FOUNDATIONS OF MATHEMATICS

With Category Theory, Mathematics Escapes From Equality

By KEVIN HARTNETT

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Two monumental works have led many mathematicians to avoid the equal sign. Their goal: Rebuild the foundations of the discipline upon the looser relationship of "equivalence." The process has not always gone smoothly.

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Ana Porta for Quanta Magazine

The equal sign is the bedrock of mathematics. It seems to make an entirely fundamental and uncontroversial statement: These things are exactly the same.

But there is a growing community of mathematicians who regard the equal sign as math's original error. They see it as a veneer that hides important complexities in the way quantities are related — complexities that could unlock solutions to an enormous number of problems. They want to reformulate mathematics in the looser language of equivalence.

"We came up with this notion of equality," said Jonathan Campbell of Duke University. "It should have been equivalence all along."

The most prominent figure in this community is Jacob Lurie. In July, Lurie, 41, left his tenured post at Harvard University for a faculty position at the Institute for Advanced Study in Princeton, New Jersey, home to many of the most revered mathematicians in the world.

Lurie's ideas are sweeping on a scale rarely seen in any field. Through his books, which span thousands of dense, technical pages, he has constructed a strikingly different way to understand some of the most essential concepts in math by moving beyond the equal sign. "I just think he felt this was the correct way to think about mathematics," said Michael Hopkins, a mathematician at Harvard and Lurie's graduate school adviser.

Lurie published his first book, *Higher Topos Theory*, in 2009. The 944-page volume serves as a manual for how to interpret established areas of mathematics in the new language of "infinity categories." In the years since, Lurie's ideas have moved into an increasingly wide range of mathematical disciplines. Many mathematicians view them as indispensable to the future of the field. "No one goes back once they've learned infinity categories," said John Francis of Northwestern University.

Jacob Lurie, a mathematician at the Institute for Advanced Study, was awarded the $3 million Breakthrough Prize in Mathematics in 2014.

John D. & Catherine T. MacArthur Foundation

Yet the spread of infinity categories has also revealed the growing pains that a venerable field like mathematics undergoes whenever it tries to absorb a big new idea, especially an idea that challenges the meaning of its most important concept. "There's an appropriate level of conservativity in the mathematics community," said Clark Barwick of the University of Edinburgh. "I just don't think you can expect any population of mathematicians to accept any tool from anywhere very quickly without giving them convincing reasons to think about it."

Although many mathematicians have embraced infinity categories, relatively few have read Lurie's long, highly abstract texts in their entirety. As a result, some of the work based on his ideas is less rigorous than is typical in mathematics.

"I've had people say, 'It's in Lurie somewhere,'" said Inna Zakharevich, a mathematician at Cornell University. "And I say, 'Really? You're referencing 8,000 pages of text.' That's not a reference, it's an appeal to authority."

Mathematicians are still grappling with both the magnitude of Lurie's ideas and the unique way in which they were introduced. They're distilling and repackaging his presentation of infinity categories to make them accessible to more mathematicians. They are performing, in a sense, the essential work of governance that must follow any revolution, translating a transformative text into day-to-day law. In doing so, they are building a future for mathematics founded not on equality, but on equivalence.

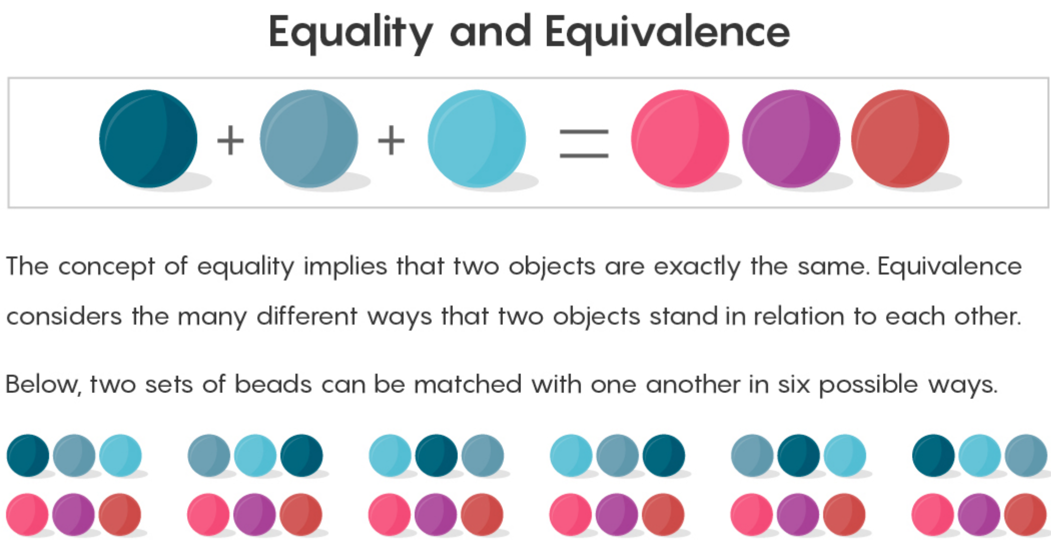
Infinite Towers of Equivalence

Mathematical equality might seem to be the least controversial possible idea. Two beads plus one bead equals three beads. What more is there to say about that? But the simplest ideas can be the most treacherous.

Since the late 19th century, the foundation of mathematics has been built from collections of objects, which are called sets. Set theory specifies rules, or axioms, for constructing and manipulating these sets. One of these axioms, for example, says that you can add a set with two elements to a set with one element to produce a new set with three elements: 2 + 1 = 3.

On a formal level, the way to show that two quantities are equal is to pair them off: Match one bead on the right side of the equal sign with one bead on the left side. Observe that after all the pairing is done, there are no beads left over.

Set theory recognizes that two sets with three objects each pair exactly, but it doesn't easily perceive all the different ways to do the pairing. You could pair the first bead on the right with the first on the left, or the first on the right with the second on the left, and so on (there are six possible pairings in all). To say that two plus one equals three and leave it at that is to overlook all the different ways in which they're equal. "The problem is, there are many ways to pair up," Campbell said. "We've forgotten them when we say equals."

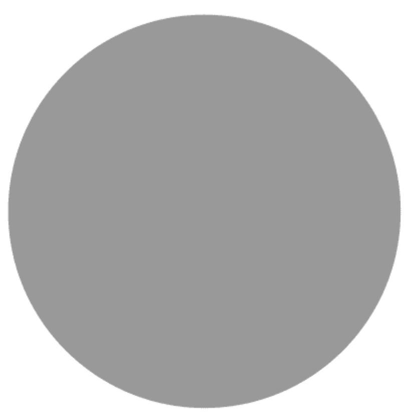


Lucy Reading-Ikkanda/Quanta Magazine

This is where equivalence creeps in. While equality is a strict relationship — either two things are equal or they're not — equivalence comes in different forms.

When you can exactly match each element of one set with an element in the other, that's a strong form of equivalence. But in an area of mathematics called homotopy theory, for example, two shapes (or geometric spaces) are equivalent if you can stretch or compress one into the other without cutting or tearing it.

From the perspective of homotopy theory, a flat disk and a single point in space are equivalent — you can compress the disk down to the point. Yet it's impossible to pair points in the disk with points in the point. After all, there's an infinite number of points in the disk, while the point is just one point.



The Point and the Disk

A flat disk and a single point in space are homotopy equivalent — you can transform the disk into the point without tearing it.

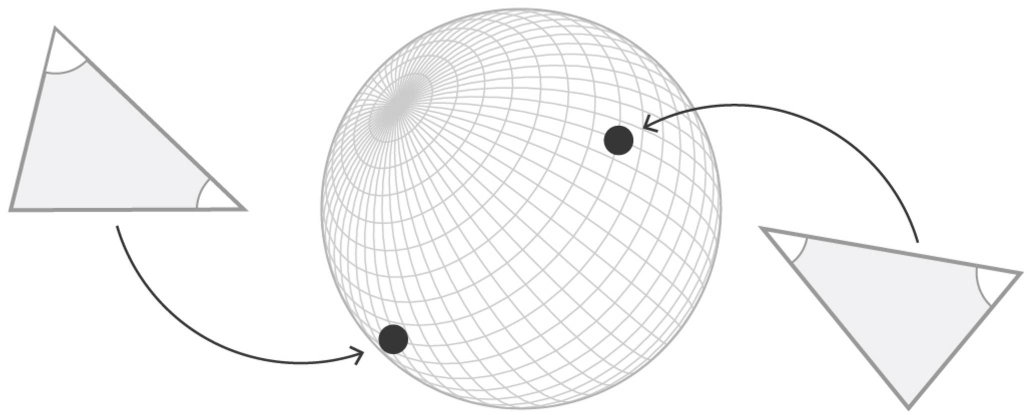
Since the mid-20th century mathematicians have tried to develop an alternative to set theory in which it would be more natural to do mathematics in terms of equivalence. In 1945 the mathematicians

Samuel Eilenberg and Saunders Mac Lane introduced a new fundamental object that had equivalence baked right into it. They called it a category.

Categories can be filled with anything you want. You could have a category of mammals, which would collect all the world's hairy, warm-blooded, lactating creatures. Or you could make categories of mathematical objects: sets, geometric spaces or number systems.

A category is a set with extra metadata: a description of all the ways that two objects are related to one another, which includes a description of all the ways two objects are equivalent. You can also think of categories as geometric objects in which each element in the category is represented by a point.

Imagine, for example, the surface of a globe. Every point on this surface could represent a different type of triangle. Paths between those points would express equivalence relationships between the objects. In the perspective of category theory, you forget about the explicit way in which any one object is described and focus instead on how an object is situated among all other objects of its type.



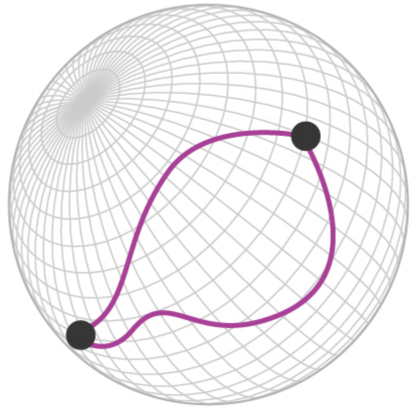
A Surface of Triangles

Every point on a surface corresponds to a different configuration of a triangle.

"There are lots of things we think of as things when they're actually relationships between things," Zakharevich said. "The phrase 'my husband,' we think of it as an object, but you can also think of it as a relationship to me. There is a certain part of him that's defined by his relationship to me."

Eilenberg and Mac Lane's version of a category was well suited to keeping track of strong forms of equivalence. But in the second half of the 20th century, mathematicians increasingly began to do math in terms of weaker notions of equivalence such as homotopy. "As math gets more subtle, it's inevitable that we have this progression towards these more subtle notions of sameness," said Emily Riehl, a mathematician at Johns Hopkins University. In these subtler notions of equivalence, the amount of information about how two objects are related increases dramatically. Eilenberg and Mac Lane's rudimentary categories were not designed to handle it.

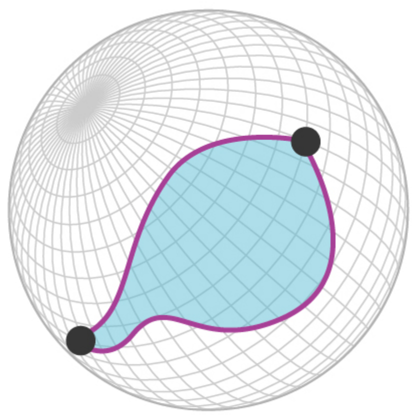
To see how the amount of information increases, first remember our sphere that represents many triangles. Two triangles are homotopy equivalent if you can stretch or otherwise deform one into the other. Two points on the surface are homotopy equivalent if there's a path linking one with the other. By studying homotopy paths between points on the surface, you're really studying different ways in which the triangles represented by those points are related.



Point Equivalence

Two points are homotopy equivalent
if there is at least one path linking one to the other.

But it’s not enough to say that two points are linked by many equal paths. You need to think about equivalences between all those paths, too. So in addition to asking whether two points are equivalent, you’re now asking whether two paths that start and end at the same pair of points are equivalent — whether there’s a path between those paths. This path between paths takes the shape of a disk whose boundary is the two paths.



Path Equivalence

Two paths are homotopy equivalent
if there is at least one surface linking one to the other.

You can keep going from there. Two discs are equivalent if there’s a path between them — and that path will take the form of a three-dimensional object. Those three-dimensional objects may themselves be connected by four-dimensional paths (the path between two objects always has one more dimension than the objects themselves).

Ultimately, you will build an infinite tower of equivalences between equivalences. By considering the entire edifice, you generate a full perspective on whatever objects you’ve chosen to represent as points on that sphere.

“It’s just a sphere, but it turns out, to understand the shape of a sphere, you need to go out to infinity in a sense,” said David Ben-Zvi of the University of Texas, Austin.

In the last decades of the 20th century, many mathematicians worked on a theory of “infinity categories” — something that would keep track of the infinite tower of equivalences between equivalences. Several made substantial progress. Only one got all the way there.

Rewriting Mathematics

Jacob Lurie’s first paper on infinity category theory was inauspicious. On June 5, 2003, the 25-year-old posted a 60-page document called “On Infinity Topoi” to the scientific preprint site arxiv.org. There, he began to sketch rules by which mathematicians could work with infinity categories.

This first paper was not universally well received. Soon after reading it, Peter May, a mathematician at the University of Chicago, emailed Lurie’s adviser, Michael Hopkins, to say that Lurie’s paper had some interesting ideas, but that it felt preliminary and needed more rigor.

“I explained our reservations to Mike, and Mike relayed the message to Jacob,” May said.

Whether Lurie took May’s email as a challenge or whether he had his next move in mind all along is not clear. (Lurie declined multiple requests to be interviewed for this story.) What is clear is that after receiving the criticism, Lurie launched into a multiyear period of productivity that has become legendary.

“I’m not inside Jacob’s brain, I can’t say exactly what he was thinking at that time,” May said. “But certainly there’s a huge difference between the draft we were reacting to and the final versions, which are altogether on a higher mathematical plane.”

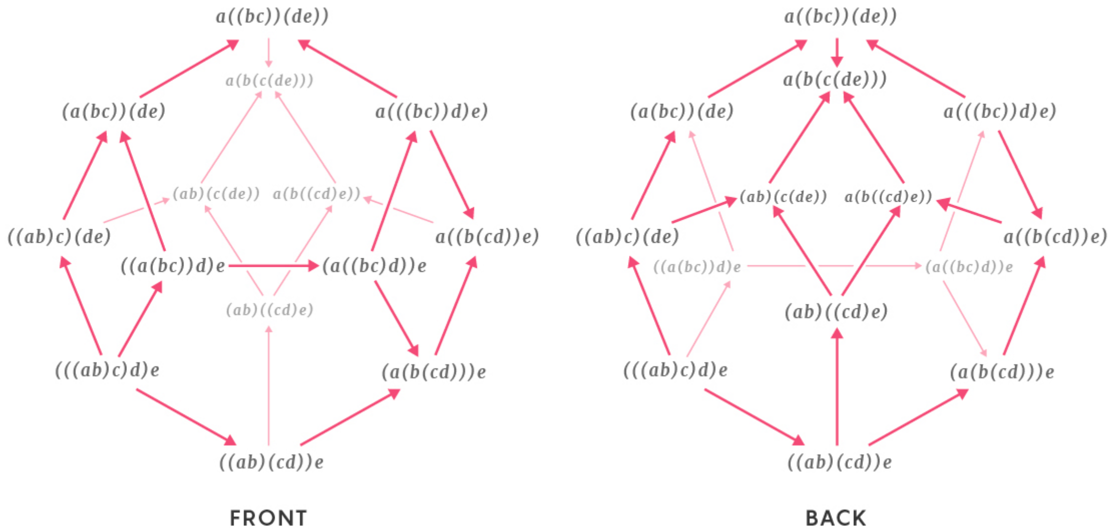
In 2006 Lurie released a draft of *Higher Topos Theory* on arxiv.org. In this mammoth work, he created the machinery needed to replace set theory with a new mathematical foundation, one based on infinity categories. “He created literally thousands of pages of this foundational machinery that we’re all now using,” said Charles Rezk, a mathematician at the University of Illinois, Urbana-Champaign, who did important early work on infinity categories. “I could not imagine producing *Higher Topos Theory*, which he produced in two or three years, in a lifetime.”

Then in 2011, Lurie followed it up with an even longer work. In it, he reinvented algebra.

Algebra provides a beautiful set of formal rules for manipulating equations. Mathematicians use these rules all the time to prove new theorems. But algebra performs its gymnastics over the fixed bars of the equal sign. If you remove those bars and replace them with the wispier concept of equivalence, some operations become a lot harder.

Take one of the first rules of algebra kids learn in school: the associative property, which says that the sum or product of three or more numbers doesn’t depend on how the numbers are grouped: $2 \times (3 \times 4) = (2 \times 3) \times 4$.

Proving that the associative property holds for any list of three or more numbers is easy when you’re working with equality. It’s complicated when you’re working with even strong notions of equivalence. When you move to subtler notions of equivalence, with their infinite towers of paths between paths, even a simple rule like the associative property turns into a thicket.



A Map of Associations

In algebra, the associative property tells you $(a \times b) \times c = a \times (b \times c)$. But when equivalence comes in, the associative property alone does not guarantee that every grouping produces the same product. This shape, called the associahedron (for five elements), records equivalences between groupings. Each grouping is represented as a vertex. Edges and faces connect groupings that are equivalent to each other under the associative property.

Omaranto

"This complicates matters enormously, in a way that makes it seem impossible to work with this new version of mathematics we're imagining," said David Ayala, a mathematician at Montana State University.

In *Higher Algebra*, the latest version of which runs to 1,553 pages, Lurie developed a version of the associative property for infinity categories — along with many other algebraic theorems that collectively established a foundation for the mathematics of equivalence.

Taken together, his two works were seismic, the types of volumes that trigger scientific revolutions. "The scale was completely massive," Riehl said. "It was an achievement on the level of Grothendieck's revolution of algebraic geometry."

Yet revolutions take time, and as mathematicians found after Lurie's books came out, the ensuing years can be chaotic.

Digesting the Cow

Mathematicians have a reputation for being clear-eyed thinkers: A proof is correct or it's not, an idea works or it doesn't. But mathematicians are also human beings, and they react to new ideas the way human beings do: with subjectivity, emotion, and a sense of personal stakes.

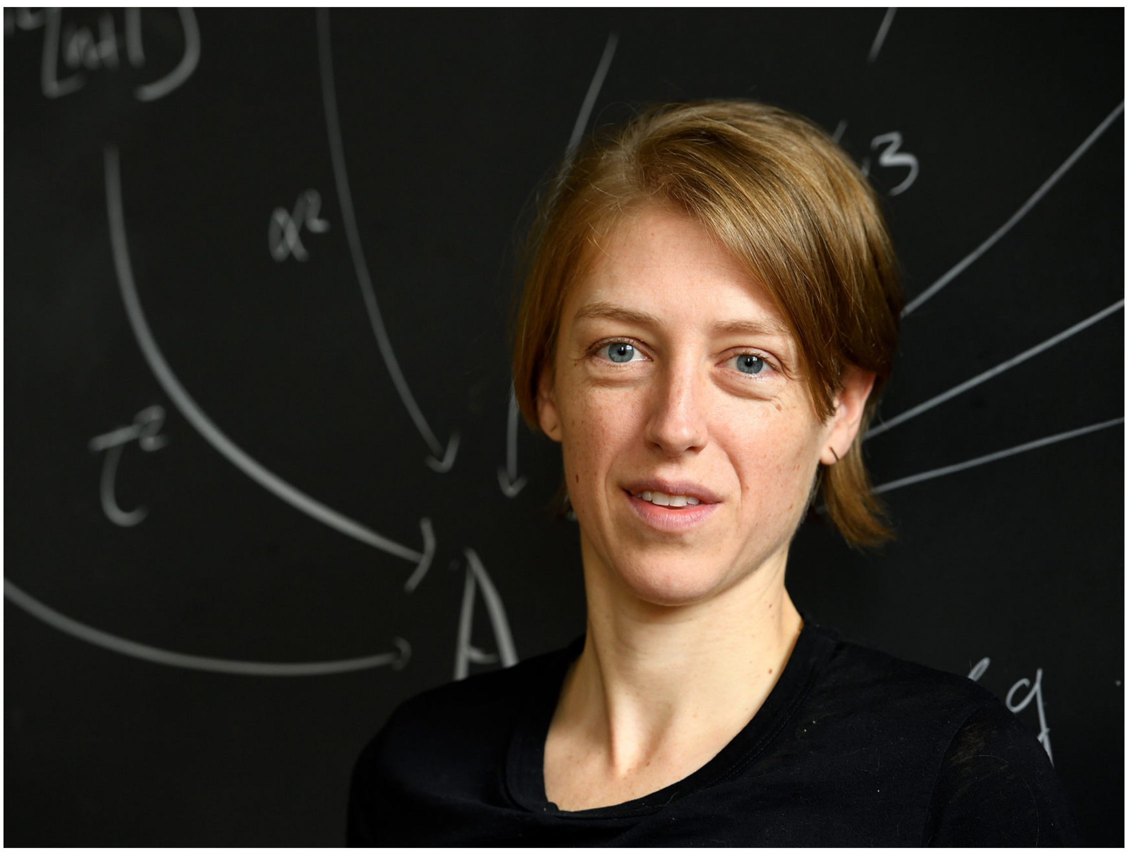
"I think a lot of writing about mathematics is done in the tone that mathematicians are searching for these glittering crystalline truths," Campbell said. "That's not how it goes. They're people with their own tastes and own domains of comfort, and they'll dismiss things they don't like for aesthetic or personal reasons."

In that respect, Lurie's work represented a big challenge. At heart it was a provocation: Here is a better way to do math. The message was especially pointed for mathematicians who'd spent their careers developing methods that Lurie's work transcended.

"There's this tension to the process where people aren't always happy to see the next generation rewriting their work," Francis said. "This is one feature affecting infinity category theory, that a lot of previous work gets rewritten."

Lurie's work was hard to swallow in other ways. The volume of material meant that mathematicians would need to invest years reading his books. That's an almost impossible requirement for busy mathematicians in midcareer, and it's a highly risky one for graduate students who have only a few years to produce results that will get them a job.

Lurie's work was also highly abstract, even in comparison with the highly abstract nature of everything else in advanced mathematics. As a matter of taste, it just wasn't for everyone. "Many people did view Lurie's work as abstract nonsense, and many people absolutely loved it and took to it," Campbell said. "Then there were responses in between, including just full-on not understanding it at all."



Emily Riehl, a mathematician at Johns Hopkins University, is helping to lead the development of higher category theory.

Will Kirk/Johns Hopkins University

Scientific communities absorb new ideas all the time, but usually slowly, and with a sense of everyone moving forward together. When big new ideas arise, they present challenges for the intellectual machinery of the community. “A lot of stuff got introduced at once, so it’s kind of like a boa constrictor trying to ingest a cow,” Campbell said. “There’s this huge mass that’s flowing through the community.”

If you were a mathematician who saw Lurie’s approach as a better way to do mathematics, the way forward was lonely. Few people had read Lurie’s work, and there were no textbooks distilling it and no seminars you could take to get your bearings. “The way you had to learn about this stuff really precisely was to just sit down and do it yourself,” said Peter Haine, a graduate student at the Massachusetts Institute of Technology who spent a year reading Lurie’s work. “I think that’s the hard part. It’s not just sit down and do it yourself — it’s sit down and do it yourself by reading 800 pages of *Higher Topos Theory*.”

Like many new inventions, *Higher Topos Theory* requires mathematicians to interact a lot with the machinery that makes the theory work. It’s like making every 16-year-old hoping for a driver’s license first learn how to rebuild an engine. “If there was a more driver-friendly version, it would become instantly more accessible to a wider mathematical audience,” said Dennis Gaitsgory, a mathematician at Harvard who has collaborated with Lurie.

As people started reading Lurie’s work and using infinity categories in their own research, other problems emerged. Mathematicians would write papers using infinity categories. Reviewers at journals would receive them and say: What is this?

“You have this situation where [papers] either come back from journals with absurd referee reports that reflect deep misunderstandings, or they just take several years to publish,” Barwick said. “It can make people’s lives uncomfortable because an unpublished paper sitting on your website for years and years starts to look a little funny.”

Yet the biggest problem was not papers that went unpublished, but papers that used infinity categories and did get published — with errors.

Lurie’s books are the single, authoritative text on infinity categories. They are completely rigorous, but hard to completely grasp. They’re especially poorly suited to serving as reference manuals — it’s difficult to look up specific theorems, or to check that a specific application of infinity categories that one might encounter in someone else’s paper really works out.

“Most people working in this field have not read Lurie systematically,” said André Joyal, a mathematician at the University of Quebec in Montreal whose earlier work was a key ingredient in Lurie’s books. “It would take a lot of time and energy, so we sort of assume what’s in his book is correct because almost every time we check on something it is correct. Actually, all the time.”

The inaccessibility of Lurie’s books has led to an imprecision in some of the subsequent research based on them. Lurie’s books are hard to read, they’re hard to cite, and they’re hard to use to check other people’s work.

“There is a feeling of sloppiness around the general infinity categorical literature,” Zakharevich said.

Despite all its formalism, math is not meant to have sacred texts that only the priests can read. The field needs pamphlets as well as tomes, it needs interpretive writing in addition to original revelation. And right now, infinity category theory still exists largely as a few large books on the shelf.

"You can take the attitude that 'Jacob tells you what to do, it's fine,'" Rezk said. "Or you can take the attitude that 'We don't know how to present our subject well enough that people can pick it up and run with it.'"

Yet a few mathematicians have taken up the challenge of making infinity categories a technique that more people in their field can run with.

A User-Friendly Theory

In order to translate infinity categories into objects that could do real mathematical work, Lurie had to prove theorems about them. And to do that, he had to choose a landscape in which to create those proofs, just as someone doing geometry has to choose a coordinate system in which to work. Mathematicians refer to this as choosing a model.

Lurie developed infinity categories in the model of quasi-categories. Other mathematicians had previously developed infinity categories in different models. While those efforts were far less comprehensive than Lurie's, they're easier to work with in some situations. "Jacob picked a model and checked that everything worked in that model, but often that's not the easiest model to work in," Zakharevich said.

In geometry, mathematicians understand exactly how to move between coordinate systems. They've also proved that theorems proved in one setting work in the others.

With infinity categories, there are no such guarantees. Yet when mathematicians write papers using infinity categories, they often move breezily between models, assuming (but not proving) that their results carry over. "People don't specify what they're doing, and they switch between all these different models and say, 'Oh, it's all the same,'" Haine said. "But that's not a proof."

For the past six years, a pair of mathematicians have been trying to make those guarantees. Riehl and Dominic Verity, of Macquarie University in Australia, have been developing a way of describing infinity categories that moves beyond the difficulties created in previous model-specific frameworks. Their work, which builds on previous work by Barwick and others, has proved that many of the theorems in *Higher Topos Theory* hold regardless of which model you apply them in. They prove this compatibility in a fitting way: "We're studying infinity categories whose objects are themselves these infinity categories," Riehl said. "Category theory is kind of eating itself here."

Riehl and Verity hope to move infinity category theory forward in another way as well. They're specifying aspects of infinity category theory that work regardless of the model you're in. This "model-independent" presentation has a plug-and-play quality that they hope will invite mathematicians into the field who might have been staying away while *Higher Topos Theory* was the only way in.

"There's a moat you have to get across to get into this world," Hopkins said, "and they are lowering the drawbridge."

Riehl and Verity expect to finish their work next year. Meanwhile, Lurie has recently started a project called Kerodon that he intends as a Wikipedia-style textbook for higher category theory. Thirteen years after *Higher Topos Theory* formalized the mathematics of equivalence, these new initiatives are an attempt to refine and promote the ideas — to make the mathematics of equivalence more universally accessible.

“Genius has an important role in developing mathematics, but actually the knowledge itself is the result of the activity of a community,” Joyal said. “It’s the real goal of knowledge to become the knowledge of the community, not the knowledge of one or two persons.”

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