Problem 1 (18 pts)

A call center records the number of calls occurring per **30-minute** interval. Assume counts in each interval are independent and follow a Poisson distribution with rate λ (per 30 minutes). That is, the probability of k calls in a given 30-minute period is:

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

for $k \geq 0$.

(a) [6 pts] Derive the likelihood $L(\lambda \mid x_1, \ldots, x_{16})$. **Simplify as far as possible** (e.g., collect terms).

(b) [6 pts] Derive the log-likelihood $\ell(\lambda) = \log L(\lambda \mid x_1, \ldots, x_{16})$. **Simplify as far as possible**.

(c) [6 pts] Find the MLE $\hat{\lambda}$. Show your derivative step and briefly justify it is a maximizer.

Problem 2 (16 pts)

The call center from Problem 1 records counts at **16** half-hour intervals:
`28, 33, 21, 27, 24, 35, 26, 30, 18, 29, 31, 22, 34, 25, 27, 32.`

(a) [4 pts] Compute $\hat{\lambda}$ using the formula from Problem 1 and report the value.

(b) [6 pts] Implement a Python function for the *negative* log-likelihood and use an optimizer to recover $\hat{\lambda}$. Compare to part (a).

(c) [6 pts] Using your $\hat{\lambda}$, what is the probability of observing *at least 60 calls in one hour* (two consecutive 30-minute intervals)? Give the formula and a numeric value.

Problem 3 (10 pts)

(a) [5 pts] Two fair dice are rolled. What is $P(\text{first die} = 2 \mid \text{sum} = 6)$?

(b) [5 pts] In an organization, let P be the event "uses a password manager" and T the event "has two-factor authentication (2FA) enabled." Suppose $P(P) = 0.55$, $P(T) = 0.45$, and $P(T \mid P) = 0.30$. Find $P(P \mid T)$.

Problem 4 (24 pts)

Three twelve-faced dice (faces 1–12) are defined by the following face *counts* (each die has a total of 24 counts, so probabilities are "count/24"). Dice A and B are *weighted* (non-uniform), while die C is *uniform.*

	1	2	3	4	5	6	7	8	9	10	11	12
A	6	5	4	3	2	1	1	1	0	0	1	0
B	3	3	3	3	3	2	2	2	1	1	0	1
C	2	2	2	2	2	2	2	2	2	2	2	2

A randomly chosen die from $\{A, B, C\}$ is rolled 8 times with the outcomes:

$$D = (4,\ 3,\ 8,\ 6,\ 2,\ 7,\ 4,\ 5).$$

(a) [8 pts] Compute $P(A \mid D)$, $P(B \mid D)$, and $P(C \mid D)$, assuming $P(A) = P(B) = P(C) = \frac{1}{3}$.

(b) [6 pts] The die is rolled once more and returns a 12. Update the posteriors.

(c) [4 pts] What are the consequences if a die assigns zero probability to any observed face?

(d) [6 pts] Carry out simulations to estimate how many rolls, on average, are needed to reach 99:1 odds between B and C, given that you are actually rolling B (and, separately, actually rolling C).

(Adapted from Exercise 3.1 from MacKay p. 47)

Problem 5 (14 pts)

A website signup experiment observes 16 independent visitors. Let π be the probability that a visitor signs up. Assume π has a *uniform prior* on $[0, 1]$.

(a) [3 pts] Write an expression for the probability of observing k signups out of 16 visitors, given π.

(b) [4 pts] Derive an expression *proportional* to the posterior distribution for π.

(c) [4 pts] Derive a point estimator for π by maximizing the (posterior) expression from part (b).

(d) [3 pts] In one run, there are 11 signups out of 16 visitors. What is the maximum likelihood estimate of π?

Problem 6 (18 pts)

Continuing the website signup experiment from Problem 5, let π be the probability a visitor signs up. Now assume a **Beta prior** for π:

$$f(\pi) = \frac{\pi^{\alpha-1}(1-\pi)^{\beta-1}}{B(\alpha, \beta)}, \qquad B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Use $n = 16$ total visitors. In one run, $k = 11$ visitors signed up. For parts (a)–(e), take $(\alpha, \beta) = (3, 2)$.

(a) **[2 pts]** Plot the prior distribution for π.

Hint: In Python, `scipy.stats.beta.pdf(x, alpha, beta)` *evaluates the Beta density on a grid* $x \in [0, 1]$.

(b) **[5 pts]** Using the Binomial likelihood from Problem 5 and the Beta prior above, derive an expression *proportional* to the posterior for π and identify its distribution.

(c) **[4 pts]** Plug in $n = 16$, $k = 11$, $(\alpha, \beta) = (3, 2)$ to get the posterior parameters α' and β'.

(d) **[2 pt]** Add the posterior $\text{Beta}(\alpha', \beta')$ to your prior plot (different color). Briefly describe the change.

(e) **[5 pts]** Using the posterior, find the MAP estimate of π and compare it to the MLE from Problem 5.