Untitled Project

Christopher Kirwan 17325310
Project Supervisor: Sinead Ryan

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Chapter 1

Introduction

1.1 The Need For Spin Models

The study of statistical mechaincs arose due to the need to connect the physical descriptions of large particle systems in the macroscopic and microscopic realms. This is done by bridging thermodynamic quantities (macroscopic) , such as temperature, with microscopic observables that fluctuate about some average. This is achieved using statistical methods and probability theory. Applying the methods developed by statistical mechanics to various mathematical models has given physicsts enormous insight into various physical phemoena and gives justification to study these models in detail.

We begin by constructing a physical model, using a substance whose individual atoms are arranged in a regular crystalline lattice. Furthermore, suppose that each atom in the lattice has a magnetic moment which we call it's spin. In this picture, we also assume that these moments tend to align with their neighbours[1] and an external magnetic field H. We introduce a measurement parameter called the magnetization, which is simply the global average of these spins.

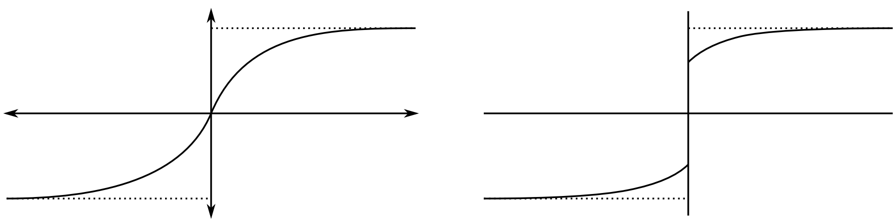

Figure 1.1

Varying the external magnetic field, we can observe two distinct behaviours around $H = 0$, called paramagnetic and ferromagnetic behaviour, first observed by Pierre Curie in 1895. In the first case, as $H \to 0$, the global order (ie: spin alignment) is lost and the magnetization tends to zero. In the ferromagnetic case, the local spin interactions are strong enough that the substance maintains a global non-zero net magnetization. The value of this magnetization depends on the direction in which the magnetic field approaches 0. Thus, a ferromagnet displays a *spontaneous magnetization* $\pm M$ at $H = 0$. This is a discontinuity in the magnetization, which corresponds to a *first order phase transition*

[1]this assumption will be made concerete later

The observations by Curie also established that a temperature dependent transition occurs in ferromagnetic materials, in which the material changes into a paramagnetic material. This transition occurs at a well-defined temperature called the Curie Temperature. A goal of 20th century thermodynamics was to be able to describe this phase transition using the framework of statistical mechanics.

In 1920, Wilhelm Lenz introduced what is now called the Ising Model (a term coined by Rudolph Peierls) to help understand that phase transition. The one-dimensional case was solved by

1.2 Defintions for the Spin $\mathrm{O}(n)$ (Symmetric) Model

We begin the discusiion of the model by defining the system on which the model "lives".

Definition 1.1. Let $d \geq 1$ and let $G = (V(G), E(G))$ be a finite, non-oriented graph. The set of vertices Λ is defined to be a subset of $\mathbb{Z}^d$ (that is, $V(G) = \Lambda$, with the coordinates $i = (i_1, \dots, i_d) \in \Lambda$ being integers). We also define the egdes of the graph $E(G)$ to be between the *nearest neighbours* of Λ. That is, pairs (i, j) with $\|i - j\|_1 = 1$. We denote nearest neignbours as $\langle i, j \rangle$.

Throughout the remainder of this text, we shall call the graph $G = (\Lambda, E)$ a *lattice* and simply refer to it as Λ. We also remark that Λ can also be $\mathbb{Z}^d$ itself. This is referred to as the *thermodynamic limit.* The need for this limit will be made apparent later.

Fixing $n \in N$, we have the *single spin configuration* as

$$\Omega_0 = \{\sigma \in \mathbb{R}^n \text{ such that } \|\sigma\|_2\} = 1 \equiv \mathbb{S}^{n-1}$$

Following from this, we have a *microstate*, often referred to as a *spin configuration*, on Λ as

$$\Omega_\Lambda = \Omega_0^\Lambda$$

and in the thermodynamic limit $\Omega = \Omega_0^{\mathbb{Z}^d}$

For each $i \in \Lambda$, we call the random variable $\boldsymbol{\sigma}_i : \Omega_\Lambda \to \mathbb{S}^{n-1}$ assigned to each point the lattice as the *spin.* We add two special restrictions onto these spins:

- That the spins themselves interact with their nearest neighbours
- This same interaction is invariant under a simultaneous rotation of the entire configuration[2]

With these restrictions, we can then assume that only a function of the inner/scalar product of the interacting spins contribute to the total energy of the system. We therefore arrive at what is *probably* the most important definition of the entire report

Definition 1.2. Let $U : [-1, 1] \to \mathbb{R}$ (usually referred to as the *potential function*) and $\mathcal{H}_\Lambda$ be the Hamiltonian of an $\mathrm{O}(n)$-symmetric model, over a lattice Λ. We thus have

$$\mathcal{H}_\Lambda = \sum_{\langle i,j \rangle \in E(G)} U(\langle \boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j \rangle) \tag{1.1}$$

In particular, if we have $U(x) = -Jx$, we then have the definiton of the Hamiltonian for the $\mathrm{O}(n)$ (or n-vector) model

$$H_\Lambda = - \sum_{\langle i,j \rangle \in E(G)} J_{ij} \langle \boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j \rangle \tag{1.2}$$

[2]The discussion of an external magnetic field will occur with the Ising Model

With $J = J_{ij} = J(i-j)$ being the *translation invariant interaction strength* between each spin on the lattice.

We can now plug in different values of n to get different models

- $n = 1$ gives the *Ising Model*, with spins taking values in $\Omega_0 = \{-1, 1\}$. The discussion on this model is given below
- $n = 2$, the spins lie along the circle with radius 1 ie, $\Omega_0 = \{x \in \mathbb{R}^2, \|x\|_2 = 1\}$ (hence O(2) symmetric). This is called the *XY Model* and is also central to the discussion.
- $n = 3, n \to \infty$ are called the *Heisenberg Model* and *Berlin–Kac Spherical Model.* $n = 4$ can be used as a "toy" model of the Higgs Sector.

Definition 1.3. At inverse temperature $\beta \in [0, \infty)$, configurations are randomly chosen from the probability measure μ defined as

$$\mathrm{d}\mu = \frac{1}{Z}\exp(\beta H(\sigma_i, \sigma_j))\mathrm{d}\sigma \tag{1.3}$$

where Z is the normalization constant called the *partition function*, given by

$$Z = \int_{\Omega_\Lambda} \exp(\beta H(\sigma_i, \sigma_j))\mathrm{d}\sigma \tag{1.4}$$

(Discussion on finite and infinte Gibbs measures/DLR formalism here??)

Usually, we restrict ourselves to a d-dimensional finite lattice, usually imposing what is called *boundary conditions* η on the lattice. This discussion is mainly oriented around the torus $\mathbb{T}_L^d$ with side length L, which is the set of vertices of G given by

$$V(\mathbb{T}_L^d) = \{0, 1, \cdots, L\}^d.$$

The edges are then defined with pairs of vertices (i, j) with $i = (i_1, \cdots, d)$ such that $\sum_{r=1}^d ((i_r - j_r) \mod L) = 1$. This boundary condition is said to be *periodic.* The configurations are then given on $\Omega_\Lambda = \Omega_0^\eta = \Omega_0^{\mathbb{T}_L^d}$.

From the definition ??, we note that the O(n) model with $\beta \in [0, \infty)$ is called *ferromagnetic.* If $U(x)$ is instead x, we call this *antiferromagnetic.* Note, however, that on biparte graphs, these relations are isomorphic through the map $f : \sigma_v \to -\sigma_v$. These definitions are actually quite different on non-biparte graphs (such as a triagonal lattice).

The above definitions of the O(n) spin models do admit some further generalizations. We can, for instance, take a vector $x \in \mathbb{R}^n$ and add a second term $\sum_{i\in\Lambda}\langle\sigma_i, x\rangle$ to the Hamiltonian defined in ??. This applys what we call an *external magnetic field* to the density in ??. We can also replace the *single site distribution* by modifying the measure $\mathrm{d}\sigma$ in ?? with another product measure on G. We can also make the Hamiltonian *anisotropic* by addding extra terms to the innner product used in the definition of the O(n) model.

1.3 The Ising Model

Using the definition of the O(n) spin moder as above, we now set $n = 1$. Thus, the spin configuration takes values in ± 1. Imposing the *periodic boundary conditions* as oulined in the previous section, we then get the Ising Model in d dimensions. We define the Hamiltonian and the Gibbs Measure/Distribution as:

Definition 1.4. The Gibbs Distribution of the Ising Model on Λ with η as the boundary condition is the probability distribution over Ω_Λ^η

$$\mathrm{d}\mu^\eta = \frac{1}{Z^\eta}\exp(\beta H(\sigma))\mathrm{d}\sigma$$

With the partition function

$$Z^\eta = \sum_{\sigma\in\Omega_\Lambda^\eta} \exp(H(\sigma))$$

Where we've replaced the integral with a summation as the single spin configuration is a discrete value. We've indexed with η to denote boundary conditions are enforced. The Hamiltonian H (without an external magnetic field) is then

$$H = -\sum_{\langle i,j\rangle} J_{ij}\sigma_i\sigma_j$$

We simplify the discussion further by allowing $J_{(ij)} = 1$ for all $(i, j) \in \Lambda$. Using this, we then define the *magnetization* to be a measurement of the whole system

$$M = \sum_{i=1}^{N} \sigma_i \Rightarrow \langle M\rangle = \frac{1}{N}M \tag{1.5}$$

with N being the total number of spins in the lattice (L^d).

1.3.1 Autocorrelations