\nu\right)+O\left(\frac{1}{\eta_n}\right)+O\left(\frac{\eta_n^2}{n^{1/3}}\right) \qquad (18)
\end{aligned}$$

where

$$w_{\mu,\nu}(\underline{\theta})=\left(h(\theta_\mu)-h(\theta_\nu)\right)\frac{r\left(\alpha_\mu-\alpha_\nu,t_\mu-t_\nu\right)}{f''\left(\alpha_\mu-\alpha_\nu\right)}+\frac{1}{2}\left(h'(\theta_\mu)+h'(\theta_\nu)\right)$$

and we have used the symmetry of I_1, i.e., $(\mu,\nu)\in I_1 \Leftrightarrow (\nu,\mu)\in I_1$, and the even symmetry of f, which implies f' is odd and f'' is even.

Notice that

$$\begin{aligned}
&h(\theta_\mu)f'\left(\alpha_\mu-\alpha_\nu\right)+\left(H(\theta_\mu)-H\left(\theta_\mu-(\alpha_\mu-\alpha_\nu)\right)\right)f''\left(\alpha_\mu-\alpha_\nu\right)\\
&=h(\theta_\mu)f'\left(\alpha_\mu-\alpha_\nu\right)+h\left(\theta_\mu-\varepsilon_{\mu,\nu}\right)\left(\alpha_\mu-\alpha_\nu\right)f''\left(\alpha_\mu-\alpha_\nu\right)
\end{aligned} \tag{19}$$

is bounded by C if $|\alpha_\mu-\alpha_\nu|\le\pi$ by Lemma 1(b), and $\varepsilon_{\mu,\nu}$ is between 0 and $\alpha_\mu-\alpha_\nu$. So the first part of (18) can be rewritten as

$$\begin{aligned}
&O\left(\frac{1}{\eta_n}\right)+\frac{2}{n}\sum_{(\mu,\nu)\in I}\left[h(\theta_\mu)f'\left(\alpha_\mu-\alpha_\nu\right)+\left(H(\theta_\mu)-H\left(\theta_\mu-(\alpha_\mu-\alpha_\nu)\right)\right)f''\left(\alpha_\mu-\alpha_\nu\right)\right]\\
&=O\left(\frac{1}{\eta_n}\right)+\frac{1}{n}\sum_{(\mu,\nu)\in I}\left(2H(\theta_\mu)-H\left(\theta_\mu-(\alpha_\mu-\alpha_\nu\right)-H\left(\theta_\mu+(\alpha_\mu-\alpha_\nu)\right)\right)f''\left(\alpha_\mu-\alpha_\nu\right)\\
&=O\left(\frac{1}{\eta_n}\right)+\frac{1}{n}\sum_{\mu}\sum_{j=1}^{n-1}\left(2H(\theta_\mu)-H\left(\theta_\mu-\frac{2j\pi}{n}\right)-H\left(\theta_\mu+\frac{2j\pi}{n}\right)\right)f''\left(\frac{2j\pi}{n}\right)
\end{aligned} \tag{20}$$

By Lemma 5, the right hand of (20) is

$$O\left(\frac{1}{\eta_n}\right)+\sum_{\mu}g(\theta_\mu)+\frac{4a}{n}\sum_{\mu}h'(\theta_\mu)+o(1)=O\left(\frac{1}{\eta_n}\right)+\sum_{\mu}g(\theta_\mu)+o(1) \tag{21}$$

where we have used the fact that $\int_0^{2\pi}h'(x)\mathrm{d}x=0$.

Now we estimate the second part of (18)

$$\frac{1}{n}\sum_{(\mu,\nu)\in I_1}w_{\mu,\nu}(\underline{\theta})\left(t_\mu-t_\nu\right)^2f''\left(\alpha_\mu-\alpha_\nu\right) \tag{22}$$

Choose a sequence $N_n<n$ and split I_1 into two sets I_1', I_1'', where

$$I_1'=\left\{(\mu,\nu)\in I_1; |\mu-\nu|<N_n\bigvee|n-(\mu-\nu)|<N_n\right\},\quad I_1''=I_1\backslash I_1'$$

Split (22) into two sums:

$$\frac{1}{n}\sum_{I_1'}\cdots+\frac{1}{n}\sum_{I_1''}\cdots \tag{23}$$

According Lemma 1(d), we see that

$$\begin{aligned}
w_{\mu,\nu}(\underline{\theta})=\frac{1}{2}\Bigg[&h'(\theta_\mu)+h'(\theta_\nu)-\frac{h(\theta_\mu)-h(\theta_\nu)}{\sin\left(\frac{\theta_\mu-\theta_\nu}{2}\right)}\\
&+\frac{h(\theta_\mu)-h(\theta_\nu)}{\sin\left(\frac{\theta_\mu-\theta_\nu}{2}\right)}\cdot\left(1+2\sin\left(\frac{\theta_\mu-\theta_\nu}{2}\right)\cdot\frac{r\left(\alpha_\mu-\alpha_\nu,t_\mu-t_\nu\right)}{f''\left(\alpha_\mu-\alpha_\nu\right)}\right)\Bigg]
\end{aligned}$$

can be bounded by $C\left(|\alpha_\mu - \alpha_\nu| + |t_\mu - t_\nu|\right)$. Therefore using Remark of Lemma 7 and Lemma 8, we have

$$\left|\frac{1}{n}\sum_{I_1'}\cdots\right| \le C\left(\frac{N_n}{n} + \frac{1}{n^{1/3}}\right)\eta_n^2$$

Since $w_{\mu,\nu}(\underline{\theta})$ is bounded, the second sum in (23) is bounded by

$$\begin{aligned}\frac{1}{n}\sum_{(\mu,\nu)\in I_1''}(t_\mu - t_\nu)^2\left|f''(\alpha_\mu - \alpha_\nu)\right| &\le \frac{C'}{n}\sum_{N_n\le|\mu-\nu|\le n-N_n}\frac{(t_\mu - t_\nu)^2}{\sin^2\left(\frac{\pi(\mu-\nu)}{n}\right)} \\ &\le \frac{C'}{n}\sum_{k=N_n}^{n-N_n}\csc^2\left(\frac{\pi k}{n}\right)\sum_\mu (t_{\mu+k} - t_\mu)^2 \le C''\left(\frac{n}{N_n}\right)\sum_\mu t_\mu^2 \end{aligned} \tag{24}$$

Combining all the estimations about (18) we get

$$|R_n(\underline{\theta})| \le C\left(\frac{1}{\eta_n} + \frac{\eta_n^2}{n^{1/3}} + \eta_n^2\left(\frac{N_n}{n}\right) + \frac{n}{N_n}\sum_\mu t_\mu^2\right) + o(1) \tag{25}$$

Choosing $N_n = n\cdot\eta_n^{-3}$ we get Lemma 9.

2.4 Appendix: The Proof of Lemma 1

(a) is obvious.

Because $\sin^2\left(\frac{x}{2}\right)f''(x)$ is smooth and even, $f''(x)$ can be rewritten as

$$\frac{\beta}{\sin^2\left(\frac{x}{2}\right)} + r(x) \tag{26}$$

where $r(x)$ is an even function on $\mathbb{T}$ and $\beta < 0$. It is not difficult to check that $r'(x), r''(x)$ both exist on the whole $\mathbb{T}$. So (b) holds if $\left|-2\beta\cot\left(\frac{x}{2}\right) + \frac{x\beta}{\sin^2\left(\frac{x}{2}\right)}\right|$ is bounded, which is easy to verify. Similarly, (c) holds.

Now turn to (d). Suppose $t \ge 0$. Because of (a) we only have to prove

$$\left|\sin^2\left(\frac{x}{2}\right)f''(x) + \frac{2}{t^2}\sin\left(\frac{x+t}{2}\right)\sin^2\left(\frac{x}{2}\right)\cdot\left(f'(x+t) - f'(x) - tf''(x)\right)\right| \le C(|x|+|t|) \tag{27}$$

We split the (27) into two inequalities

$$\left|1 + \frac{2}{t^2}\sin\left(\frac{x+t}{2}\right)\sin^2\left(\frac{x}{2}\right)\left(-2\cot\left(\frac{x+t}{2}\right) + 2\cot\frac{x}{2} - \frac{t}{\sin^2\left(\frac{x}{2}\right)}\right)\right| \le C(|x|+|t|) \tag{28}$$

and

$$\left|\sin^2\left(\frac{x}{2}\right)r(x) + \frac{2}{t^2}\sin\left(\frac{x+t}{2}\right)\sin^2\left(\frac{x}{2}\right)\int_x^{x+t}(r(s) - r(x))\mathrm{d}s\right| \le C(|x|+|t|) \tag{29}$$

Both of (28) and (29) are easy to verify.

2.5 Appendix II: Proofs of Lemma 7, 8 and 10

We provide proofs of some lemmas in this section. They are mostly cited from [Joh88] and we only want to make the reader easier to verify our lemmas. So this part is not included in this section.

Proof of Lemma 7 Equation (6) and the definition of E_n shows that if $\underline{\alpha}+\underline{t} \in E_n$ then

$$-\sum_{\mu\neq\nu}\int_0^1 (1-\tau)\left(t_\mu-t_\nu\right)^2 f''\left(\alpha_\mu-\alpha_\nu+\tau\left(t_\mu-t_\nu\right)\right)\mathrm{d}\tau \leq Kn \tag{30}$$

Define

$$I_1=\left\{(\mu,\nu)\in I:\left|t_\mu-t_\nu\right|\cdot\left|\sin\frac{\pi(\mu-\nu)}{n}\right|^{-1}\leq 2\eta_n\right\}$$

and $I_2=I\backslash I_1$. If $(\mu,\nu)\in I_1$ we see that

$$\left|\frac{\sin\left(\frac{\pi(\mu-\nu)}{n}+\tau\left(t_\mu-t_\nu\right)\right)}{\sin\left(\frac{\pi(\mu-\nu)}{n}\right)}\right|=\left|\frac{\sin\left(\frac{\pi(\mu-\nu)}{n}\right)+\tau\left(t_\mu-t_\nu\right)\cos\left(\frac{\pi(\mu-\nu)}{n}+\epsilon_{\mu,\nu}\right)}{\sin\left(\frac{\pi(\mu-\nu)}{n}\right)}\right|\leq 3\eta_n$$

where $\epsilon_{\mu,\nu}$ is between 0 and $\tau\left(t_\mu-t_\nu\right)$.

Therefore,

$$\begin{aligned}-f''\left(\alpha_\mu-\alpha_\nu+\tau\left(t_\mu-t_\nu\right)\right)&\geq C_1\csc^2\left(\frac{\pi(\mu-\nu)}{n}+\tau\left(t_\mu-t_\nu\right)\right)\\&\geq\frac{C_1}{9\eta_n^2}\csc^2\left(\frac{\pi(\mu-\nu)}{n}\right)\geq-\frac{C_1}{9C_2\eta_n^2}f''\left(\alpha_\mu-\alpha_\nu\right)\end{aligned}$$

Interpolating this into (30) we obtain (a). $\forall(\mu,\nu)\in I_2$, we have

$$\alpha_{\mu,\nu}\triangleq\left|\sin\frac{\pi(\mu-\nu)}{n}\right|\cdot\frac{1}{\left|t_\mu-t_\nu\right|}\leq\frac{1}{2\eta_n}\leq\frac{1}{2}$$

Combining this with the fact that $|\sin(x+y)/\sin x|\leq 1+|y/\sin x|$ yields

$$\begin{aligned}C_2Kn&\geq\sum_{(\mu,\nu)\in I_2}\int_0^{\alpha_{\mu,\nu}}\frac{(1-\tau)\left(t_\mu-t_\nu\right)^2}{\sin^2\left(\frac{\pi(\mu-\nu)}{n}+\tau\left(t_\mu-t_\nu\right)\right)}\mathrm{d}\tau\\&\geq\sum_{(\mu,\nu)\in I_2}\int_0^{\alpha_{\mu,\nu}}\frac{\left(1-\frac{1}{2\eta_n}\right)\left(t_\mu-t_\nu\right)^2}{\left(1+\frac{\tau}{\alpha_{\mu,\nu}}\right)^2\sin^2\left(\frac{\pi(\mu-\nu)}{n}\right)}\mathrm{d}\tau\geq C\sum_{(\mu,\nu)\in I_2}\frac{\left|t_\mu-t_\nu\right|}{\left|\sin\left(\frac{\pi(\mu-\nu)}{n}\right)\right|}\geq C\eta_n\# I_2\end{aligned}\tag{31}$$

So (b) follows.

Proof of Lemma 8 (30) implies

$$\sum_{\mu\neq\nu}\int_0^1(1-\tau)\left(t_\mu-t_\nu\right)^2C_1\mathrm{d}\tau\leq Kn$$

So (a) holds for $C\geq 2K/C_1$.

Now suppose that $t_\mu \geq a > 0$ for some constant a and some μ. In $E_n \subseteq H_n, \varphi_\mu > \varphi_\nu$ and consequently $t_{\mu+1} - t_\mu > -2\pi/n$. It follows that $t_\nu \geq a/2$ for $\mu \leq \nu \leq \mu + [na/4\pi]$. Hence

$$C \geq \sum_\mu t_\mu^2 \geq \frac{a^2}{4}\left[\frac{na}{4\pi}\right] \geq C'a^3 n \Rightarrow a \leq C\frac{1}{n^{1/3}}$$

We thus have $t_\mu \leq C\frac{1}{n^{1/3}}$ and in the same way we get $t_\mu \geq -C\frac{1}{n^{1/3}}$ so (b) follows.

Proof of Lemma 10 We consider a particular $g(\theta) = \pm\cos(m\theta), \pm\sin(m\theta)$ and analyze how the bounds of $R_n(\underline{\theta})$ and $T_n(\underline{\theta})$ depend on m in E_n. Then we get

$$|G_n(\underline{\theta})| \leq Cm^3, \quad \underline{\theta} \in E_n \tag{32}$$

with a constant C independent of m. So

$$\begin{aligned}\exp\left(\mathrm{E}\left|\sum_\mu g(\theta_\mu)\right|\right) &\leq \mathrm{E}\left(\exp\left|\sum_\mu g(\theta_\mu)\right|\right)\\ &\leq \mathrm{E}\left(\exp\left(-\sum_\mu g(\theta_\mu)\right) + \exp\left(\sum_\mu g(\theta_\mu)\right)\right) \leq C'\exp\left(Cm^3\right)\end{aligned}$$

for our particular g. We have thus proved that

$$\mathrm{E}\left|\sum_\mu \cos\left(m\theta_\mu\right)\right| \leq Cm^3; \quad \mathrm{E}\left|\sum_\mu \sin\left(m\theta_\mu\right)\right| \leq Cm^3$$

where the constant C does not depend on m.

Set

$$A_m^{(n)} = \left\{\underline{\theta}; \left|\sum_\mu \cos\left(m\theta_\mu\right)\right| > m^3 n^{1/4}\right\} \qquad B_m^{(n)} = \left\{\underline{\theta}; \left|\sum_\mu \sin\left(m\theta_\mu\right)\right| > m^3 n^{1/4}\right\}$$

for $m = 1$, $q_n = \left[n^{1/12}\right]$, and define

$$E_n' = E_n \Big\backslash \left(\bigcup_{m=1}^{q_n}(A_m^{(n)} \cup B_m^{(n)})\right)$$

It is easy to verify (a) follows. And we have

$$\left|\sum_\mu \cos\left(m\theta_\mu\right)\right| \leq m^3 n^{1/4}, \qquad \left|\sum_\mu \sin\left(m\theta_\mu\right)\right| \leq m^3 n^{1/4} \tag{33}$$

if $\underline{\theta} \in E_n'$ for $m = 1, \ldots, q_n$.

If we combine the inequality

$$\sum_{\mu\neq\nu} \frac{(t_\mu - t_\nu)^2}{\left|\sin\left(\frac{\pi(\mu-\nu)}{n}\right)\right|} \leq Cn \tag{34}$$

which can be deduced from Lemma 7(a) and (31), (b) can be deduced by using discrete Fourier expansion of $\{t_\mu\}$. Specific calculations can be found in [Joh88].

3 Fourth-Order Convergence in Szegő's Asymptotic Formula and Random Matrix Theory

In this section, we consider the function satisfying the fourth-order conditions. For convenience we donate it again as $f(x)$. We will start our discussion from the simplest situation where

$$f''(x) = \frac{C}{\sin^4 \frac{x}{2}}, \qquad C \text{ is a negative constant}$$

If we put such expression into (3), a simple calculation would uncover that γ_k vanishes so that Szegö's asymptotic formula might turn to be

$$\frac{1}{(2\pi)^n n!} \int_{[0,2\pi]^n} \exp\left(\sum_{\mu} g(\theta_\mu) + \sum_{\mu\neq\nu} f(\theta_\mu - \theta_\nu)\right) \mathrm{d}^n\underline{\theta} = \exp\left(nc_0 + o(1)\right)$$

Integrating the function $f''(x)$ twice we obtain one of the antiderivatives as

$$f(x) = 4\log\left|2\sin\frac{x}{2}\right| - \frac{1}{\sin^2\frac{x}{2}}$$

in the situation that $C = -3/2$. We attempt to follow Johansson's method in his proof of Szegö's formula and present a major result in the following section.

3.1 Main Result

To get started with the problem, we initially estimate the normalization constant A_n expressed in (1). The key idea is to estimate the exponent of integrand

$$F_n(\underline{\theta}) := \sum_{\mu\neq\nu} f(\theta_\mu - \theta_\nu) \tag{35}$$

On threshold, we are going to give an upper bound of A_n.

Proposition 1. *For every fixed point $\theta = (\theta_0, \theta_1, \ldots, \theta_{n-1}) \in H_n$, write $\theta_\mu = \frac{2\pi\mu}{n} + t_\mu + \sigma_0, \mu = 0, \ldots, n-1,$ where σ_0 is chosen so that $\sum_\mu t_\mu = 0$. Let $\underline{\alpha} = \left(\sigma_0, \sigma_0 + \frac{2\pi}{n}, \ldots, \sigma_0 + \frac{2(n-1)\pi}{n}\right)$. Then we have $F_n(\underline{\alpha}) \geq F_n(\underline{\theta})$.*

We omit the proof since it is elementary; see [Joh88] for the details. Here we obtain the fact that $F_n(\alpha)$ is the maximum value regardless of where θ posits. Therefore we could give an upper bound of A_n:

$$A_n = n! \int_{H_n} \exp\left(\sum_{\mu\neq\nu} f(\theta_\mu - \theta_\nu)\right) \mathrm{d}\underline{\theta} \leq n! \int_{H_n} \exp\left(F_n(\underline{\alpha})\right) \mathrm{d}\underline{\theta} = (2\pi)^n \exp\left(F_n(\underline{\alpha})\right)$$

We highlight our attention on the assessment of lower bound which might reveal more complexity. Consider a half of n-dimensional cube W_n inside H_n(its length of edge denoted as Δ_n):

$$W_n = \left\{\theta \in H_n : \left|\theta_\mu - \frac{2\pi\mu}{n}\right| < \frac{\Delta_n}{2} \quad \text{for every } \mu = 0, 1, \ldots, n-1\right\}$$

Here we pick $\Delta_n \leq \frac{\pi}{n}$ so that the half cube is contained within H_n.

Proposition 2. *We have the following lower bound for A_n:*

$$A_n \geq \frac{1}{2} \exp\Big(F_n(\underline{\alpha}) - C_1 n^4 \Delta_n - C_2 n^6 \Delta_n^2 + \log n! - n \log \frac{1}{\Delta_n}\Big)$$

where C_1 and C_2 are both positive constants.

Proof of Proposition 2. Let

$$\Delta_{\mu,\nu} = \theta_\mu - \theta_\nu - \frac{2\pi(\mu-\nu)}{n}$$

and we have

$$|\Delta_{\mu,\nu}| \leq \Big|\theta_\mu - \frac{2\pi\mu}{n}\Big| + \Big|\theta_\nu - \frac{2\pi\nu}{n}\Big| \leq \Delta_n$$

Thus

$$\begin{aligned}\frac{A_n}{n!} &= \int_{H_n} \exp\left(\sum_{\mu\neq\nu} f(\theta_\mu - \theta_\nu)\right) \mathrm{d}\underline{\theta} \geq \int_{W_n} \exp\left(\sum_{\mu\neq\nu} f\left(\frac{2\pi(\mu-\nu)}{n} + \Delta_{\mu,\nu}\right)\right) \mathrm{d}\underline{\theta} \\ &= \int_{W_n} \exp\left(\sum_{\mu\neq\nu} f\left(\frac{2\pi(\mu-\nu)}{n}\right) + \sum_{\mu\neq\nu} f'\left(\frac{2\pi(\mu-\nu)}{n}\right)\Delta_{\mu,\nu} + \frac{1}{2}\sum_{\mu\neq\nu} f''(\xi_{\mu,\nu})\Delta_{\mu,\nu}^2\right) \mathrm{d}\underline{\theta} \\ &\geq \int_{W_n} \exp\left(\sum_{\mu\neq\nu} f\left(\frac{2\pi(\mu-\nu)}{n}\right) - \sum_{\mu\neq\nu} \left|f'\left(\frac{2\pi(\mu-\nu)}{n}\right)\right|\Delta_n - \frac{1}{2}\sum_{\mu\neq\nu} \big|f''(\xi_{\mu,\nu})\big|\Delta_n^2\right) \mathrm{d}\underline{\theta}\end{aligned}$$

The eqquality is established by Lagrange's Mean Value Theorem, where $\xi_{\mu,\nu} \in \Big(\frac{2\pi(\mu-\nu)}{n} - \Delta_n, \frac{2\pi(\mu-\nu)}{n} + \Delta_n\Big)$. The inequality is simplified as

$$A_n \geq n! \int_{W_n} \exp\big(F_n(\underline{\alpha}) - \mathrm{I} - \mathrm{II}\big)\mathrm{d}\underline{\theta} \tag{36}$$

letting

$$\mathrm{I} = \sum_{\mu\neq\nu}\left|f'\left(\frac{2\pi(\mu-\nu)}{n}\right)\right|\Delta_n \qquad \mathrm{II} = \frac{1}{2}\sum_{\mu\neq\nu}\big|f''(\xi_{\mu,\nu})\big|\Delta_n^2$$

We are now to give the upper bounds for both terms. Nevertheless, we only cite the result below for the I term and put the details into appendix for reader's convenience:

$$\mathrm{I} \;\leq\; C_1 n^4 \Delta_n \qquad \text{for a certain positive constant } C_1 \tag{37}$$

Currently we set about to estimate the term II. We declare that

$$\mathrm{II} \;\leq\; C_2 n^6 \Delta_n^2 \qquad \text{for } C_2 = \frac{12}{\pi^4} \tag{38}$$

$f''(x) = \frac{3}{2\sin^4\frac{x}{2}}$ and for each pair $1 \leq \mu, \nu \leq n (\mu \neq \nu)$,

$$|\xi_{\mu,\nu}| \geq \frac{2\pi|\mu-\nu|}{n} - \Delta_n \geq \frac{2\pi}{n} - \Delta_n \geq \frac{\pi}{n} \qquad\qquad (\text{Notice that } \Delta_n \leq \frac{\pi}{n})$$

and

$$|\xi_{\mu,\nu}| \le \frac{2\pi|\mu-\nu|}{n} + \Delta_n \le \frac{2(n-1)\pi}{n} + \Delta_n \le \frac{(2n-1)\pi}{n}$$

Hence we have the estimation of term II:

$$\mathrm{II} \le \frac{3}{4}\sum_{\mu\neq\nu} \frac{\Delta_n^2}{\sin^4\left|\frac{\xi_{\mu,\nu}}{2}\right|} \le \frac{3}{4}n(n-1)\Big(\frac{2n}{\pi}\Big)^4 \Delta_n^2 \le \frac{12}{\pi^4} n^6 \Delta_n^2$$

Take the results into (36), we obtain

$$\begin{aligned} &A_n \ge n! \int_{W_n} \exp\left(F_n(\underline{\alpha}) - C_1 n^4 \Delta_n - C_2 n^6 \Delta_n^2\right) \mathrm{d}\underline{\theta} \\ = &\frac{1}{2}\Delta_n^n \exp\left(F_n(\underline{\alpha}) - C_1 n^4\Delta_n - C_2 n^6 \Delta_n^2 + \log n!\right) \qquad \text{(Note that the volume of half cube } W_n \text{ is } \frac{\Delta_n^n}{2}\text{)} \\ = &\frac{1}{2}\exp\left(F_n(\underline{\alpha}) - C_1 n^4\Delta_n - C_2 n^6 \Delta_n^2 + \log n! - n\log\frac{1}{\Delta_n}\right) \end{aligned}$$

Consequently, we prove the proposition. □

We attempt to estimate the left-hand term of (2):

$$\frac{1}{A_n}\int_{E_n} \exp\left(\sum_\mu g(\theta_\mu) + F_n(\underline{\theta})\right) \mathrm{d}\underline{\theta}$$

Enlightened by [Joh88] and our previous analysis in Section 2, we might find a smaller region H_n so that the integration could be asymptotically expressed by the integration on the subset. For this, we define E_n in form of

$$E_n = \{\underline{\theta} \in H_n : F_n(\underline{\alpha}) - F_n(\underline{\theta}) \le L_n\} \tag{39}$$

in which $\{L_n\}$ are non-negative numbers that would be determined later. Thereby in E_n the value of $F_n(\underline{\theta})$ is more close to its maximum point.

Let $E_n^c = H_n \backslash E_n$, we have the following

$$\begin{aligned} &\frac{1}{A_n}\int_{E_n^c} \exp\Big(\sum_\mu g(\theta_\mu) + F_n(\underline{\theta})\Big) \mathrm{d}\underline{\theta} \\ \le &\frac{1}{A_n}\int_{E_n^c} \exp\Big(\sum_\mu g(\theta_\mu) + F_n(\underline{\alpha}) - L_n\Big) \mathrm{d}\underline{\theta} \le \frac{(2\pi)^n}{n! A_n} \exp\left(n||g||_\infty + F_n(\underline{\alpha}) - L_n\right) \end{aligned}$$

Our goal is to render

$$\frac{1}{A_n}\int_{E_n^c} \exp\Big(\sum_\mu g(\theta_\mu) + F_n(\underline{\theta})\Big) \mathrm{d}\underline{\theta} \to 0 \qquad \text{when } n \to \infty$$

We conclude our results as the following

Theorem 3. *Suppose that $E_n \subseteq H_n$ is presented as* (39)*. By picking $L_n = 2n\log n$, we get*

$$\frac{1}{A_n}\int_{E_n^c} \exp\Big(\sum_\mu g(\theta_\mu) + F_n(\underline{\theta})\Big) \mathrm{d}\underline{\theta} = o(1), \qquad n \to \infty \tag{40}$$

Proof of Theorem 3. According to Proposition 2, the calculation continues as

$$\frac{1}{A_n}\int_{E_n^c} \exp\Big(\sum_{\mu} g(\theta_\mu) + F_n(\underline{\theta})\Big)\mathrm{d}\underline{\theta} \tag{41}$$

$$\leq \frac{(2\pi)^n}{n!} 2\exp\bigg(\big(-F_n(\underline{\alpha}) + C_1 n^4\Delta_n + C_2 n^6 \Delta_n^2 - \log n! + n\log\frac{1}{\Delta_n}\big) + \big(n||g||_\infty + F_n(\underline{\alpha}) - L_n\big)\bigg)$$

After simplification, the right hand of (41) turns out to be

$$2\exp\bigg((C_3 n - 2\log n!) + \big(C_1 n^4 \Delta_n + C_2 n^6 \Delta_n^2 + n\log\frac{1}{\Delta_n}\big) - L_n\bigg)$$

where we let $C_3 = \log(2\pi) + ||g||_\infty > 0$. Using Stirling's formula, it is simple to find out that the initial term in the bracket goes to negative infinity at an order of $-2n\log n$ when $n \to \infty$. The second-bracket one is related to Δ_n, a sequence of numbers which runs to infinitesimal as n grows.

In the second bracket, we select Δ_n sufficiently small so as to satisfy that the first two terms are negligible in the exponent; yet the selection ought not to be too small, so the last term is controlled by the order $n\log n$. Let $\Delta_n = \frac{1}{n^3}$ (the assumption $\Delta_n \leq \frac{\pi}{n}$ is obviously guaranteed), the right hand of (41) turns out to be

$$2\exp\big((C_3 n - 2\log n!) + \big(C_1 n + C_2 + 3n\log n\big) - L_n\big)$$

and it does not exceed

$$2\exp\Big(n\log n + \widetilde{C}n - L_n\Big)$$

when we make $\widetilde{C} = C_1 + C_2 + C_3 + 2$, in which we appeal to a direct corollary of inequality $n! \geq \left(\frac{n}{e}\right)^n$ when $n \geq 1$. Therefore by taking $L_n = 2n\log n$, (40) is successfully established. □

3.2 Appendix

We are going to demonstrate the estimation of term I in the proof of Proposition 2. Recall from the proof that

$$\mathrm{I} = \sum_{\mu\neq\nu}\left|f'\left(\frac{2\pi(\mu-\nu)}{n}\right)\right|\Delta_n$$

We overlooked the last Δ_n term, and hence we have

$$\begin{aligned}\sum_{\mu\neq\nu}\left|f'\left(\frac{2\pi(\mu-\nu)}{n}\right)\right| &= \sum_{\mu\neq\nu}\left|\frac{2\cos\left(\frac{\pi(\mu-\nu)}{n}\right)}{\sin\left(\frac{\pi(\mu-\nu)}{n}\right)}\right| + \sum_{\mu\neq\nu}\left|\frac{\cos\left(\frac{\pi(\mu-\nu)}{n}\right)}{\sin^3\left(\frac{\pi(\mu-\nu)}{n}\right)}\right| \\ &= \sum_{\mu\neq\nu}\left|\frac{2}{\sin\left(\frac{\pi(\mu-\nu)}{n}\right)}\right| + \sum_{\mu\neq\nu}\left|\frac{1}{\sin^3\left(\frac{\pi(\mu-\nu)}{n}\right)}\right| \end{aligned} \tag{42}$$

For the first addend of (42), we have

$$\sum_{\mu\neq\nu}\left|\frac{2}{\sin\left(\frac{\pi(\mu-\nu)}{n}\right)}\right| = n\sum_{k=1}^{n-1}\frac{1}{\sin\frac{k\pi}{n}} = \frac{2n}{\sin\frac{\pi}{n}} + \sum_{k=2}^{n-2}\frac{1}{\sin\frac{k\pi}{n}} \leq \frac{2n}{\sin\frac{\pi}{n}} + \frac{n^2}{\pi}\int_{\frac{\pi}{n}}^{\frac{(n-1)\pi}{n}}\frac{\mathrm{d}x}{\sin x}$$

$$= \frac{2n}{\sin\frac{\pi}{n}} + \frac{n^2}{\pi}\log\frac{1+\cos\frac{\pi}{n}}{1-\cos\frac{\pi}{n}} \le \frac{2n}{\sin\frac{\pi}{n}} + \frac{2n^2}{\pi}\log\frac{1}{\sin\frac{\pi}{2n}} \le K_1 n^3$$

And for the second term of (42),

$$\begin{aligned}
\sum_{\mu\neq\nu}\left|\frac{1}{\sin^3\left(\frac{\pi(\mu-\nu)}{n}\right)}\right| &= \frac{n}{2}\sum_{k=1}^{n-1}\frac{1}{\sin^3\frac{k\pi}{n}} = \frac{n}{\sin^3\frac{\pi}{n}} + \frac{1}{2}\sum_{k=2}^{n-2}\frac{1}{\sin^3\frac{k\pi}{n}} \\
&\le \frac{n}{\sin^3\frac{\pi}{n}} + \frac{n^2}{2\pi}\int_{\frac{\pi}{n}}^{\frac{(n-1)\pi}{n}}\frac{\mathrm{d}x}{\sin^3 x} = \frac{n}{\sin^3\frac{\pi}{n}} + \frac{n^2}{2\pi}\left(\frac{\cos\frac{\pi}{n}}{\sin^2\frac{\pi}{n}} + \frac{1}{2}\log\frac{1+\cos\frac{\pi}{n}}{1-\cos\frac{\pi}{n}}\right) \\
&\le \frac{n}{\sin^3\frac{\pi}{n}} + \frac{n^2}{2\pi}\left(\frac{1}{\sin^2\frac{\pi}{n}} + \frac{1}{2}\log\frac{1}{\sin^2\frac{\pi}{2n}}\right) \le K_2 n^4
\end{aligned}$$

Here K_1 and K_2 are two positive constants. Combining the two conclusions it is obviously seen that

$$\mathrm{I} \ \le\ C_1 n^4 \Delta_n$$

if we let $C_1 = K_1 + K_2$. Therefore the proof is accomplished. □

4 Conclusions

In this paper, we have extended Szegő's asymptotic formula to a broader class of integrals involving smooth, periodic convex functions, relaxing the initial smoothness and analyticity assumptions. By leveraging Johansson's change of variables method and incorporating fourthorder convergence effects, we derived new estimates for these generalized integrals. Our results reveal deeper insights into the asymptotic behavior of Toeplitz determinants, particularly in the context of random matrix theory.

Specifically, we demonstrated that integrals of the form

$$\int_{[0,2\pi]^n} \exp\left(\sum_{\mu} g(\theta_\mu) + \sum_{\mu\neq\nu} f(\theta_\mu - \theta_\nu)\right) \mathrm{d}^n\underline{\theta}$$

asymptotically equal

$$\exp\left(nc_0 + \sum_{k=1}^{\infty}\gamma_k c_k c_{-k} + o(1)\right)$$

where g is a smooth periodic function, f satisfies convexity conditions, c_k are the Fourier coefficients of g, and γ_k are constants depending on f. In cases such as $f(x) = \log\left|\sin\left(\frac{x}{2}\right)\right|$, our result recovers Szegő's original asymptotic formula.

This generalization not only advances the applicability of Szegö's theorem but also opens new research directions in random matrix theory, statistical mechanics, and asymptotic analysis in mathematical physics. Our exploration of higher-order convergence rates highlights key connections between Toeplitz determinants, matrix eigenvalue distributions, and phenomena analogous to the Central Limit Theorem in high-dimensional spaces.

Future research may further relax smoothness conditions, explore other integral types, or investigate the practical implications of these extended results through numerical studies. The findings of this work contribute to a richer theoretical framework for random matrix theory and high-dimensional asymptotic behavior, offering exciting possibilities for advancing both theoretical and applied fields.

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