1

1.(a) 设 $f(x)=\frac{1}{2}x^TGx+b^Tx+c$

$g(x)=Gx+b$

对于最优点 x^* 有 $Gx^*+b=0$

$$g(x^{(0)})=G(x^*+\mu s)+b$$
$$=Gx^*+b+G\mu s$$
$$=G\mu s$$

由特征向量的性质得 $Gs=\lambda s$

$\therefore g^{(0)}=\mu\lambda s$

$$x^{(1)}=x^{(0)}-\frac{1}{\lambda}g^{(0)}=x^{(0)}-\mu s=x^*$$

∴一次迭代后终止

(b) 设 $G=aI$, $x^{(0)}=x^*+p$

$$g^{(0)}=G(x^*+p)+b$$
$$=Gp$$

$Gp=ap$

$$x^{(1)}=x^{(0)}-\frac{1}{a}g^{(0)}=x_0-p=x^*$$

∴一次迭代后终止

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2.(a)
$$\begin{cases}\frac{\partial q}{\partial x_1}=10x_1-x_2-11=0\\ \frac{\partial q}{\partial x_2}=10x_2-x_1+11=0\end{cases}$$

$\therefore x^*=(1,-1)^T$

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(b) $G=\begin{bmatrix}10 & -1\\ -1 & 10\end{bmatrix}$

∵ G 正定

∴ $q(x)$ 是凸函数

(c) G 的特征值 λ 满足 $(\lambda-10)^2+1=0$

$\therefore \lambda=9,11$

$$\frac{f(x_{k+1})-f(x^*)}{f(x_k)-f(x^*)}\le\left(\frac{11-9}{11+9}\right)^2=0.01$$

∴收敛因子最大不超过0.01

3

3. 经共轭梯度法求得 $x^*=(-0.4472, -1.8944, -3.3416, -2.7889)^T$

$f(x^*) = -6.2361$

$g^{(0)} = Gx^{(0)}+b = (-1, 0, 2, \sqrt{5})^T$

$Gg^{(0)} = (-2, -1, 4-\sqrt{5}, -2+2\sqrt{5})^T$

$$G^2g^{(0)} = \begin{pmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{pmatrix}(-1, 0, 2, \sqrt{5})^T = (-3, -4+\sqrt{5}, 11-4\sqrt{5}, -8+5\sqrt{5})^T$$

$g^{(0)}$ 和 $Gg^{(0)}$ 显然线性无关

$G^2g^{(0)} = \lambda_0 g^{(0)} + \lambda_1 Gg^{(0)}$ 可求得 $\lambda_0 = -5+2\sqrt{5}$，$\lambda_1 = 4-\sqrt{5}$

$G^3g^{(0)} = G\cdot G^2g^{(0)} = \lambda_0 Gg^{(0)} + \lambda_1 G^2 g^{(0)} = \lambda_0\lambda_1 g^{(0)} + (\lambda_0+\lambda_1^2)Gg^{(0)}$

∴向量组 $g^{(0)}, Gg^{(0)}, G^2g^{(0)}, G^3g^{(0)}$ 中只有两个独立向量

(b) 迭代次数与 $g^{(0)}, Gg^{(0)}, G^2g^{(0)}, G^3g^{(0)}$ 中独立向量的个数相等

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4

4.(a) $H^{(0)}=I$, $g=\begin{bmatrix}20x_1\\2x_2\end{bmatrix}$, $G=\begin{bmatrix}20&0\\0&2\end{bmatrix}$, $x^{(0)}=(0.1,1)^T$

$$g^{(0)}=\begin{bmatrix}2\\2\end{bmatrix}$$

$$p^{(0)}=-H_0g_0=\begin{bmatrix}-2\\-2\end{bmatrix}$$

$$\alpha^{(0)}=\frac{g^{(0)T}p^{(0)}}{p^{(0)T}Gp^{(0)}}=\frac{8}{88}=\frac{1}{11}$$

$$x^{(1)}=x^{(0)}+\alpha^{(0)}p^{(0)}=(-\frac{9}{110},\frac{9}{11})^T$$

$$g^{(1)}=(-\frac{18}{11},\frac{18}{11})^T$$

$$s^{(0)}=x^{(1)}-x^{(0)}=(-\frac{2}{11},-\frac{2}{11})^T$$

$$y^{(0)}=g^{(1)}-g^{(0)}=(-\frac{40}{11},-\frac{4}{11})^T$$

$$H^{(1)}=H^{(0)}+\frac{s^{(0)}s^{(0)T}}{s^{(0)T}y^{(0)}}-\frac{H^{(0)}y^{(0)}y^{(0)T}H^{(0)}}{y^{(0)T}H^{(0)}y^{(0)}}=I+\begin{pmatrix}\frac{1}{22},&\frac{1}{22}\\\frac{1}{22},&\frac{1}{22}\end{pmatrix}-\begin{pmatrix}\frac{100}{101},&\frac{10}{101}\\\frac{10}{101},&\frac{1}{101}\end{pmatrix}$$

$$=\frac{1}{2222}\begin{bmatrix}123&-119\\-119&2301\end{bmatrix}$$

$$p^{(1)}=-H^{(1)}g^{(1)}=\frac{1}{101}(18,-180)^T$$

$$\alpha^{(1)}=\frac{g^{(1)T}p^{(1)}}{p^{(1)T}Gp^{(1)}}=\frac{101}{220}$$

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$$x^{(2)}=x^{(1)}+\alpha^{(1)}p^{(1)}=(0,0)^T$$

$$g^{(2)}=(0,0)^T$$

$$s^{(1)}=x^{(2)}-x^{(1)}=(\frac{9}{110},-\frac{9}{11})^T$$

$$y^{(1)}=g^{(2)}-g^{(1)}=(\frac{18}{11},-\frac{18}{11})^T$$

$$H^{(2)}=\begin{bmatrix}\frac{1}{20}&0\\0&\frac{1}{2}\end{bmatrix}$$

∴ 2次一维搜索后达到最优解，且 $H^{(2)}=G^{-1}$

(b) $p^{(0)} = -g^{(0)} = (-2, -2)^T$

$$t = -\frac{p^{(0)T} g^{(0)}}{p^{(0)T} G p^{(0)}} = \frac{1}{11}$$

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$$x^{(1)} = x^{(0)} + t p^{(0)} = \left(-\frac{9}{110}, \frac{9}{11}\right)^T$$

$$g^{(1)} = \left(-\frac{18}{11}, \frac{18}{11}\right)^T$$

$$\alpha = \frac{g^{(1)T} G p^{(0)}}{p^{(0)T} G p^{(0)}} = \frac{81}{121}$$

$$p^{(1)} = -g^{(1)} + \alpha p^{(0)} = \left(\frac{36}{121}, -\frac{360}{121}\right)^T$$

$$t = -\frac{p^{(1)T} g^{(1)}}{p^{(1)T} G p^{(1)}} = \frac{11}{40}$$

$$x^{(2)} = x^{(1)} + t p^{(1)} = (0, 0)^T$$

$$g^{(2)} = (0, 0)^T$$

$\therefore$ 该方法与共轭梯度法等价