Category

From Encyclopedia of Mathematics

A concept formalizing a number of algebraic properties of collections of morphism between mathematical objects of the same type (sets, topological spaces, groups, etc.) under the condition that these collections contain the identity mappings and are closed with respect to successive composition (or product) of mappings. A category $\mathfrak{C}$ consists of a class $\mathrm{Ob}\ \mathfrak{C}$, whose elements are called objects of the category, and a class $\mathrm{Mor}\ \mathfrak{C}$, the elements of which are called morphisms of the category. These classes must satisfy the following conditions:

1) to each ordered pair of objects A, B is associated a set $H_{\mathfrak{C}}(A, B)$ (also denoted by $\mathfrak{C}(A, B)$, $\mathrm{Hom}\ (A, B)$, or $H(A, B)$) of members of Mor. If $\alpha \in H(A, B)$, then A is called the source, or domain (of definition), of the morphism α, and B the target, or range (of values), or codomain of α; instead of $\alpha \in H(A, B)$ one often writes $\alpha : A \rightarrow B$ or $A \rightarrow^{\alpha} B$;

2) each morphism of the category $\mathfrak{C}$ belongs to only one set $H_{\mathfrak{C}}(A, B)$;

3) in the class $\mathrm{Mor}\ \mathfrak{C}$, the following partial composition (product) rule is given: The product of two morphisms $\alpha : A \rightarrow B$ and $\beta : C \rightarrow D$ is defined if and only if $B = C$, in which case it belongs to $H(A, D)$; the product of α and β is denoted by $\alpha\ \beta$, or, depending on one's choice of conventions, by $\beta\ \alpha$;

4) for any morphisms $\alpha : A \rightarrow B$, $\beta : B \rightarrow C$ and $\gamma : C \rightarrow D$, the associativity law holds:

$$(\alpha\ \beta)\ \gamma = \alpha\ (\beta\ \gamma);$$

5) each set $H_{\mathfrak{C}}(A, A)$ contains a (distinguished) morphism 1_A such that $\alpha\ .1_A = \alpha$ and $1_A.\ \beta = \beta$ for any morphisms $\alpha : X \rightarrow A$ and $\beta : A \rightarrow Y$; the morphisms 1_A are called units, identities or ones.

The notion of a class occurring in the definition of a category presupposes the use of axioms from set theory which distinguish between sets and classes. The most commonly used is the axiom scheme of Gödel–Bernays–von Neumann.

Sometimes, in the definition of a category, it is not required that the classes $H(A, B)$ be sets. Sometimes, instead of using classes, one assumes the existence of a universal set and requires that the classes $\mathrm{Ob}\ \mathfrak{C}$ and $\mathrm{Mor}\ \mathfrak{C}$ belong to a fixed universal set.

Since there is a bijective correspondence between the identities of a category $\mathfrak{C}$ and the class $\mathrm{Ob}\ \mathfrak{C}$, a category $\mathfrak{C}$ can be defined as a class of morphisms with a partial product satisfying additional conditions (see, for example, [6], [9]).

The notion of a category was introduced in 1945 [8]. The origins of category theory and the initial stimulus for its development came from algebraic topology. Subsequent investigation revealed the unifying role of the notion of a category and the notion of a functor related to it for many branches of mathematics.

Contents

Examples of categories.

1) The category of sets Ens. The class $\mathrm{Ob\,Ens}$ consists of all sets; the class $\mathrm{Mor\,Ens}$ consists of all functions between sets; composition of morphism is composition of functions (see Sets, category of).

2) The category of topological spaces Top (or $\mathfrak{T}$). The class $\mathrm{Ob\,Top}$ consists of all topological spaces, the class $\mathrm{Mor\,Top}$ of all continuous mappings, while composition is again composition of mappings.

3) The category of groups Gr (or $\mathfrak{G}$). The class $\mathrm{Ob\,Gr}$ consists of all groups, the class $\mathrm{Mor\,Gr}$ of all group homomorphisms, and composition again is composition of homomorphisms (see Category of groups). One defines in a similar fashion the categories of vector spaces over some field, the category of rings, etc.

4) The category of binary relations between sets $\mathrm{Rel\,Ens}$ (or $R(\mathfrak{S})$). The class of objects of this category is $\mathrm{Ob\,Ens}$, but as morphisms from A to B one takes all binary relations, that is, all subsets of the Cartesian product $A \times B$; composition is composition of binary relations (cf. Binary relation).

5) A monoid (semi-group with identity) is a category with a single object; conversely, every category consisting of a single object is a monoid.

6) A pre-ordered set N can be regarded as a category $\mathfrak{N}$ for which $\mathrm{Ob}\,\mathfrak{N} = N$, $\mathrm{Mor}\,\mathfrak{N} = \{(a, b) : a, b \in N, \ \square\, a \leq b\}$, while the product is defined by the equality $(a, b)(b, c) = (a, c)$.

All categories listed above admit an "isomorphic imbedding" into the category of sets. A category with this property is called a concrete category. Not every category is concrete; for example, the category with as objects all topological spaces and whose morphisms are defined to be homotopy classes of continuous mappings [10].

The number of examples of categories can be considerably enlarged by means of various constructions; foremost, by means of functors of categories or categories of diagrams.

A mapping $F: \mathfrak{C} \to \mathfrak{C}'$ between two categories $\mathfrak{C}, \mathfrak{C}'$ is called a covariant functor (cf. also Functor) if for each object $A \in \mathrm{Ob}\,\mathfrak{C}$, $F(A) \in \mathrm{Ob}\,\mathfrak{C}'$, for each morphism $\alpha \in H_{\mathfrak{C}}(A, B)$, its image $F(\alpha) \in H_{\mathfrak{C}'}(F(A), F(B))$, if $F(1_A) = 1_{F(A)}$, and if, finally, $F(\alpha\beta) = F(\alpha)F(\beta)$ whenever the product $\alpha\beta$ is defined. If the objects of $\mathfrak{C}$ constitute a set, then one can construct the category of diagrams $\mathrm{Funct}\,(\mathfrak{C}, \mathfrak{C}')$ or $F(\mathfrak{C}, \mathfrak{C}')$, the objects of which are all covariant functors from $\mathfrak{C}$ to $\mathfrak{C}'$ and with as morphisms all natural transformations of these functors.

To each category $\mathfrak{C}$ one can associate its dual category $\mathfrak{C}$, or $\mathfrak{C}^T$, or $\mathfrak{C}^{op}$, for which $\mathrm{Ob}\,\mathfrak{C}^* = \mathrm{Ob}\,\mathfrak{C}$ and $H_{\mathfrak{C}^*}(A, B) = H_{\mathfrak{C}}(B, A)$ for any $A, B \in \mathrm{Ob}\,\mathfrak{C}$. A covariant functor from $\mathfrak{C}^*$ to $\mathfrak{C}'$ is called a contravariant functor from $\mathfrak{C}$ to $\mathfrak{C}'$. Along with functors of one argument one can consider many-placed functors or functors of several arguments; cf. Functor.

For each statement in category theory there is a dual statement, obtained by a formal "reversal of the arrows" . In this connection the so-called duality principle holds: A statement p is true in category theory if and only if the dual statement p^* is true.

Many concepts and results in mathematics turn out to be dual to others from a category-theoretic point of view: injectivity and projectivity, nilpotency and the notion of category of a topological space in the sense of Lyusternik–Shnirel'man, varieties and radicals in algebra, etc.

A category-theoretic analysis of the foundations of homology theory led to the introduction in the mid-fifties of so-called Abelian categories (cf. Abelian category). Within this framework it proved possible to realize the basic constructions of homological algebra [2]. In the 1960s, interest in non-Abelian categories grew, as a result of problems in logic, general algebra, topology, and algebraic geometry. An intensive development of universal algebra and the axiomatic construction of homotopy theory marked the beginnings of various lines of investigation: the category-theoretic study of varieties of universal algebras, the theory of isomorphisms of direct decompositions, the theory of adjoint functors, and the theory of duality of functors. The latter development uncovered the existence of relations between these areas of study. As a result of the recent theory of relative categories, which makes wide use of the techniques of adjoint functors and closed categories, a duality has been established between homotopy theory and the theory of universal algebras; this is based on the interpretation of the categoric definitions of a monoid and a comonoid in suitable categories of functors (see, for example, [7]). Along with the development of the general theory of relative categories special classes of such categories were introduced: 2-categories or formal categories; categories with an involution or $\mathfrak{I}$-categories, including, in particular, the category of binary relations; etc. A special case of 2-categories is the category of small categories, which can be placed at the basis of an axiomatic construction of mathematics.

The classes of categories listed above are characterized by the fact that their sets of morphisms $H(A, B)$ possess an additional structure. Another method of introducing an additional structure in a category is to provide the category with a Grothendieck topology and to construct the category of sheaves over the topologized category (so-called topoi; cf. Topos).

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Comments

The notion of a category was introduced in 1942 by S. Eilenberg and S. MacLane [a1], and has since found numerous applications in algebra, topology and the foundations of mathematics. The intuitive idea is that a category consists of all the objects in some "universe of mathematical discourse" together with all the mappings between them. By identifying an object with its identity morphism it is possible to define the notion of a category in terms of morphisms alone. It is intuitively clearer and also more customary to use both objects and morphisms. The two approaches are mixed-up to some extent in the article above. The object and morphism definition is as follows. A category $\mathfrak{C}$ consists of

A1) a class $\mathrm{Ob}\ \mathfrak{C}$ whose elements are called objects of $\mathfrak{C}$, and a class $\mathrm{Mor}\ \mathfrak{C}$ whose elements are called morphisms or arrows of $\mathfrak{C}$;

A2) operations assigning to each morphism α of $\mathfrak{C}$ a pair of objects $(d_0 (\alpha), d_1 (\alpha))$, called the domain and codomain (or source and target) of α. One writes "a: A B" or "AaB" to mean "a is a morphism with domain A and codomain B" ; this rephrases 1) above;

A3) an operation assigning to each object A of $\mathfrak{C}$ a morphism $1_A : A \rightarrow A$, called the identity morphism on A; this is the precise meaning of part of 5) above;

A4) a partial (binary) product operation for morphisms, the product $\alpha \beta$ (called the composite of α and β) being defined if and only if $d_0 (\alpha) = d_1 (\beta)$, and satisfying $d_1 (\alpha \beta) = d_1 (\alpha)$ and $d_0 (\alpha \beta) = d_0 (\beta)$ whenever it is defined; this rephrases 3);

these data being subject to the axioms

A5) composition is associative, i.e. $(\alpha \beta) \gamma = \alpha (\beta \gamma)$ whenever both sides are defined; this rephrases 4) above;

A6) identity morphisms are units for composition, i.e. $\alpha 1_A = \alpha$ and $1_A \beta = \beta$ whenever the composites are defined; this and A3) rephrase 5).

The classes $\mathrm{Ob}\ \mathfrak{C}$ and $\mathrm{Mor}\ \mathfrak{C}$ are not required to be sets, and in many of the leading examples (see, e.g. the main text above and the examples (A1)–(A4), (A7) below) they are not sets. However, most examples have the property that, for each pair of objects (A, B), the collection of morphisms α with $d_0 (\alpha) = A$ and $d_1 (\alpha) = B$ forms a set (usually denoted by $\mathfrak{C}(A, B)$ or $\mathrm{Hom}\ (A, B)$); such categories are sometimes called locally small, although other writers include this condition as part of the definition of a category. A category $\mathfrak{C}$ is said to be small if $\mathrm{Ob}\ \mathfrak{C}$ and $\mathrm{Mor}\ \mathfrak{C}$ are sets (cf. Small category); it turns out that many of the fundamental mathematical structures may be regarded as small categories (e.g. (A6) and (A7) below).

It follows from the definition that each object in a category has a unique identity morphism; thus it is possible to identify objects with their identity morphisms, leading to an axiomatization of categories in which "morphism" and "composite" are the only primitive notions (see [9]).

Examples of categories.

(A1) See the main article. The category $\mathbf{Ens}$ of sets is more often denoted by $\mathbf{Set}$.

(A3) Next to the category $\mathbf{Gr}$ one may consider the categories $\mathbf{Ab}$ of Abelian groups, $\mathbf{Mod}_R$ of (right) modules over a fixed ring R, etc.

(A4) The category of binary relations between sets is usually denoted by $\mathbf{Rel}$.

(A5) A semi-group with identity M is also called a monoid. It defines a category with one object $*$, the elements of M being interpreted as morphisms $* \rightarrow *$.

(A6) A partially ordered set $(A, \leq)$ defines a category whose objects are the elements of A, and whose morphisms are the instances of the order-relation: that is, there is just one morphism $a \rightarrow b$ if $a \leq b$, and none otherwise.

(A7) Similarly, an oriented graph (or diagram scheme) can be interpreted as a category of which the objects are the vertices and the morphism from v_1 to v_2 are the oriented paths from v_1 to v_2 including the trivial identity paths from v to v for all vertices. Inversely a category can be seen as a (very large) oriented graph with loops and multiple edges together with an equivalence relation identifying certain paths, cf. [11], Section II.7.

(A8) The homotopy category $\mathbf{Htpy}$ or $\mathbf{Htp}$ has the same objects as $\mathbf{Top}$, but its morphisms are homotopy classes of continuous mappings. This category can be proved to be nonconcrete.

In the study of categories, functors (morphisms of categories) play an essential role. A functor $F: \mathfrak{C} \rightarrow \mathfrak{D}$ consists of two functions, one assigning to each object A of $\mathfrak{C}$ an object FA of $\mathfrak{D}$, and the other assigning to each morphism α of $\mathfrak{C}$ a morphism $F\alpha$ of $\mathfrak{D}$, in such a way that the categorical structure is preserved: $d_i(F\alpha) = F(d_i(\alpha))$ $(i = 0, 1)$, $F(1_A) = 1_{FA}$ and $F(\alpha\beta) = (F\alpha)(F\beta)$ whenever $\alpha\beta$ is defined. There is also a third level of structure: if F and G are both functors $\mathfrak{C} \rightarrow \mathfrak{D}$, a natural transformation (or functorial morphism $\eta: F \rightarrow G$ is a function assigning to each object A of $\mathfrak{C}$ a morphism $\eta_A : FA \rightarrow GA$ in $\mathfrak{D}$, such that for every $A \rightarrow^{\alpha} B$ in $\mathfrak{C}$ the diagram

$$\begin{array}{ccc} FA & \stackrel{F\alpha}{\rightarrow} & FB \\ \eta_A \downarrow & \square & \downarrow \eta_B \\ GA & \underset{G\alpha}{\rightarrow} & GB \end{array}$$

commutes (i.e. $(G\alpha)(\eta_A) = (\eta_B)(F\alpha)$). Functors may be composed (and every category has an identity functor); thus there is a category $\mathcal{Cat}$ of (small) categories and functors between them. Natural transformations may be composed; thus, given two categories $\mathfrak{C}$ and $\mathfrak{D}$, there is a category $[\mathfrak{C}, \mathfrak{D}]$ (or $\mathfrak{D}^{\mathfrak{C}}$) of functors $\mathfrak{C} \rightarrow \mathfrak{D}$ and natural transformations between them. This is one important way in which new categories are constructed from existing ones. The resulting categories are called categories of functors or categories of diagrams. The latter name is especially understandable if $\mathfrak{C}$ is the category corresponding to a diagram scheme (oriented graph). Indeed, then a functor $\mathfrak{C} \rightarrow \mathfrak{D}$ "is" a diagram in $\mathfrak{D}$. Other important constructions to obtain new categories are: taking quotients (cf. Quotient category), taking localizations (cf. Localization in categories) and constructing relative and comma categories. The important notion of a derived category involves several of these constructions. If $\mathfrak{C}$ is a category and B an object of $\mathfrak{C}$, then the relative category $\mathfrak{C} / B$ of objects over B has as objects all morphisms $\alpha : A \rightarrow B$ of $\mathfrak{C}$ into B and a morphism in $\mathfrak{C} / \mathfrak{B}$ of α to $\alpha' : A' \rightarrow B$ is a morphism $\phi : A \rightarrow A'$ such that $\alpha' \circ \phi = \alpha$. Dually there is the notion of the relative category of objects in $\mathfrak{C}$ under a given object. Intuitively an object $\alpha : A \rightarrow B$ of $\mathfrak{C} / B$ is a family of objects of $\mathfrak{C}$ parametrized by B or a fibre object (fibred object). The systematic consideration of these relative objects, i.e. fibre objects (and their duals) combined with base change and deformation ideas has become a most important technique in many parts of mathematics, especially in algebra (notably homological algebra), algebraic and differential geometry, topology, and differential and algebraic topology. It is especially important to find the right fibrewise versions of definitions, theorems and concepts. (A second additional not unrelated major trend involves finding the right equivariant versions in those case in which there is a group of symmetries present (as well).)

The idea of a comma category generalizes that of categories of objects over or under a given object. Let there be given three categories $\mathfrak{C}, \mathfrak{B}, \mathfrak{A}$ and two functors S, T arranged as follows

$$\mathfrak{A} \xrightarrow{T} \mathfrak{C} \xleftarrow{S} \mathfrak{B}.$$

Then the comma category (T, S) or $(T \downarrow S)$ has as objects all triples (a, f, b) consisting of an object $a \in \mathfrak{A}$, an object $b \in \mathfrak{B}$ and a morphism $f : TA \to SB$ in $\mathfrak{C}$. A morphism (α , β) from (a, f, b) to (a', f', b') consists of a pair of morphisms $\alpha : A \to A'$ and $\beta : B \to B'$ such that $f' \circ T\alpha = T\beta \circ f$.

Examples of functors. ITEM {(A9} The forgetful functor $\mathbf{Top} \to \mathbf{Set}$ sends each topological space to its underlying set, and each continuous function to itself ("forgetting" the continuity). Similarly, one has forgetful functors $\mathbf{Gr} \to \mathbf{Set}, \mathbf{Mod}_R \to \mathbf{Ab}$, etc.

(A10) If $\mathfrak{C}$ is a locally small category, then for each $A \in \operatorname{Ob} \mathfrak{C}$ there is a functor $\mathfrak{C}(A, -) : \mathfrak{C} \to \mathbf{Set}$ sending B to the set $\mathfrak{C}(A, B)$ and $B \to^{\alpha} B'$ to the function which sends β to $\alpha \ \beta$. Such functors (or functors isomorphic to them in $[\mathfrak{C}, \mathbf{Set}]$) are called representable (cf. Representable functor).

(A11) There is a functor F: $\mathbf{Set} \to \mathbf{Gr}$ sending a set A to the free group generated by A, and a function $A \to^{\alpha} B$ to the unique homomorphism $FA \to FB$ sending the generator a of FA to $\alpha \ (a) \in FB$, for each $a \in A$.

(A12) The (singular) homology groups of spaces (cf. Homology group) define functors $\mathbf{Htpy} \to \mathbf{AB}$ (one for each dimension ≥ 0).

(A13) A functor between monoids, considered as categories, is just a monoid homomorphism.

(A14) A functor between partially ordered sets, considered as categories, is just an order-preserving mapping.

(A15) If G is a group, considered as a category, a functor $G \to \mathbf{Set}$ (respectively, $G \to \mathbf{Mod}_R$) is a permutation (respectively, an R-linear representation) of G (cf. Representation of a group).

A functor $F: \mathfrak{C} \to \mathfrak{D}$ is said to be faithful (cf. Faithful functor) if it is "injective on Hom-sets" ; i.e. if, given two morphisms $A \to^{\alpha} B$ and $A \to^{\beta} B$ with the same domain and codomain in $\mathfrak{C}$, $F\alpha = F\beta$ implies $\alpha = \beta$. F is said to be full if it is "surjective on Hom-sets" in a similar sense. Forgetful functors (as in (A9) above) are always faithful. The property that a category $\mathfrak{C}$ be concrete can now be rephrased as: There is a faithful functor $\mathfrak{C} \to \mathbf{Set}$.

Given a category $\mathfrak{C}$, one can form its opposite or dual category $\mathfrak{C}^{op}$ by keeping the same objects as $\mathfrak{C}$ and reversing all the morphisms. The category $\mathbf{Rel}$ is isomorphic to its opposite, though most familiar categories are not. A functor $\mathfrak{C}^{op} \to \mathfrak{D}$ is sometimes called a contravariant functor from $\mathfrak{C}$ to $\mathfrak{D}$; for emphasis, functors $\mathfrak{C} \to \mathfrak{D}$ are then called covariant. (For example, if $\mathfrak{C}$ is locally small, one may define a contravariant functor $\mathfrak{C}(-, A)$ from $\mathfrak{C}$ to $\mathbf{Set}$, by analogy with the covariant functor $\mathfrak{C}(A, -)$ of example (A10) above.) The duality principle for categories is essentially the assertion that something which is true for all categories is true for the duals of all categories.

S. MacLane [a4] introduced the idea that Cartesian products can be characterized in categorical terms, by a universal property; this gave rise to the general categorical notion of limit (and the dual notion of colimit), which includes products as a special case (cf. Limit and Universal problems).

The key notion of adjunction came latter [a5]: given functors $F: \mathfrak{C} \to \mathfrak{D}$ and $G: \mathfrak{D} \to \mathfrak{C}$, one says that F is left adjoint to G (written $F \dashv G$) if there is a bijection between morphisms $FA \to B$ and morphisms $A \to GB$ which is natural in A and B; this is equivalent to the existence of natural transformations $\eta : 1_{\mathfrak{C}} \to GF$ and $\epsilon : FG \to 1_{\mathfrak{D}}$ satisfying certain identities [11] (cf. Adjoint functor). For example, the free group functor (example (A11) above) is left adjoint to the forgetful functor $\mathbf{Gr} \to \mathbf{Set}$; Galois connections (cf. Galois correspondence) are examples of (contravariant) adjunctions between partially ordered sets. A functor which has a left adjoint preserves all limits; the converse implication is valid under suitable "smallness conditions" (the adjoint functor theorem, see [9]).

Given an adjunction $(F \dashv G)$ as above, the composite functor $T = GF: \mathfrak{C} \to \mathfrak{C}$ is equipped with natural transformations $\eta : 1_{\mathfrak{C}} \to T$ and $\mu = G\epsilon_F : TT \to T$ satisfying certain identities; these data define the notion of a monad or triple on a category, which played a central role in much categorical research in the 1960's and later years.

The identities which a triple (T, μ, η) on a category $\mathfrak{C}$ is required to satisfy are the following: $\mu_A \circ T(\mu_A) = \mu_A \circ \mu_T(A)$, $\mu_A \circ T(\eta_A) = 1_{T(A)}$, $\mu_A \circ \eta_{T(A)} = 1_{T(A)}$. An algebra for the triple T, or T-algebra, is an object X of $\mathfrak{C}$ together with a morphism $\xi : TX \rightarrow X$ such that the following identities hold: $\xi \circ \eta_X = 1_X$, $\xi \circ T(\xi) = \xi \circ \mu_X$. It is a good idea to write out these requirements in terms of commutative diagrams. They are reminiscent of associativity and unit requirements.

Dually, i.e. reversing all arrows, there is the notion of a cotriple and the corresponding notion of a co-algebra over such a cotriple. An important example of a cotriple in the category $\mathbf{Ring}$ of commutative rings with unit element is the functor W: $\mathbf{Ring} \longrightarrow \mathbf{Ring}$ of the big Witt vectors together with the structure of a special λ-ring on $W(R)$. The co-algebras for this cotriple are precisely the special λ-algebras (cf. Witt vector and λ-ring).

Important examples of triples arise from adjunctions involving forgetful functors. For example, let V: $\mathbf{Ring} \longrightarrow \mathbf{Set}$ be the forgetful functor from the category of commutative rings with unit element to the category of sets. This one has an adjoint F: $\mathbf{Set} \longrightarrow \mathbf{Ring}$ which assigns to a set $E \in \mathbf{Set}$ the free commutative ring with generator E, i.e. the ring $\mathbf{Z}[X_e : e \in E$ of commutative polynomials over $\mathbf{Z}$ in the variables $X_e, e \in E$. The freeness property of $\mathbf{Z}[X_e : e \in E$, i.e. the property that for every ring A and every collection of elements $(a_e)_{e \in E}$ of A there is precisely one homomorphism of rings $\phi : \mathbf{Z}[X_e : e \in E] \longrightarrow A$ such that $\phi(X_e) = a_e$ for all $e \in E$, precisely expresses the fact that V and F are adjoint functors: $\mathbf{Set}(E, V(A)) \cong \mathbf{Ring}(FE, A)$. The corresponding natural transformation $\mathrm{id} \longrightarrow VF$ is given by $e \mapsto X_e$ (cf. also Adjoint functor).

Every monad and comonad can be induced by an adjunction; in fact there is a "best possible" such adjunction, in which $\mathfrak{D}$ is taken to be the category of (Eilenberg–Moore) algebras for the monad, [a7]. A general adjunction $(F \dashv G)$ is said to be monadic (or $\mathfrak{D}$ is said to be monadic over $\mathfrak{C}$) if $\mathfrak{D}$ is (canonically) equivalent to the category of algebras for the induced monad on $\mathfrak{C}$. The adjunction between the $\mathbf{Set}$ and $\mathbf{Gr}$, mentioned above, is monadic; more generally, the categories which are monadic over $\mathbf{Set}$ can be characterized [a8] as those which arise from varieties of universal algebras (provided one allows infinitary as well as finitary algebraic operations; the finitary case can also be characterized in categorical terms [9], using the notion of algebraic theory). See also Variety of universal algebras.

Another phrase that is used to denote a triple (T, μ, η) is algebraic theory (in monad form) over the category $\mathfrak{C}$. It is so to speak the theory of the category of T-algebras. There are, at least, two more equivalent ways in which this notion is approached. One is as follows [a6]. An algebraic theory in clone form $(T, \eta, \circ)$ consists of an "object assignment function" $A \mapsto TA$ ($= T$-terms with variables in A) for all objects A, an "insertion of variables mapping" $\eta_A : A \longrightarrow TA$ for all A and a "clone-composition function" $\mathfrak{C}(B, TC) \times \mathfrak{C}(A, TB) \longrightarrow^{\circ} \mathfrak{C}(A, TC)$ for each ordered triple (A, B, C) of objects of $\mathfrak{C}$. For each $f : A \longrightarrow B$ in $\mathfrak{C}$ let $f^{\#}$ be the composite $A \longrightarrow B \longrightarrow^{\eta_B} TB$. Then the data $(T, \eta, \circ)$ are supposed to satisfy the following axioms. For all $\alpha : A \longrightarrow TB$, $\beta : B \longrightarrow TC$, $\gamma : C \longrightarrow TD$ and $f : A \longrightarrow B$,

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha),$$
$$\eta_B \circ \alpha = \alpha,$$
$$\alpha \circ f^{\#} = \alpha f .$$

This defines a new category $\mathfrak{C}_T$, the Kleishi category of $(T, \eta, \circ)$. The objects of $\mathfrak{C}_T$ are the objects of $\mathfrak{C}$, $\mathfrak{C}_T(A, B) = \mathfrak{C}(A, TB)$, composition is given by $\circ$, and the identity morphisms are the η_A in $\mathfrak{C}_T(A, A) = \mathfrak{C}(A, TA)$.

A simple example of an algebraic theory in clone form is as follows. Let R be a ring with unit. For a set A let TA be the vector space $R^{(A)} = \{(r_\alpha)_{\alpha \in A} : r_\alpha \in R \square \textbf{and only finitely many } \square r_{\alpha\square} \textbf{ are different from zero}\}$. A matrix with columns indexed by B and rows indexed by A is a mapping $\alpha : B \longrightarrow TA$, i.e. a morphism in $\mathbf{Set}$;

$\alpha(b)$ is the b-th column of the matrix. Given an $A \times B$ matrix $\alpha : B \rightarrow TA$ and a $B \times C$ matrix $\beta : C \rightarrow TB$, define their composite $\alpha \circ \beta : C \rightarrow TA$ by the usual matrix product, i.e. $(\alpha \circ \beta)(c)$ is the A-vector with components

$$(\alpha \circ \beta)(c)_a = \sum_b \alpha(b)_a \beta(c)_b .$$

The insertion of variables assignment $\eta : A \rightarrow TA$ is defined by $\eta(a)_{a'} = \delta_{a,a'}$ where δ denotes the Kronecker delta (cf. Kronecker symbol). It is easily checked that the axioms above are satisfied.

Let $(T, \eta, \circ)$ be an algebraic theory in clone form. For $f : A \rightarrow B$ in $\mathfrak{C}$ define $Tf : TA \rightarrow TB$ as the composite $f^{\#} \circ \mathrm{id}_{TA}$. It follows readily that T is then a functor and that $\eta : \mathrm{id} \rightarrow T$ is a natural transformation. Further define $\mu_A : TTA \rightarrow TA$ as the composite $1_{TA} \circ 1_{TTA}$. Then μ is also a natural transformation and (T, μ, η) is a triple. Moreover, this construction yielding a triple for each algebraic theory in clone form is a bijection. For a discussion of the algebraic theories (in clone and monad form) coming from a universal algebra and a third categorical way of viewing universal algebras see Universal algebra.

The language of categories and functors was originally introduced to meet the needs of algebraic topology and homological algebra [a1], [a4]. In the 1950's and early 1960's much attention was focused on Abelian categories (cf. Abelian category), which may be defined as categories satisfying all the elementary properties of $\mathbf{Ab}$; it was shown in [2] that they provide an adequate foundation for the development of homological algebra, and in [a11] that every small Abelian category admits a full imbedding, preserving finite limits and colimits, into $\mathbf{Mod}_R$ for some R.

In an Abelian category $\mathfrak{C}$, the "Hom-sets" $\mathfrak{C}(A, B)$ have a natural Abelian group structure; this observation provided one of the incentives for developing the theory of enriched (or relative) categories [a12], that is, categories whose "Hom-sets" are objects of some "base category" $\mathfrak{B}$. Categories enriched over themselves (such as $\mathbf{Ab}$ and $\mathbf{Cat}$) are called closed categories [a13] (cf. Closed category); an important class of closed categories (including $\mathbf{Cat}$ but not $\mathbf{Ab}$) consists of those where the closed structure (the "internal Hom") is related by an adjunction to the categorical product structure — such categories are called Cartesian closed. The notion of a Cartesian closed category played an important role in F.W. Lawvere's axiomatization of the category of small categories as a foundation for mathematics [a14], and in his latter development with M. Tierney of the notion of an elementary topos, which has dominated much of categorical research in the 1970's and 1980's (see [a15]). Cartesian closed categories are also of importance in logic, since they provide models for the (typed) λ-calculus (see [a16]).

Categories enriched over $\mathbf{Cat}$ (commonly called 2-categories) have also received a good deal of attention in recent years. They are distinguished from the general run of enriched categories by the possibility of considering diagrams within them which commute "up to isomorphism" but not exactly; the weaker notion of a bicategory [a17] is a further expression of this idea. 3-categories and higher-dimensional categories have also been studied, and have proved to be of importance in the algebraic study of homotopy types [a18]. In these areas of category theory coherence theorems play an important part: these are theorems which allow one to deduce the commutativity of a large class of diagrams from that of certain particular diagrams (see [a19], for example).

Further areas of category theory in which much work has been done in recent years include the theory of fibred categories [a2] (which, together with enriched category theory, is an expression of the idea that $\mathbf{Set}$ can be replaced by some more general base category as a foundation for much of mathematics), and the theory of topological categories [a3] (which is concerned with the study of concrete categories whose forgetful functors to $\mathbf{Set}$ have good infinitary properties, similar to those of the forgetful functor $\mathbf{Top} \rightarrow \mathbf{Set}$, see also Topologized category).

In addition to the books [9] and [11], [12] and [13] may also be recommended as general accounts of category theory.

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