Stellar Fusion Model

Chris Kilday

For my final project, I decided to write a computer simulation modeling nuclear fusion inside of a star, specifically Hydrogen burning through the proton-proton chain. The computation involves a few notable simplifying assumptions and other approximations, so is not meant to accurately reproduce observed results; the purpose is instead to investigate the causes and effects of Hydrogen burning in a star. For simplicity, all stars in the model are assumed to be ideal gasses of pure ionized hydrogen.

To further simplify the model, every star is uniquely determined by its mass and its radius is given by the observational relation

$$\frac{R}{R_\odot} \approx \left(\frac{M}{M_\odot}\right)^{0.6}$$

Though the reaction rate for the proton-proton chain is in general difficult to solve for, it can be it can be approximated by the simple formula

$$\varepsilon_{pp}(r) \propto \rho(r) T(r)^4$$

The pressure of the star is constrained by the requirement of hydrostatic equilibrium and treating the star as an ideal gas allows us to solve for the temperature.

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho(r)$$

$$T(r) = \frac{\mu m_H}{k}\frac{P(r)}{r}$$

Where $\mu = \frac{1}{2}$ is the mean molecular weight of ionized hydrogen gas.

The only free parameter left is the density, which, in the general case, can only be solved numerically by integrating through time. Instead, I chose to approximate it as a linear decay

$$\rho(r) \approx \rho_{central}\left(1 - \frac{r}{R}\right)$$

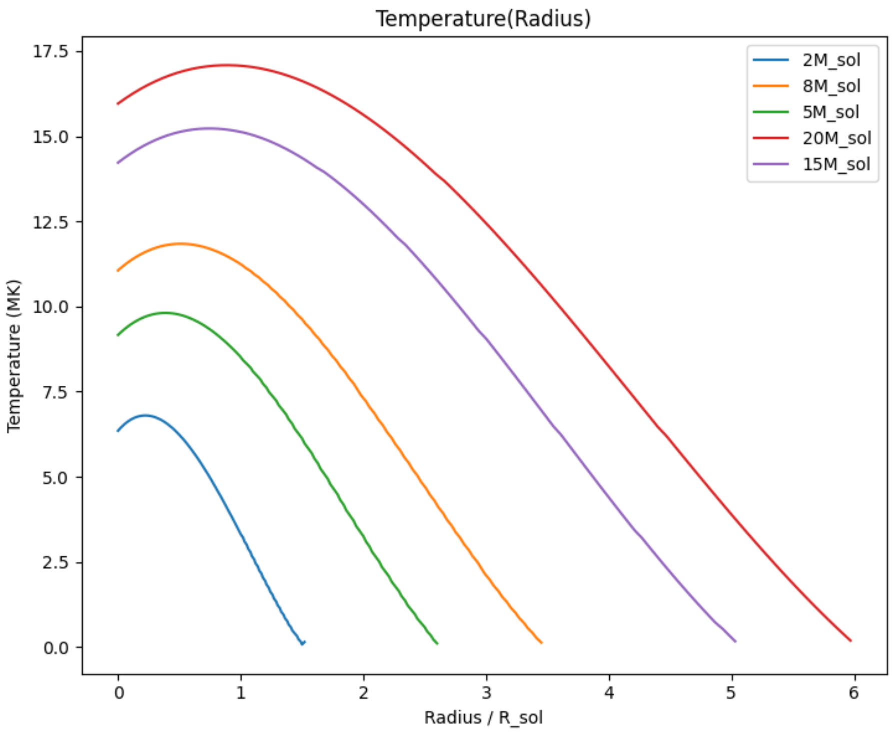
$$\rho_{central} = \frac{3M}{\pi R^3}$$

Where $\rho_{central}$ is normalized to the total mass *M*. Integrating with the boundary condition *P(R) = 0* yields

$$\mathrm{P}(r) = P_{central} - \frac{1}{36}\mathrm{G}\pi\mathrm{R}^2\rho_{central}^2\left[9\frac{r^4}{R^4} - 28\frac{r^3}{R^3} + 24\frac{r^2}{R^2}\right]$$

$$P_{central} = \frac{5}{36}G\pi R^2\rho_{central}^2$$

This produces the following temperature profile.

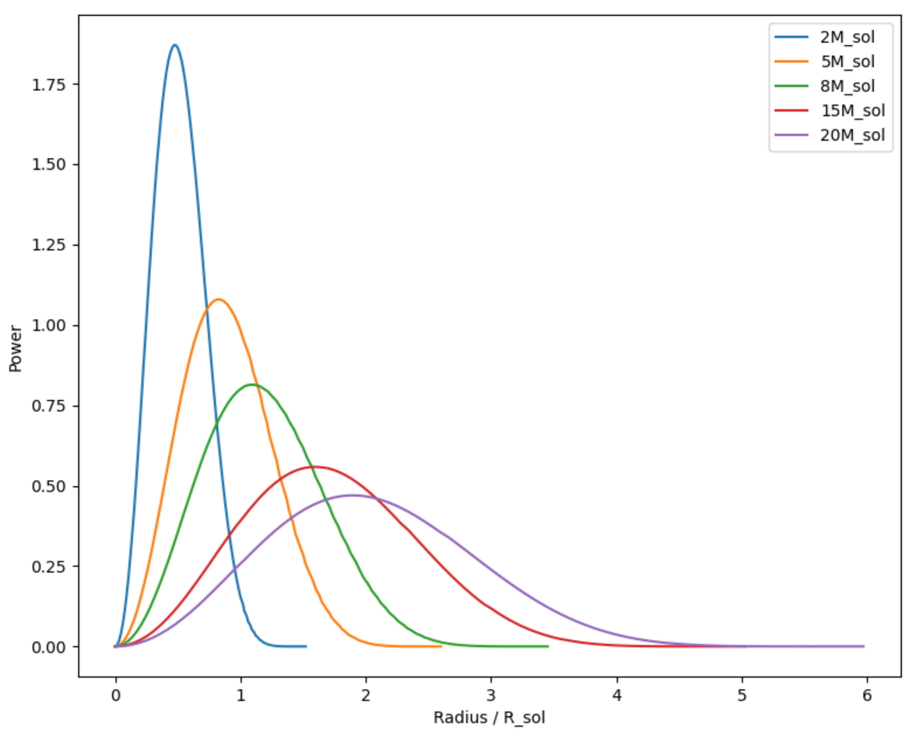


The hydrogen burning rate ε_{pp} gives an energy production rate per unit mass, so integrating over the mass of the star yields the luminosity.

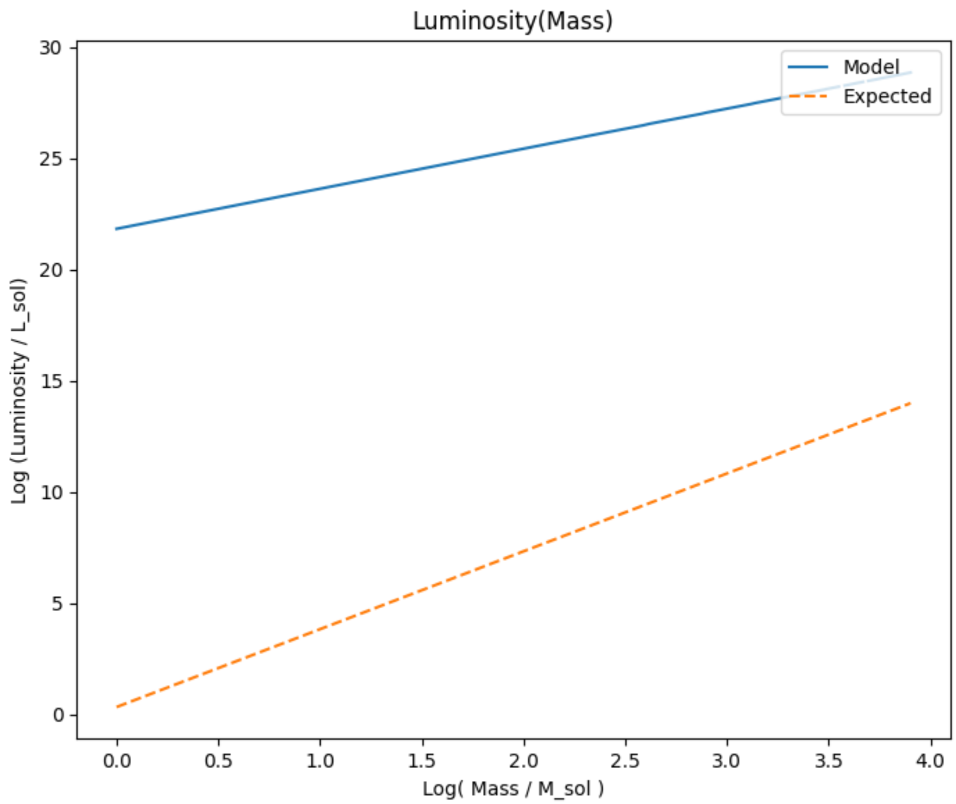
$$L = \int_0^M \varepsilon(m) dm = \int_0^R 4\pi r^2 \rho(r)\varepsilon(r) dr$$

$$\propto \int_0^R r^2 \rho(r)^2 T(r)^4 dr$$

Plotting the integrand as a function of *r* and normalizing to the total luminosity shows the radial distribution of energy production in the star.



One of my goals of the project was to try and show how nearly all the fusion happens in the core of the star. While the distribution seems reasonable for lower mass stars, the higher mass stars clearly show much more burning at a further radius than observed. I'm sure this is mainly due to the poor density approximation. A more accurate profile would have the density drop off much more rapidly, and consequently the luminosity would drop significantly, as it is proportional to ρ^2. The inaccuracy is shown clearly when plotting the luminosity as a function of the mass of the star.



The dashed line shows the observed relation $L \propto M^{3.5}$. Clearly the stars in my model produce way more energy than is observed. The pure hydrogen assumption probably also aids to the inaccuracy, as not only does metallicity affect luminosity, but I completely ignored hydrogen burning through the CNO cycle as the stars in my model contained no Carbon, Nitrogen, or Oxygen.