AVERAGE VARIANCE BOUNDS FOR INTEGER POINTS ON THE SPHERE

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Abstract. Consider the integer points lying on the sphere of fixed integer radius projected onto the unit sphere. Duke showed that, on congruence conditions for the radius, these points equidistribute. To further this study of equidistribution, we consider the variance of the number of points in a spherical cap. An asymptotic for this variance was conjectured by Bourgain-Rudnick-Sarnak. We prove an upper bound on the average (over radii) of these variances of the correct size.

1. Introduction

Assume $n \in \mathcal{N} := \{n \in \mathbb{N} \ : \ n \not\equiv 0, 4, 7 (\mathrm{mod}\ 8)\}$ and let

$$\mathcal{E}(n) = \{\mathbf{x} \in \mathbb{Z}^3 \ : \ |\mathbf{x}|^2 = n\},$$

further let

$$\widehat{\mathcal{E}}(n) := \frac{1}{\sqrt{n}}\mathcal{E}(n) \subset \mathbb{S}^2.$$

Assuming the generalized Riemann hypothesis, Linnik [Lin68] showed that these points equidistribute on the sphere. This was then proved unconditionally by Duke [Duk88] and Golubeva-Fomenko [GF90] following breakthrough work of Iwaniec [Iwa87].

Given this equidistribution result, a natural question one can ask is whether the fine-scale statistics of the points $\widehat{\mathcal{E}}(n)$ converge to what one expects for uniformly distributed random variables on the sphere. This is exactly what Bourgain, Rudnick and Sarnak [BRS17] ask in their seminal work on the subject. One of the statistics they consider is the variance; to that end, let $N_n := \#\mathcal{E}_n$ then, given a set $\Omega \subset \mathbb{S}^2$, we let

$$Z(n, \Omega) := \#(\widehat{\mathcal{E}}(n) \cap \Omega).$$

In which case, we define the variance to be

$$\mathrm{Var}(\Omega, n) := \int_{\mathbb{S}^2} |Z(n, \Omega + \zeta) - N_n\sigma(\Omega)|^2 \, \mathrm{d}\sigma(\zeta),$$

where σ is the normalized area measure on the sphere. Bourgain-Rudnick-Sarnak posed the following conjecture:

Conjecture 1 ([BRS17, Conjecture 1.6]). *Let Ω_n be a sequence of spherical caps. If $N_n^{-1+\varepsilon} \ll \sigma(\Omega_n) \ll N_n^{-\varepsilon}$ as $n \to \infty$, with $n \in \mathcal{N}$, then*

$$\mathrm{Var}(\Omega_n, n) \sim N_n\sigma(\Omega_n). \tag{1.1}$$

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A spherical cap is a set $\Omega_n = \{\mathbf{x} \in \mathbb{S}^2 \ : \ \mathrm{dist}(\mathbf{x}, \mathbf{y}) \le r_n\}$. The conjecture is also stated for annuli. While, proving this conjecture in full appears to be out of reach, Bourgain, Rudnick and Sarnak establish an upper bound [BRS17, Theorem 1.7] (assuming the Lindelöf hypothesis for $\mathrm{GL}(2)/\mathbb{Q}$ L-functions) of

$$\mathrm{Var}(\Omega_n, n) \ll n^{\varepsilon} N_n \sigma(\Omega_n), \qquad \forall \varepsilon > 0, \tag{1.2}$$

provided n is square-free.

Following this work, Humphries and Radziwiłł [HR22] proved the conjecture by Bourgain-Rudnick-Sarnak, provided n is square-free, in the microscopic regime (where $\sigma(\Omega_n)$ is small). The regime where $\sigma(\Omega_n) \sim n^{-\varepsilon}$ remains open and, as noted by Humphries-Radziwiłł, is equivalent to the Lindelöf hypothesis for a family of L-functions. Following this, Shubin proved a similar theorem, assuming GRH for $\mathrm{GL}(2)/\mathbb{Q}$ [Shu23]. Moreover, Lubotzky, Phillips, Sarnak [LPS86] and Ellenberg, Michel, and Venkatesh [EMV13], among many others, have studied similar equidistribution problems.

The purpose of this paper is to provide unconditional progress towards this conjecture, without assuming n is square-free by averaging over the different levels n. For that, fix a spherical cap $\Omega_X \subset \mathbb{S}^2$ and let

$$\mathcal{A}_{X,H} := \frac{1}{H} \sum_{\substack{n \in \mathcal{N} \\ X \le n \le X+H}} \mathrm{Var}(\Omega_X, n).$$

Further let $\mathcal{A}_X := \mathcal{A}_{1,X-1}$ denote the average from $n = 1$ to X. The following result is the main result in the paper

Theorem 2. *Fix an integer X, and a spherical cap Ω_X with area $\sigma(\Omega_X) = cN_X^{\delta}$, where $-1 < \delta < 0$ and $c > 0$ a constant. Then we have*

$$\mathcal{A}_X \ll X^{1/2} \sigma(\Omega_X). \tag{1.3}$$

Moreover, for any $\frac{X^{3/4}}{\sigma(X)^{3/4}} < H < \infty$, we have

$$\mathcal{A}_{X,H} \ll X^{1/2} \sigma(\Omega_X). \tag{1.4}$$

Note that $X^{1/2-\varepsilon} \ll N_X \ll X^{1/2+\varepsilon}$, thus Theorem 2 is evidence towards Conjecture 1.

1.1. **Strategy of proof:** First we smooth the variance, this will provide us with better control in the frequency aspect. Then, by decomposing the variance we can relate it to the Fourier coefficients of the theta sums associated to the spherical harmonics. With that, the average of variances is related to the average value of Fourier coefficients of some half-integer weight, holomorphic cusp forms. It's well known that such averages can be well estimated, however for our purpose we need to bound these averages uniformly in both the weight/frequency and in n. We achieve this by carefully analyzing a Rankin-Selberg type L-function.

1.2. **Smoothing.** Rather than work with the discrete count $Z(n, \Omega)$, it is more convenient to smooth, thus introducing a parameter, $\rho > 0$ which we can choose at the end to optimize our bounds. To that end, given $z, \zeta \in \mathbb{S}^2$, let

$$\chi_\Omega(z, \zeta) : \begin{cases} 1 & \text{if } z \in \Omega + \zeta, \\ 0 & \text{otherwise,} \end{cases}$$

denote the indicator function of $\Omega+\zeta$. Further, let

$$k_\rho(z,\zeta) := \begin{cases} \frac{1}{2\pi(1-\cos\rho)} & \text{if } d(z,\zeta)<\rho \\ 0 & \text{otherwise,} \end{cases}$$

where d denotes the distance on the sphere. Now let

$$k_\rho(\Omega,\zeta,z) = (\chi_\Omega(\cdot,\zeta) * k_\rho)(z) = \int_{\mathbb{S}^2} k_\rho(z,\xi)\chi_\Omega(\xi,\zeta)d\sigma(\xi).$$

The smooth count at a point, is then

$$Z_\rho(n,\Omega+\zeta) = \sum_{\mathbf{x}\in\widehat{\mathcal{E}}(n)} k_\rho(\Omega,\zeta,\mathbf{x}),$$

and the smooth variance can be expressed as

$$\mathrm{Var}_\rho(\Omega,n) := \int_{\mathbb{S}^2} |Z_\rho(n,\Omega+\zeta) - N_n\sigma(\Omega)|^2 \,\mathrm{d}\sigma(\zeta).$$

Given a spherical cap Ω_X, let

$$A_{X,\rho} := \sum_{\substack{n\in\mathcal{N}\\ n\in[1,X]}} \mathrm{Var}_\rho(\Omega_X,n).$$

Further, let

$$A_{X,H,\rho} := \sum_{\substack{n\in\mathcal{N}\\ n\in[X,X+H]}} \mathrm{Var}_\rho(\Omega_X,n).$$

The following is a smooth version of Theorem 2,

Theorem 3. *Fix an integer X, a smoothing parameter $\rho>0$, and a spherical cap Ω_X with area $\sigma(\Omega_X)=cN_X^\delta$, where $-1<\delta<0$ and $c>0$ is a constant. Then, for any $\varepsilon>0$ we have the bound*

$$A_{X,\rho} \ll X^{1/2}\sigma(\Omega_X) + X^{1/2}\rho\sigma(\Omega_X)^{1/2} + X^{\varepsilon}\sigma(\Omega_X)^{1/2}\rho^{-1/2+\varepsilon}. \tag{1.5}$$

and moreover

$$A_{X,H,\rho} \ll X^{1/2}\sigma(\Omega_X) + X^{1/2}\rho\sigma(\Omega_X)^{1/2} + \frac{X^{1+\varepsilon}}{H}\sigma(\Omega_X)^{1/2}\rho^{-1/2-\varepsilon}. \tag{1.6}$$

With Theorem 3 at hand, Theorem 2 follows somewhat immediately:

Proof of Theorem 2. To prove Theorem 2, we need to relate the variance to the smoothed variance. Consider the variance:

$$\begin{aligned}\mathrm{Var}(\Omega,n) &= \int_{\mathbb{S}^2} |Z(n,\Omega+\zeta)-\sigma(\Omega)N_n|^2\,\mathrm{d}\sigma(\zeta)\\ &\le \mathrm{Var}_\rho(\Omega,n) + \int_{\mathbb{S}^2} |Z(n,\Omega+\zeta)-Z_\rho(n,\Omega+\zeta)|^2\,\mathrm{d}\sigma(\zeta).\end{aligned}$$

The latter quantity can then be written

$$\int_{\mathbb{S}^2} |Z(n,\Omega+\zeta) - Z_\rho(n,\Omega+\zeta)|^2 \,\mathrm{d}\sigma(\zeta) = \int_{\mathbb{S}^2} \left| \sum_{\mathbf{x}\in\widehat{\mathcal{E}}(n)} (\chi_\Omega(\mathbf{x},\zeta) - k_\rho(\Omega,\zeta,z)) \right|^2 \mathrm{d}\sigma(\zeta)$$

$$= \int_{\mathbb{S}^2} \left| \sum_{\mathbf{x}\in\widehat{\mathcal{E}}(n)} \int_{\mathbb{S}^2} k_\rho(\mathbf{x},\xi)(\chi_\Omega(\mathbf{x},\zeta) - \chi_\Omega(\xi,\zeta)) \right|^2 \mathrm{d}\sigma(\zeta)$$

$$= \sum_{\mathbf{x},\mathbf{y}\in\widehat{\mathcal{E}}(n)} \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} k_\rho(\mathbf{x},\xi)(\chi_\Omega(\mathbf{x},\zeta) - \chi_\Omega(\xi,\zeta))\mathrm{d}\sigma(\xi) \int_{\mathbb{S}^2} k_\rho(\mathbf{y},\xi)(\chi_\Omega(\mathbf{y},\zeta) - \chi_\Omega(\xi,\zeta))\mathrm{d}\sigma(\xi)\mathrm{d}\sigma(\zeta)$$

$$\ll \frac{N_n\sigma(\Omega)}{\rho^2} \sum_{\mathbf{x}\in\widehat{\mathcal{E}}(n)} \int_{\mathbb{S}^2}\int_{d(\mathbf{x},\xi)<\rho} (\chi_\Omega(\mathbf{x},\zeta) - \chi_\Omega(\xi,\zeta))\mathrm{d}\sigma(\xi)\mathrm{d}\sigma(\zeta)$$

$$\ll \frac{N_n\sigma(\Omega)}{\rho^2} \sum_{\mathbf{x}\in\widehat{\mathcal{E}}(n)} \int_{d(\mathbf{x},\xi)<\rho} (\rho\sigma(\Omega_X)^{1/2})\mathrm{d}\sigma(\xi)$$

$$\ll N_n^2\sigma(\Omega)^{3/2}\rho.$$

Hence we require $\rho < \frac{1}{X^{1/2}\sigma(\Omega_X)^{1/2}}$, for example, choose $\rho = \frac{1}{X^{1/2+\varepsilon}\sigma(\Omega_X)^{1/2}}$.

With that, we have that

$$\begin{aligned} A_X &\ll A_{X,\rho} + X^{1/2-\varepsilon}\sigma(\Omega_X) \\ &\ll X^{1/2}\sigma(\Omega_X) + \frac{1}{X^\varepsilon} + X^{1/4+\varepsilon}\sigma(\Omega_X)^{3/4}. \end{aligned}$$

Which is the correct size if $X^{1/2}\sigma(\Omega_X) > X^{1/4+\varepsilon}\sigma(\Omega_X)^{3/4}$ or rather $\sigma(\Omega_X) > X^{-1+\varepsilon}$ which is always the case. (1.4) follows similar lines with the same choice of ρ.

□

2. Reducing to bounds on Fourier coefficients of automorphic forms

Consider the smooth variance:

$$\mathrm{Var}_\rho(\Omega_X, n) = \int_{\mathbb{S}^2} (Z_\rho(n;\Omega_X+\zeta) - N_n\sigma(\Omega_X))^2 \,\mathrm{d}\sigma(\zeta);$$

we expand this into spherical harmonics: For $m = 0,1,\ldots$ let $\phi_{m,j}$ for $j = 1,2,\ldots,2m+1$, denote an orthonormal basis of eigenfunctions for Δ on $\mathbb{S}^2$. Define the Weyl sum:

$$W_{m,j}(n) := \sum_{\mathbf{x}\in\widehat{\mathcal{E}}(n)} \phi_{m,j}(\mathbf{x}). \tag{2.1}$$

Now, note that $k_\rho(\Omega_X,\zeta,z)$ is a function of $d(\zeta,z)$, let $k_\rho(t)$ denote the same function on $\mathbb{R}$. Then define:

$$h_X(m) := 2\pi\int_0^1 k_\rho(t)P_m(t)\mathrm{d}t,$$

where $P_m(t)$ is the m^{th} Legendre polynomial then the smooth variance can be written as

$$\mathrm{Var}_\rho(\Omega_n, n) = \sum_{m=1}^{\infty} h_X(m)^2 \sum_{j=1}^{2m+1} |W_{m,j}(n)|^2 .$$

While [BRS17] use that the Weyl sums can be expressed in terms of special values of L-functions, we return to the original proof of Linnik's conjecture by Duke to express these Weyl sums in terms of Fourier coefficients of half integer weight modular forms.

For a spherical harmonic of degree m, let $\theta_{m,j}(z) := \sum_{\ell\in\mathbb{Z}^3} \phi_{m,j}(\ell)e(z|\ell|^2)$, then $\theta_{m,j}$ is a holomorphic cusp form of weight $k := 3/2 + m$ for the group $\Gamma := \Gamma_0(4)$. Let $a_{m,j}(n)$ denote the n^{th} Fourier coefficient of $\theta_{m,j}$. Then

$$W_{m,j}(n) = \frac{a_{m,j}(n)}{n^{m/2}},$$

(see [Duk88] for more details).

With all that, we can express the average of variances as

$$A_{X,\rho} = \frac{1}{X} \sum_{m=1}^{\infty} h_X(m)^2 \sum_{j=1}^{2m+1} \sum_{\substack{n\in\mathcal{N}\\ n\in[1,X]}} \frac{|a_{m,j}(n)|^2}{n^m}.$$

Note that the Ramanujan conjecture for half-integral weight cusp forms suffices to achieve our desired bound.

To simplify matters, and since we only need an upper bound, we can complete the sum in n. That is, let

$$A_{X,\rho} \ll \frac{1}{X} \sum_{m=1}^{\infty} h_X(m)^2 \sum_{j=1}^{2m+1} \sum_{n\in[1,X]} \frac{|a_{m,j}(n)|^2}{n^{k-3/2}},$$

(we drop the ρ from the notation since it will be implicit in all that follows).

2.1. **Fourier coefficients of holomorphic cusp forms.** Working more generally, let

$$f(z) = \sum_{i=1}^{\infty} a_{m,j}(n)e(nz)$$

be a holomorphic cusp form of weight $k := m + 3/2$ for $\Gamma_0(4)$. Then an immediate bound is the following

$$a_{m,j}(n) \ll_k n^{k/2-1/4}\tau(n), \tag{2.2}$$

where τ is the divisor function.

It is more convenient to normalize the Fourier coefficients by $n^{-(k-1)/2}$, to that end let $b_{m,j}(n) := a_{m,j}(n)n^{-(k-1)/2}$. In which case, we want to bound the following

$$A_{X,\rho} \ll \frac{1}{X} \sum_{m=1}^{\infty} h_X(m)^2 \sum_{j=1}^{2m+1} \sum_{n\in[1,X]} n^{1/2} |b_{m,j}(n)|^2 .$$

By positivity, we then have that

$$A_{X,\rho} \ll \frac{1}{X^{1/2}} \sum_{m=1}^{\infty} h_X(m)^2 \mathcal{F}_X, \tag{2.3}$$

where

$$\mathcal{F}_X := \sum_{j=1}^{2m+1} \sum_{n\in[1,X]} |b_{m,j}(n)|^2 .$$

Hence the problem reduces to finding bounds on $\mathcal{F}_X$.

2.2. **Uniform bound.** Before bounding $\mathcal{F}_X$, we require the following lemma which gives a uniform bound on the Fourier coefficients $a_{m,j}(n)$, taking into account the n and k dependence, and the normalization. For a weight k modular form, f, we denote the L^2 mass by

$$\|f\|_2 = \left(\int_{\Gamma\backslash\mathbb{H}} y^{k-2} |f(z)|^2 \, \mathrm{d}x\mathrm{d}y\right)^{1/2} .$$

Lemma 4. *The Fourier coefficients of the theta series $\theta_{m,j}$ satisfy the following bound for any $\varepsilon > 0$ if $k < 2\pi e n$ then*

$$\sum_{j=1}^{2m+1} \frac{|a_{m,j}(n)|^2}{\|\theta_{m,j}\|_2^2} \ll \frac{(4\pi)^k n^{k-1/2+\varepsilon}}{\Gamma(k-1)k^{1/2}}. \tag{2.4}$$

Otherwise, if $k > 2\pi e n$ we have

$$\sum_{j=1}^{2m+1} \frac{|a_{m,j}(n)|^2}{\|\theta_{m,j}\|_2^2} \ll \frac{(4\pi n)^{k-1}}{\Gamma(k-1)}. \tag{2.5}$$

Proof. First normalize $\theta_{m,j}$ by the weight k, L^2-mass. Now, since the $\theta_{m,j}$ are orthogonal, $\theta_{m,j}/\|\theta_{m,j}\|_2$ can be taken to be an element of an orthonormal basis of $\mathcal{S}_k$, the space of weight k cusp forms. Let $\{g_\ell\}_{\ell=1}^L$ denote such a basis with $g_j = \theta_{m,j}/\|\theta_{m,j}\|_2$ for $i = 1, \dots 2m+1$. Then, using Petersson's formula for the Fourier coefficients of the Poincaré series in terms of generalized Kloosterman sums, one can derive the following (see [Iwa87, Lemma 1])

$$\sum_{\ell=1}^{L} |\widehat{g_\ell}(n)|^2 = \frac{(4\pi n)^{k-1}}{\Gamma(k-1)} \left(1 + 2\pi i^{-k} \sum_{c\equiv 0 (\mathrm{mod}\ 4)} c^{-1} J_{k-1}\left(\frac{4\pi n}{c}\right) K(n,n;c)\right), \tag{2.6}$$

where J_{k-1} is the Bessel function of order $k-1$ and K denotes the generalized Kloosterman sum:

$$K(m,n;c) := \sum_{d (\mathrm{mod}\ c)} \varepsilon_d^{-2k} \left(\frac{c}{d}\right) e\left(\frac{m\overline{d} + nd}{c}\right)$$

(see [Iwa87, 389] for the notation in the above definition). For our purposes the bound

$$|K(n,n;c)| \le (n,c)^{1/2} c^{1/2} \tau(c)$$

is all we need from K. As for the Bessel function, for $c < n$ we use that the Bessel function is bounded by 1, and for $c \ge n$ insert the bound $J_{k-1}(z) \le |z/2|^{k-1} / \Gamma(k)$ (see [Bat53, p14

(4)]), to deduce

$$\sum_{\ell=1}^{L} |\widehat{g}_\ell(n)|^2 \ll \frac{(4\pi n)^{k-1}}{\Gamma(k-1)} \left(1 + \sum_{c=1}^{X} c^{-1/2+\varepsilon} + \sum_{X}^{\infty} c^{-1/2+\varepsilon} \left|2\pi n/c\right|^{k-1} /\Gamma(k)\right),$$
$$\ll \frac{(4\pi n)^{k-1}}{\Gamma(k-1)} \left(1 + X^{1/2+\varepsilon} + \frac{(2\pi n)^{k-1}}{\Gamma(k)k} X^{3/2-k+\varepsilon}\right).$$

Using Sterling's formula we find that $X = \frac{n}{k}$ gives the optimal saving when $n \gg k$.

In the other case, suppose $k > 2\pi e n$, then the same calculation yields

$$\sum_{\ell=1}^{L} |\widehat{g}_\ell(n)|^2 \ll \frac{(4\pi n)^{k-1}}{\Gamma(k-1)} \frac{(2\pi n)^{k-1}}{\Gamma(k)} + \frac{(4\pi n)^{k-1}}{\Gamma(k-1)}.$$

Now applying Stirling's formula to $\Gamma(k)$ and using that $k > 2\pi e n$ we arrive at

$$\sum_{\ell=1}^{L} |\widehat{g}_\ell(n)|^2 \ll \frac{(4\pi n)^{k-1}}{\Gamma(k-1)} \frac{(2\pi n)^{k-3/2}}{(k/e)^k} + \frac{(4\pi n)^{k-1}}{\Gamma(k-1)}$$
$$\ll \frac{(4\pi n)^{k-5/2}}{\Gamma(k-1)} + \frac{(4\pi n)^{k-1}}{\Gamma(k-1)} \ll \frac{(4\pi n)^{k-1}}{\Gamma(k-1)}.$$

□

2.3. **L^2 bounds on $\theta_{m,j}$.** Another ingredient of the proof that we need is the following estimate on the L^2 norm of $\theta_{m,j}$:

Lemma 5. *For any $k \geq 5/2$, we have*

$$\int_{\Gamma\backslash\mathbb{H}} y^{k-2} \left|\theta_{m,j}(z)\right|^2 \mathrm{d}x\mathrm{d}y \ll \frac{\Gamma(k)}{(4\pi)^k} \left(\left|W_m(1)\right|^2 + \frac{1}{2^{k-3/2-\varepsilon}}\right) \ll \frac{\Gamma(k)}{(4\pi)^k} k. \tag{2.7}$$

Proof. The proof uses a standard trick to estimate L^2 norms of theta series. Namely, consider the Eisenstein series on $\Gamma = \Gamma_0(4)$:

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \Im(\gamma z)^s,$$

and consider the integral

$$\int_{\Gamma\backslash\mathbb{H}} \left|\theta_{m,j}(z)\right|^2 E(z,s) y^{k-2} \mathrm{d}x\mathrm{d}y.$$

Unfolding the Eisenstein series then yields

$$\int_{\Gamma_\infty\backslash\mathbb{H}} \left|\theta_{m,j}(z)\right|^2 y^{s+k-2} \mathrm{d}x\mathrm{d}y.$$

We have that $\theta_{m,j}(z) = \sum_{\ell \in \mathbb{Z}^3} \phi_{m,j}(\ell) e(z|\ell|^2)$. Inserting this definition and using the orthogonality of exponentials with integral frequency we arrive at

$$\int_0^\infty \sum_{L \in \mathbb{N}} \sum_{\substack{\ell, \widetilde{\ell} \in \mathbb{Z}^3 \\ |\ell|^2 = |\widetilde{\ell}|^2 = L}} \phi_{m,j}(\ell) \overline{\phi_{m,j}(\widetilde{\ell})} e^{-4\pi L y} y^{s+k-2} \mathrm{d}y.$$

Applying the change of variables $y \mapsto 4\pi L y$ we then have

$$\Gamma(s+k-1) \sum_{L\in\mathbb{N}} \frac{1}{(4\pi L)^{s+k-1}} \left| \sum_{\substack{\ell\in\mathbb{Z}^3 \\ |\ell|^2=L}} \phi_{m,j}(\ell) \right|^2 .$$

Note that the sum being squared, is exactly the Weyl sum (2.1), thus we have

$$\Gamma(s+k-1) \sum_{L\in\mathcal{N}} \frac{1}{(4\pi L)^{s+k-1}} \left|W_{m,j}(L)\right|^2 .$$

Thus we arrive at the bound

$$\int_{\Gamma\backslash\mathbb{H}} \left|\theta_{m,j}(z)\right|^2 E(z,s) y^{k-2} \mathrm{d}x\mathrm{d}y \ll \frac{\Gamma(s+k-1)}{(4\pi)^{s+k}} \sum_{L\in\mathcal{N}} \frac{1}{L^{s+k-1}} \left|W_{m,j}(L)\right|^2 .$$

Taking the residue at $s=1$ yields the desired bound

$$\int_{\Gamma\backslash\mathbb{H}} \left|\theta_{m,j}(z)\right|^2 y^{k-2} \mathrm{d}x\mathrm{d}y \ll \frac{\Gamma(k)}{(4\pi)^k} \sum_{L\in\mathcal{N}} \frac{1}{L^k} \left|W_{m,j}(L)\right|^2 .$$

Using the sup norm bound on spherical harmonics $\|\phi_{m,j}\|_\infty \ll k^{1/2}$ (see e.g [SW71, Chapter IV Cor. 2]) we can bound the Weyl sum by $W_{m,j}(L) \ll N_L k^{1/2} \ll L^{1/2+\varepsilon} k^{1/2}$ giving

$$\begin{aligned}\int_{\Gamma\backslash\mathbb{H}} \left|\theta_{m,j}(z)\right|^2 y^{k-2} \mathrm{d}x\mathrm{d}y &\ll \frac{\Gamma(k)}{(4\pi)^k} \left(\left|W_{m,j}(1)\right|^2 + \sum_{L\geq 2} \frac{1}{L^{k-1/2-\varepsilon}} k \right) \\ &\ll \frac{\Gamma(k)}{(4\pi)^k} \left(\left|W_{m,j}(1)\right|^2 + \frac{1}{2^{k-3/2-\varepsilon}} \right).\end{aligned}$$

Now to get the second inequality in (2.7) we again use the sup norm bound $\sup(\phi_{m,j}) \ll k^{1/2}$. □

3. Bounds on L-series

Given our theta series $\theta_{m,j}(z)$, we wish to achieve bounds on the average of Fourier coefficients. To that end, define the normalized Dirichlet series

$$L_{m,j}(s) = \sum_{n\geq 1} \frac{|b_{m,j}(n)|^2}{\|\theta_{m,j}\|^2 n^s}.$$

Further, define the complete series as

$$\Lambda(s) = \zeta(2s)(2\pi)^{-2(s+k-1)}\gamma(k,s)L_{m,j}(s),$$

where the gamma factor is given by

$$\gamma(k,s) := \Gamma(s)\Gamma(s+k-1).$$

Then we have the following classical lemma

Lemma 6. *The complete series Λ can be meromorphically continued to $\mathbb{C}$ with a simple pole at $s=1$. Moreover, it satisfies the functional equation*

$$\Lambda(s) = \Lambda(1-s). \tag{3.1}$$

Proof. Our aim is to use the standard Rankin-Selberg trick, namely, define the Eisenstein series

$$E_\Gamma(z,s) = \sum_{\Gamma_\infty \backslash \Gamma} \Im(\gamma z)^s. \tag{3.2}$$

Then, by Parseval's identity, we have

$$(4\pi)^{-(s+k-1)} \sum_{n\geq 1} \frac{|a_{m,j}(n)|^2}{n^{s+k-1}} \Gamma(s+k-1) = \int_0^1 \int_0^\infty |\theta_{m,j}(z)|^2 \, y^k y^s \frac{\mathrm{d}y}{y^2}.$$

Now note that by the modularity of the theta series, we have that $|f(z)|^2 \, y^k$ is Γ-invariant. Thus, refolding the integral, we arrive at

$$(4\pi)^{-(s+k-1)} \sum_{n\geq 1} \frac{|a_{m,j}(n)|^2}{n^{s+k-1}} \Gamma(s+k-1) = \int_{\Gamma\backslash\mathbb{H}} |\theta_{m,j}(z)|^2 \, \Im(z)^k E_\Gamma(z,s) \mu(\mathrm{d}z).$$

Now multiply both sides of the equation by $\pi^{-(s+k-1)}\Gamma(s)\zeta(2s)$. In which case, the functional equation for the Eisenstein series means that we obtain a functional equation for $s \mapsto 1-s$.

□

3.1. **Bounds on the critical line.** The next ingredient we will need is a bound on $L_{m,j}$ on the critical line $\Re(s) = 1/2$. For this, we will derive an approximate functional equation as done in [Blo20] which is itself motivated by [IK04, Theorem 5.3, Proposition 5.4], from which we derive the following bound

Lemma 7. *For any $\varepsilon > 0$ and for $|t| \ll k^\varepsilon X^\varepsilon$*

$$\sum_{j=1}^{2m+1} L_{m,j}(\frac{1}{2}+it) \ll \frac{(4\pi)^k}{\Gamma(k-1)} X^\varepsilon k^{1/2+\varepsilon}, \tag{3.3}$$

with the explicit constant depending only on ε.

Proof. Consider the integral

$$W_+(s) := \frac{1}{2\pi i} \int_{(1)} e^{u^2} \Lambda(u+s) \frac{\mathrm{d}u}{u},$$

Shift the contour to (-1), thus picking up the residues at $u = 0$ and $u = 1-s$:

$$W_+(s) = \frac{1}{2\pi i} \int_{(-1)} e^{u^2} \Lambda(u+s) \frac{\mathrm{d}u}{u} + \Lambda(s) + \frac{e^{(1-s)^2}}{1-s} \operatorname{Res}(\Lambda, 1).$$

Now consider this expression at $s = \frac{1}{2} + it$, and apply the functional equation for Λ to the integral on $u = -1 + i\tau$, giving

$$\begin{aligned}\Lambda(\frac{1}{2}+it) &= W_+(s) + \frac{1}{2\pi i} \int_{(-1)} e^{u^2} \Lambda(-\frac{1}{2}+it+i\tau) \frac{\mathrm{d}u}{u} + \frac{e^{(1-s)^2}}{1-s} \operatorname{Res}(\Lambda, 1) \\ &= W_+(s) + W_-(s) + \frac{e^{(1-s)^2}}{1-s} \operatorname{Res}(\Lambda, 1),\end{aligned}$$

where

$$W_-(s) := \frac{1}{2\pi i}\int_{(-1)} e^{u^2}\Lambda(\frac{3}{2}+it+i\tau)\frac{\mathrm{d}u}{u}.$$

We calculate the residue in Lemma 5, giving us the bound

$$\Lambda(\frac{1}{2}+it) \ll W_+(s)+W_-(s)+\left|\frac{e^{(1/2-it)^2}}{1/2-it}\right|\frac{1}{\pi^k}.$$

Now consider

$$\frac{W_+(s)}{\gamma(k,s)} \ll \left|\sum_{n\geq 1}\frac{|b_{m,j}(n)|^2}{\|\theta_{m,j}\|^2 n^s}\int_{(1)}\frac{e^{u^2}}{n^u}\frac{\gamma(k,u+s)}{\gamma(k,s)}\zeta(2(u+s))(2\pi)^{-2(u+s+k)}\frac{\mathrm{d}u}{u}\right|.$$

To bound $W_+/\gamma(k,s)$ we separate the sum over n into two regimes. Fix $T=\lceil 2\pi e k\rceil$, W_1 will sum over $n<T$ and W_2 will sum over $n\geq T$.

Consider first

$$W_2(s) := \left|\sum_{n>k^{1+\varepsilon}}\frac{|b_{m,j}(n)|^2}{\|\theta_{m,j}\|^2 n^s}\int_{(1)}\frac{e^{u^2}}{n^u}\frac{\gamma(k,u+s)}{\gamma(k,s)}\zeta(2(u+s))(2\pi)^{-2(u+s+k)}\frac{\mathrm{d}u}{u}\right|.$$

Shift the contour to $\Re(u)=A$. Then we have

$$W_2(s) := \left|\sum_{n>T}\frac{|b_{m,j}(n)|^2}{\|\theta_{m,j}\|^2 n^s}\int_{\mathbb{R}}\frac{e^{A^2-\tau^2+2iA\tau}}{n^{A+i\tau}}\frac{\gamma(k,A+i\tau+s)}{\gamma(k,s)}\zeta(2(A+i\tau+s))(2\pi)^{-2(A+i\tau+s+k)}\frac{\mathrm{d}\tau}{A+i\tau}\right|.$$

The exponential causes the integral to be rapidly converging, thus we can bound $\tau \ll (k+|t|)^{\varepsilon}$. Moreover we can bound the zeta function by $(\tau+|t|)^{\varepsilon} \ll (|t|+k)^{\varepsilon}$, and bound the gamma factor using Stirling's formula:

$$\frac{\gamma(k,A+i\tau+s)}{\gamma(k,s)} \ll \exp(\pi|A+i\tau|)(|s|+3)^A(|s+k|+3)^A \tag{3.4}$$

(see [IK04, p 100]). Thus we arrive at the bound

$$\begin{aligned} W_2(s) &\ll_A (k+|t|)^{\varepsilon}(2\pi)^{-2k}\sum_{n>T}\frac{|b_{m,j}(n)|^2}{\|\theta_{m,j}\|^2 n^{1/2}}\int_{|\tau|\leq(k+|t|)^{\varepsilon}}\left|\frac{e^{-\tau^2}}{n^A}\frac{\gamma(k,A+i\tau+s)}{\gamma(k,s)}\right|\mathrm{d}\tau \\ &\ll_A (k+|t|)^{\varepsilon}(1+|t|)^A(2\pi)^{-2k}\sum_{n>T}\frac{|b_{m,j}(n)|^2\,k^A}{\|\theta_{m,j}\|^2 n^{A+1/2}}. \end{aligned}$$

From here we apply Lemma 4 to conclude:

$$\begin{aligned} \sum_{j=1}^{2m+1}\frac{W_2(\frac{1}{2}+it)}{\zeta(1+2it)(2\pi)^{-k}} &\ll (kX)^{\varepsilon}\sum_{n>T}\frac{k^A}{n^{A+1/2}}\sum_{j=1}^{2m+1}\frac{|b_{m,j}(n)|^2}{\|\theta_{m,j}\|^2} \\ &\ll \frac{(4\pi)^k}{\Gamma(k-1)k^{1/2}}(kX)^{\varepsilon}k^A\sum_{n>T}\frac{1}{n^{A-\varepsilon}} \\ &\ll \frac{(4\pi)^k}{\Gamma(k-1)}(kX)^{\varepsilon}k^{1/2+\varepsilon}. \end{aligned}$$

Turning to W_1 we shift the contour to $\Re(u) = \frac{1}{2} + \sigma$. Hence

$$W_1(s) := \left| \sum_{n<T} \frac{|b_{m,j}(n)|^2}{\|\theta_{m,j}\|^2 n^s} \int_{(\frac{1}{2}+\sigma)} \frac{e^{u^2}}{n^u} \frac{\gamma(k,u+s)}{\gamma(k,s)} \zeta(2(u+s))(2\pi)^{-2(u+s+k)} \frac{\mathrm{d}u}{u} \right|.$$

we can again use the oscillatory term to bound the integral such that $\tau := \Im(u) \ll (|t|+k)^\varepsilon$

$$\begin{aligned} W_1(s) &\ll \sum_{n<T} \frac{|b_{m,j}(n)|^2}{\|\theta_{m,j}\|^2 n^{1/2}} \left| \int_{|\tau|<(|t|+k)^\varepsilon} \frac{e^{u^2}}{n^u} \frac{\gamma(k,u+s)}{\gamma(k,s)} \zeta(2(u+s))(2\pi)^{-2(u+s+k)} \mathrm{d}\tau \right| \\ &\ll (2\pi)^{-2k} (|t|+k)^\varepsilon \sum_{n<T} \frac{|b_{m,j}(n)|^2}{\|\theta_{m,j}\|^2 n^{1+\sigma}} \int_{|\tau|<(|t|+k)^\varepsilon} e^{u^2} \frac{\gamma(k,u+s)}{\gamma(k,s)} \mathrm{d}\tau \\ &\ll (2\pi)^{-2k} (|t|+k)^\varepsilon (1+|t|)^{1/2+\sigma} |k|^{1/2+\sigma} \sum_{n<T} \frac{|b_{m,j}(n)|^2}{\|\theta_{m,j}\|^2 n^{1+\sigma}}. \end{aligned}$$

From here we apply Lemma 4 to conclude:

$$\begin{aligned} \sum_{j=1}^{2m+1} \frac{W_1(\frac{1}{2}+it)}{\zeta(1+2it)(2\pi)^{-k}} &\ll (1+|t|)^{1/2+\sigma+\varepsilon} |k|^{1/2+\sigma+\varepsilon} \sum_{n<T} \frac{1}{n^{1+\sigma}} \sum_{j=1}^{2m+1} \frac{|b_{m,j}(n)|^2}{\|\theta_{m,j}\|^2} \\ &\ll (Xk)^\varepsilon |k|^{1/2+\sigma+\varepsilon} \frac{(4\pi)^{k-1}}{\Gamma(k-1)}. \end{aligned}$$

Set $\sigma = \varepsilon$ to yield the desired bound.

Finally, we note that W_- can be bounded using the same technique. Together this yields (3.3).

□

3.2. **Bounds on $\mathcal{F}_X$.** To achieve the desired bound on $\mathcal{F}_X$ the following lemma allows us to express $\mathcal{F}_X$ in terms of the residue of $L_{m,j}$ at $s = 1$.

Lemma 8. *For $\mathcal{F}_X$ as above, as $X \to \infty$ we have*

$$\mathcal{F}_X = \sum_{j=1}^{2m+1} C \frac{\|\theta_{m,j}\|_2^2}{(4\pi)^{-k}\Gamma(k)} X + O(X^{1/2+\varepsilon} k^{5/2+\varepsilon}). \tag{3.5}$$

Proof. By Perron's formula we have

$$\sum_{j=1}^{2m+1} \sum_{n \le X} |b_{m,j}(n)|^2 = \sum_{j=1}^{2m+1} \left(\frac{1}{2\pi i} \int_{(c)} L_{m,j}(s) \frac{X^s}{s} \mathrm{d}s \|\theta_{m,j}\|_2^2 \right) + \frac{1}{2} \sum_{j=1}^{2m+1} |b_{m,j}(X)|^2,$$

where the integral takes place on $\Re(s) = c > 1$. Using Lemmas 4 and 5 we can bound the last term by $X^{1/2+\varepsilon} k^{3/2+\varepsilon}$. Pushing the contour to the left, we pick up the contribution from the pole at $s = 1$. Thus

$$\sum_{j=1}^{2m+1} \sideset{}{'}\sum_{n \le X} |b_{m,j}(n)|^2 = \sum_{j=1}^{2m+1} \|\theta_{m,j}\|_2^2 \left(\mathrm{Res}(L_{m,j}, 1) X + \frac{1}{2\pi i} \int_{(1/2)} L_{m,j}(s) \frac{X^s}{s} \mathrm{d}s \right) \tag{3.6}$$

(where the $\sum'$ denotes that the last term is multiplied by $1/2$). For the residue of $L_{m,j}$, we can use the Rankin-Selberg method to write this as

$$\operatorname{Res}(L_{m,j},1) = C\frac{1}{(4\pi)^{-k}\Gamma(k)}, \tag{3.7}$$

where the constant does not depend on the weight k (see [Kow, Proposition 3.6.2] for details).

Thus, it remains to estimate the error term

$$\begin{aligned} R &:= \sum_{j=1}^{2m+1} \|\theta_{m,j}\|_2^2 \int_{(1/2)} L_{m,j}(s)\frac{X^s}{s}\mathrm{d}s \\ &\ll \sum_{j=1}^{2m+1} \|\theta_{m,j}\|_2^2 \left(\int_{1/2-iT}^{1/2+iT} L_{m,j}(s)\frac{X^s}{s}\mathrm{d}s + \left| \int_{1/2+iT}^{\infty} L_{m,j}(s)\frac{X^s}{s}\mathrm{d}s \right| \right). \end{aligned}$$

To bound the first term we insert (3.3), and for the second term we integrate by parts

$$\begin{aligned} R \ll & X^{1/2}\left(\frac{(4\pi)^k}{\Gamma(k-1)}X^{\varepsilon}k^{1/2+\varepsilon}\right)\max_j(\|\theta_{m,j}\|_2^2)\int_0^T t^{1+\varepsilon}\frac{\frac{1}{2}+t}{\frac{1}{4}+t^2}\mathrm{d}t \\ & + \sum_{j=1}^{2m+1} X^{1/2}\left|\int_T^{\infty} L'_{m,j}(\frac{1}{2}+it)\frac{e^{i\log(t)}}{(\frac{1}{2}+it)\log(t)}\mathrm{d}t\right|. \end{aligned}$$

Setting $T = X^{\varepsilon}$ yields

$$\begin{aligned} R \ll & X^{1/2}\left(\frac{T(4\pi)^k k^{1/2+\varepsilon}}{\Gamma(k-1)}\right)\max_j(\|\theta_{m,j}\|_2^2) \\ & + X^{1/2}\frac{Tk}{\log T} + \sum_{j=1}^{2m+1} X^{1/2}\left|\int_T^{\infty} L'_{m,j}(\frac{1}{2}+it)\frac{e^{i\log(t)}}{(\frac{1}{2}+it)\log(t)}\mathrm{d}t\right|. \end{aligned}$$

Now, from Lemma 5, we have that $\|\theta_{m,j}\|_2^2 \ll \frac{\Gamma(k)}{(4\pi)^k}k$. Thus

$$R \ll X^{1/2}Tk^{5/2+\varepsilon} + X^{1/2}\frac{Tk}{\log T} + \sum_{j=1}^{2m+1} X^{1/2}\left|\int_T^{\infty} L'_{m,j}(\frac{1}{2}+it)\frac{e^{i\log(t)}}{(\frac{1}{2}+it)\log(t)}\mathrm{d}t\right|.$$

Upper bounding the remaining two integrals, we arrive at (for $T \gg 1$)

$$R \ll X^{1/2}k^{5/2}T. \tag{3.8}$$

Putting (3.8) and (3.7) together, with $T = X^{\varepsilon}$, we conclude (3.5).

□

3.3. **Proof of Theorem 3.** The advantage of having smoothed is that h_X satisfies the bound:

$$h_X(m) \ll m^{-3/2}\sigma(\Omega_X)^{1/4}\min\left(1, \frac{(\sin\rho)^{1/2}}{m^{3/2}(1-\cos\rho)}\right), \tag{3.9}$$

which is derived in [LPS86, (2.13)].

Turning to our average of variances, this gives:

$$A_{X,\rho} \ll \frac{1}{X^{1/2}} \sum_{m=1}^{\infty} h_X(m)^2 \mathcal{F}_X.$$

Apply Lemma 8 to extract the main term:

$$A_{X,\rho} \ll \frac{1}{X^{1/2}} \sum_{m=1}^{\infty} h_X(m)^2 \left(\sum_{m=1}^{2m+1} \frac{\|\theta_{m,j}\|_2^2}{(4\pi)^{-k}\Gamma(k)} X + X^{1/2+\varepsilon} k^{5/2+\varepsilon} \right).$$

To the main term (left most term in the brackets) we apply Lemma 5, giving

$$\mathcal{M} := X^{1/2} \sum_{m=1}^{\infty} h_X(m)^2 \sum_{j=1}^{2m+1} \frac{\|\theta(\cdot,\phi_{m,j})\|_2^2}{(4\pi)^{-k}\Gamma(k)} \ll X^{1/2} \sum_{m=1}^{\infty} h_X(m)^2 \sum_{j=1}^{2m+1} (|W_{m,j}(1)|^2 + \frac{1}{2^k}).$$

The term involving $W_{m,j}(1)$ is exactly $X^{1/2}\operatorname{Var}_\rho(\Omega_X, 1)$, which can be bounded by $X^{1/2}\sigma(\Omega_X)+ X^{1/2}\rho\sigma(\Omega_X)^{1/2}$. The other term can be fairly easily bounded using (3.9) giving

$$\mathcal{M} := X^{1/2}\sigma(\Omega_X) + X^{1/2}\rho\sigma(\Omega_X)^{1/2}.$$

Bounding the error term:

$$\mathcal{E} := X^{\varepsilon} \sum_{m=1}^{\infty} h_X(m)^2 k^{5/2+\varepsilon},$$

is more delicate. For this we again insert the bounds (3.9)

$$\begin{aligned}
\mathcal{E} &\ll X^{\varepsilon}\sigma(\Omega_X)^{1/2} \sum_{m=1}^{\infty} m^{-1/2+\varepsilon} \min\left(1, \frac{\sin(\rho)^{1/2}}{m^{3/2}(1-\cos\rho)}\right)^2 \\
&\ll X^{\varepsilon}\sigma(\Omega_X)^{1/2} \left(\sum_{m=1}^{1/\rho} m^{-1/2+\varepsilon} + \frac{1}{\rho^3} \sum_{m=1/\rho}^{\infty} \frac{1}{m^{7/2-\varepsilon}} \right) \\
&\ll X^{\varepsilon}\sigma(\Omega_X)^{1/2}\rho^{-1/2-\varepsilon}.
\end{aligned}$$

Thus we have the bound

$$A_{X,\rho} \ll X^{1/2}\sigma(\Omega_X) + X^{1/2}\rho\sigma(\Omega_X)^{1/2} + X^{\varepsilon}\sigma(\Omega_X)^{1/2}\rho^{-1/2+\varepsilon}.$$

□

3.4. **Averages over windows.** All that remains, is to bound the smoothed average:

$$\begin{aligned}
A_{X,H,\rho} &= \frac{1}{H} \sum_{m=1}^{\infty} h_X(m)^2 \sum_{j=1}^{2m+1} \sum_{n\in[X,X+H]} \frac{|a_{m,j}(n)|^2}{n^{k-3/2}} \\
&= \frac{1}{H} \sum_{m=1}^{\infty} h_X(m)^2 \sum_{j=1}^{2m+1} \left(\sum_{n\in[1,X+H]} n^{1/2}\,|b_{m,j}(n)|^2 - \sum_{n\in[1,X]} n^{1/2}\,|b_{m,j}(n)|^2 \right).
\end{aligned}$$

Up to a negligible error, we can write this as

$$\begin{aligned} A_{X,H,\rho} &= \frac{X^{1/2}}{H}\sum_{m=1}^{\infty} h_X(m)^2 \sum_{j=1}^{2m+1}\left(\sum_{n\in[1,X+H]}|b_{m,j}(n)|^2 - \sum_{n\in[1,X]}|b_{m,j}(n)|^2\right)(1+o(1)) \\ &= \frac{X^{1/2}}{H}\sum_{m=1}^{\infty} h_X(m)^2 \left(\mathcal{F}_{X+H}-\mathcal{F}_X\right)(1+o(1)). \end{aligned}$$

Now to this we apply Lemma 8 yielding

$$A_{X,H,\rho} = \frac{X^{1/2}}{H}\sum_{m=1}^{\infty} h_X(m)^2 \left(\sum_{j=1}^{2m+1} \frac{\|\theta_{m,j}\|_2^2}{(4\pi)^{-k}\Gamma(k)}H + O(X^{1/2+\varepsilon}k^{5/2+\varepsilon})\right)(1+o(1)).$$

To this we can apply the same methodology as section 3.3

$$\begin{aligned} A_{X,H,\rho} &\ll X^{1/2}\sum_{m=1}^{\infty} h_X(m)^2 \left(\sum_{j=1}^{2m+1} \frac{\|\theta_{m,j}\|_2^2}{(4\pi)^{-k}\Gamma(k)}\right) + O\left(\frac{X^{1+\varepsilon}}{H}\sum_{m=1}^{\infty} h_X(m)^2 k^{5/2+\varepsilon}\right) \\ &\ll X^{1/2}\sigma(\Omega_X) + X^{1/2}\rho\sigma(\Omega_X)^{1/2} + \frac{X^{1+\varepsilon}}{H}\sigma(\Omega_X)^{1/2}\rho^{-1/2-\varepsilon}. \end{aligned}$$

□

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