Financial Software Engineering

Take Home Exam 2019

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Question 1.1

In the Cournot Duopoly two firms compete over a fixed market demand. The game models the strategic interaction of the game. The produce produced between the firms is very similar and can be said to be "strategic substitutes" of each other. Firm F_1 and F_2 produce quantities q_1 and q_2. The strategy is to determine what quantity each firm will produce.

In this game the utility function models the price of one good in terms of the other players production. $P(q_1, q_2) = A - B(q_1 + q_2)$ where P is price/Unit and A and B are constants. Price decrease with quantity. Cost per unit to each firm is defined by C such that
Total cost to Firm $F_1 = Cq_1$ and total cost to Firm $F_2 = Cq_2$.

The payoffs for each firm can be defined in terms of the other companies' production as:

$$Payoff = U_1(q_1, q_2) = Total\ revenue - total\ cost$$
$$= \frac{price}{unit} - totalcost$$
$$= \big(A - B(q_1 + q_2)\big)q_2 - Cq_1$$
$$= \big(A - C - B(q_1 + q_2)\big)q_1$$
$$\therefore U_2(q_1, q_2) = \big(A - C - B(q_1 + q_2)\big)q_2$$

The payoffs for each firm depend on both your production and the production of the other firm. To find the best response S_1^* of F_1 we differentiate WRT S_1 and set the equation = 0

$$U_1(q_1, q_2) = q_1\big(A - C - B(q_1 + q_2)\big)$$
$$= Aq_1 - Cq_1 - Bq_1^2 - Bq_1q_2$$
$$\frac{\partial U_1}{\partial q_1} = A - C - 2Bq_1 - Bq_2 = 0$$
$$\therefore q_1^* = \frac{A - C - Bq_2}{2B} = BR_1(q_2)$$
$$and\ q_2^* = \frac{A - C - Bq_2}{2B}$$

These values q_1^* and q_2^* are the best responses (BR) of each firm given the actions of the *other* firm. From this we can find the NE where these values intercept.

$$q_1^* = BR_1(q_2^*)$$
$$q_2^* = BR_2(q_1^*)$$
$$\therefore q_1^* = \frac{A - C - Bq_2^*}{2B}\ and\ q_2^* = \frac{A - C - Bq_1^*}{2B}\ only\ at\ NE$$

We can therefore solve simultaneously.

$$\therefore q_1^* = \frac{A - C}{2B} - \frac{1}{2}q_2^* = \frac{A - C}{2B} - \frac{1}{2}\left(\frac{A - C}{2B} - \frac{1}{2}q_1^*\right)$$
$$\Longrightarrow \frac{3}{4}q_1 = \frac{A - C}{4B}$$

$$\therefore q_1^* = q_2^* = \frac{A-C}{3B}$$

This is therefore the Nash equilibrium for this game: $NE = \left(\frac{A-C}{3B}, \frac{A-C}{3B}\right)$. This is the value given in the question as $q_c = \frac{a-c}{3}$.

Next, the monopoly quantity is considered. The is seen as the aggregate profit between the two firms as $q_1 + q_2 = q_m$

$$\begin{gathered} U_1(q_1, q_2) + U_2(q_1, q_2) \\ = (q_1 + q_2)[A - C - B(q_1 + q_2)] \\ = q_m(A - C - Bq_m) \end{gathered}$$

The utility can be found for the monopoly as:

$$\begin{gathered} U_m(q_m) = q_m(A - C - Bq_m) \\ = (A - C)q_m - Bq_m^2 \end{gathered}$$

As before, maximize through differentiation:

$$\frac{\partial U_m}{\partial q_m} = (A - C) - 2Bq_m = 0$$

$$\therefore q_m^* = \frac{A-C}{2B}$$

To find the value for each firm we set $q_1 = q_2$, as with the tragedy of the commons where each firm is assumed to produce half of this quantity q_m^*

$$\therefore q_m^1 = q_m^2 = \frac{A-C}{4B}$$

This quantity is lower than the value before, meaning that each respective company would need to produce less product to be in a combined monopoly maximization position.

In this configuration each's individual utility payoff is higher than before in the true NE configuration. That is $U_1\left(\frac{q_m}{2}, \frac{q_m}{2}\right) > U_1(q_c, q_c)$. However, despite the individual payoffs being higher in this configuration where the firms cooperate the Nash payoff is not Pareto optimum. This means that one of the firms can deviate and increase their quantity to increase their payoff. If Each player plays this Pareto outcome, they will have an incentive to deviate such they maximize their own personal outcome. The other firm will therefore do the same in response to try and maximize their outcome. Ultimately this coverages back to the NE of q_c where no individual has an incentive to deviate.

At this point the outcome is similar to the prisoner's dilemma because the outcome that maximizes the collective payoff is not the same as that that maximizes the individual payoff so there is an incentive to deviate. This can be shown mathematically by looking at the payoffs and profits earned by each firm. The profits for Firm 1 (which are the same for firm 2) are:

$$\begin{gathered} \pi_{mm}^1 = (P - c)q_m^1 = (a - Q - c)q_m^1 = \left(a - \frac{a-c}{2} - c\right)q_m^1 \\ = \frac{a-c}{2}\,\frac{a-c}{4} = \frac{(a-c)^2}{8} = 0.13(a-c)^2 \end{gathered}$$

If both are playing the Cournot equilibrium quantity, then the profit earned by Firm 1(2) are as follows:

$$\pi_{cc}^1 = (P - c)q_c^1 = \left(a - \frac{2(a-c)}{3} - c\right)q_c^1 = \left(\frac{a+2c}{3} - c\right)q_c^1$$

$$= \frac{(a-c)^2}{9} = 0.11(a-c)^2$$

If one of the firms(say the firm 1) plays the Cournot quantity and the other plays the monopoly quantity then firm 1's profits are:

$$\pi_{cm}^1 = (P-c)q_c^1 = \left(\frac{5a}{12} + \frac{7c}{12} - c\right)\frac{a-c}{3}$$

$$= \frac{5}{36}(a-c)^2 = 0.14(a-c)^2$$

And then firm 2's profits are:

$$\pi_{cm}^2 = (P-c)q_m^2 = \left(\frac{5a}{12} + \frac{7a}{12} - c\right)\frac{a-c}{4}$$

$$= \frac{5}{48}\,(a-c)^2 = 0.1(a-c)^2$$

This notation can be simplified by:

$$\alpha = (a-c)^2$$

Through symmetry their profits are reverse when the roles swapped. A payoffs table can be formed as:

		Player 2	
		w	n
Player 1	w	$0.13\alpha, 0.13\alpha$	$0.10\alpha, 0.14\alpha$
	n	$0.14\alpha, 0.10\alpha$	$0.11\alpha, 0.11\alpha$

Which has the same shape as that of the prisoner's dilemma. Regardless of the other firm's choice both firm's would maximize their payoffs by choosing to produce the Cournot quantity. Each firm has a strictly dominated strategy $\frac{q_m}{2}$ and they are both worse off in equilibrium (where they were making $0.11(a-c)^2$ in profits) than they would have been had they cooperated by producing q_m together (which would have earned them $0.13(a-c)^2$

Question 1.2

This game has 1 final subgame outcome in which firm 2 and 3 simultaneously choose their quantities. There is also an improper subgame which is the game itself. In this improper game firm 1 chooses its quantities given its beliefs about the outcome of the competition between firm 2 and 3 in the final subgame. This outcome can be split into the final (proper) and initial (improper) subgames.

Final (proper) subgame: Firm 2's payoff is given by $(a - Q - m)q_2$ where $Q = q_1 + q_2 + q_3$. Firm 2's best response function can be found then as:

$$BR_2(q_1, q_3) = \frac{a - m - q_1 - q_3}{2}$$

Firm 2 and firm 3 best response cross at this point such that:

$$\left(\frac{a - m - q_1}{3}; \frac{a - m - q_1}{3}\right)$$

Therefore, the initial subgame can be found. Firm 1 chooses quantity q_1 that maximizes its payoff given by $(a - Q - m)q_1$ where $Q = \frac{2(a-m-q_1)}{3} + q_1$. Therefore, the unique Subgame perfect equilibrium is:

$$\left(\frac{a - m}{2}; \frac{a - m - q_1}{3}; \frac{a - m - q_1}{3}\right)$$

And from this the unique subgame perfect outcome is:

$$\left(\frac{a - m}{2}; \frac{a - m}{6}; \frac{a - m}{6}\right)$$

Question 1.3

Strategy in a repeated game:

A repeated game is one where a stage game (a strategic form game) is played at each date for some duration of T periods. The two main kinds of repeated games are finite and infinitely repeated games. In many repeated games the preferred strategy is not to play a Nash strategy of the stage game, but to cooperate and play a socially optimum strategy. In a repeated game the outcome of each round is known to all players before the next round takes place and this is used to inform the strategy. This strategy will evolve as each round of the game is played wherein the previous rounds information is used as inputs to a function to define the strategy of the next round. Therefore a strategy in a repeated game is a function of all past rounds. The set of strategies for player i in a repeated game is referred to as $\sum^i$. Each player will determine their strategies or moves taking into account all previous moves up until that moment.

Subgame in a repeated game:

A subgame is a subset of any game that includes an initial node (which has to be independent from any information set) and all its successor nodes.

Subgame-perfect Nash equilibrium:

A subgame-perfect equilibrium is an equilibrium not only overall, but also for each subgame.

Question 1.4

Stage payoffs are given by the two matrices. $R > D > r > D/2$. In this setup the banks act as an intermediary between lenders and borrowers. Each deposit has an initial value D. the bank aims to transform the maturity structures of the deposits. Deposits are short term but loans are long term.

The deposits are applied towards an investment project which has a return of 2r if the project is liquidated after one period and 2R if liquidated after two periods. The return is 2D if the project is continued for more than two terms. If either of the depositor asks for a return of its deposit, then the project is liquidated, and the withdrawing depositor either gets D or his share in liquidation proceeds, whichever is larger. Denote the first stage actions by w for withdrawing and n for not withdrawing. Let the second period actions for the two players be W and N respectively.

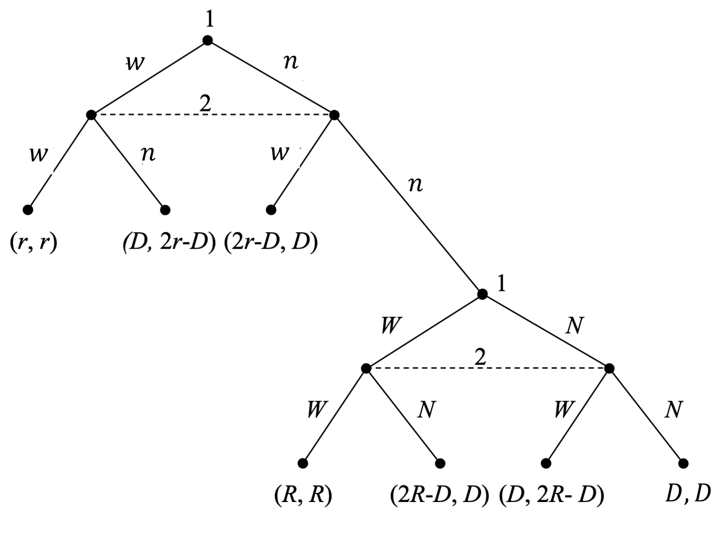
	w	n
w	r, r	$D, 2r - D$
n	$2r - D, D$	Second period

First period

	W	N
W	R, R	$2R - D, D$
N	$D, 2R - D$	D, D

Second period

This can be expressed as a normal form as well as:



Two pure strategy subgame perfect equilibria of the game are:

$$s_1^* = \{w, W\}, s_2^*\{w, W\}$$
$$s_1^* = \{n, W\}, s_2^*\{n, W\}$$

Question 1.5
Firm i's action space: $A_i = \{p: p \geq 0\}$
Firm i's type space: $T_i = \{H, L\}$
Firm i's beliefs: $hb_H + (1-h)b_L$
Firm i's strategy space: $S_i = \{(p_{iH}, p_{iL}): p_{iH}, p_{iL} \in A_i\}$

In order to find the pure-strategy Bayesian Nash equilibrium an assumption needs to be made that: $hb_H + (1-h)b_L < 2$

Firm i's utility function (for type t): $\left[a - p_{it} + b_t\left(hp_{jH} + (1-h)p_{jL}\right)\right]p_{it}$

We can therefore find Firm i's maximization problem as:

$$\max_{p_{it}}\left[a - p_{it} + b_t\left(hp_{jH} + (1-h)p_{jL}\right)\right]p_{it}$$

Then by taking the first order derivative and setting to zero(first order condition)

$$a - 2p_{it} + b_t\left(hp_{jH} + (1-h)p_{jL}\right) = 0$$

Therefore for $i = 1,2$,

$$p_{iH} = \frac{a}{2} + \frac{b_H\left(hp_{jH} + (1-h)p_{jL}\right)}{2}$$
$$p_{iL} = \frac{a}{2} + \frac{b_L\left(hp_{jH} + (1-h)p_{jL}\right)}{2}$$

Next, let $b = hb_H + (1-h)b_L$. From this we can generate:

$$p_{iH} = \frac{a}{2} + \frac{ab_H}{4} + bb_H\frac{hp_{iH} + (1-h)p_{iL}}{4}$$
$$p_{iL} = \frac{a}{2} + \frac{ab_L}{4} + bb_L\frac{hp_{iH} + (1-h)p_{iL}}{4}$$

And lastly taking the values of $i = 1,2$ For the two firms it can be shown that:

$$p_{iH} = \frac{1}{1 - \frac{1}{4}b^2}\left[\frac{1}{2}a\left(1 + \frac{1}{2}b_H\right) + \frac{1-h}{8}ab(b_H - b_L)\right]$$
$$p_{iH} = \frac{1}{1 - \frac{1}{4}b^2}\left[\frac{1}{2}a\left(1 + \frac{1}{2}b_L\right) + \frac{h}{8}ab(b_H - b_L)\right]$$

Question 1.6

There are n players in the game.

Define the type spaces by the range of the distribution as $T_i = [0,1]$ such that there is a valuation at each $t_i \in T_i$

Define the action spaces over the same range of distribution as $A_i = [0,1]$ such that there is a value for every $a_i \in A_i$

Therefore, the strategy spaces are defined as: $S_i = \{s_i: T_i \to A_i\}$

The payoffs can be defined for each player i as:

$$u_i(a_i, a_{-i}, t_i) = \begin{cases} t_i - a_i, & if\ a_i > a_j, \forall j \neq i \\ \dfrac{t_i - a_i}{k}, & if\ a_i\ is\ one\ of\ the\ k\ largest\ bids \\ 0, & otherwise \end{cases}$$

At this point we want to show that for a set of strategies $(s_1^*, s_2^*, \ldots, s_n^*)$ is a Bayesian Nash equilibrium where each strategy is $s_i^*(t_i) = \frac{n-1}{n} t_i$

In order to prove this one can simply show that for each player i and their associated type t_i and strategy $s_i^*(t_i)$ solves the maximization problem:

$$\max_{a_i \in A_i} \mathbb{E}_{\mathrm{t-i}}(s_{-i}^*(t_{-i}), a_i; t_i)$$

Where this can be expressed as:

$$\mathbb{E}_{\mathrm{t-i}}(s_{-i}^*(t_{-i}), a_i; t_i) = \sum_{t_{-i} \in T_{-i}} \mathrm{P}_i(t_{-i}|t_i) \times u_i(s_{-i}^*(t_{-i}), a_i; t_i)$$
$$= (t_i - a_i) \times Prob\big(a_i > s_j^*(t_j), \forall j \neq i\big)$$
$$+ \sum_{k=2}^{n} \frac{t_i - a_i}{k} \times Prob(a_i\ is\ one\ of\ the\ k\ largest\ bids)$$

Working though this computation yields that:

$$Prob(a_i\ is\ one\ of\ the\ k\ largest\ bids)$$
$$\leq Prob(Player\ i\ shares\ the\ winner\ of\ the\ auction\ with\ another\ player\ j)$$
$$= Prob\big(s_j^*(t_j) = a_i\big) = Prob\left(t_j = a_i \frac{n}{n-1}\right) = 0$$

However, we know that $Prob\big(t_j = \ell\big) = 0$ for any value of $\ell \epsilon [0,1]$ as a result of the fact that t_j is uniformly distributed on $[0,1]$.

It can also be shown that:

$$Prob\big(a_i > s_j^*(t_j), \forall j \neq i\big)$$
$$= Prob\left(a_i > \frac{n-1}{n} t_j, \forall j \neq i\right)$$

This part is our definition of $s_j^*(t_j)$ and this equation can be expressed as:

$$= \prod_{j \neq i} Prob\left(a_i > \frac{n-1}{n} t_j\right)$$

Based off the independence of the expressions. This can also be shown as:

$$= \prod_{j \neq i} Prob\left(t_j < \frac{n}{n-1} a_i\right)$$

From this it can be shown that when $a_i \geq \frac{n-1}{n}, \prod_{j \neq i} Prob\left(t_j < \frac{n}{n-1} a_i\right) = 1$ and so Player $i's$ expected payoff can be found as $t_i - a_i$, which yields an overall maximizer of $\frac{n-1}{n}$

Therefore, for a given value of $a_i \leq \frac{n-1}{n}$

$$\prod_{j \neq i} Prob\left(t_j < \frac{n}{n-1} a_i\right)$$

$$= \prod_{j \neq i} \left(\frac{n}{n-1} a_i\right) = \left(\frac{n}{n-1} a_i\right)^{n-1}$$

Which follows a uniform distribution.

As a result the expected payoff for player i is:

$$\left(\frac{n}{n-1}\right)^{n-1} a_i^{n-1}(t_i - a_i)$$

With the unique maximizer of $\frac{n-1}{n} t_i = s_i^*(t_i)$

As a result the global maximizer is $\frac{n-1}{n} t_i = s_i^*(t_i)$ and every Player i's strategy $s_i^*(t_i) = \frac{n-1}{n} t_i$ are defined by a symmetric Bayesian Nash Equilibrium.

Question 1.7

Each bidder i would choose her bid $b = B(v_i)$ which will maximize her expected payoff such that:

$$\pi_i = (v_i - b_i)\Pr\big(b(v_j) < b_i\big) + \frac{1}{2}(v_i - b_i)\Pr\big(b(v_j) = b_i\big)$$
$$= (v_i - b_i)\Pr\big(b(v_j) < b_i\big)$$
$$= (v_i - b_i)F\big(B^{-1}(b_i)\big),$$

Where F represents the cumulative distribution function of the valuations. She would choose b_i such that $\frac{\partial \pi_i}{\partial b_i} = 0$. By differentiating this expression π_i WRT v_i one can obtain:

$$\frac{d\pi_i}{dv_i} = \frac{\partial \pi_i}{\partial v_i} + \left(\frac{\partial \pi_i}{\partial b_i}\right)\frac{db_i}{dv_i} = \frac{\partial \pi_i}{\partial v_i}$$

From this an optimally chosen bid value b_i must therefore satisfy:

$$\frac{d\pi_i}{dv_i} = \frac{\partial \pi_i}{\partial v_i} = F\big(B^{-1}(b_i)\big)$$

Next, we used the symmetry assumption. This states that if two bidders with the same valuation should submit the same bid. The equilibrium condition implies that bidder i's optimal bid must be the bid implied by the decision rule B. This means that at equilibrium, $b_i = B(v_i)$. As a result, when substituting this equilibrium condition into the equation above we get

$$\frac{d\pi_i}{dv_i} = F(v_i)$$

This differential can be solved for π_i through integration. The bounty condition $B(0) = 0$ is used.

$$\pi_i(v_i) = \int_0^{v_i} F(x)dx$$

This is then combined with the definition of the expected pay off equation which is found from the bidder's strategy defined by:

$$(v_i - b_i)F(v_i) = \int_0^{v_i} F(x)dx$$

$$b_i = B(v_i) = v_i - \frac{\int_0^{v_i} F(x)dx}{F(v_i)} \; for \; I = 1,2$$