Setting:

M: real analytic manifold

$\Lambda \subseteq S^*M$: closed subanalytic isotropic subset

"isotropic" meanings it is stratified by isotropic submanifolds.

Theorem (Ganatra-Pardon-Shende):

$\exists$ a equivalence of categories

$$\mathrm{Perf}\, \mathcal{W}(T^*M, \Lambda)^{op} \simeq Sh_\Lambda(M)^c$$

+ description on certain generators

<u>Rmk</u>: Proved by matching generators (functorially)

Conj.: $\exists$ a decomposition by a priori defined functors:

$$
\begin{tikzcd}
0.\ \mathrm{Perf}\,\mathcal{W}(T^*M,\Lambda)^{op} \arrow[rr, equal] \arrow[dr, dashed, "\text{not yet defined}"'] & & 1.\ Sh_\Lambda(M)^c \\
& 2.\ wsh_\Lambda(M) \arrow[ur, "3.\ w_\Lambda^+", "\sim"'] &
\end{tikzcd}
$$

0. $\mathcal{W}(T^*M, \Lambda)$:

$\alpha \quad r^i dx_i$

In priciple,

Obj. : Lagrangians which are exact, cylindrical at ∞, disjoint from Λ, ...

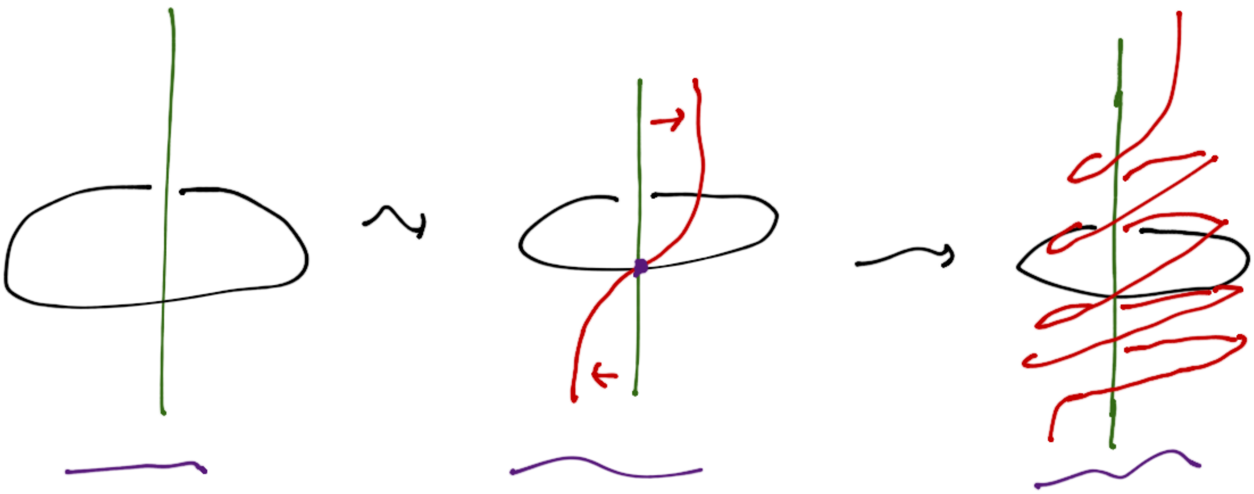
Mor. : Wrapped Floer cohomology after H;

$$HW^*(L, K) \simeq \operatorname*{colim}_{L \rightsquigarrow L^w} HF^*(L^w, K).$$

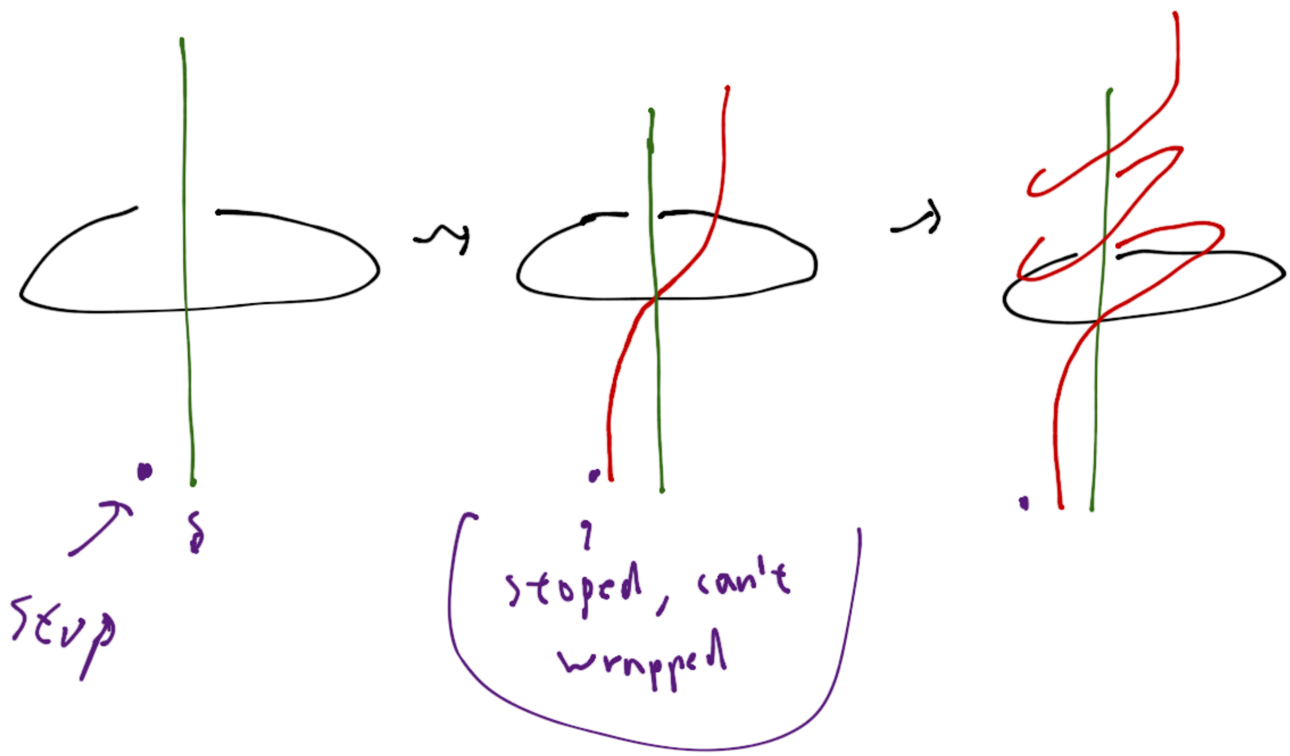
positive wrappings

$\alpha(\partial_t L_t) \geq 0$

e.g. $\mathcal{W}(T^*S^1, \emptyset)$, $HW^*(T_0^*S^1, T_0^*S^1) = \mathbb{Z}[t, t^{-1}]$



$\mathcal{W}(T^*S^1, s_0 \in S^1)$, $HW^*(T_0^*S^1, T_0^*S^1) = \mathbb{Z}[t]$



Quick tour of sheaf theory

$$\begin{tikzcd} Sh(M) \arrow[rr, hook, bend right] \arrow[rr, phantom, "\perp"] & & PSh(M) \arrow[ll, bend right, "(-)^{\dagger}"'] \end{tikzcd}$$

.. A presheaf $\mathcal{F}$ is a functor

$$\mathcal{F} : Op_M^{op} \longrightarrow Ch(\mathbb{Z}), \text{ i.e.}$$

Ch(k), Sp, $\mathcal{C}$: dg cat's / ∞ stable cat's

$\forall$ $U \subseteq V$ $\exists$ $\mathcal{F}(V) \longrightarrow \mathcal{F}(U)$

.. $\mathcal{F} \in PSh(M)$ is a sheaf if it satisfying certain gluing condition.

.. One can sheafify presheaf.

e.g. $M = *$, $PSh(*) = Sh(*) = Ch(\mathbb{Z})$.

<u>Def'n</u>: The stalk of $\mathcal{F}$ at $x \in M$ is

$$\mathcal{F}_x := \underset{U \ni x}{\operatorname{colim}} \mathcal{F}(U). \in Ch(\mathbb{Z})$$

<u>lemma</u>: A morphism $\alpha : F \to G$ in $Sh(M)$ is an isom. iff $\alpha_x : F_x \to G_x$ is an isom. $\forall x \in M$,

e.g. For $\underline{A} \in Ch(\mathbb{Z})$, the constant sheaf $\underline{A}_M$ is the sheafication of

$$(\underline{U} \subseteq V) \longmapsto \underline{A} \xrightarrow{=} \underline{A}.$$

$$(\underline{A}_M)_x = A \quad \forall x \in M.$$

Some ways to build more sheaves (six-functor formalism)

$*$-adjunction:

Let $f: M \to N$ be a map. $\exists$

$$f_* : Sh(M) \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} Sh(N) : f^*$$

.. For $F \in Sh(M)$, $(f_* F)(V) := F(f^{-1}V)$

.. For $G \in Sh(N)$, $f^* G$ is the sheafification

of $\quad U \longmapsto \operatorname{colim}_{V \supseteq f(U)} G(V)$.

Facts:

1. $(f^* G)_x = G_{f(x)}$

2. Let $i_x : \{x\} \hookrightarrow M$. Then $i_x^* F = F_x$.

3. Let $p : M \to \{*\}$. Then $p^* A = A_M$.

$!$-push : Consider the map $f : M \to N$ again.

$\exists$ the proper pushforward $f_! : Sh(M) \to Sh(N)$.

Facts:

1. Let $p : M \to \{*\}$. Then $p_! F = \Gamma_c(M; F)$, the compactly supported sections.

2. For $y \in N$, $(f_! F)_y = \Gamma_c(f^{-1}(y); F|_{f^{-1}(y)})$.

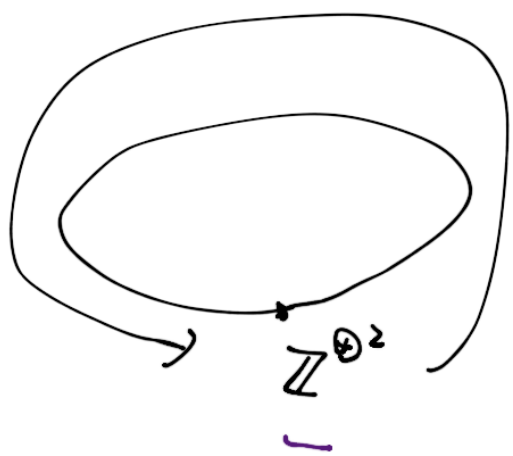
e.g. $p: \coprod_{n\in\mathbb{Z}} \{n\} \longrightarrow \{*\}$

For $F \in Sh\left(\coprod_{n\in\mathbb{Z}} \{n\}\right)$, $p_* F = \prod_n F_n$ while

$p_! F = \bigoplus_n F_n$.

e.g. $f : S^1 = \mathbb{R}^1/\mathbb{Z}^1 \longrightarrow S^1$ by $f(\bar{x}) = \overline{2x}$.

$f_* \mathbb{Z}_{S^1} = f_! \mathbb{Z}_{S^1} =$



$m(a_1, a_2)$
$= (a_2, a_1)$

Not constant.

Def'n: A sheaf $\mathcal{F} \in Sh(M)$ is local constant or a local system if $\exists$ cover $\{U\}_i$ of M s.t.

$\mathcal{F}|_{U_i} = A_{U_i}$ for some $A \in Ch(\mathbb{Z})$.

Denote them by $Loc(M)$.

Let $\mathcal{S} = \{M_\alpha\}_\alpha$ be a stratification. $M = \coprod M_\alpha$

Def'n: ① A sheaf $\mathcal{F} \in Sh(M)$ is $\mathcal{S}$-constructible if $\mathcal{F}|_{M_\alpha} \in Loc(M_\alpha)$.

② $\mathcal{F}$ is constructible if it is $\mathcal{S}$-constructible for some $\mathcal{S}$

($i : Z \hookrightarrow M$ (loc closed), $A_Z := i_! A_Z$)

e.g. Write $(-\infty, 0] \overset{i}{\hookrightarrow} \mathbb{R}^1 \overset{j}{\hookleftarrow} (0,\infty)$.

$\mathbb{Z}_{(-\infty,0]}$: $\mathbb{Z} = \mathbb{Z} \to 0$

$\mathbb{Z}_{(0,\infty)}$: $0 = 0 \to \mathbb{Z}$

are both constructible.

• Microsupport:

For $F \in Sh(M)$, one can assign a conic closed subset $SS(F) \subseteq T^*M$.

.. Complicated definition for general F

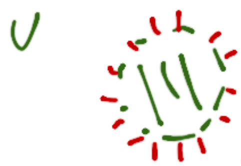
.. $SS(F)$ can be recover from

$\mathrm{supp}(F)$ and $SS^\infty(F) := (SS(F) \setminus 0_M)/\mathbb{R}_{>0}$

$= SS(F) \cap 0_M$

e.g. Let $U \subseteq M$ be an open set with smooth boundary. Then

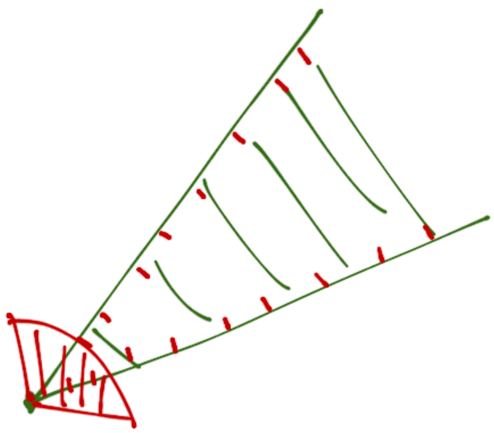
$$SS(\mathbb{Z}_U) = N^*_{out}(U), \quad SS(\mathbb{Z}_{\bar{U}}) = N^*_{in}(U).$$

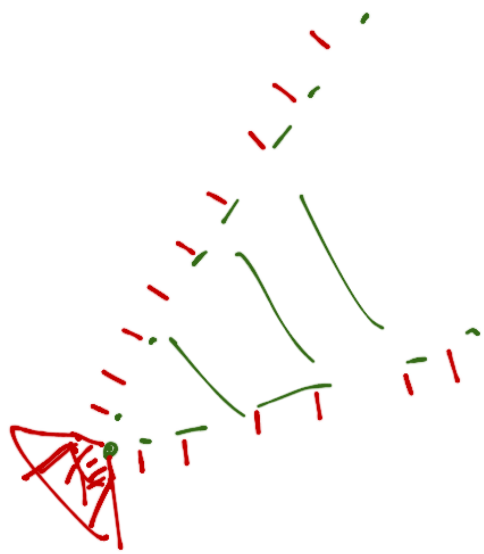


e.g. If $\gamma \subseteq \mathbb{R}^n$ is a closed cone,

then $\quad SS(\mathbb{Z}_\gamma) \cap T_0^* \mathbb{R}^n = \gamma^\circ$

<u>Rmk</u>: In practice, one obtains bounds on SS from simple ones plus compatibility with standard functors.



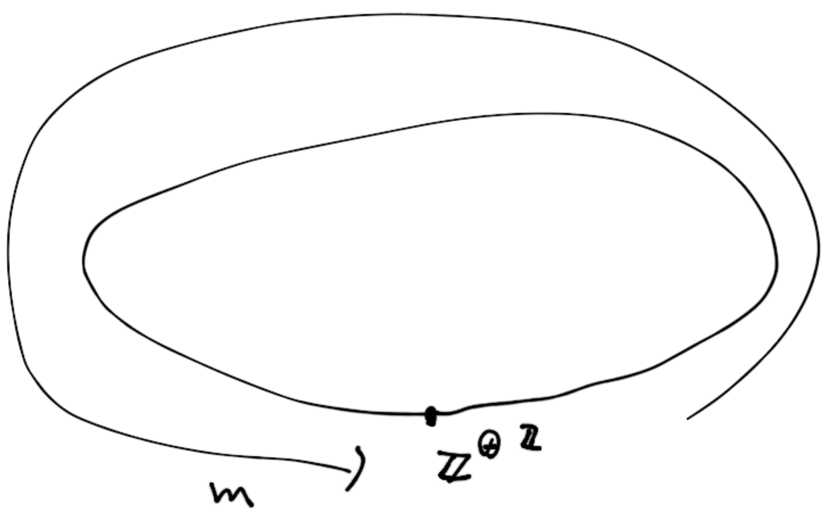


Theorem (Kashiwara–Schapira) Let $F \in Sh(M)$. Assume $SS^\infty(F)$ is subanalytic. Then F is constructible iff $SS^\infty(F)$ is isotropic.

Cor. $Sh_\Lambda(M) := \{ F \in Sh(M) \mid SS^\infty(F) \subseteq \Lambda \}$ consists of constructible sheaves

e.g. Consider $Loc(S^1) = Sh_\emptyset(S^1)$. $k^{\oplus \mathbb{Z}}$

$\pi_! \mathbb{Z}_{\mathbb{R}^1} \in Loc(S^1)$, $(\pi_! \mathbb{Z}_{\mathbb{R}^1})_0 = \Gamma_c(\coprod_{n\in\mathbb{Z}} \{n\}; \mathbb{Z}) = \mathbb{Z}^{\oplus \mathbb{Z}}$



0-th

$m(\underline{\cdots, a_{-1}, a_0, a_1, \cdots})$

0-th

$= (\underline{\cdots, a_{-2}, a_{-1}, a_0, \cdots})$

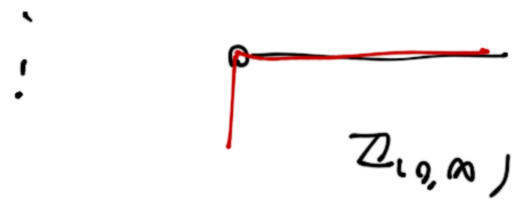
$$\mathrm{Hom}(\pi_! \mathbb{Z}_{\mathbb{R}^1}, \pi_! \mathbb{Z}_{\mathbb{R}^1}) = \mathrm{Hom}(\mathbb{Z}_{\mathbb{R}^1}, \underline{\pi^* \pi_! \mathbb{Z}_{\mathbb{R}^1}}) = \underline{\mathbb{Z}[t, t^{-1}]}$$

$$\| $$

$$\Gamma(\mathbb{Z}_{\mathbb{R}^1}; \underline{\pi^* \pi_! \mathbb{Z}_{\mathbb{R}^1}})$$

e.g. $\Lambda = S^*_{n.1} S^1$.

Recall $SS(\mathbb{Z}_{[0,\infty)})$:



$\Rightarrow$ $SS(\pi_! \mathbb{Z}_{[0,\infty)})$:

As a comparison to $W(T^*S^1, \Lambda)$,

one should think $\pi_! \mathbb{Z}_{[0,\infty)}$ as

Furthermore, one can compute

$$\mathrm{Hom}(\pi_! \mathbb{Z}_{(0,\infty)}, \pi_! \mathbb{Z}_{(0,\infty)}) \simeq \mathbb{Z}[t] .$$

· Isotopies of sheaves

·· We first define convolution :

Let $F \in Sh(M)$, $K \in Sh(M \times N)$.

$$p_1^* F \to K \otimes p_1^* F$$

Write
$$\begin{tikzcd} & M \times N \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ F \quad M & & N \quad \cdot \quad p_{2!}(K \otimes p_1^* F) \end{tikzcd}$$

Then K induces a functor

$$K \overset{*}{\circ} (-) : Sh(M) \longrightarrow Sh(N) \text{ by}$$

$$F \longmapsto K \circ F := p_{2!}(K \otimes p_1^* F).$$

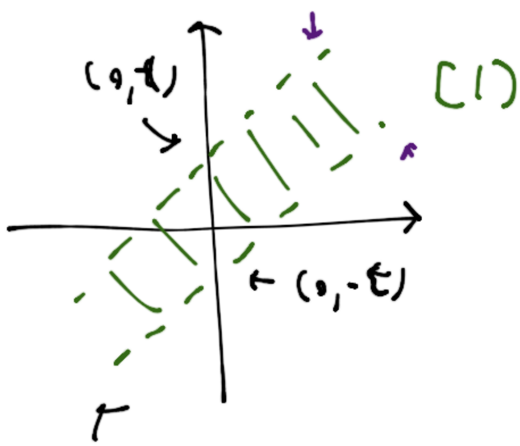
e.g.① Consider $M = N$ and $K = \mathbb{Z}_\Delta$, then

$$K \circ F = p_{2!}\left(\mathbb{Z}_\Delta \otimes p_1^* F \right)$$

$$= p_{2!}\,\Delta_! \circ \Delta^* p_1^* F$$

$$= (p_2 \circ \Delta)_! \circ (p_1 \circ \Delta)^* F = F$$

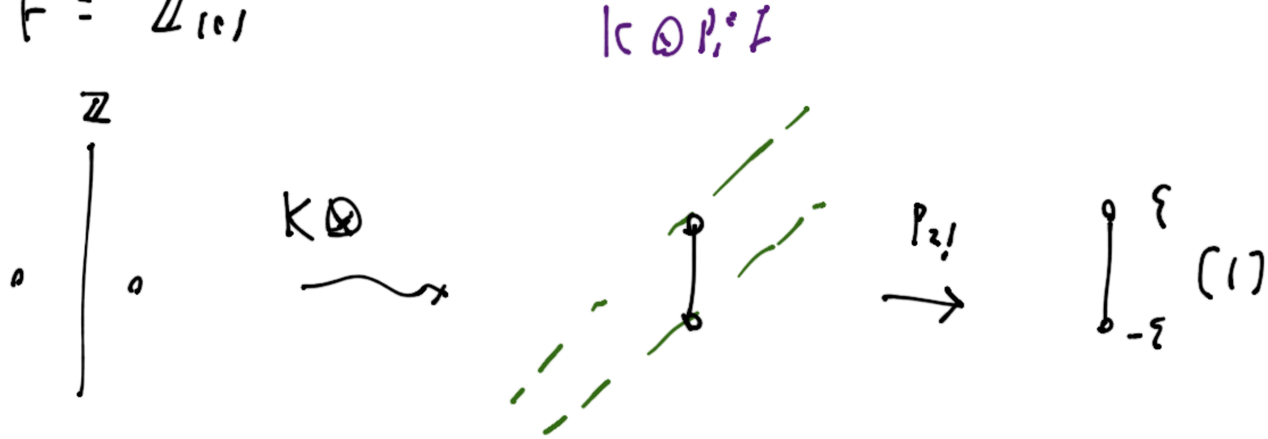
e.g.② Consider $M = N = \mathbb{R}^1$, and

$K =$ 

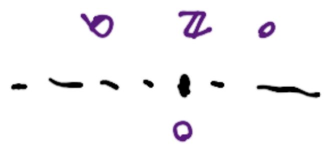
$= \mathbb{Z}_{\{x-t<y<x+t\}}[1]$

When $F = \mathbb{Z}_{\{0\}}$

$p_1^* F$

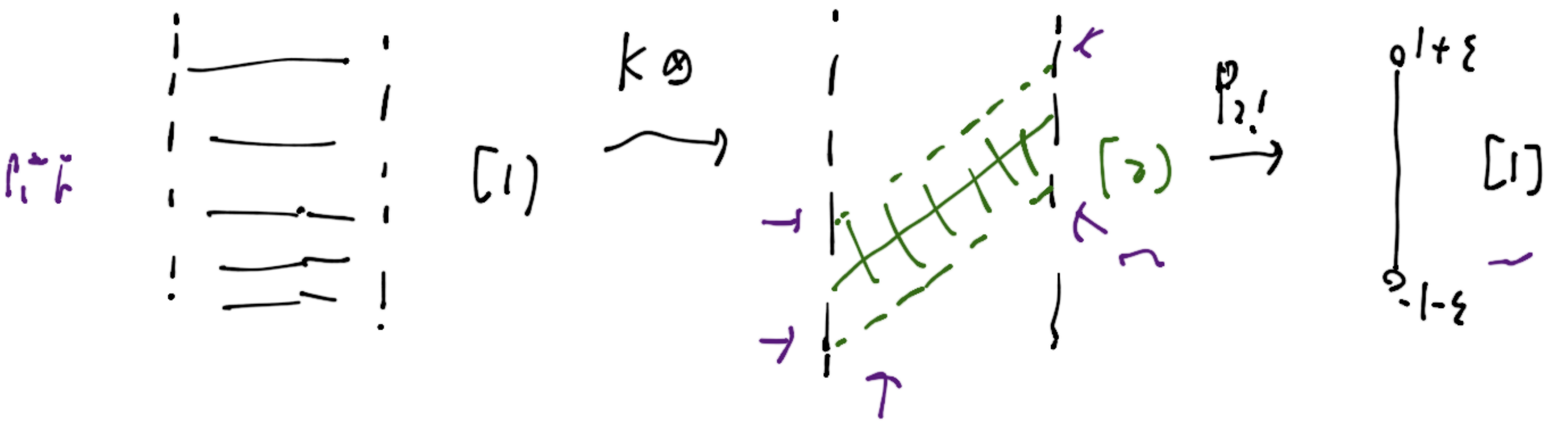


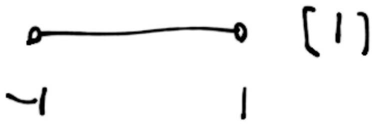
F



That is, $\mathbb{Z}_{\{x-\varepsilon<y<x+\varepsilon\}}[1] \circ \mathbb{Z}_{\{0\}} = \mathbb{Z}_{(-\varepsilon,\varepsilon)}[1]$

When $F = \mathbb{Z}_{(-1,1)}[1]$





Rmk: The last arrow is a combination

of ① $(p_{2!}G)_y = \Gamma_c(\mathbb{R}\times\{y\}; G|_{\mathbb{R}\times y})$ and

② $\Gamma_c(\mathbb{R}^1; \mathbb{Z}) = \mathbb{Z}[-1]$

$\Gamma_c(\mathbb{R}^n; \mathbb{Z}) = \mathbb{Z}[-n]$

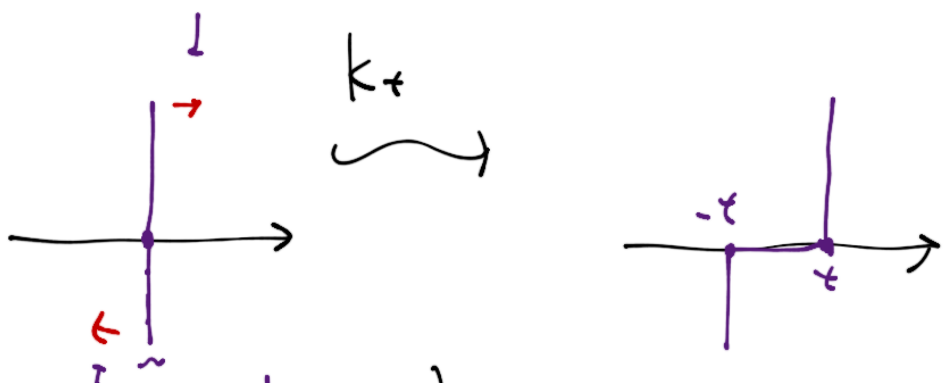
Observation:

Write $K_t = \mathbb{Z}_{\{x-t<y<x+t\}}[1]$, then

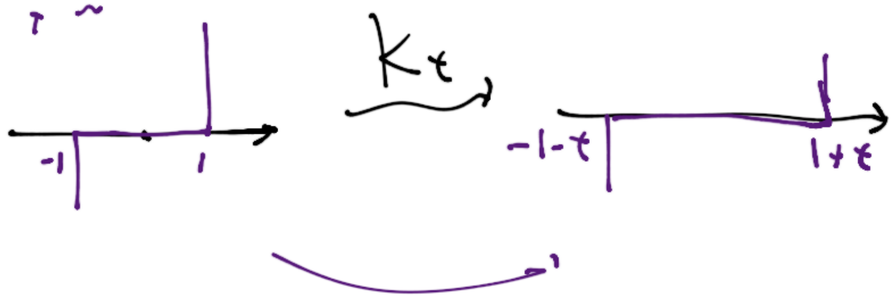
K_t moves the microsupport positively by t

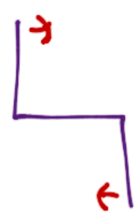
That is,

$SS(\mathbb{Z}_{\{0\}}) =$



$SS(\mathbb{Z}_{(-1,1)}) =$





More precisely, let Φ denote the contact isotopic $\Phi(x,\xi,t) := \begin{cases} (x+t,\xi), & \xi > 0 \\ (x-t,\xi), & \xi < 0 \end{cases}$

$\varphi_t := \Phi(\cdot,-,t)$

One notice that $\underbrace{\varphi_t\, SS^\infty(\mathbb{Z}_{\{0\}})}\ = SS(\underbrace{K_t \circ}\mathbb{Z}_{\{0\}})$.

In general,

Thm (Kashiwara - Schapira):

Let M be a manifold and $\underbrace{\Phi : S}^*M \times I \to S^*M$ a contact isotopy. Then $\exists!\ K(\Phi) \in Sh(M \times M \times I)$ s.t. ① $SS^\infty(K(\Phi)) \subseteq \Lambda_\Phi$: certain Legendrian determined by Φ

② $K(\Phi)|_{t=0} = \mathbb{Z}_\Delta$.

Def'n For a $F \in Sh(M)$, call the sheaf

$$K(\Phi) \circ F := p_{2!}\left(K(\Phi) \otimes p_1^* F \right)$$

where

$$\begin{tikzcd} \overset{x}{M} \times \overset{y}{M} \times \overset{t}{I} \arrow[r, "p_2"] \arrow[d, "p_1"'] & \overset{y}{M} \times \overset{t}{I} \\ \overset{x}{M} & \end{tikzcd}$$

the isotopy of F by Φ.

Let $i_t : M \times \{t\} \hookrightarrow M \times I$.

Since $i_t^*(K(\Phi) \circ F) = K(\Phi)|_t \circ F$,

write $F_t := K(\Phi)|_t \circ F$.

Uniqueness of $K(\Phi) \Rightarrow SS^\infty(F_t) = \varphi_t(SS^\infty(F))$

· Wrapped sheaves:

<u>Rmk</u>: When Φ is positive, i.e., $\alpha(\partial_t \Phi) \geq 0$,

$\exists$ canonical map $K(\Phi)|_a \to K(\Phi)|_b$

for $a \leq b$.

$\Rightarrow$ For $F \in Sh(M)$, $\exists$ continuous map (def'n)

$F_a \longrightarrow F_b$ for $a \leq b$.

We mimic the definition of $WF(T^*M, \Lambda)$

to define $wsh_\Lambda(M)$:

That is, let

$$\widetilde{wsh}_\Lambda(M) := \{ F \in Sh(M) \mid \underline{supp(F) : compact},$$

$SS^\infty(F)$ is isotopic to a subanalytic isotropic, away from Λ

F_x is perfect $\forall x$ $\}$.

Let $C_{M,\Lambda} := \{ F \xrightarrow{c} F^w \mid F \in \widetilde{wsh}_\Lambda(M) \}$

Here we use w to denote any positive wrapping ^compactly supported away from Λ.

Def'n

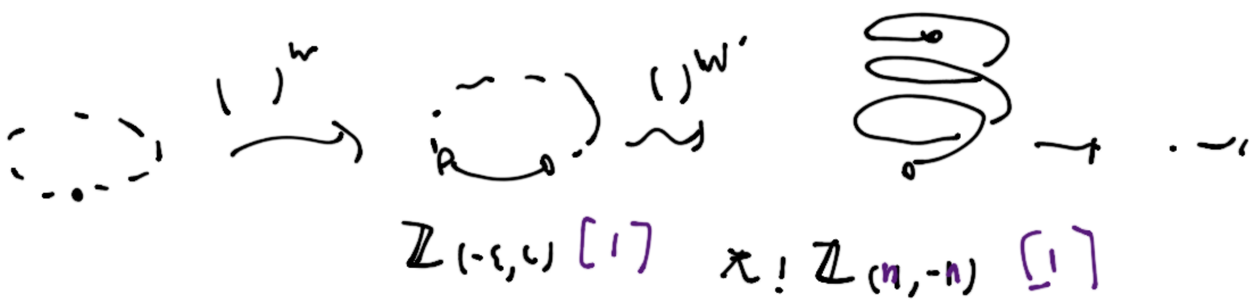
$$wsh_\Lambda(M) := C_{M,\Lambda}^{-1}\, \widetilde{wsh}_\Lambda(M),$$ i.e., we formally invert the continuation maps.

Lemma: Let $F, G \in wsh_\Lambda(M)$.

Then $\mathrm{Hom}_w(G, F) \simeq \operatorname*{colim}_{F \to F^w} \mathrm{Hom}(G, F^w)$.

e.g. $wsh_\emptyset(S^1)$:

Take $\mathbb{Z}_{(0)} \in wsh_\emptyset(S^1)$.



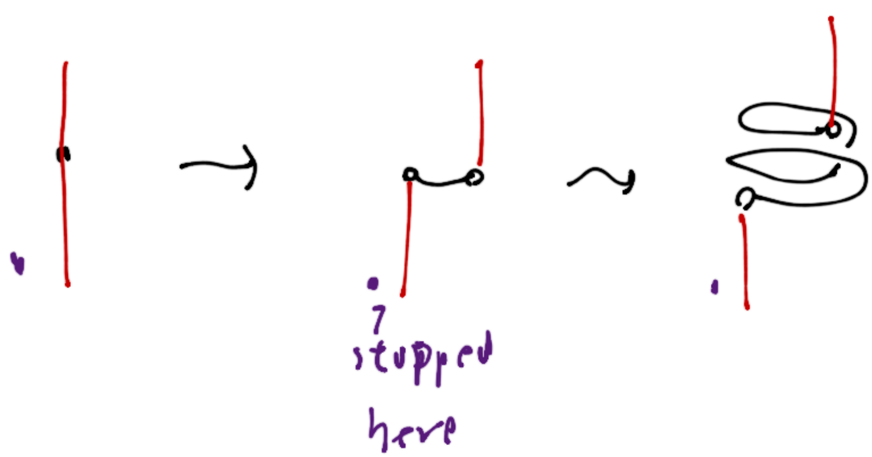
$$\mathrm{Hom}_w(\mathbb{Z}_{\{0\}}, \mathbb{Z}_{\{0\}}) = \mathrm{Hom}_w(\mathbb{Z}_{(-\varepsilon,\varepsilon)}[1], \mathbb{Z}_{(-\varepsilon,\varepsilon)}[1])$$

cancel each other

$$= \operatorname*{colim}_{n \in \mathbb{N}} \mathrm{Hom}(\mathbb{Z}_{(-\varepsilon,\varepsilon)}, \mathbb{Z}_{(n-\varepsilon, n+\varepsilon)})$$

$$= \operatorname*{colim}_{n \in \mathbb{N}} \mathbb{Z}^{\oplus 2n+1} = \mathbb{Z}^{\oplus \mathbb{Z}} = \mathbb{Z}[t, t^{-1}].$$

Similarly, if $\Lambda = S^{\cdot}_{0,\le} S^1$, being with $\mathbb{Z}_{\{\delta\}}$, $\delta > 0$,



and $\mathrm{Hom}_w(\mathbb{Z}_{\{\delta\}}, \mathbb{Z}_{\{\delta\}}) = \mathbb{Z}[t]$.

Now note that abstract reason implies $\exists$ adjunction

$$\begin{tikzcd} Sh_\Lambda(M) \arrow[rr, shift right, "\iota_{\Lambda *}"'] \arrow[rr, phantom, "\perp"] & & Sh(M) \arrow[ll, bend right, "\iota_\Lambda^*"'] \end{tikzcd}$$

Prop. (K, previously by Shende with constructible condition.)

$$\iota_\Lambda^* F = w_\Lambda^+ F := \operatorname{colim}_{w:\Lambda^c} F^w.$$

That is, taking colimits over continuation maps away from Λ blows away microsupport.

Since $F \xrightarrow{c} F^w$ becomes an isom. after $\mathcal{W}_\Lambda^+$,

$\mathcal{W}_\Lambda^+$ tautologically descends to

can show it easily

$$\mathcal{W}_\Lambda^+ : wsh_\Lambda(M) \longrightarrow Sh_\Lambda(M)^c.$$

Thm (K). $\mathcal{W}_\Lambda^+$ is an equivalence.

e.g. $\mathcal{W}_\emptyset^+ : wsh_\emptyset(S^1) \xrightarrow{\sim} Loc(S^1)^c$

$$\mathbb{Z}_{\{\delta\}} \longmapsto \pi_! \mathbb{Z}_{\mathbb{R}^1}[1]$$

$\Lambda = S^* S^1_{0,+}$

$$\mathcal{W}_\Lambda^+ : wsh_\Lambda(S^1) \xrightarrow{\sim} Sh_\Lambda(S^1)^c$$

$$\mathbb{Z}_{\{\delta\}} \longmapsto \pi_! \mathbb{Z}_{(0,\infty)}[1].$$