



# higher category theory

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## Context

## Higher category theory

## 1. Idea

*Higher category theory* is the generalization of [category theory](#) to a context where there are not only [morphisms](#) between [objects](#), but generally [k-morphisms](#) between  $(k - 1)$ -morphisms, for all  $k \in \mathbb{N}$ .

Higher category theory studies the generalization of  [\$\infty\$ -groupoids](#) – and hence, via the [homotopy hypothesis](#), of [topological spaces](#) – to that of [directed spaces](#) and their *combinatorial or algebraic models*. It is to the theory of  [\$\infty\$ -groupoids](#) as [category theory](#) is to the theory of [groupoids](#) (and hence of [groups](#)).

These [combinatorial](#) or [algebraic](#) models are known as [n-categories](#) or, when  $n \rightarrow \infty$ , as  [\$\infty\$ -categories](#) or  [\$\omega\$ -categories](#), or, in more detail, as [\(n,r\)-categories](#):

- the natural number  $n$  denotes the maximal dimension of *non-trivial* cells in the model,
- while the natural number  $r$  denotes the maximal dimension of the *directed* cells.

So an ordinary [topological space](#) or  [\$\infty\$ -groupoid](#) is an [\( \$\infty, 0\$ \)-category](#): it has cells of arbitrary dimension and all of them are reversible.

In contrast to that, a [combinatorial](#) or [algebraic](#) model for a [directed space](#) in which the 1-dimensional paths may not all be reversible is an [\( \$\infty, 1\$ \)-category](#): it still has cells of arbitrary dimension, but only those of dimension greater than 1 are guaranteed to be reversible.

Often it is convenient in practice to consider the case where the possible dimension  $n$  of non-trivial cells is finite. This is familiar from how a [topological space](#) that happens to have vanishing [homotopy groups](#) in dimension above some  $n$  – a [homotopy  \$n\$ -type](#) – is modeled by an  [\$n\$ -groupoid](#). A fully directed version of this is an  [\$n\$ -category](#), which is short for [\( \$n, n\$ \)-category](#): non-trivial cells up to and including dimension  $n$ , and all of them allowed to be non-reversible. Actually, it is possible to go on to an  $(n, n + 1)$ -category, or  $(n + 1)$ -[poset](#); you can either consider that the  $n$ -cells are ordered, or else consider that there are irreversible  $(n + 1)$ -cells which are indistinguishable. (Reversible indistinguishable  $(n + 1)$ -cells are all identities and so might as well not exist.)

For low  $n$  very explicit [algebraic models](#) for  $n$ -categories are available but in their full generality become quickly rather untractable as  $n$  increases: the series starts with [bicategory](#), [tricategory](#) and [tetracategory](#). While bicategories have found plenty of applications, already the axioms of tricategories are rather involved and their theory mainly serves to produce the statement that there is a good [semi-strictifications](#) of tricategories called [Gray-categories](#).

Indeed, there are many *strictified* models for higher categories: combinatorial or algebraic models that sacrifice full generality for a better concrete control. Notably there is a useful combinatorial/algebraic model for [strict  \$\infty\$ -categories](#) which, while falling short, already goes a long way towards describing general higher categorical structures. In fact, by [Simpson's conjecture](#) every  [\$\infty\$ -category](#) is equivalent to one that looks like a [strict  \$\infty\$ -category](#) except for possibly having weak unit laws.

The challenge of describing fully general  [\$\infty\$ -categories](#) is to achieve a combinatorial or algebraic control of all the higher composition rules of higher cells. One can distinguish roughly two orthogonal approaches to dealing with the problem:

in the [algebraic definition of higher category](#) an algebraic machinery is set up that allows to concretely handle the explicit *choices* of composites of cells. Such machinery usually involves [operadic](#) tools in one way or other. The most sophisticated definitions of this kind are the closely related [Batanin  \$\infty\$ -category](#) and [Trimble  \$\infty\$ -category](#).

On the other hand, in the [geometric definition of higher category](#) a combinatorial machinery is set up that allows to guarantee *existence* of composites of cells. In the [simplicial models for weak  \$\infty\$ -categories](#) higher categories are characterized as [simplicial sets](#) with the extra [property](#) that certain composites exist. The issue here is to characterize these existence laws correctly.

The basic example for such “existence laws” is the *Kan-filler condition* that characterizes simplicial sets that are [Kan complexes](#), the models for  $(\infty, 0)$ -categories. More general higher categories are obtained by relaxing the Kan condition in just the right way. For instance by simply restricting the Kan-condition to just a certain sub-set of all cells yields the definition of simplicial sets that are called [quasi-categories](#). These model  $(\infty, 1)$ -categories.

The right further relaxation of the (weak) Kan filler condition is more involved. An approach to capture this has been given by [Dominic Verity](#)’s definition of simplicial sets that are called [complicial sets](#) and [weak complicial sets](#).

One expects that every algebraic definition of higher categories admits a construction called a [nerve](#) that maps it to a [simplicial set](#) that would constitute the corresponding geometric model.

Another approach to handle the geometric definition of higher categories is a recursive one that uses  $n$ -fold simplicial sets. This is based on the notion of [complete Segal space](#), which is essentially a variation of the concept of [quasi-category](#). Its advantage is that its definition may be applied recursively to yield the notion of [n-fold complete Segal spaces](#). These model  $(\infty, n)$ -categories for finite  $n$ .

Finally, a large supply of further models exists for  $(\infty, 1)$ -categories in terms of [enriched category theory](#). [Simplicially enriched model categories](#) are a highly-developed toolkit for handling [presentable  \$\(\infty, 1\)\$ -categories](#). [Pretriangulated dg-enriched categories](#) and [A- \$\infty\$  categories](#) are a comparably highly developed toolkit for handling [stable  \$\(\infty, 1\)\$ -categories](#).

## 2. Basic concepts

The basic concept on which higher category theory is built is the notion of [k-morphism](#) for all  $k \in \mathbb{N}$ , equipped with a notion of composition, such that [coherence laws](#) are satisfied.

This is what it’s all about.

## 3. Basic constructions

### *Higher presheaves*

- [higher topos theory](#)

### *Higher universal constructions*

- [2-limit](#)
- [\$\(\infty, 1\)\$ -adjunction](#)

- [\$\(\infty, 1\)\$ -Kan extension](#)
  - [\$\(\infty, 1\)\$ -limit](#)
- [\$\(\infty, 1\)\$ -Grothendieck construction](#)

## 4. Basic theorems

- [homotopy hypothesis](#)-theorem
- [delooping hypothesis](#)-theorem
- [periodic table](#)
- [stabilization hypothesis](#)-theorem
- [exactness hypothesis](#)
- [holographic principle](#)

## 5. Applications

See

- [applications of \(higher\) category theory](#).

### *Extended cobordisms*

One major application of higher category theory and one of the driving forces in developing it has been [extended functorial quantum field theory](#). This has recently led to what may become one of the central theorems of higher category theory, the proof of the [cobordism hypothesis](#). This roughly characterizes the  [\$\(\infty, n\)\$ -category of cobordisms](#)  $\text{Bord}_n$  as the free  [\$\(\infty, n\)\$ -category](#) with duals on a single generator.

## 6. Models

There are many different *models* for bringing the abstract notion of higher category onto paper.

- [\$\(n \times k\)\$ -category](#)
- [n-fold category](#)
- [\$\(n, r\)\$ -category](#)
  - [Theta-space](#)
  - [\$\infty\$ -category/ \$\infty\$ -category](#)
  - [\$\(\infty, n\)\$ -category](#)

- n-fold complete Segal space
- $(\infty, 2)$ -category
- $(\infty, 1)$ -category
  - quasi-category
    - algebraic quasi-category
  - simplicially enriched category
  - complete Segal space
  - model category
  - internal category in homotopy type theory
- $(\infty, 0)$ -category/ $\infty$ -groupoid
  - Kan complex
    - algebraic Kan complex
    - simplicial T-complex
- n-category =  $(n, n)$ -category
  - 2-category,  $(2, 1)$ -category
  - 1-category
  - 0-category
  - $(-1)$ -category
  - $(-2)$ -category
- n-poset =  $(n-1, n)$ -category
  - 2-poset
- n-groupoid =  $(n, 0)$ -category
  - 2-groupoid, 3-groupoid
- categorification/decategorification
- geometric definition of higher category
  - Kan complex
  - quasi-category
  - simplicial model for weak  $\infty$ -categories
    - complicial set
    - weak complicial set
- algebraic definition of higher category
  - bicategory
  - bigroupoid

- [tricategory](#).
- [tetracategory](#).
- [strict  \$\infty\$ -category](#).
- [Batanin  \$\infty\$ -category](#).
- [Trimble  \$\infty\$ -category](#).
- [Grothendieck-Maltsiniotis  \$\infty\$ -categories](#)
- [stable homotopy theory](#).
  - [symmetric monoidal category](#).
  - [symmetric monoidal  \$\(\infty, 1\)\$ -category](#).
  - [stable  \$\(\infty, 1\)\$ -category](#).
    - [dg-category](#).
    - [A- \$\infty\$  category](#).
    - [triangulated category](#).

## *1-categorical models*

- [homotopical category](#).
- [model category theory](#).
- [enriched category theory](#).

## 7. Related concepts

- [opetopic type theory](#).

### [higher category theory](#).

- [\$\(0, 1\)\$ -category theory](#).
- [1-category theory](#),  [\$\(\infty, 1\)\$ -category theory](#).
- [2-category theory](#),  [\$\(\infty, 2\)\$ -category theory](#).
- [\$\(\infty, n\)\$ -category theory](#).

## 8. References

For a very gentle introduction to higher category theory, try [The Tale of  \$n\$ -Categories](#), which begins in “week73” of This Week’s Finds and goes on from there ...; keep clicking the links.

For a slightly more formal but still pathetically easy introduction, try:

- [John Baez](#), [An Introduction to n-Categories](#), in

*7th Conference on Category Theory and Computer Science*, eds.

E. Moggi and G. Rosolini, Springer Lecture Notes in Computer Science vol. 1290, Springer, Berlin, 1997.

For a free introductory text on  $n$ -categories that's *full of pictures*, try this:

- [Eugenia Cheng](#) and [Aaron Lauda](#), [Higher-Dimensional Categories: An Illustrated Guidebook](#).

[Tom Leinster](#) has written about “comparative  $\infty$ -categoriology” (to [borrow](#) a term):

- Tom Leinster, *A Survey of Definitions of  $n$ -Category* ([arXiv](#))
- Tom Leinster, *Higher Operads, Higher Categories* ([arXiv](#))

A grand picture of the theory of higher categories is drawn in

- [Carlos Simpson](#), *Homotopy Theory of Higher Categories* ([pdf](#))

Another collection of discussions of definitions of higher categories is given at

- [John Baez](#), [Peter May](#), [Approaching Higher Category Theory](#).

A brief useful survey of approaches to the definition of higher categories is provided by the set of slides

- [Andre Joyal](#), [Tim Porter](#), [Peter May](#), *Weak categories* ([pdf](#))

The theory of [quasi-categories](#) as  [\$\(\infty, 1\)\$ -categories](#) has reached a point where it is well developed and being applied to a wealth of problems with

- [Jacob Lurie](#), *Higher Topos Theory* ([arXiv](#))

There's a lot more to add here, even if we restrict ourselves to very general texts. (More specialized stuff should go under more specialized subcategories!)

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