Serre presentation for $\imath$-QCG

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Goal: describe a Serre presentation for quasi-split $\imath$-quantum covering groups $\mathbf{U}^\imath_\pi$.

Outline:

1. Serre presentation for $\imath$-quantum groups.
2. Brief intro to quantum covering groups.
3. Serre presentation for $\imath$-quantum covering groups.
4. Applications to $\imath$-canonical basis.

Preliminary report; will be on arXiv soon.

Background: $\imath$-Quantum Groups $\mathbf{U}^\imath$

Let $(I, \cdot)$ be a Cartan datum, and let $(Y, X, \langle \cdot, \cdot \rangle, \dots)$ be a root datum of type $(I, \cdot)$. A permutation τ of the set I is called an *involution* of $(I, \cdot)$ if $\tau^2 = \mathrm{id}$ and $\tau i \cdot \tau j = i \cdot j$ for $i, j \in I$.
Let $\mathbf{U}$ be a (Drinfeld-Jimbo) quantum group with generators E_i, F_i, K_h for $i \in I$ for $h \in Y$, and let $\widetilde{K}_i := K_{h_i}^{\epsilon_i}$ and $q_i := q^{\epsilon_i}$ where $\epsilon_i = \frac{i \cdot i}{2}$ for all $i \in I$.

Definition ([Letzter 99],[Kolb 14],[Chen-Lu-Wang 18])

The *quasi-split $\imath$-quantum group* $\mathbf{U}^\imath$ is the $\mathbb{Q}(q)$-subalgebra of $\mathbf{U}$ generated by

$$B_i := F_i + q_i^{-1} E_{\tau i} \widetilde{K}_i^{-1} \quad (i \in I), \qquad K_\mu \quad (\mu \in Y^\imath),$$

where $Y^\imath = \{h \in Y | \tau(h) = -h\}$.

Background: $\imath$-Quantum Groups $\mathbf{U}^\imath$

- [Bao-Wang 18a] quasi-split $\mathbf{U}^\imath$ of type AIII/AIV,
  $\rightsquigarrow$ $\imath$-canonical basis for finite-dimensional simple $\mathbf{U}^\imath$-modules and tensor products.
  $\rightsquigarrow$ Application: Kazhdan-Lusztig theory for super type B.
- [Bao-Wang 18b,c] finite type and Kac-Moody generality
  $\rightsquigarrow$ $\imath$-canonical bases for the highest weight integrable $\mathbf{U}$-modules and their tensor products regarded as $\mathbf{U}^\imath$-modules.
  $\rightsquigarrow$ $\imath$-canonical basis for $\dot{\mathbf{U}}^\imath$ the modified $\imath$-quantum group.

$\imath$-divided powers

Recall: Lusztig's divided powers $F_i^{(n)} := \frac{F_i^n}{[n]^!_{q_i}}$, where $[n]^!_{q_i} := \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$.
$\rightsquigarrow$ canonical basis for simple $\mathbf{U}$-modules.
Does not descend to a canonical basis for $\mathbf{U}^\imath$. Instead we need

Definition ([Bao-Wang 18a],[Chen-Lu-Wang 18])

For $i \in I$ with $\tau i = i$, the $\imath$-*divided powers* are defined to be

$$B_{i,\bar{1}}^{(m)} = \frac{1}{[m]^!_{q_i}} \begin{cases} B_i \prod_{j=1}^k (B_i^2 - [2j-1]^2_{q_i}) & \text{if } m = 2k+1, \\ \prod_{j=1}^k (B_i^2 - [2j-1]^2_{q_i}) & \text{if } m = 2k; \end{cases}$$

$$B_{i,\bar{0}}^{(m)} = \frac{1}{[m]^!_{q_i}} \begin{cases} B_i \prod_{j=1}^k (B_i^2 - [2j]^2_{q_i}) & \text{if } m = 2k+1, \\ \prod_{j=1}^k (B_i^2 - [2j-2]^2_{q_i}) & \text{if } m = 2k. \end{cases}$$

$\imath$-divided powers

Example ($m = 2$)

$$B_{\bar{0}}^{(2)} = \frac{B^2}{[2]^!}, \qquad \text{and} \qquad B_{\bar{1}}^{(2)} = \frac{B^2-1}{[2]^!}$$

Example ($m = 3$)

$$B_{\bar{0}}^{(3)} = \frac{B(B^2-[2]^2)}{[3]^!}, \qquad \text{and} \qquad B_{\bar{1}}^{(3)} = \frac{B(B^2-1)}{[3]^!}$$

These closed-form formulas were first conjectured in [Bao-Wang 18a] and studied in depth in [Berman-Wang 17] $\rightsquigarrow$ $\imath$-*canonical basis* for certain based $\mathbf{U}$-modules including simple $\mathbf{U}$-modules and their tensor products, viewed as $\mathbf{U}^\imath$-modules ([BW18b,c]).

Serre presentation for quasi-split $\mathbf{U}^\imath$

Two key relations show up in the Serre presentation for quasi-split $\mathbf{U}^\imath$ of arbitrary Kac-Moody type:

Proposition ([Chen-Lu-Wang 18] $\imath$-Serre relation)

$$\text{If } \tau i = i \neq j, \quad \sum_{n=0}^{1-a_{ij}} (-1)^n B^{(n)}_{i,\overline{a_{ij}}+\overline{p_i}} B_j B^{(1-a_{ij}-n)}_{i,\overline{p}_i} = 0$$

Proposition ([Balagovic-Kolb 15] 'q-Serre correction relation')

$$\text{If } \tau i \neq i, \quad \sum_{n=0}^{1-a_{i,\tau i}} (-1)^n B_i^{(n)} B_{\tau i} B_i^{(1-a_{i,\tau i}-n)} = \frac{q_i^{-1}}{q_i - q_i^{-1}}$$
$$\cdot \Big(q_i^{a_{i,\tau i}} (q_i^{-2}; q_i^{-2})_{-a_{i,\tau i}} B_i^{(-a_{i,\tau i})} \widetilde{K}_i \widetilde{K}_{\tau i}^{-1} - (q_i^2; q_i^2)_{-a_{i,\tau i}} B_i^{(-a_{i,\tau i})} \widetilde{K}_{\tau i} \widetilde{K}_i^{-1} \Big)$$

where $(x;x)_0 = 1$ and $(x;x)_n = \prod_{i=1}^n (1 - x^i)$ for $n \geq 1$.

Serre presentation for quasi-split $\mathbf{U}^\imath$

Theorem 1 (Serre presentation for quasi-split $\mathbf{U}^\imath$)

[Chen-Lu-Wang 18] The $\mathbb{Q}(q)$-algebra $\mathbf{U}^\imath$ has a presentation with generators B_i $(i \in I)$, K_μ $(\mu \in Y^\imath)$ and the relations (1)–(6) below: for $\mu, \mu' \in Y^\imath$ and $i \neq j \in I$,

$$K_\mu K_{-\mu} = 1, \quad K_\mu K_{\mu'} = K_{\mu+\mu'}, \tag{1}$$

$$K_\mu B_i - q_i^{-\langle \mu, \alpha_i \rangle} B_i K_\mu = 0, \tag{2}$$

$$B_i B_j - B_j B_i = 0, \quad \text{if } a_{ij} = 0 \text{ and } \tau i \neq j, \tag{3}$$

$$\sum_{n=0}^{1-a_{ij}} (-1)^n B_i^{(n)} B_j B_i^{(1-a_{ij}-n)} = 0, \quad \text{if } j \neq \tau i \neq i, \tag{4}$$

$$q\text{-Serre correction relation}, \quad \text{if } \tau i \neq i, \text{ and} \tag{5}$$

$$\imath\text{-Serre relation}, \quad \text{if } \tau i = i \neq j. \tag{6}$$

Serre presentation for quasi-split $\mathbf{U}^\imath$

Remarks:

1. Various cases Theorem 1 were known earlier e.g. finite type [Letzter 03]; Kac-Moody type, $a_{ij} \in \{0, -1, -2\}$ in [Kolb 14, Theorems 7,4 7,8], $a_{ij} = -3$ in [Balagovic-Kolb 15], this features a 'Serre correction' relation for $\tau i = i$ that can be written in terms of $\imath$-divided powers:

Example ($a_{12} = -1$)

$B_1^2 B_2 - [2]_1 B_1 B_2 B_1 + B_2 B_1^2 = B_2$ (cf. [Ko14, (7.10)]) can be written as

$$\Big(\frac{B_1^2 - 1}{[2]_1}\Big) B_2 - B_1 B_2 B_1 + B_2 \Big(\frac{B_1^2}{[2]_1}\Big) = 0 \text{ OR } \Big(\frac{B_1^2}{[2]_1}\Big) B_2 - B_1 B_2 B_1 + B_2 \Big(\frac{B_1^2 - 1}{[2]_1}\Big) = 0$$

2. Writing the $\imath$-Serre relation in terms of $\imath$-divided powers allowed the authors of [CLW18] to leverage expansion formulas in $\imath$-divided powers, and reduce its proof to a q-binomial identity (More later).

Serre presentation for quasi-split $\mathbf{U}^\imath$

Remarks:

1. Various cases of the Serre presentation of [CLW18] were known earlier e.g. finite type due to [Letzter 03], Kac-Moody type with $a_{ij} \in \{0, -1, -2\}$ in [Kolb 14, Theorems 7,4 7,8], and $a_{ij} = -3$ in [Balagovic-Kolb 15], featuring a 'Serre correction' relation for $\tau i = i$ that can be written in terms of $\imath$-divided powers:

Example ($a_{12} = -1$)

$B_1^2 B_2 - [2]_1 B_1 B_2 B_1 + B_2 B_1^2 = B_2$ (cf. [Ko14, (7.10)]) can be written as

$$B_{1,\bar{1}}^{(2)} B_2 - B_1 B_2 B_1 + B_2 B_{1,\bar{0}}^{(2)} = 0 \text{ OR } B_{1,\bar{0}}^{(2)} B_2 - B_1 B_2 B_1 + B_2 B_{1,\bar{1}}^{(2)} = 0$$

2. Writing the $\imath$-Serre relation in terms of $\imath$-divided powers allowed the authors of [CLW18] to leverage expansion formulas in $\imath$-divided powers, and reduce its proof to a q-binomial identity (More on this and its π-analogue later).

Quantum Covering Groups $\mathbf{U}_\pi$

[Clark-Hill-Wang 14]: Key features – two parameters q ('quantum') and π ('covering') such that $\pi^2 = 1$. When $\pi = 1$, $\mathbf{U}_\pi$ specializes to the familiar quantum group of Drinfeld and Jimbo; when $\pi = -1$, $\mathbf{U}_\pi$ specializes to an anisotropic quantum supergroup.

Definition (and some notation)

- Let $(I, \cdot)$ be a Cartan datum. If I can be decomposed as $I = I_{\bar{0}} \coprod I_{\bar{1}}$ such that (a) $I_{\bar{1}} \neq \emptyset$ and (b) $2i \cdot j / i \cdot i \in 2\mathbb{Z}$ for all $i \in I_{\bar{1}}$, we say that $(I, \cdot)$ is a *super Cartan datum*.
- Let $p(i)$ denote the parity of i e.g. $p(i) = \bar{1}$ for $i \in I_{\bar{1}}$.
- For (q, π)-integers we will denote $[n]_i := \frac{(\pi_i q_i)^n - q_i^{-n}}{\pi_i q_i - q_i^{-1}}$.

Quantum Covering Groups $\mathbf{U}_\pi$

Definition

Let $(Y, X, \langle\cdot,\cdot\rangle, \dots)$ be a root datum of type $(I, \cdot)$. The quantum covering group $\mathbf{U}_\pi$ is the $\mathbb{Q}(q)^\pi$-algebra with generators E_i, F_i, K_h, J_h for all $i, j \in I$ and $h \in Y$ subject to the relations (Q1)-(Q5) below for all $i, j \in I, \mu, \mu' \in Y$:

$$K_0 = 1, K_\mu K_{\mu'} = K_{\mu+\mu'}, \quad J_{2\mu} = 1, J_\mu J_{\mu'} = J_{\mu+\mu'}, \quad J_\mu K_{\mu'} = K_{\mu'} J_\mu, \tag{Q1}$$

$$K_\mu E_i = q^{\langle \mu, i' \rangle} E_i K_\mu, \quad J_\mu E_i = \pi^{\langle \mu, i' \rangle} E_i J_\mu, \tag{Q2}$$

$$K_\mu F_i = q^{-\langle \mu, i' \rangle} F_i K_\mu, \quad J_\mu F_i = \pi^{-\langle \mu, i' \rangle} F_i J_\mu, \tag{Q3}$$

$$E_i F_j - \pi^{p(i)p(j)} F_j E_i = \delta_{ij} \frac{\widetilde{J}_i \widetilde{K}_i - \widetilde{K}_{-i}}{\pi_i q_i - q_i^{-1}}, \tag{Q4}$$

Quantum Covering Groups $\mathbf{U}_\pi$

Definition ($\mathbf{U}_\pi$, continued)

$$S_{ij}(E_i, E_j) = S_{ij}(F_i, F_j) = 0, \qquad ((q,\pi)\text{-Serre relations}) \qquad \text{(Q5)}$$

where $S_{ij}(\theta_i, \theta_j) = \sum_{k=0}^{1-a_{ij}} (-1)^k \pi_i^{kp(j)+\binom{k}{2}} \theta_i^{(1-a_{ij}-k)} \theta_j \theta_i^{(k)}$.

Proposition ([CHW 13, Section 2.2])

$\mathbf{U}_\pi$ is a Hopf superalgebra, with comultiplication given by

$$\Delta(E_i) = E_i \otimes 1 + \widetilde{J}_i \widetilde{K}_i \otimes E_i \quad (i \in I),$$
$$\Delta(F_i) = F_i \otimes \widetilde{K}_{-i} + 1 \otimes F_i \quad (i \in I),$$
$$\Delta(K_\mu) = K_\mu \otimes K_\mu, \qquad \Delta(J_\mu) = J_\mu \otimes J_\mu \quad (\mu \in Y).$$

where $\widetilde{K}_i = K_i^{\epsilon_i}$ and $\widetilde{J}_i = J_i^{\epsilon_i}$ (informally, $\widetilde{K}_i$ ‘is’ q^{h_i} and $\widetilde{J}_i$ ‘is’ π^{h_i}).

$\imath$-Quantum Covering Groups $\mathbf{U}^\imath_\pi$

Definition

The *quasi-split $\imath$-quantum covering group* $\mathbf{U}^\imath_\pi$ is the $\mathbb{Q}(q)$-subalgebra of $\mathbf{U}_\pi$ generated by

$$B_i := F_i + q_i^{-1} E_{\tau i} \widetilde{K}_i^{-1}, \quad \widetilde{J}_i \quad (i \in I), \qquad K_\mu \quad (\mu \in Y^\imath).$$

In the $\imath$-QCG setting, we have π-analogues of preceding formulas and results:

Definition ($\imath^\pi$-divided powers)

For $i \in I$ with $\tau i = i$, the $\imath^\pi$-divided powers are defined to be

$$B^{(m)}_{i,\bar{1}} = \frac{1}{[m]^!_{\pi_i,q_i}} \begin{cases} B_i \prod_{j=1}^k (B_i^2 - [2j-1]^2_{q_i} \widetilde{J}_i) & \text{if } m = 2k+1, \\ \prod_{j=1}^k (B_i^2 - [2j-1]^2_{q_i} \widetilde{J}_i) & \text{if } m = 2k; \end{cases} \tag{7}$$

$$B^{(m)}_{i,\bar{0}} = \frac{1}{[m]^!_{\pi_i,q_i}} \begin{cases} B_i \prod_{j=1}^k (B_i^2 - \pi_i [2j]^2_{q_i} \widetilde{J}_i) & \text{if } m = 2k+1, \\ \prod_{j=1}^k (B_i^2 - \pi_i [2j-2]^2_{q_i} \widetilde{J}_i) & \text{if } m = 2k, \end{cases} . \tag{8}$$

$\imath^\pi$-divided powers

Example ($m = 2$)

$$B_{\bar{0}}^{(2)} = \frac{B^2}{[2]^!}, \qquad \text{and} \qquad B_{\bar{1}}^{(2)} = \frac{B^2 - \widetilde{J}}{[2]^!}$$

Example ($m = 3$)

$$B_{\bar{0}}^{(3)} = \frac{B(B^2 - \pi[2]^2\widetilde{J})}{[3]^!}, \qquad \text{and} \qquad B_{\bar{1}}^{(3)} = \frac{B(B^2 - \widetilde{J})}{[3]^!}$$

Serre presentation for quasi-split $\mathbf{U}^{\imath}_{\pi}$

1. π-version of the q-Serre correction relation:

Proposition ([C. 19] (q,π)-Serre correction relation)

If $\tau i \neq i$,
$$\sum_{n=0}^{1-a_{i,\tau i}} (-1)^n \pi_i^{n+\binom{n}{2}} B_i^{(n)} B_{\tau i} B_i^{(1-a_{i,\tau i}-n)} = \frac{q_i^{-1}}{\pi_i q_i - q_i^{-1}}$$
$$\cdot \Big(q_i^{a_{i,\tau i}} (\pi_i q_i^{-2}; \pi_i q_i^{-2})_{-a_{i,\tau i}} B_i^{(-a_{i,\tau i})} \mathcal{Z}_i - (\pi_i q_i^2; \pi_i q_i^2)_{-a_{i,\tau i}} B_i^{(-a_{i,\tau i})} \mathcal{Z}_{\tau i} \Big)$$
where $\mathcal{Z}_j := \widetilde{J}_j \widetilde{K}_j \widetilde{K}_{\tau j}^{-1}$.

2. π-version of $\imath$-Serre relation

Proposition ([C. 19] $\imath^{\pi}$-Serre relation)

$$\text{If } \tau i = i \neq j, \quad \sum_{n=0}^{1-a_{ij}} (-1)^n \pi_i^{n+\binom{n}{2}} B_{i,\overline{a_{ij}}+\overline{p_i}}^{(n)} B_j B_{i,\overline{p}_i}^{(1-a_{ij}-n)} = 0$$

Remarks on the proof of the $\imath^\pi$-Serre relation

Remarks on the proof

1. Action of the LHS on $\mathbf{1}_\lambda := \sum_{\mu;\langle h_i,\mu\rangle=\lambda} \mathbf{1}_\mu \in \dot{\mathbf{U}}$ vanishes for any λ via expansion formulas e.g. for even $\imath$-divided power with $m = 2n$:

Formula ([C. 19])

For $n \geq 1$ and $\lambda \in \mathbb{Z}$, we have

$$B_{i,\bar{0}}^{(2n)} \mathbf{1}_{2\lambda} = \sum_{c=0}^{n} \sum_{a=0}^{2n-2c} \pi_i^a (\pi_i q_i)^{2(a+c)(n-a-\lambda)-2ac-\binom{2c+1}{2}} \begin{bmatrix} n-c-a-\lambda \\ c \end{bmatrix}_{q^2} E_i^{(a)} F_i^{(2n-2c-a)} \mathbf{1}_{2\lambda}.$$

(when $\pi_i = 1$, these results were known to [Berman-Wang 17])

2. Ultimately reduces to a (q, π)-identity; a version of [Chen-Lu-Wang 18, Theorem 3.10] with $\sqrt{\pi} q$ substituted for q.

Serre presentation for quasi-split $\mathbf{U}^\imath_\pi$

Theorem 2 ([C. 19])

The $\mathbb{Q}(q)^\pi$-algebra $\mathbf{U}^\imath$ has a presentation with generators B_i, $\widetilde{J}_i$ $(i \in I)$, K_μ $(\mu \in Y^\imath)$ and the relations (R1)–(R6) below: for $\mu, \mu' \in Y^\imath$ and $i \neq j \in I$,

$$K_\mu K_{-\mu} = 1, \quad K_\mu K_{\mu'} = K_{\mu+\mu'}, \tag{R1}$$

$$K_\mu B_i - q_i^{-\langle \mu, \alpha_i \rangle} B_i K_\mu = 0, \quad \widetilde{J}_i \text{ is central} \tag{R2}$$

$$[B_i, B_j] = 0, \quad \text{if } a_{ij} = 0 \text{ and } \tau i \neq j, \tag{R3}$$

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \pi_i^{np(j)+\binom{n}{2}} B_i^{(n)} B_j B_i^{(1-a_{ij}-n)} = 0, \quad \text{if } j \neq \tau i \neq i, \tag{R4}$$

$$(q,\pi)\text{-Serre correction relation}, \quad \text{if } \tau i \neq i, \text{ and} \tag{R5}$$

$$\imath^\pi\text{-Serre relation.} \quad \text{if } \tau i = i \neq j \tag{R6}$$

Application: Canonical basis for quasi-split $\mathbf{U}^\imath_\pi$

Constructing $\imath$-canonical basis:

1. bar-involution.
2. quasi K-matrix.
3. $\imath$-canonical basis for simple $\mathbf{U}_\pi$-modules and their tensor products regarded as $\mathbf{U}^\imath_\pi$-modules (a la [Bao-Wang 18c]).
4. $\imath$-canonical basis for the algebra $\dot{\mathbf{U}}^\imath_\pi$.

Thank you for your attention

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