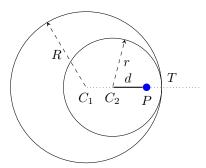
NOTES ON HYPOTROCHOIDS AND EPITROCHOIDS

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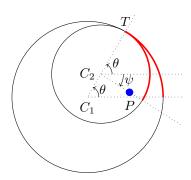
1. Hypotrochoids

A hypotrochoid is the curve obtained by tracing the positions taken by a point P rigidly attached to a circle C_2 of centre C_2 and radius r, P being at a distance d from C_2 , with C_2 rolling around the inside of another circle C_1 of centre C_1 and radius R. To compute the equation of the curve, one assumes that C_1 is located at the origin of the plane, so has coordinates (0,0), and C_1 , C_2 and P are horizontally aligned, in that order from left to right, as shown in the following picture.



As C_2 rotates clockwise and moves anticlockwise around the inside of C_1 , when $\overrightarrow{C_1C_2}$ has gone from an angle of 0 to a positive angle of θ , and $\overrightarrow{C_2P}$ from an angle of 0 to a negative angle of ψ , the point of contact T between both circles has travelled the same distance along both circles—represented in red in the picture below—, namely, θR on C_1 , and $(\theta - \psi)r$ on C_2 . Hence:

$$\psi = -\frac{R-r}{r}\theta$$



At this stage, since $\overrightarrow{C_1P} = \overrightarrow{C_1C_2} + \overrightarrow{C_2P}$, the point P has coordinates:

$$x = (R - r)\cos(\theta) + d\cos(-\psi)$$

$$y = (R - r)\sin(\theta) + d\sin(-\psi)$$

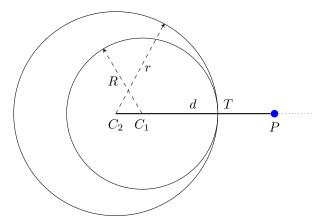
that is:

$$x = (R - r)\cos(\theta) + d\cos\left(\frac{R - r}{r}\theta\right)$$

$$y = (R - r)\sin(\theta) - d\sin(\frac{R - r}{r}\theta)$$

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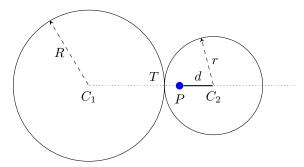
Note that P can "stick out" of C_2 , that is, d can be larger than r, as shown in the following picture, which also illustrates that C_2 can be larger than C_1 , that is, r can be greater than R; that does not change the above reasoning and the equations still hold.



The *period* of a hypotrochoid is the number of times T gets back to its original position, as C_2 keeps rotating around the inside of C_1 , for P to get back to its original position. It is equal to the least strictly positive integer ρ such that $\rho \times 2\pi R$ is a multiple of $2\pi r$; hence it is equal to $\frac{r}{\gcd(r,R)}$.

2. Epitrochoids

If we let C_2 roll around the outside rather than the inside of C_1 , then the curve obtained by tracing the positions taken by P is called an *epitrochoid*. To compute the equation of the curve, one assumes that C_1 , C_2 and P are horizontally aligned, with C_2 to the right of C_1 and with P to the left of C_2 , and also to the left of C_1 in case d is greater than R + r; the following picture illustrates the case where r < R and d < r.



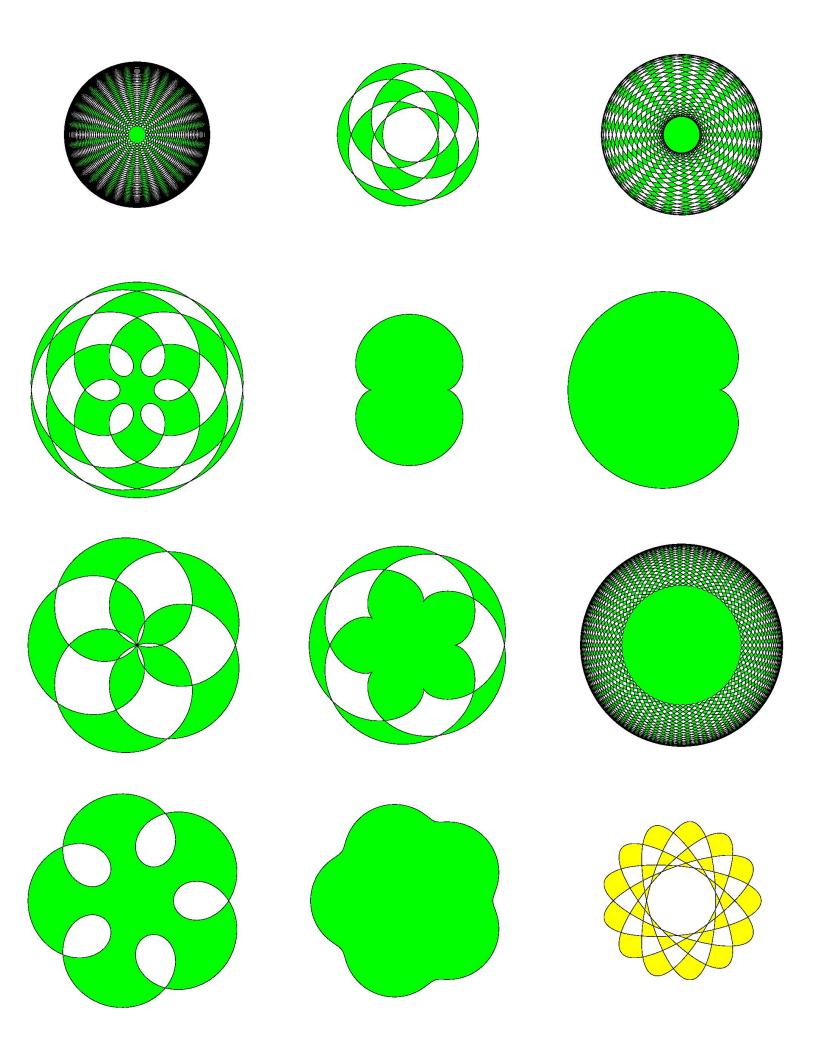
The reasoning that yields the equations for hypotrochoids can be immediately adapted to epitrochoids and result in the following equations:

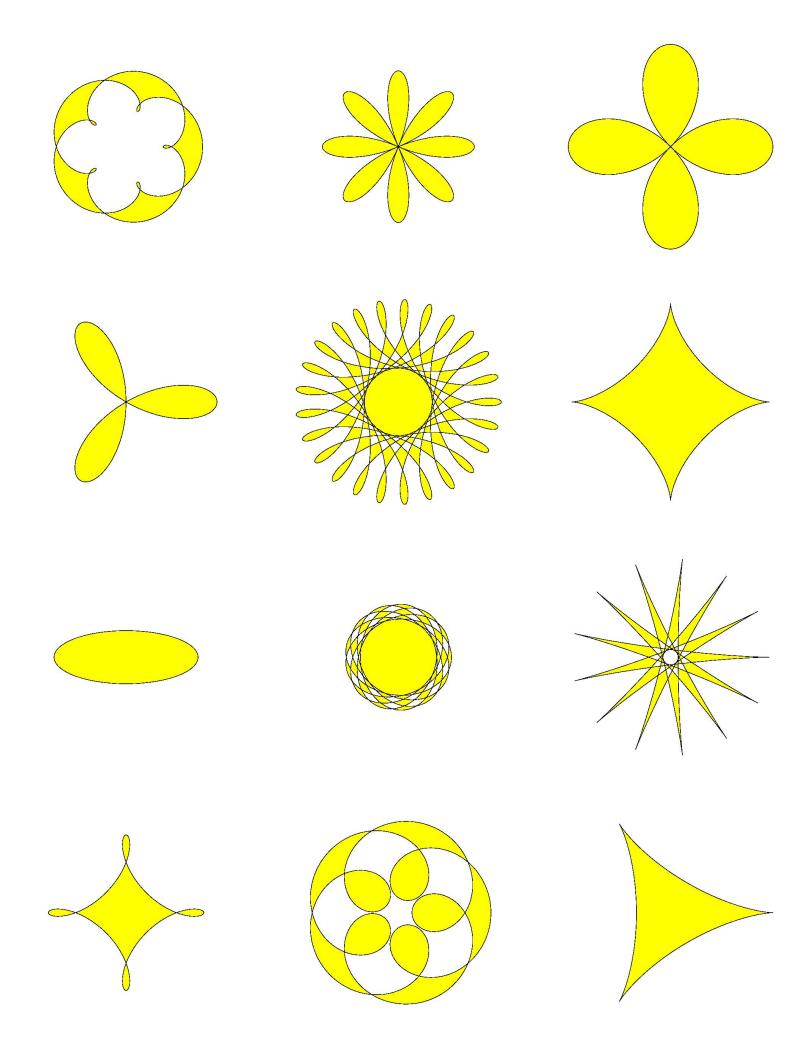
$$x = (R+r)\cos(\theta) - d\cos\left(\frac{R+r}{r}\theta\right)$$
$$y = (R+r)\sin(\theta) - d\sin\left(\frac{R+r}{r}\theta\right)$$

The period of an epitrochoid is also equal to $\frac{r}{\gcd(r,R)}$.

3. Particular cases

Ellipse, deltoid, astroid, nephroid, cardioid and roses are amongst the following pictures of epitrochoids (with a green filling) and hypotrochoids (with a yellow filling).





The following table shows how ellipse, deltoid, astroid, nephroid and a few other particular cases are obtained. When d is equal to r, hypotrochoids are also called hypocycloids, and epitrochoids are also called epicycloids.

| | Hypotrochoids | | | | | Epitrochoids | |
|-------|-------------------|---------------------------------------|---------------------------------------|--------------------|----------|-------------------|----------------|
| | $r = \frac{R}{2}$ | $r \in \{\frac{R}{3}, \frac{2R}{3}\}$ | $r \in \{\frac{R}{4}, \frac{3R}{4}\}$ | $r = \frac{3R}{2}$ | r = 2R | $r = \frac{R}{2}$ | r = R |
| d=r | ellipse | deltoid | astroid | nephroid | cardioid | nephroid | cardioid |
| d = 0 | segment | nent circle | | | | | |
| Any d | | | | | | | Pascal limaçon |

To be complete, one should let R be ∞ ; then \mathcal{C}_1 is a line and the associated curves are called *trochoids*, with *cycloids* as a particular case when d = r...

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