Introduction to Slice Sampler

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October 26, 2017

Outline

Why Slice Sampler?

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- Slice sampler is a type of Markov Chain Monte Carlo (MCMC) methods.
- In computational statistics especially in Bayesian simulation-based inference methods, Slice sampler can sample target distributions directly.
- By introducing auxiliary random variables, slice sampler is quite simple to derive.

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- Suppose that we want to draw samples from some distribution $p(\theta)$, where $p(\theta) = f(\theta)/k$, where k may not be known or is very difficult to obtain.

- In Bayesian statistics, for example, $p(\theta) = \Pi_{i=1}^{n} f_i(\theta)/k, f_i(\theta) > 0$.

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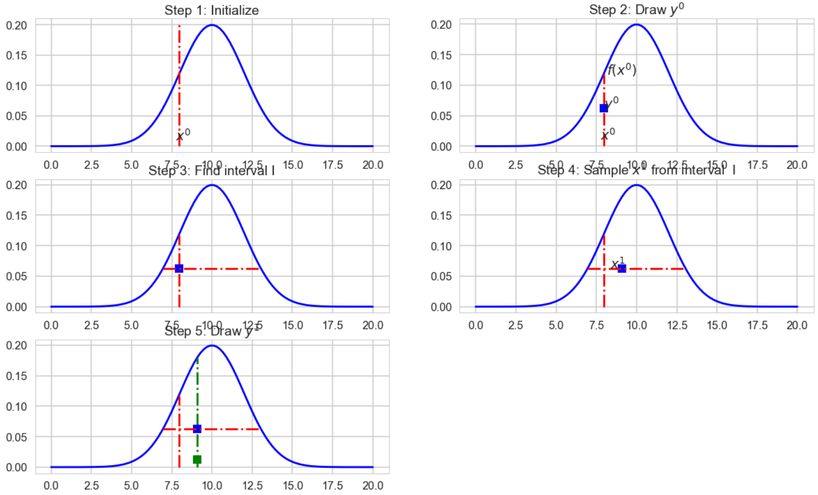
- Sample uniformly from underneath the curve $f(x)$ without the need to reject any points.
- Slice sampler – a simplest form

0. Choose an initial value x for which $f(x) > 0$.
1. Sample a y value uniformly from $(0, f(x))$.
2. Draw a horizontal line across the curve at the y position.
3. Sample x uniformly from the line segment within the curve.
4. Go to **1**.

- An auxiliary variable y is introduced.

How slice sampler works? (Cont'd)



Slice samplers – a more general form

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0. Choose an initial value x_1 and set $t = 1$.
1. for $i = 1$ to n: Draw $u_i \sim U(0,1)$ and let $y_i = u_i \times f_i(x_i)$
2. Draw x_{t+1} uniformly from the set $\{x : f_i(x_i) > y_i, i = 1, ..., n\}$
3. Stop if a stopping criterion is met; Otherwise, set $t = t + 1$ and repeat from **1**.

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- The generated sample $\{x_t, t = 0, 1, ...\}$ is approximately distributed according to $p(x)$.

Example 1 – sample a Gamma distribution

- Suppose that $X \sim Gamma(\alpha, \beta)$ with the density function $p(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} \propto x^{\alpha-1} e^{-x/\beta}$.

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0. Choose an initial value x_1 and set $t = 1$.
1. (1) Draw $u_1 \sim U(0, 1)$ and let $u_1 = u_1 \times x_t^{\alpha-1}$ and set $u_1 < x^{\alpha-1}$, then we have $x > u_1^{1/(\alpha-1)}$.
(2) Draw $u_2 \sim U(0, 1)$ and let $u_2 = u_2 \times e^{-x_t/\beta}$ and set $u_2 < e^{-x/\beta}$, then we have $x < -\beta \log(u_2)$.
2. Draw x_{t+1} uniformly from the interval $(u_1^{1/(\alpha-1)}, -\beta \log(u_2))$
3. Stop if a stopping criterion is met; Otherwise, set $t = t + 1$ and repeat from **1**.

- The generated sample $\{x_t, t = 0, 1, ...\}$ is approximately distributed according to the distribution $Gamma(\alpha, \beta)$.

Example 1 – sample a Gamm distribution (Cont'd)

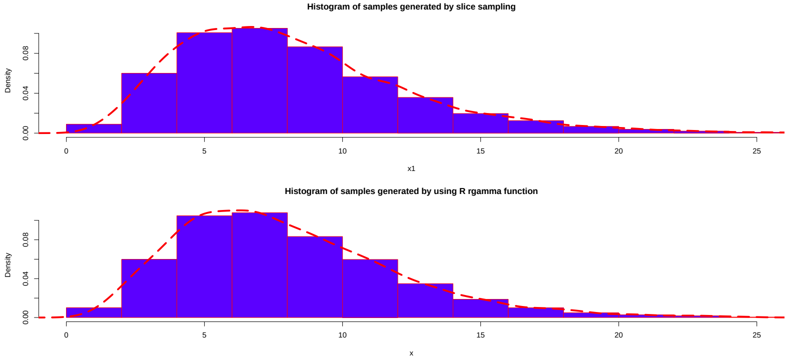

Figure: The slice sampler was iterated 10,000 times to sample $Gamma(4, 2)$.

Example 2 – sample a Student-t distribution

- Let Y be a random variable following a Student-t distribution $t(v)$.

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- Let Y be a random variable following a Student-t distribution $t(v)$.
- The density function is

$$
\begin{aligned}
p(y|v) &= \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\,\Gamma\left(\frac{v}{2}\right)}\left(1+\frac{y^2}{v}\right)^{-(v+1)/2} \\
&\propto \left(1+\frac{y^2}{v}\right)^{-(v+1)/2}
\end{aligned}
$$

Example 2 – A slice sampler to sample the Student-t distribution

0. Given y_t, the sampled point after the t-th iteration of the slice sampler.

1. Draw $u_1 \sim \mathcal{U}(0,1)$. Let $u_2 = u_1 \times \left(1 + \frac{y_t^2}{v}\right)^{-\frac{v+1}{2}}$ and let

$$u_2 \le \left(1 + \frac{y^2}{v}\right)^{-\frac{v+1}{2}}.$$

Then we have

$$-\sqrt{(u_2^{-2/(v+1)} - 1)v} \le y \le \sqrt{(u_2^{-2/(v+1)} - 1)v}.$$

2. Draw $y_{t+1} \sim \mathcal{U}\left(-\sqrt{(u_2^{-2/(v+1)} - 1)v}, \sqrt{(u_2^{-2/(v+1)} - 1)v}\right)$.

3. Repeat **1** and **2**.

Example 2 – sample a Student-t distribution

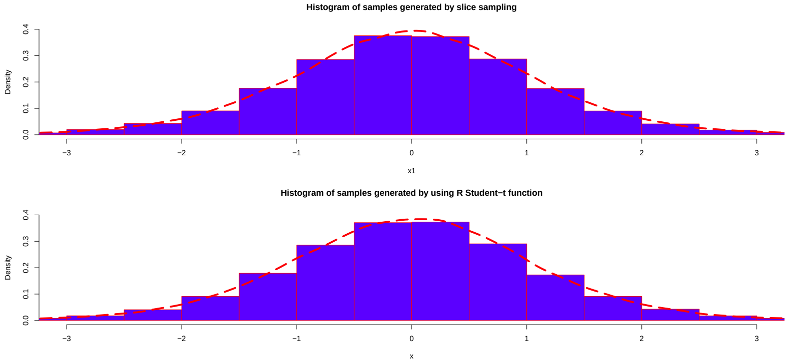

Figure: The slice sampler was iterated 10,000 times to sample a Student-t distribution with 10 degrees of freedom.

Example 3 – sample a more general density function.

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- The density function is

$$p(x) = c\frac{xe^{-x}}{1+x}, x > 0$$

  where c is an unknown normalizing constant.
- When using the slice sampler, we define $f_1(x) = x/(1+x)$ and $f_2(x) = e^{-x}$.

Example 3 – sample a more general density function (Cont'd).

0. Given x_t, the sampled point after the t-th iteration of the slice sampler.
1. Draw $u_1 \sim \mathcal{U}(0,1)$. Let $u_2 = u_1 \times \frac{x_t}{1+x_t}$ and let $u_2 < \frac{x}{1+x}$ then we have

$$x > \frac{u_2}{1-u_2}$$

2. Draw $u_3 \sim \mathcal{U}(0,1)$. Let $u_4 = u_3 \times e^{-x_t}$, then we have

$$x < -\log(u_4)$$

3. Draw $x_{t+1} \sim \mathcal{U}\left(\frac{u_2}{1-u_2}, -\log(u_4)\right)$.
4. Repeat **1** to **3**.

Example 2 – sample a Student-t distribution

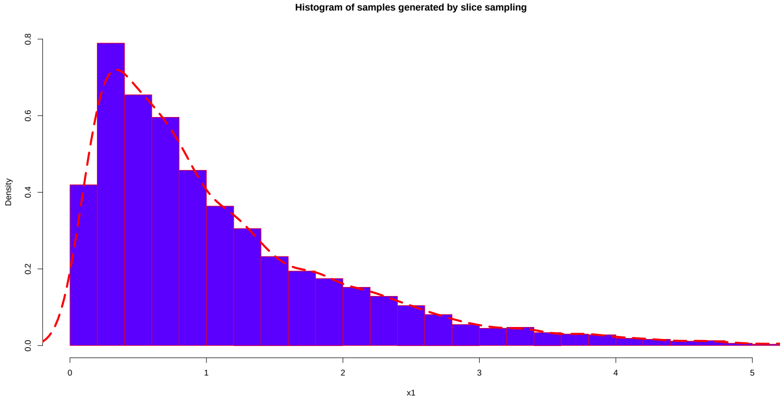

Figure: The slice sampler was iterated 10,000 times to sample a general distribution with pdf $p(x) = c\frac{xe^{-x}}{1+x}, x > 0$.

Summary

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- Three examples were given to illustrate how slice sampler works.

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- Introduced slice sampler.
- Three examples were given to illustrate how slice sampler works.
- Slice sampler is very powerful if the target density function is complicated or known up to a unknown constant.

Thank you!

References

[1] Robert, C. and G., Casella. 2013. Monte Carlo Statistical Methods: Edition 2. London: Chapman and Hall.

[2] Neal, R. N. 2003. Slice sampling. *The Annals of Statistics* 31: 705-767.

[3] Chib, S. and Greenberg E. 1995. Understanding the Metropolis-Hastings Algorithm. *The American Statistician* Vol. 49, No. 4: 327-335.