Mixed method for linear elastic thin-plate flexure: variational form

PROBLEM DESCRIPTION

Consider a thin plate loaded with some force $f = f(x, y)$. Let $\mathbf{M}$ denote the moment tensor and $\eta = \eta(x, y)$ represent the vertical deflection in response to f. With λ_1 and λ_2 some appropriately-defined scalar-valued functions in x and y, the governing equation and constitutive relation on the plate are:

$$-\nabla \cdot (\nabla \cdot \mathbf{M}) = f \tag{1a}$$

$$\mathbf{M} = \lambda_1 \nabla\nabla\eta + \lambda_2 tr\left(\nabla\nabla\eta\right)\mathbf{I} \tag{1b}$$

($\mathbf{I}$ the two-by-two identity). The goal is to separately put Equations 1a and 1b into variational form.

USEFUL IDENTITIES

Some useful identities which will help in attaining the variational forms for Equations 1a and 1b. In the identities below, α and β will denote scalar quantities, $\vec{a}$ and $\vec{b}$ are vectors, and $\mathbf{A}$ and $\mathbf{B}$ are tensors. Ω represents the domain of the problem, Γ is the boundary, and $\hat{n}$ is the unit, outward-pointing vector normal to Γ.

$$\nabla \cdot (\alpha\mathbf{A}) = \alpha(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot \nabla\alpha \tag{2}$$

$$\nabla \cdot (\alpha\vec{a}) = \alpha(\nabla \cdot \vec{a}) + \vec{a} \cdot \nabla\alpha \tag{3}$$

$$\int_\Omega \nabla \cdot \vec{a} = \int_\Gamma \vec{a} \cdot \hat{n} \tag{4}$$

...hopefully some more useful identities will become apparent...

PUTTING EQUATION 1a INTO VARIATIONAL FORM

Multiply both sides of the governing equation by arbitrary scalar function $w = w(x, y)$. Then

$$-w\nabla \cdot (\nabla \cdot \mathbf{M}) = wf$$

Note that the left-hand side is the product of the scalar variable $-w$ with the divergence of the vector $(\nabla \cdot \mathbf{M})$. Therefore, by the product rule of Equation 3, the equation above can be rewritten so that

$$(\nabla \cdot \mathbf{M}) \cdot \nabla w - \nabla \cdot [w(\nabla \cdot \mathbf{M})] = wf.$$

Integrating over the domain Ω, we have that

$$\int_\Omega [(\nabla \cdot \mathbf{M}) \cdot \nabla w] - \int_\Omega \nabla \cdot [w(\nabla \cdot \mathbf{M})] = \int_\Omega wf.$$

Finally, by the Divergence Theorem of Equation 4, it follows that

$$\int_\Omega [(\nabla \cdot \mathbf{M}) \cdot \nabla w] - \int_\Gamma [w(\nabla \cdot \mathbf{M}) \cdot \hat{n}] = \int_\Omega wf \tag{5}$$

PUTTING EQUATION 1b INTO VARIATIONAL FORM

Let $\mathbf{N}$ be an arbitrary tensor function. Then the constitutive equation of Equation 1b requires that

$$\int_\Omega \mathbf{N} : \mathbf{M} = \int_\Omega \lambda_1 \mathbf{N} : \nabla\nabla\eta + \int_\Omega \lambda_2 \mathbf{N} : tr(\nabla\nabla\eta)\mathbf{I}$$

I'll split the right-hand side up into two pieces:

The first term

First, we'll try writing the expression

$$\int_\Omega \lambda_1 \mathbf{N} : \nabla\nabla\eta \tag{6}$$

in terms of only first derivatives...

The second term

Finally, put the expression

$$\int_{\Omega} \lambda_2 \mathbf{N} : tr(\nabla\nabla\eta)\mathbf{I} \tag{7}$$

in terms of only first derivatives...

THE 1-D VERSION

Okay that was too hard! Time to simplify: now we will solve the governing equation and constitutive equations of the form

$$-M'' = f \tag{8a}$$

$$M = (\lambda_1 + \lambda_2)\eta'' \tag{8b}$$

Now we only need the basic simplifying identities - namely, the product rule and the Fundamental Theorem of Calculus. Multiplying both sides of Equation 8a by arbitrary $w(x, y)$ and integrating over x, we see that

$$-\int_x wM'' = \int_x f.$$

By the product rule, $(wM')' = w'M' + wM''$, and so

$$\int_x w'M' - \int_x (wM')' = \int_x f.$$

By the FTC, this is equivalent to the statement

$$\int_x w'M' - wM'|_{x_0}^{x_1} = \int_x f. \tag{9}$$

Next, do the same for the constitutive relation of Equation 8b, with arbitrary function $v(x, y)$:

$$\int_x vM = \int_x (\lambda_1 + \lambda_2)v\eta''$$

Then applying the product rule yields

$$\int_x vM = \int_x \big[(\lambda_1 + \lambda_2)v\eta'\big]' - \int_x [(\lambda_1 + \lambda_2)v]'\eta'$$

and, by the FTC,

$$\int_x vM = (\lambda_1 + \lambda_2)v\eta'|_{x_0}^{x_1} - \int_x [(\lambda_1 + \lambda_2)v]'\eta'. \tag{10}$$