# Double Mass Spring-Damper Robust Control

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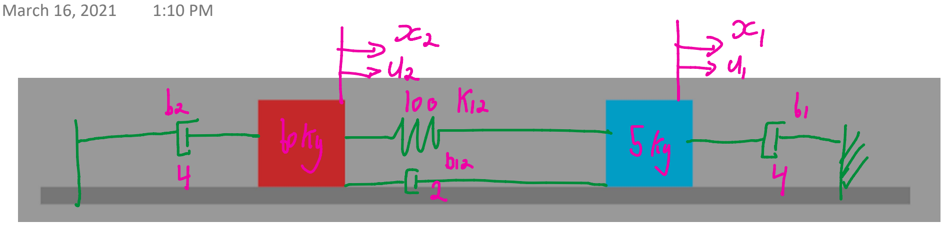


$$60\ddot{x}_2 = u_2 - 4\dot{x}_2 - (\dot{x}_2 - \dot{x}_1)2 - 100(x_2 - x_1)$$

$$\ddot{x}_2 = \frac{1}{60}\left[-4\dot{x}_2 - 2\dot{x}_2 + 2\dot{x}_1 - 100x_2 + 100x_1 + u_2\right]$$

$$= \frac{1}{60}\left[-6\dot{x}_2 + 2\dot{x}_1 - 100x_2 + 100x_1 + u_2\right]$$

$$= \frac{100}{60}x_1 - \frac{100}{60}x_2 + \frac{2}{60}\dot{x}_1 - \frac{6}{60}\dot{x}_2 + \frac{u_2}{60}$$

$$5\ddot{x}_1 = u_1 - 4\dot{x}_1 - 2(\dot{x}_1 - \dot{x}_2) - 100(x_1 - x_2)$$

$$\ddot{x}_1 = \frac{1}{5}\left[-4\dot{x}_1 - 2\dot{x}_1 + 2\dot{x}_2 - 100x_1 + 100x_2 + u_1\right]$$

$$= \frac{1}{5}\left[-6\dot{x}_1 + 2\dot{x}_2 - 100x_1 + 100x_2 + u_1\right]$$

$$= \frac{-100}{5}x_1 + \frac{100}{5}x_2 - \frac{6}{5}\dot{x}_1 + \frac{2}{5}\dot{x}_2 + \frac{u_1}{5}$$

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$$m_2\ddot{x}_2 = u_2 - b_{12}(\dot{x}_2 - \dot{x}_1) - K_{12}(x_2 - x_1) - b_2\dot{x}_2$$

$$= u_2 - b_{12}\dot{x}_2 + b_{12}\dot{x}_1 - K_{12}x_2 + K_{12}x_1 - b_2\dot{x}_2$$

$$= K_{12}x_1 - K_{12}x_2 + b_{12}\dot{x}_1 - (b_{12}+b_2)\dot{x}_2 + u_2$$

$$m_1\ddot{x}_1 = u_1 - b_1\dot{x}_1 - b_{12}(\dot{x}_1 - \dot{x}_2) - K_{12}(x_1 - x_2)$$

$$= u_1 - b_1\dot{x}_1 - b_{12}\dot{x}_1 + b_{12}\dot{x}_2 - K_{12}x_1 + K_{12}x_2$$

$$= u_1 - K_{12}x_1 + K_{12}x_2 - (b_1 + b_{12})\dot{x}_1 + b_{12}\dot{x}_2$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} K_{12} & -K_{12} \\ -K_{12} & K_{12} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} (b_1+b_{12}) & -b_{12} \\ -b_{12} & (b_{12}+b_2) \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

## Define our Model Error Limits

$$\underbrace{\begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{bu}$$

direct forces apply to masses

$$\hat{f} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{K_{12}}{m_1}\cdot x_1 + \frac{K_{12}}{m_1}\cdot x_2 - \frac{(b_1+b_{12})}{m_1}\cdot\dot{x}_1 + \frac{b_{12}}{m_1}\cdot\dot{x}_2 \\ \frac{K_{12}}{m_2}\cdot x_1 - \frac{K_{12}}{m_2}\cdot x_2 + \frac{b_{12}}{m_2}\cdot\dot{x}_1 - \frac{(b_{12}+b_2)}{m_2}\cdot x_2 \end{bmatrix}$$

$$f = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{K_{12}}{m_1}\cdot x_1 + \frac{K_{12}}{m_1}\cdot x_2 - \frac{(b_1+b_{12})}{m_1}\cdot\dot{x}_1 + \frac{b_{12}}{m_1}\cdot\dot{x}_2 \\ \frac{K_{12}}{m_2}\cdot x_1 - \frac{K_{12}}{m_2}\cdot x_2 + \frac{b_{12}}{m_2}\cdot\dot{x}_1 - \frac{(b_{12}+b_2)}{m_2}\cdot x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$F = |\hat{f} - f| = \begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

## Define our Sliding Surface

$$\hat{u} = [m_1]\left[-\hat{f} + \ddot{x}_d - \lambda\dot{\tilde{x}}\right]$$

## Define our Sliding Surface

$$\hat{u} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}\left[-\hat{f} + \ddot{x}_d - \lambda \dot{\tilde{x}}\right]$$

$$S = \dot{\tilde{x}} + \lambda \tilde{x}$$

$$\begin{aligned}
\dot{S} = \ddot{\tilde{x}} + \lambda \dot{\tilde{x}} &= (\ddot{x} - \ddot{x}_d) + \lambda \dot{\tilde{x}} \\
&= (f + bu) - \ddot{x}_d + \lambda \dot{\tilde{x}} \\
&= (f + b\hat{u}) - \ddot{x}_d + \lambda \dot{\tilde{x}} \\
&= \left(f + \begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}(-\hat{f} + \ddot{x}_d - \lambda \dot{\tilde{x}})\right) - \ddot{x}_d + \lambda \dot{\tilde{x}} \\
&= f - \hat{f} + \cancel{\ddot{x}_d} \cancel{- \lambda \dot{\tilde{x}}} \cancel{- \ddot{x}_d} \cancel{+ \lambda \dot{\tilde{x}}} \\
&= f - \hat{f}
\end{aligned}$$

We want our System to obey this Sliding Condition

$$\frac{1}{2}\frac{d}{dt}S^2 < -\eta |S|$$

$$S \cdot \dot{S} < -\eta |S|$$

$\star$ Consider if $u = u_0 - \underbrace{K\,\mathrm{Sgn}(S)}_{\text{Robust Term}}$

$$\dot{S} = f - \hat{f} - K\,\mathrm{Sgn}(S)$$

$$S \cdot (f - \hat{f} - K \cdot \mathrm{Sgn}(S)) < -\eta |S|$$

$$S \cdot (f - \hat{f}) - S \cdot K \cdot \mathrm{Sgn}(S) < -\eta |S| \quad // \; S \cdot K \cdot \mathrm{Sgn}(S) = K \cdot |S|$$

$$S \cdot (f - \hat{f}) - K \cdot |S| < -\eta |S| \quad // \; S \cdot (f - \hat{f}) = S \cdot F$$

$$S \cdot F - K|S| < -\eta |S|$$

$$S \cdot F + \eta |S| < K|S| \quad // \Longrightarrow \; K = F + \eta$$

// where

$$F = \begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} \frac{d_1}{m_1} \\ \frac{d_2}{m_2} \end{bmatrix}$$

$$u = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\left[-\hat{f} + \ddot{x}_d - \lambda \dot{\tilde{x}}\right] - \left[F + \eta\right] \cdot \mathrm{Sgn}(S)$$

$\in \mathbb{R}^{2\times1}$