Counting in Lattice Orbits

Alex Kontorovich and Christopher Lutsko

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Abstract

Given a lattice subgroup, $\Gamma < \mathrm{SL}_m(\mathbb{R})$, and a point $o \in \mathbb{R}^m$, let $N_\Gamma(T)$ denote the number of points in the orbit $o \cdot \Gamma$ whose (Euclidean) distance from the origin is bounded by T. In this paper we obtain effective asymptotic estimates for $N_\Gamma(T)$ using an abstract spectral method previously developed by the authors.

1 Introduction

In this paper, we study the general orbital counting problem in real space, by which we mean the following. Fix $m \geq 2$ and a base point $o \in \mathbb{R}^m$, and let $\Gamma < G := \mathrm{SL}_m(\mathbb{R})$ be a discrete lattice such that the orbit $\mathcal{O} := o \cdot \Gamma \subset \mathbb{R}^m$ is discrete and infinite. Then the orbital counting problem is to obtain sharp asymptotic estimates for

$$N_\Gamma(T) := \#\{z \in \mathcal{O} \ : \ \|z\| \leq T\}, \tag{1.1}$$

where $\|z\|^2 := z_1^2 + \cdots + z_m^2$ (or another archimedean norm). For a maximal compact $K \cong \mathrm{SO}(m)$, the locally symmetric space $X := \Gamma\backslash G/K$ is endowed with a Riemannian metric corresponding to the Killing form on the Lie algebra of G; let $\mathcal{C}$ denote the Laplace operator (quadratic Casimir) on X; see §2.3 for details and normalization. Then the spectrum of $\mathcal{C}$ below 1 consists of finitely many "exceptional" eigenvalues (each with finite multiplicity)

$$0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_k < 1.$$

Our normalization of the Casimir operator is such that the tempered spectrum starts at 1. Further, let s_i be the positive root $s_i = m\sqrt{\lambda_i}$. Our main theorem is then the following

Theorem 1. *For any lattice $\Gamma < \mathrm{SL}_m(\mathbb{R})$, there exist constants $c_0 > 0$, $c_1, \ldots, c_k$ such that*

$$N_\Gamma(T) = c_0 T^m + c_1 T^{(m-s_1)} + \cdots + c_k T^{(m-s_k)} + O(T^{m-\eta_m}), \tag{1.2}$$

with

$$\eta_m = \frac{2m}{(m+2)(m-1)+4}.$$

For example $\eta_2 = 1/2$, $\eta_3 = 3/7$, $\eta_4 = 4/11$, $\eta_5 = 5/16$, *and* $\eta_6 = 3/11$.

Remark. In the general case of an arbitrary lattice subgroup, Γ, and base point o, the best known result appears to still be the ground-breaking work of Duke-Rudnick-Sarnak [DRS93]. However, due to the general context in which they work, their error estimate (which is not calculated explicitly) is far from optimized. Thus, our η_m is certainly larger.

Remark. In the case that $\Gamma = \mathrm{SL}_m(\mathbb{Z})$ and $o = e_m = (0, \ldots, 0, 1)$, this problem amounts to counting primitive lattice points in the T-ball. That is, let $r_m^*(n)$ denote the number of ways to express an integer, n, as the sum of m squares which share no common factor. That is

$$r_m^*(n) := \#\{\mathbf{a} \in \mathbb{Z}^m \ : \ \|\mathbf{a}\|^2 = n \text{ with } (a_1, \ldots, a_m) = 1\},$$

where $\|\cdot\|$ denotes the Euclidean norm. A classic problem, is to obtain an asymptotic formula for the number

$$N_m(T) := \sum_{n=1}^{T^2} r_m^*(n).$$

The main term is known to be $c_m T^m$, where $c_m = 1/\zeta(m)$ [Chr56] however finding optimal estimates for the error term is a challenging problem. With that, let

$$N_m(T) = c_m T^m + O(T^{m-\eta_m}). \tag{1.3}$$

When $m = 2$ the best known error term is due to Wu [Wu02] (assuming the Riemann hypothesis) that (1.3) holds with $\eta_2 < \frac{387}{304} \approx 1.273\ldots$. For $m = 3$ the best known result is that of Goldfeld-Hoffstein [GH85] that $\eta_3 < 39/32 \approx 1.219\ldots$, which follows from the fact that N_3 can be related to the first moment of the quadratic Dirichlet L-function $L(\frac{1}{2}, \chi_{8d})$ (similar estimates were achieved by Young [You09] in the smooth case). For $m \geq 4$, one can use Möbius inversion to compare the primitive lattice point count to the Gauss circle problem. Then one can easily show that the asymptotic estimate above holds for any value of $\eta_m < 1$. All of these results are much stronger that what one can achieve in the generality of Theorem 1, and are possible due to the arithmetic nature of the lattice $\mathrm{SL}_m(\mathbb{Z})$.

Remark. In smooth form (see Theorem 6 below), our error term exhibits square-root cancellation, that is, has size $T^{m/2}$, which is optimal in the sense that it reaches the tempered spectrum.

Our method is based on an abstract counting method developed by the authors in [Kon09, KL22, Lut22]. The main technical innovation in this paper is an analysis of the Lie algebra structure, explicit Casimir operator, and Haar measure resulting from a parametrization of the group suitable for the application.

1.1 Plan of paper

In Section 2, we present some preliminaries in Lie algebras, groups, and decompositions thereof, along with the main Structure Theorem (see Theorem 2) for the Haar measure and Casimir operator in these coordinates. In Section 3 we prove Theorem 1 in the case where $m = 3$. Finally, in Section 4 we explain how to generalize the proof for arbitrary $m \geq 3$, the strategy being similar.

2 Preliminaries

Without loss of generality (conjugating Γ), we can choose our base point to be $o = e_m := (0, \ldots, 0, 1) \in \mathbb{R}^m$. Let

$$G := \mathrm{SL}_m(\mathbb{R}) := \left\{ g = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ & \vdots & \vdots & \\ x_{(m-1)1} & x_{(m-1)2} & \cdots & x_{(m-1)m} \\ a_1 & a_2 & \cdots & a_m \end{pmatrix} \ : \ \det g = 1 \right\},$$

with coordinates as specified. Further let

$$H := \mathrm{ASL}_{m-1}(\mathbb{R}) \ = \ \{ g \in G \ : \ a_1 = \cdots = a_{m-1} = 0 \ , \ a_m = 1 \} \, .$$

Now fix Γ and let $\Gamma_H := \Gamma \cap H$. By the assumed discreteness of the orbit $\mathcal{O}$, the stabilizer Γ_H is a lattice in H. Then our count can be expressed as

$$N_\Gamma(T) := \#\{ \gamma \in \Gamma_H \backslash \Gamma \ : \ a_1^2 + \cdots + a_m^2 \leq T^2 \}.$$

2.1 Group decomposition for $m = 3$

For the reader's benefit, we first explicate everything in the case $m = 3$. Let $\mathfrak{g} := \mathfrak{sl}_3(\mathbb{R})$ be the Lie algebra associated to G. It is convenient to decompose $\mathfrak{g}$ according to the following basis:

$$X_{H,1} := \begin{pmatrix} & & \\ & & 1 \\ & & \end{pmatrix}, \quad X_{H,2} := \begin{pmatrix} & 1 & \\ & & \\ & & \end{pmatrix}, \quad X_{H,3} := \begin{pmatrix} & & 1 \\ & & \\ & & \end{pmatrix},$$

$$X_{H,4} := \begin{pmatrix} 1 & & \\ & -1 & \\ & & \end{pmatrix}, \quad X_{H,5} := \begin{pmatrix} & 1 & \\ -1 & & \\ & & \end{pmatrix},$$

$$X_A := \begin{pmatrix} -1/2 & & \\ & -1/2 & \\ & & 1 \end{pmatrix}$$

$$X_{K,1} := \begin{pmatrix} & & 1 \\ & & \\ -1 & & \end{pmatrix}, \quad X_{K,2} := \begin{pmatrix} & & \\ & & 1 \\ & -1 & \end{pmatrix}.$$

The basis elements $X_{H,i}$ generate $\mathfrak{h} = \mathrm{Lie}(H)$, and we denote their matrix exponentials by $n_1(x_1) = \exp(x_1 X_{H,1}), n_2(x_2), n_3(x_3), a_H(t), k(\theta)$ respectively. We denote the exponential of X_A by $\widetilde{a}(t) = \exp(X_A t)$. Rather than work with the t variable, we prefer to work with $r = e^t > 0$; thus we set

$$a(r) := \widetilde{a}(\log r) = \begin{pmatrix} r^{-1/2} & & \\ & r^{-1/2} & \\ & & r \end{pmatrix}.$$

Finally $X_{K,1}$ and $X_{K,2}$ correspond to two rotations, and we denote their exponentials by $k_1(\theta_1) = \exp(\theta_1 X_{K,1})$ and similarly for $k_2(\theta_2)$. Thus, given a $g \in G$, we can write

$$g = n(\mathbf{x}) a_H(t) k(\theta) a(r) k_1(\theta_1) k_2(\theta_2),$$

where $n(\mathbf{x}) = n_1(x_1) n_2(x_2) n_3(x_3)$. Note that $a(r)$ commutes with $k(\theta)$, and if we move k to the right, then $k k_1 k_2$ generate $\mathrm{SO}(3)$.

2.2 Group decomposition for general m

In general, we decompose G into: $H \cong \mathrm{ASL}_{m-1}(\mathbb{R}) \times$ a one-parameter diagonal subgroup $\times$ a product of $(m-1)$ one-parameter rotations. That is, we again let

$$a(r) = \mathrm{diag}(r^{-1/(m-1)}, \ldots, r^{-1/(m-1)}, r).$$

Let $X_{K,i}$ be the element of the Lie algebra with $(X_{K,i})_{mi} = -(X_{K,i})_{im} = 1$ for $i = 1, \ldots, m-1$, and let $k_i(\theta_i) = \exp(\theta_i X_{K,i})$. Then for any $g \in G$ we can write

$$g = ha(r)k_1(\theta_1) \cdots k_{m-1}(\theta_{m-1}),$$

for some $h \in H$, $r > 0$, and $\theta_i \in [0, 2\pi)$. We denote by $k(\boldsymbol{\theta}) := k_1(\theta_1) \cdots k_{m-1}(\theta_{m-1})$.

We further decompose H into a product of an upper triangular matrix $n(\mathbf{x})$, where $\mathbf{x}$ has dimension $\frac{m(m-1)}{2}$ $\times$ a diagonal matrix $a_H(\mathbf{t})$ (of dimension $m-2$) $\times$ an element of $\mathrm{SO}(m-1)$ that we denote $k_H(\boldsymbol{\varphi})$ (of dimension $\frac{(m-1)(m-2)}{2}$). Thus we write

$$g = n(\mathbf{x})a_H(\mathbf{t})k_H(\boldsymbol{\varphi})a(r)k(\boldsymbol{\theta}). \tag{2.1}$$

Crucially, note that k_H commutes with $a(r)$. This allows us to multiply together the a_H and $a(r)$ matrices and change coordinates to the more standard Iwasawa coordinates (see [Gol06]) where the Haar measure and Casimir operator are known, resulting in the following.

Theorem 2 (Structure Theorem of the Haar measure and Casimir operator)**.** *The Haar measure on G in the coordinates of (2.1) is given by*

$$\mathrm{d}g = r^{m-1}\rho_1(\mathbf{x}, \mathbf{t}, \boldsymbol{\varphi})\rho_2(\boldsymbol{\theta})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{t}\mathrm{d}\boldsymbol{\varphi}\mathrm{d}r\mathrm{d}\boldsymbol{\theta}, \tag{2.2}$$

where ρ_1 is the Haar measure density on $\mathrm{ASL}_{m-1}(\mathbb{R})$ and ρ_2 is bounded. Meanwhile the quadratic Casimir operator, acting on left-H-invariant functions, is given in these coordinates by

$$\mathcal{C}f(r, \boldsymbol{\theta}) = \frac{4}{m^2}(r^2\partial_{rr} + r\partial_r)f(r, \boldsymbol{\theta}). \tag{2.3}$$

Proof. The Haar measure is well-known in standard Iwasawa coordinates, see, e.g. [Gol06, Theorem 1.6.1]. Thus to prove (2.2), all that is needed is a change of coordinates and an inductive argument.

As for the Casimir operator, let $X_{H,1}, \ldots, X_{H,(m-1)m}$ be a basis for $\mathfrak{h}$, let X_A be the Lie element $\mathrm{diag}(\frac{1}{m-1}, \ldots, \frac{1}{m-1}, 1)$, and let $X_{K,1}, \ldots, X_{K,m-1}$ be the basis elements corresponding to k_i.

The quadratic Casimir operator (as an element of the universal enveloping algebra of $\mathfrak{g}$) is then given by:

$$\mathcal{C} = \sum_{i=1}^{(m-1)m} X_{H,i}^* X_{H,i} + X_A^* X_A + \sum_{i=1}^{m-1} X_{K,i}^* X_{K,i},$$

where X^* is the dual element (that is $B(X_i, X_j^*) = \delta_{i,j}$ where B is the Killing form). Each basis element, X corresponds to a differential operator given by

$$D_X f(g) = \frac{d}{dt} f(g \exp(tX))\bigg|_{t=0}.$$

Using the fact that $a(r)$ commutes with the almost all of H, one can show that the only contrbution to the Casimir (when acting on H-invariant function) is given by $X_A^* X_A = c_m X_A^2$, for some constant c_m. Note that for the differential operator X_A, we can use the method in [BKS10, Proof of Lemma 2.7] (also used in [KL22, Proof of Theorem 8]) to compute X_A. The normalization can be derived by acting on the I function [Gol06, Definition 2.4.1] and matching eigenvalues. $\square$

2.3 Decomposition of $L^2(\Gamma\backslash G/K)$ into irreducibles and the spectral theorem

The Riemannian metric on the locally symmetric space $\Gamma\backslash G/K$ has an associated Laplace-Beltrami operator. With respect to the right-regular representation of G on $L^2(\Gamma\backslash G)$, the quadratic Casimir operator $\mathcal{C}$ agrees on the subspace $\mathscr{H} := L^2(\Gamma\backslash G/K)$ of right K-invariant functions, with the Laplacian. This operator, $\mathcal{C}$, is positive and self-adjoint, and thus its spectrum lies in $\mathbb{R}_{\geq 0}$. We have the following abstract spectral theorem (see e.g., [Rud73, Ch. 13])

Theorem 3 (Abstract Spectral Theorem)**.** *There exists a spectral measure $\widehat{\mu}$ on $\mathbb{R}_{\geq 0}$ and a unitary spectral operator $\widehat{\ }: \mathscr{H} \to L^2([0,\infty), d\widehat{\mu})$ such that:*

i) Abstract Parseval's Identity: for $\phi_1, \phi_2 \in \mathscr{H}$

$$\langle \phi_1, \phi_2\rangle_{\mathscr{H}} = \langle \widehat{\phi_1}, \widehat{\phi_2}\rangle_{L^2([0,\infty),d\widehat{\mu})}, \tag{2.4}$$

and

ii) The spectral operator is diagonal with respect to L: for $\phi \in \mathscr{H}$ and $\lambda \geq 0$,

$$\widehat{L\phi}(\lambda) = \lambda\widehat{\phi}(\lambda). \tag{2.5}$$

Moreover, if λ is in the point specturm of L with associated eigenspace $\mathscr{H}_\lambda$, then for any $\psi_1, \psi_2 \in \mathscr{H}$ one has

$$\widehat{\psi_1}(\lambda)\overline{\widehat{\psi_2}}(\lambda) = \langle \operatorname{Proj}_{\mathscr{H}_\lambda} \psi_1, \operatorname{Proj}_{\mathscr{H}_\lambda} \psi_2\rangle, \tag{2.6}$$

where Proj refers to the projection to the subspace $\mathscr{H}_\lambda$. In the special case that $\mathscr{H}_\lambda$ is one-dimensional and spanned by the normalized eigenfunction ϕ_λ, we have that

$$\widehat{\psi_1}(\lambda)\widehat{\overline{\psi}_2}(\lambda) = \langle \psi_1, \phi_\lambda \rangle \langle \phi_\lambda, \psi_2 \rangle. \tag{2.7}$$

The group G acts by right regular representation on $\mathscr{H}$. The Hilbert space $\mathscr{H}$ decomposes into components as follows

$$\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_k \oplus \mathscr{H}^{tempered}$$

where $\mathscr{H}_i$ is a finite dimensional eigenspace with $\mathcal{C}$-eigenvalue λ_i and $\mathscr{H}^{tempered}$ denotes the tempered spectrum.

3 Counting orbits in $m = 3$ dimensions

In this section we restrict to the case $m = 3$. Theorem 1 will follow from Theorem 4 below, which achieves a similar asymptotic for a *smoothed* count. To that end let

$$\chi_T(g) := \begin{cases} 1 & \text{if } r \le T \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$F_T(g) = \sum_{\gamma \in \Gamma_H \backslash \Gamma} \chi_T(\gamma g),$$

hence $N_\Gamma(T) = F_T(e)$.

Let $\psi_\varepsilon \in L^2(\Gamma_H \backslash G/K)$ be a smooth approximation to the identity. More specifically, let $\psi_{1,\varepsilon}(x)$ have total mass one and be supported in a ball of radius ε around 0. Let $\psi_{2,\varepsilon}(r)$ be a unit mass bump function supported in a ball of radius ε around 1. Then

$$\psi_\varepsilon(h(\mathbf{x})a_H(t)k_1a(r)k_2k_3) = \psi_{1,\varepsilon}(x_1)\psi_{1,\varepsilon}(x_2)\psi_{1,\varepsilon}(x_3)\psi_{2,\varepsilon}(t)\psi_{2,\varepsilon}(r).$$

Let $\Psi \in L^2(\Gamma \backslash G/K)$ denote the Γ-average of ψ_ε

$$\Psi(g) := \sum_{\gamma \in \Gamma_H \backslash \Gamma} \psi_\varepsilon(\gamma g).$$

Then our smoothed count is given by

$$\widetilde{N}(T) := \langle F_T, \Psi \rangle_\Gamma.$$

For this smooth count, we have the following asymptotic

Theorem 4. *Let* $\Gamma < \mathrm{SL}_3(\mathbb{Z})$ *be co-finite. Then*

$$\widetilde{N}(T) = c_0(\varepsilon)T^3 + c_1(\varepsilon)T^{3-s_1} + \cdots + c_k(\varepsilon)T^{3-s_k} + O(T^{3/2}\varepsilon^{-5/2}),$$

where $c_i(\varepsilon) = c + O(\varepsilon)$.

3.1 Proof of Theorem 1 with $m = 3$

Theorem 4 leads rather immediately to a proof of Theorem 1. To see this, note that, after unfolding

$$\begin{aligned}\widetilde{N}(T) &= \sum_{\Gamma_H\backslash\Gamma} \int_{\Gamma_H\backslash G/K} \chi_T(\gamma g)\psi_\varepsilon(g)\mathrm{d}g \\ &= \sum_{\Gamma_H\backslash\Gamma} \int_{\Gamma_H\backslash G/K} \chi_T(\gamma n(\mathbf{x})a_H(t)k_1 a(r))\psi_\varepsilon(\mathbf{x}, t, r)\mathrm{d}g.\end{aligned}$$

Now note that since $a(r)$ commutes with k_1, we have that χ_T is right k_1 invariant. Thus

$$\widetilde{N}(T) = \sum_{\Gamma_H\backslash\Gamma} \int_{\Gamma_H\backslash G/K} \chi_T(\gamma n(\mathbf{x})a_H(s)a(r))\psi_\varepsilon(n(\mathbf{x})a(\mathbf{y}))\mathrm{d}g.$$

Since x_1, x_2, x_3, t and r are all in balls of radius ε around 0 or 1, we can expand $n(\mathbf{x})a_H(t)a(r) = I + O(\varepsilon)M$ for some matrix M with bounded entries. Thus, after some matrix manipulation it is not hard to see that there exists a constant c such that

$$\chi_T(\gamma n(\mathbf{x})a_H(t)a(r)) = \begin{cases} 1 & \text{if } r_\gamma < \frac{T}{1+c\varepsilon} \\ 0 & \text{if } r_\gamma > \frac{T}{1-c\varepsilon}. \end{cases}$$

From which it follows that

$$\widetilde{N}(T(1-c\varepsilon)) \leq N_\Gamma(T) \leq \widetilde{N}(T(1+c\varepsilon)).$$

Now Theorem 1 follows by choosing ε appropriately to optimize the error terms

$$T^3\varepsilon = T^{3/2}\varepsilon^{-5/2}$$

That is $\varepsilon = T^{-3/7}$.

□

Hence, the remainder of the Section 3 is devoted to proving Theorem 4.

3.2 Unfolding and the differential equation

By unfolding, our smooth count becomes

$$\begin{aligned}\widetilde{N}(T) &= \int_{\Gamma\backslash G/K} \sum_{\gamma\in\Gamma_H\backslash\Gamma} \chi_T(\gamma g)\Psi(g)\mathrm{d}g \\ &= \int_{\Gamma_H\backslash G/K} \chi_T(g)\Psi(g)\mathrm{d}g \\ &= \int_0^\infty \chi_T(r)r^2 \left(\int_{\Gamma_H\backslash H} \Psi(ha(r))\mathrm{d}h\right)\mathrm{d}r,\end{aligned}$$

note that Ψ is right K invariant. Let $f(r) := \int_{\Gamma_H\backslash H} \Psi(ha(r))\mathrm{d}h$ denote the quantity inside the brackets.

Then using Theorem 2 we know that, for any value of λ, f satisfies the differential equation

$$(\frac{4}{9}(r^2\partial_{rr} + r\partial_r) - \lambda)f(r) = g(r)$$

with $g(r) := \int_{\Gamma_H\backslash H}(\mathcal{C}-\lambda)\Psi(ha(r))\mathrm{d}h$. The same equation holds for any $\Psi \in L^2(\Gamma\backslash G/K)$ and any value λ. If $g \equiv 0$ and $\lambda \neq 0$ then there are two homogeneous solutions given by r^s where $\lambda = \frac{4}{9}s^2$.

Assume $s \geq 0$ and write

$$\begin{aligned}\alpha_\pm(T) &:= \int_0^\infty \chi_T(r)r^{2\pm s}\mathrm{d}r \\ &= \frac{1}{3\pm s}T^{3\pm s}\end{aligned}$$

since $|\Re(s)| \leq 3/2$. Using the same proof in [Kon09, Proof of Lemma 3.3] we can write

$$\widetilde{N}(T) = A_+\alpha_+ - A_-\alpha_-(T) + O(\|(\mathcal{C}-\lambda)\Psi\|). \tag{3.1}$$

However, note that we can trivially bound $\widetilde{N}(T) \ll T^3$, thus we can in fact write

$$\widetilde{N}(T) = A\alpha(T) + O(\|(\mathcal{C}-\lambda)\Psi\|), \tag{3.2}$$

where $\alpha(T) = \alpha_-(T)$. Hence we can solve for A and write

$$\widetilde{N}(T) = K_T(\lambda)\widetilde{N}(1) + O(\|(\mathcal{C}-\lambda)\Psi\|). \tag{3.3}$$

with $K_T(\lambda) = \frac{\alpha(T)}{\alpha(1)}$. The following theorem states that, since (3.3) holds for any Ψ and any λ we can in fact show that the error vanishes. Since the proof is identical to the proof of [Kon09, Proposition 3.5] we omit it. Note that we can create an operator $K_T(\mathcal{C})$ via a power series expansion of K_T.

Theorem 5 (Main Identity)**.** *For T large enough we have*

$$F_T(g) = K_T(\mathcal{C})F_1(g) \tag{3.4}$$

almost everywhere. Moreover

$$K_T(\lambda) = \begin{cases} cT^{3-s} & \text{if } s < 3/2 \\ cT^{3/2} & \text{if } s = 3/2 + it. \end{cases} \tag{3.5}$$

3.3 Proof of Theorem 4

With the main identity at hand, we can proceed with the proof of Theorem 4. By Parseval's identity (2.4)

$$\begin{aligned} \widetilde{N}(T) &= \langle F_T, \Psi \rangle_\Gamma \\ &= \langle \widehat{F}_T, \widehat{\Psi} \rangle_{\mathrm{Spec}(\Gamma)} \\ &= \widehat{F_T}(\lambda_0)\widehat{\Psi}(\lambda_0) + \widehat{F_T}(\lambda_1)\widehat{\Psi}(\lambda_1) + \cdots + \widehat{F_T}(\lambda_k)\widehat{\Psi}(\lambda_k) \\ &\qquad + \int_{\mathcal{S}^{temp}} \widehat{F_T}(\lambda)\widehat{\Psi}(\lambda)\mathrm{d}\widehat{\mu}(\lambda). \end{aligned}$$

Now for each point in the exceptional spectral, $\widehat{\Psi}(\lambda_i)$ is the projection onto the i^{th} eigenspace, which is finite dimensional. Thus

$$\widehat{\Psi}(\lambda_i) = \langle \Psi, \phi_i \rangle,$$

by the mean value theorem we have that

$$\widehat{\Psi}(\lambda_i) = C + O(\varepsilon).$$

Moreover using (3.5) we have that

$$\begin{aligned} \widehat{F_T}(\lambda_i) &= T^{3-s_i}\langle F_1, \phi_i \rangle, \\ &= c_i T^{3-s_i}, \end{aligned}$$

for some constants c_i.

As for the error term, we can use the fact that the Casimir operators act like scalars on irreducibles and (3.5) to achieve the following bound

$$\begin{aligned}\int_{\mathcal{S}^{temp}} \widehat{F_T}(\lambda)\widehat{\Psi}(\lambda)\mathrm{d}\widehat{\mu}(\lambda) &= \int_{\mathcal{S}^{temp}} \widehat{K_T(\mathcal{C})F_T}(\lambda)\widehat{\Psi}(\lambda)\mathrm{d}\widehat{\mu}(\lambda)\\ &= \int_{\mathcal{S}^{temp}} K_T(\mathcal{C})\widehat{F_1}(\lambda)\widehat{\Psi}(\lambda)\mathrm{d}\widehat{\mu}(\lambda)\\ &= \int_{\mathcal{S}^{temp}} K_T(\lambda)\widehat{F_1}(\lambda)\widehat{\Psi}(\lambda)\mathrm{d}\widehat{\mu}(\lambda)\\ &\ll T^{3/2}\int_{\mathcal{S}^{temp}} \widehat{F_1}(\lambda)\widehat{\Psi}(\lambda)\mathrm{d}\widehat{\mu}(\lambda).\end{aligned}$$

From here we again apply Parseval's identity and Cauchy-Schwarz yielding

$$\int_{\mathcal{S}^{temp}} \widehat{F_T}(\lambda)\widehat{\Psi}(\lambda)\mathrm{d}\widehat{\mu}(\lambda) \ll T^{3/2}\|F_1\|\|\Psi\|.$$

Since $\Gamma\backslash G$ has finite volume and F_1 is bounded, the L^2 norm of F_1 is bounded. Moreover we can bound

$$\|\Psi\|_\Gamma \ll 1/\varepsilon^{5/2}.$$

From which we conclude

$$\widetilde{N}(T) = c_0(\varepsilon)T^3 + c_1(\varepsilon)T^{3-s_1} + \cdots + c_k(\varepsilon)T^{3-s_k} + O(T^{3/2}\varepsilon^{-5/2}). \tag{3.6}$$

4 Counting orbits of arbitrary $m \geq 3$ dimensions

For $m \geq 3$, the proof is more or less identical, just with more complex notation. We give below the main differences in the argument.

4.1 Smoothing

We begin by smoothing the count. To that end, let

$$\chi_T(g) := \begin{cases} 1 & \text{if } r \leq T \\ 0 & \text{otherwise.}\end{cases}$$

Let

$$F_T(g) = \sum_{\gamma\in\Gamma_H\backslash\Gamma} \chi_T(\gamma g),$$

hence $N_\Gamma(T) = F_T(e)$. Let

$$\psi_\varepsilon(g) = \left(\prod_{i=1}^{m(m-1)/2} \psi_{1,\varepsilon}(x_i)\right)\left(\prod_{j=1}^{m-2} \psi_{2,\varepsilon}(t_j)\right)\psi_{2,\varepsilon}(r)$$

and let $\Psi(g) := \sum_{\gamma\in\Gamma_H\backslash\Gamma} \psi_\varepsilon(\gamma g)$. We then define the smooth count to be

$$\widetilde{N}(T) := \langle F_T, \Psi\rangle.$$

Now Theorem 1 follows from the following

Theorem 6. *For any $\Gamma < \mathrm{SL}_m(\mathbb{Z})$ of finite co-volume we have*

$$\widetilde{N}(T) = c_0(\varepsilon)T^m + c_1(\varepsilon)T^{m-s_1} + \cdots + c_k(\varepsilon)T^{m-s_k} + O(\varepsilon^{-(m+2)(m-1)/4}T^{m/2}), \tag{4.1}$$

where for any $i = 1, \ldots, k$ we have that $c_i(\varepsilon) = C_i(1 + O(\varepsilon))$, where C_i are independent of ε.

From here, optimizing ε by choosing $\varepsilon = T^{\frac{-2m}{(m+2)(m-1)+4}}$ leads to the error term in Theorem 1 in exactly the same way as was done in Section 3.1. The remainder of the paper is devoted to the proof of Theorem 6.

4.2 Unfolding and differential equation

By unfolding, we have

$$\begin{aligned}\widetilde{N}(T) &= \int_{\Gamma\backslash G/K} \sum_{\gamma\in\Gamma_H\backslash\Gamma} \chi_T(\gamma g)\Psi(g)\mathrm{d}g \\ &= \int_0^\infty \chi_T(r)r^{m-1}\left(\int_{\Gamma_H\backslash H} \Psi(ha_r)\mathrm{d}h\right)\mathrm{d}r.\end{aligned}$$

Let $f(r) := \int_{\Gamma_H\backslash H} \Psi(ha_r)\mathrm{d}h$ denote the quantity inside the brackets.

By Theorem 2, for any value of λ, f satisfies the differential equation

$$(\frac{4}{m^2}(r^2\partial_{rr} + r\partial_r) - \lambda)f(r) = g(r)$$

with $g(r) := \int_{\Gamma_H\backslash H}(\mathcal{C} - \lambda)\Psi(ha(r))\mathrm{d}h$. The same equation holds for any $\Psi \in L^2(\Gamma\backslash G/K)$ and any vector λ. If $g \equiv 0$ and $\lambda \neq 0$ then there are two homogeneous solutions given by r^s where $\lambda = \frac{4}{m^2}s^2$.

Assume $s \geq 0$ and write

$$\begin{aligned}\alpha_\pm(T) &:= \int_0^\infty \chi_T(r)r^{m-1\pm s}\mathrm{d}r \\ &= \frac{1}{m \pm s}T^{m\pm s}\end{aligned}$$

since $|\Re(s)| \le \frac{m}{2}$. We can then write

$$\widetilde{N}(T) = A_+\alpha_+ - A_-\alpha_-(T) + O(\|(\mathcal{C}-\lambda)\Psi\|). \tag{4.2}$$

Note that we can trivially bound $\widetilde{N}(T) \ll T^m$, thus we can in fact write

$$\widetilde{N}(T) = A\alpha(T) + O(\|(\mathcal{C}-\lambda)\Psi\|), \tag{4.3}$$

where $\alpha(T) = \alpha_-(T)$. Solving for A we conclude

$$\widetilde{N}(T) = K_T(\lambda)\widetilde{N}(1) + O(\|(\mathcal{C}-\lambda)\Psi\|). \tag{4.4}$$

with $K_T(\lambda) = \frac{\alpha(T)}{\alpha(1)}$. From here, the same main identity holds

Theorem 7 (Main Identity)**.** *For T large enough we have*

$$F_T(g) = K_T(\mathcal{C})F_1(g) \tag{4.5}$$

almost everywhere. Moreover

$$K_T(\lambda) = \begin{cases} cT^{m-s} & \text{if } s < m/2 \\ cT^{m/2} & \text{if } s = m/2 + it. \end{cases} \tag{4.6}$$

4.3 Proof of Theorem 4

With the main identity at hand, we can proceed with the proof of Theorem 6. By Parseval's identity (2.4)

$$\widetilde{N}(T) = \widehat{F_T}(\lambda_0)\widehat{\Psi}(\lambda_0) + \widehat{F_T}(\lambda_1)\widehat{\Psi}(\lambda_1) + \cdots + \widehat{F_T}(\lambda_k)\widehat{\Psi}(\lambda_k) \\ + \int_{\mathcal{S}^{temp}} \widehat{F_T}(\lambda)\widehat{\Psi}(\lambda)\mathrm{d}\widehat{\mu}(\lambda).$$

As when $m = 3$ we can again use (4.6) to extract the T dependence from the main terms and bound the contribution from the continuous spectrum giving

$$\widetilde{N}(T) = cT^m(C + O(\varepsilon)) + cT^{m-s_1}(C + O(\varepsilon)) + \cdots + cT^{m-s_k}(C + O(\varepsilon)) \\ + T^{m/2}\|F_1\|\|\Psi\|.$$

Now $\|F_1\|$ is bounded, and $\|\Psi\| \ll 1/\varepsilon^{(m+2)(m-1)/2}$ (since we are smoothing in $m(m-1)/2 + m - 1$ directions). This yields (4.1).

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Department of Mathematics, Rutgers University, Hill Center - Busch Campus, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA. *E-mail:* ***alex.kontorovich@rutgers.edu***

Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland. *E-mail:* ***christopher.lutsko@math.uzh.ch***