Appendix A
Topoi and Logic

In this section, we will explore the tight connection between topos theory and logic. In particular, to each topos there is associated a language for expressing the internal language of the topos. The converse is also true: given a language one can define a corresponding topos.

A.1 First Order Languages

A language, in its most raw definition, comprises a collection of atomic variables, and a collection of primitive operations called logical connectives, whose role is to combine together such primitive variables transforming them into formulas or sentences. Moreover, in order to reason with a given language, one also requires rules of inference, i.e. rules which allow you to generate other valid sentences from the given ones.

The semantics or meaning of the logical connectives, however, is not given by the formulae and sentences themselves, but it is defined through a so called evaluation map, which is a map from the set of atomic variables and sentences to a set of truth values. Such a map enables one to determine when a formula is true and, thus, defines its semantics/meaning.

In this perspective it turns out that the meaning of the logical connectives is given in terms of some set of objects which represent the truth values. The logic that a given language will exhibit will depend on what the set of truth values is considered to be. In fact, what was said above is a very abstract characterisation of what a language is. To actually use it as a deductive system of reasoning, one needs to define a mathematical context in which to represent this abstract language. In this way the elementary and compound propositions will be represented by certain mathematical objects and the set of truth values will itself be identified with an algebra.

For example, in standard classical logic, the mathematical context used is **Sets** and the algebra of truth values is the Boolean algebra of subsets of a given set.

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However, in a general topos, the internal logic/algebra will not be Boolean but will be a generalisation of it, namely a Heyting algebra.

In order to get a better understanding of what a language is, we will start with a very simple language called propositional language $P(l)$.

A.2 Propositional Language

The propositional language $P(l)$ contains a set of symbols and a set of formation rules.

Symbols of $P(l)$

(i) An infinite list of symbols $\alpha_0, \alpha_1, \alpha_2, \ldots$ called *primitive propositions*.
(ii) A set of symbols $\neg$, $\vee$, $\wedge$, $\Rightarrow$ which, for now, have no explicit meaning.
(iii) Brackets), (.

Formation Rules

(i) Each primitive proposition $\alpha_i \in P(l)$ is a sentence.
(ii) If α is a sentence, then so is $\neg\alpha$.
(iii) If α_1 and α_2 are sentences, then so are $\alpha_1 \wedge \alpha_2$, $\alpha_1 \vee \alpha_2$ and $\alpha_1 \Rightarrow \alpha_2$.

Note also that $P(l)$ does not contain the quantifiers $\forall$ and $\exists$. This is because it is only a propositional language. To account for quantifiers one has to go to more complicated languages called higher-order languages, which will be described later.

The inference rule present in $P(l)$ is the modus ponens (the 'rule of detachment') which states that, from α_i and $\alpha_i \Rightarrow \alpha_j$, the sentence α_j may be derived. Symbolically this is written as

$$\frac{\alpha_i, \alpha_i \Rightarrow \alpha_j}{\alpha_i}. \tag{A.1}$$

We will see, later on, what exactly the above expression means.

In order to use the language $P(l)$ one needs to represent it in a mathematical context. The choice of such a context will depend on what type of system we want to reason about. For now we will consider a classical system, thus the mathematical context in which to represent the language $P(l)$ will be **Sets**. In **Sets**, the truth object (object in which the truth values lie) will be the Boolean set $\{0, 1\}$, thus the truth values will undergo a Boolean algebra. This, in turn, implies that the logic of the language $P(l)$, as represented in **Sets**, will be Boolean.

The rigorous definition of a representation of the language $P(l)$ is as follows:

Definition A.1 Given a language $P(l)$ and a mathematical context τ, a representation of $P(l)$ in τ is a map π from the set of primitive propositions to elements in the algebra in question: $\alpha \to \pi(\alpha)$.

As we will see, the specification of the algebra will depend on what type of theory we are considering, i.e. classical or quantum.

In classical physics, propositions are represented by the Boolean algebra of all (Borel) subsets of the classical state space, thus, given a representation π, we can define the semantics of $P(l)$ as follows:

$$\begin{aligned}
\pi(\alpha_i \vee \alpha_j) &:= \pi(\alpha_i) \vee \pi(\alpha_j) \\
\pi(\alpha_i \wedge \alpha_j) &:= \pi(\alpha_i) \wedge \pi(\alpha_j) \\
\pi(\alpha_i \Rightarrow \alpha_j) &:= \pi(\alpha_i) \Rightarrow \pi(\alpha_j) \\
\pi(\neg\alpha_i) &:= \neg\big(\pi(\alpha_i)\big)
\end{aligned} \tag{A.2}$$

where, on the left hand side, the symbols $\{\neg, \wedge, \vee, \Rightarrow\}$ are elements of the language $P(l)$, while on the right hand side they are the logical connectives in algebra, in which the representation takes place. It is in such an algebra that the logical connectives acquire meaning.

For the classical case, since the algebra in which a representation lives, is the Boolean algebra of subsets, the logical connectives on the right hand side of (A.2) are defined in terms of set-theoretic operations. In particular, we have the following associations:

$$\pi(\alpha_i) \vee \pi(\alpha_j) := \pi(\alpha_i) \cup \pi(\alpha_j) \tag{A.3}$$

$$\pi(\alpha_i) \wedge \pi(\alpha_j) := \pi(\alpha_i) \cap \pi(\alpha_j)$$

$$\neg\big(\phi(\alpha_i)\big) := \pi(\alpha_i)^c$$

$$\pi(\alpha_i) \Rightarrow \pi(\alpha_j) := \pi(\alpha_i)^c \cup \pi(\alpha_j). \tag{A.4}$$

So far, we have seen how logical connectives are represented in the topos **Sets**. However, it is possible to give a general definition of logical connectives in terms of arrows. Such a definition would then be valid for any topos. To retrieve the logical connectives for the classical case, in which the topos is **Sets**, we then simply replace, in the definitions that will follow, the general truth object Ω with the set $\{0, 1\} = 2$.

Logical connectives in a general topos τ are defined as follows:

- **Negation**
  We will now describe how to represent negation as an arrow in a given topos τ. Let us assume that the τ-arrow representing the value true is $\top : 1 \to \Omega$, which is the arrow used in the definition of the sub-object classifier. Given such an arrow, negation is identified with the unique arrow $\neg : \Omega \to \Omega$, such that the following

diagram is a pullback:

$$\begin{tikzcd}
1 \arrow[r, "\perp"] \arrow[d] & \Omega \arrow[d, "\neg"] \\
1 \arrow[r, "\top"'] & \Omega
\end{tikzcd}$$

Where $\perp$ is the topos analogue of the arrow *false* in **Sets**, i.e. $\perp$ is the character of $!_1 : 0 \to 1$:

$$\begin{tikzcd}
0 \arrow[r, "!_1"] \arrow[d, "!_1"'] & 1 \arrow[d, "\perp"] \\
1 \arrow[r, "\top"'] & \Omega
\end{tikzcd}$$

- **Conjunction**
  Conjunction is identified with the following arrow:

$$\cap : \Omega \times \Omega \to \Omega$$

which is the character of the product arrow $\langle \top, \top \rangle : 1 \to \Omega \times \Omega$, such that the following diagram is a pullback:

$$\begin{tikzcd}
1 \arrow[r, "\langle \top, \top \rangle"] \arrow[d, "id_1"'] & \Omega \times \Omega \arrow[d, "\cap"] \\
1 \arrow[r, "\top"'] & \Omega
\end{tikzcd}$$

where $\langle \top, \top \rangle$ is defined as follows:

$$\begin{tikzcd}
& 1 \arrow[r, "\top"] & \Omega \\
1 \arrow[ur, "I_1"] \arrow[dr, "I_1"'] \arrow[rr, "\langle \top, \top \rangle"] & & \Omega \times \Omega \arrow[u, "pr_1"'] \arrow[d, "pr_2"] \\
& 1 \arrow[r, "\top"'] & \Omega
\end{tikzcd}$$

- **Disjunction**
  Disjunction is identified with the arrow

$$\cup : \Omega \times \Omega \to \Omega \tag{A.5}$$

which is the character of the image of the arrow

$$\big[\langle \top, id_\Omega \rangle, \langle id_\Omega, \top \rangle\big] : \Omega + \Omega \to \Omega \times \Omega \tag{A.6}$$

such that the following diagram commutes:

$$\begin{tikzcd}
\Omega + \Omega \arrow[r, "{[\langle \top, 1_\Omega \rangle, \langle 1_\Omega, \top \rangle]}"] \arrow[d, "!_{\Omega+\Omega}"'] & \Omega \times \Omega \arrow[d, "\cup"] \\
1 \arrow[r, "\top"] & \Omega
\end{tikzcd}$$

- **Implication**
  Given two arrows

$$\begin{tikzcd}
\Omega \times \Omega \arrow[r, shift left, "\cap"] \arrow[r, shift right, "pr_1"'] & \Omega
\end{tikzcd}$$

their equaliser is some map

$$e : \left(\leq := \big\{ \langle x, y \rangle | x \leq y \text{ in } \Omega \big\} \right) \to \Omega \times \Omega \tag{A.7}$$

such that $\cap \circ e = pr_1 \circ e$.
Implication is then defined as the character of e, i.e. as the map

$$\Rightarrow : \Omega \times \Omega \to \Omega \tag{A.8}$$

such that the following diagram is a pullback:

$$\begin{tikzcd}
\leq \arrow[r, "e"] \arrow[d, "!_e"'] & \Omega \times \Omega \arrow[d, "\Rightarrow"] \\
1 \arrow[r, "\top"] & \Omega
\end{tikzcd}$$

In order to complete the definition of a propositional language in a general topos τ, we also need to define the valuation functions (which give us the semantics) in terms of arrows in that topos. We recall from the definition of the sub-object classifier that a truth value in a general topos τ is given by a map $1 \to \Omega$ (in **Sets** we have $1 \to \{0, 1\} = 2 = \Omega$). The collection of such arrows $\tau(1, \Omega)$ represents the collection of all truth values. Thus, a valuation map in a general topos is defined to be a map $V : \{\pi(\alpha_i)\} \to \tau(1, \Omega)$ such that the following equalities hold:

$$V\big(\neg\big(\pi(\alpha_i)\big)\big) = \neg \circ V\big(\pi(\alpha_i)\big) \tag{A.9}$$

$$V\big(\pi(\alpha_i) \vee \pi(\alpha_j)\big) = \vee \circ \big\langle V\big(\pi(\alpha_i)\big), V\big(\pi(\alpha_j)\big)\big\rangle \tag{A.10}$$

$$V\big(\pi(\alpha_i) \wedge \pi(\alpha_j)\big) = \wedge \circ \big\langle V\big(\pi(\alpha_i)\big), V\big(\pi(\alpha_j)\big)\big\rangle \tag{A.11}$$

$$V\big(\pi(\alpha_i) \Rightarrow \pi(\alpha_j)\big) = \Rightarrow \circ\big\langle V\big(\pi(\alpha_i)\big), V\big(\pi(\alpha_j)\big)\big\rangle. \tag{A.12}$$

A.2.1 Example in Classical Physics

We have stated above that classical physics uses the topos **Sets**. We now want to represent, in **Sets**, the propositional language $P(l)$, as defined for a classical system S. So now the elementary propositions will be propositions pertaining the physical system S. From now on, we will denote a language referred to a particular system S by $P(l)(S)$. Since S is a (classical) physical system, the elementary propositions which $P(l)(S)$ will contain will be of the form $A \in \Delta$ meaning " the quantity A which represents some physical observable, has value in a set Δ". We now define the representation map for this language as follows:

$$\begin{aligned}
\pi_{cl} : P(l)(S) &\to \mathcal{O}(S) \\
A \in \Delta &\mapsto \big\{s \in S | \tilde{A}(s) \in \Delta\big\} = \tilde{A}^{-1}(\Delta)
\end{aligned} \tag{A.13}$$

where S is the classical state space, $\mathcal{O}(S)$ is the Boolean algebra of subsets of S which lives in **Sets** and $\tilde{A} : S \to \mathbb{R}$ is the map from the state space to the reals, which identifies the physical quantity A. We now define the truth values for propositions.

Normally, such truth values are state-dependent, i.e. they depend on the state with respect to which we are preforming the evaluation. In classical physics, states are simply identified with elements s of the state space S. Thus, for all $s \in S$, we define the truth value of the proposition $\tilde{A}^{-1}(\Delta)$ as follows:

$$v(A \in \Delta; s) = \begin{cases} 1 & \text{iff } s \in \tilde{A}^{-1}(\Delta) \\ 0 & \text{otherwise.} \end{cases} \tag{A.14}$$

Therefore the truth values lie in the Boolean algebra $\Omega = \{0, 1\}$.

It is interesting to note that the application of the propositional language $P(l)$ for quantum theory, fails. This is because in quantum theory propositions are identified with projection operators, thus the representation map will be

$$\begin{aligned} \pi_q : \{\alpha_i\} &\to P(\mathcal{H}) \\ A \in \Delta &\mapsto \pi_q(A \in \Delta) := \hat{E}[A \in \Delta] \end{aligned} \tag{A.15}$$

where $\hat{E}[A \in \Delta]$ is the projection operator which projects onto the subset Δ of the spectrum of $\hat{A}$. Now the problem with this construction is that the set of all projection operators undergoes a logic which is not distributive, but the logic of the propositional language is distributive with respect to the logical connectives $\vee$ and $\wedge$. Therefore, such a representation will not work. To solve this problem we need to introduce a higher order language which we will examine in the next section.

A.3 The Higher Order Type Language *l*

We will now define a more complex language called higher order type language and denoted by l. Such a language consists of a set of symbols and terms.

Symbols

1. A collection of "sorts" or "types". If $T_1, T_2, \ldots, T_n$, $n \geq 1$, are type symbols, then so is $T_1 \times T_2 \times \cdots \times T_n$. If $n = 0$ then $T_1 \times T_2 \times \cdots \times T_n = 1$.
2. If T is a type symbol, then[1] so is PT.
3. Given any type T there are countable many variables of type T.
4. There is a special symbol $*$.
5. A set of function symbols for each pair of type symbols, together with a map which assigns to each function its type. This assignment consists of a finite, non-empty list of types. For example, if we have the pair of type symbols (T_1, T_2), the associated function symbol will be $F_l(T_1, T_2)$. An element $f \in F_l(T_1, T_2)$ has type T_1, T_2. This is indicated by writing $f : T_1 \to T_2$.

[1] PT indicates the collection of all subobjects of T.

6. A set of relation symbols R_i together with a map which assigns the type of the arguments of the relation. This consists of a list of types. For example, a relation taking an argument $x_1 \in T_1$ of type T_1 and an argument $x_2 \in T_2$ of type T_2 is denoted as $R = R(x_1, x_2) \subseteq T_1 \times T_2$.

Terms

1. The variables of type T are terms of type T, $\forall T$.
2. The symbol $*$ is a term of type 1.
3. A term of type Ω is called a formula. If the formula has no free variables, then we call it a sentence.
4. Given a function symbol $f : T_1 \to T_2$ and a term t of type T_1, then $f(t)$ is a term of type T_2.
5. Given $t_1, t_2, \dots, t_n$ which are terms of type $T_1, T_2, \dots, T_n$, respectively, then $\langle t_1, t_2, \dots, t_n \rangle$ is a term of type $T_1 \times T_2 \times \cdots \times T_n$.
6. If x is a term of type $T_1 \times T_2 \times \cdots \times T_n$, then for $1 \leq i \leq n$, x_i is a term of type T_i.
7. If ω is a term of type Ω and α is a variable of type T, then $\{\alpha | \omega\}$ is a term of type PT.
8. If x_1, x_2 are terms of the same type, then $x_1 = x_2$ is a term of type Ω.
9. If x_1, x_2 are terms of type T and PT respectively, then $x_1 \in x_2$ is a term of type Ω.
10. If x_1, x_2 are terms of type PT and PPT respectively, then $x_1 \in x_2$ is a term of type Ω.
11. If x_1, x_2 are both terms of type PT, then $x_1 \subseteq x_2$ is a term of type Ω.

The entire set of formulas in the language l are defined, recursively, through repeated applications of formation rules, which are the analogues of the standard logical connectives. In particular, we have *atomic formulas* and *composite formulas*. The former are:

1. The set of relation symbols.
2. Equality terms defined above.
3. Truth $\top$ is an atomic formula with no free variables.
4. False $\bot$ is an atomic formula with no of free variables.

We can now build more complicated formulas through the use of the logical connectives $\vee$, $\wedge$, $\Rightarrow$ and $\neg$. These are the *composite formulas*:

1. Given two formulas α and β, then $\alpha \vee \beta$ is a formula for which, the set of free variables is defined to be the union of the free variables in α and β.
2. Given two formulas α and β, then $\alpha \wedge \beta$ is a formula for which, the set of free variables is defined to be the union of the free variables in α and β.
3. Given a formula α its negation $\neg\alpha$ is still a formula with the same number of free variables.
4. Given two formulas α and β, then $\alpha \Rightarrow \beta$ is a formula with free variables given by the union of the free variables in α and β.

It is interesting to note that the logical operations just defined can actually be expressed in terms of the primitive symbols as follows:

1. $\top := (* = *)$.
2. $\alpha \wedge \beta := (\langle \alpha, \beta \rangle = \langle \top, \top \rangle) = \langle * = *, * = * \rangle$.
3. $\alpha \Leftrightarrow \beta := (\alpha = \beta)$.
4. $\alpha \Rightarrow \beta := ((\alpha \wedge \beta) \Leftrightarrow \alpha) := (\langle \alpha, \beta \rangle = \langle \top, \top \rangle = \alpha)$.
5. $\forall x \alpha := (\{x : \alpha\} = \{x : \top\})$.[2]
6. $\perp := \forall w w = (\{w : w\} = \{w : \top\})$.
7. $\neg \alpha := \alpha \Rightarrow \perp$.
8. $\alpha \vee \beta := \forall w[(\alpha \Rightarrow w \wedge \beta \Rightarrow w) \Rightarrow w]$.
9. $\exists x \alpha := \forall w[\forall x(\alpha \Rightarrow w) \Rightarrow w]$.[3]

A.4 Representation of *l* in a Topos

We now want to show how a representation of the first order language l takes place in a topos. The main idea is that of identifying each of the terms in l an arrow in a topos. In particular we have:

Definition A.2 Given a topos τ the interpretation/representation (M) of the language l in τ consists of the following associations:

1. To each type $T \in l$ an object $T^{\tau_M} \in \tau$.
2. To each relation symbol $R \subseteq T_1 \times T_2 \times \cdots \times T_n$ a sub-object $R^{\tau_M} \subseteq T_1^{\tau_M} \times T_2^{\tau_M} \times \cdots \times T_n^{\tau_M}$.
3. To each function symbol $f : T_1 \times T_2 \times \cdots \times T_n \to X$ a τ-arrow $f^{\tau_M} : T_1^{\tau_M} \times T_2^{\tau_M} \times \cdots \times T_n^{\tau_M} \to X$.
4. To each constant c of type T a τ-arrow $c^{\tau_M} : 1^{\tau_M} \to T^{\tau_M}$.
5. To each variabe x of type T a τ-arrow $x^{\tau_M} : T^{\tau_M} \to T^{\tau_M}$.
6. The symbol Ω is represented by the sub-object classifier Ω^{τ_M}.
7. The symbol 1 is represented by the terminal object 1^{τ_M}.

Now that we understand how the basic symbols of the abstract language l are represented in a topos, we can proceed to understand also how the various terms and formulas are represented. Needless to say, these are all defined in a recursive manner.

Given a term $t(x_1, x_2, \ldots, x_n)$ of type Y with free variables x_i of type T_i, i.e. $t(x_1, x_2, \ldots, x_n) : T_1 \times \cdots \times T_n \to Y$, then the representative in a topos of this term

[2] $\forall x \alpha$ means:“ for all x with the property α”, while $\{x : y\}$ indicates the set of all x, such that y

[3] $\exists x \alpha$ means:“ there exists an x with the property α”.

would be a τ-map

$$t(x_1, x_2, \ldots, x_n) : T_1^{\tau_M} \times \cdots \times T_n^{\tau_M} \to Y^{\tau_M}. \tag{A.16}$$

Formulas in the language are interpreted as terms of type Ω. In the topos τ this type Ω is identified with the sub-object classifier Ω^{τ_M}. In particular, a term of type Ω of the form $\phi(t_1, t_2, \ldots, t_n)$ with free variables t_i of type T_i is represented by an arrow

$$\phi(t_1, t_2, \ldots, t_n)^{\tau_M} : T_1^{\tau_M} \times \cdots \times T_n^{\tau_M} \to \Omega^{\tau_M}.$$

On the other hand, a term ϕ of type Ω with no free variables is represented by a global element $\phi : 1^{\tau_M} \to \Omega^{\tau_M}$. As we will see, these arrows will represent the truth values.

The reason why, in a topos, formulas are identified with arrows with codomain Ω rests on the fact that sub-objects, of a given object in a topos, are in bijective correspondence with maps from that object to the sub-object classifier. In fact, by construction, formulas single out sub-objects of a given object X in terms of a particular relation which they satisfy, i.e. they define elements of $Sub(X)$. Such sub-objects are in bijective correspondence with maps $X \to \Omega$.

In particular, given a formula $\phi(x_1, \ldots, x_n)$ with free variables x_i of type T_i, which in the language l is associated with the subset $\{x_i|\phi\} \subseteq \prod_i T_i$, we obtain the topos representation

$$\{(x_1, \ldots, x_n)|\phi\}^{\tau_M} \subseteq T_1^{\tau_M} \times \cdots \times T_n^{\tau_M} \tag{A.17}$$

which, through the *Omega Axiom* (see Axion 8.4), gets identified with the map

$$\{(x_1, \ldots, x_n)|\phi\}^{\tau_M} \to T_1^{\tau_M} \times \cdots \times T_n^{\tau_M} \xrightarrow{\chi_{\{(x_1,\ldots,x_n)|\phi\}^{\tau_M}}} \Omega^{\tau_M}. \tag{A.18}$$

To understand how formulas are represented in a topos τ, let us consider the formula stating that two terms are the same, i.e. $t(x_1, x_2, \ldots, x_n) = t'(x_1, x_2, \ldots, x_n)$. The representation of such a formula in a topos τ is identified with the equalizer of the two τ-arrows representing the terms $t(x_1, x_2, \ldots, x_n)$ and $t'(x_1, x_2, \ldots, x_n)$. In particular, we have

$$\begin{tikzcd} \{x_1, x_2, \ldots, x_n|t = t'\}^{\tau_M} \arrow[r, tail] & T_1^{\tau_M} \times \cdots \times T_n^{\tau_M} \arrow[r, shift left, "t^{\tau_M}"] \arrow[r, shift right, "t'^{\tau_M}"'] & Y^{\tau_M} \end{tikzcd}$$

Instead, if we consider a relation $R(t_1, \ldots, t_n)$ of terms t_i of type Y_i with variables x_j of type T_j, then the formula pertaining this relation $\{(x_1 \ldots x_n)|R(t_1, \ldots, t_n)\}$ is represented, in τ, by pulling back the sub-object $R^{\tau_M} \subseteq Y_1^{\tau_M} \times \cdots \times Y_n^{\tau_M}$ (representing the relation $R(t_1, \ldots, t_n)$) along the term arrow $\langle t_1^{\tau_M}, \ldots, t_n^{\tau_M}\rangle : T_1^{\tau_M} \times \cdots \times T_n^{\tau_M} \to$

$Y_1^{\tau M} \times \cdots \times Y_n^{\tau M}$:

$$\begin{tikzcd}
\{(x_1, x_2, \ldots, x_n) | R(t_1, \ldots, t_n)\}^{\tau M} \arrow[r] \arrow[d] & R^{\tau M} \arrow[d] \\
T_1^{\tau M} \times \cdots \times T_n^{\tau M} \arrow[r, "\langle t_1^{\tau M}, \ldots, t_n^{\tau M} \rangle"] & Y_1^{\tau M} \times \cdots \times Y_n^{\tau M}
\end{tikzcd}$$

The atomic formulas meaning truth and false ($\top$ and $\perp$, respectively) will be represented in a topos τ by the greatest and lowest elements of the Heyting algebra of the sub-objects of any object in the topos. For example we have that

$$\{x_1 \ldots x_n | \top\}^{\tau M} = T_1^{\tau M} \times T_2^{\tau M} \times \cdots \times T_n^{\tau M} \tag{A.19}$$

$$\{x_1 \ldots x_n | \perp\}^{\tau M} = \emptyset^{\tau M}. \tag{A.20}$$

So far we have established how to represent formulas in a topos. Next, we will explain how to represent logical connectives between formulas in a topos. In particular, given a collection of formulas represented as sub-objects of the type object $\prod_i T_i$, the logical connectives between these are represented by the corresponding operations in the Heyting algebra of sub-objects of the object $\prod_i T_i^{\tau M}$ in τ. Since we are dealing with sub-objects, we can also represent the logical connectives in terms of τ-arrows with codomain Ω as follows: For example, consider two formulas ϕ, ρ, of type Ω with free variables x_1 and x_2 of type $T_1^{\tau M}$ and $T_2^{\tau M}$, respectively. The conjunction $\phi \wedge \rho$ is represented by the arrow

$$\phi \wedge \rho : T_1^{\tau M} \times T_2^{\tau M} \xrightarrow{\langle \phi, \rho \rangle} \Omega^{\tau M} \times \Omega^{\tau M} \xrightarrow{\wedge} \Omega^{\tau M}. \tag{A.21}$$

Similarly, we have

$$\phi \vee \rho : T_1^{\tau M} \times T_2^{\tau M} \xrightarrow{\langle \phi, \rho \rangle} \Omega^{\tau M} \times \Omega^{\tau M} \xrightarrow{\vee} \Omega^{\tau M} \tag{A.22}$$

$$\phi \Rightarrow \rho : T_1^{\tau M} \times T_2^{\tau M} \xrightarrow{\langle \phi, \rho \rangle} \Omega^{\tau M} \times \Omega^{\tau M} \xrightarrow{\Rightarrow} \Omega^{\tau M} \tag{A.23}$$

$$\neg \rho : T_2^{\tau M} \xrightarrow{\rho} \Omega^{\tau M} \xrightarrow{\neg} \Omega^{\tau M}. \tag{A.24}$$

We now give two very important notions: that of a *theory in l* and that of a *model of l*.

Definition A.3 Given a language l, a theory $\mathcal{T}$ in l is a set of formulas which are called the axioms of $\mathcal{T}$.

Definition A.4 A model of a theory is a representation M in which all the axioms of $\mathcal{T}$ are valid. Such axioms are, then, represented by the arrow true : $1 \to \Omega$.

An example of this is given by the theory of abelian groups which can be seen as model of a theory in a given language. The language required will only contain one type of elements G, no relations, two function symbols

$$+ : G \times G \to G \tag{A.25}$$

$$- : G \to G \tag{A.26}$$

and a constant 0. A representation of this language, which will lead us to the theory of groups, will be defined in the topos **Sets**. Such a representation of G will be identified as a set G^M, on which the function symbols act upon:

$$+^M : G^M \times G^M \to G^M \tag{A.27}$$

$$\langle g_1, g_2 \rangle \mapsto g_1 g_2 \tag{A.28}$$

and

$$-^M : G^M \to G^M \tag{A.29}$$

$$g \mapsto -g. \tag{A.30}$$

The constant 0 will be an element $0^M \in G^M$.

Such a representation will be a model for the theory of abelian groups if the function symbols satisfy the axioms of abelian groups, i.e. if the following should hold in the representation M:

$$(g_1 + g_2) + g_3 = g_1 + (g_2 + g_3) \tag{A.31}$$

$$g_1 + g_2 = g_2 + g_1 \tag{A.32}$$

$$g_1 + 0 = g_1 \tag{A.33}$$

$$g_1 + (-g_1) = 0. \tag{A.34}$$

Given two models M and M' of a theory $\mathcal{T}$ in a language l, we say that these two models are homomorphic if there is a homomorphism of the respective interpretations of the model, i.e. for each symbol type X in l, these maps are homomorphisms:

$$H_X : X^M \to X^{M'} \tag{A.35}$$

where X^M and $X^{M'}$ are the representations of the symbol type X of l in the representation M and M', respectively. Such a map is called a homomorphism if it respects all relation symbols, function symbols and constants.

In the example of abelian groups, model homomorphisms would simply be group homomorphisms.

The definition of homomorphic representations gives rise to a category $\mathcal{I}$, whose objects are all possible representations of a given language l in a topos τ, and whose morphisms are the above mentioned homomorphisms of representations. Given such a category, each theory $\mathcal{T}$ gives rise to a full subcategory of $\mathcal{I}$ called $Mod(\mathcal{T}, \tau)$, whose objects are models of the theory $\mathcal{T}$ in the topos τ, and whose morphisms are homomorphisms of models.

In this section, we have seen how, given a first order type language l, it is possible to represent such a language in a topos τ. However, interestingly enough, the converse is also true, namely: given a topos τ, it has associated to it an internal first order language l, which enables one to reason about τ in a set theoretic way, i.e. using the notion of elements.

Definition A.5 Given a topos τ, its internal language $l(\tau)$ has a type symbol $\ulcorner A \urcorner$ for each object $A \in \tau$, a function symbol $\ulcorner f \urcorner : \ulcorner A_1 \urcorner \times \ulcorner A_2 \urcorner \times \cdots \times \ulcorner A_n \urcorner \rightarrow \ulcorner B \urcorner$ for each map $f : A_1 \times A_2 \times \cdots \times A_n \rightarrow B$ in τ and a relation $\ulcorner R \urcorner \subseteq \ulcorner A_1 \urcorner \times \ulcorner A_2 \urcorner \times \cdots \times \ulcorner A_n \urcorner$ for each sub-object $R \subseteq A_1 \times A_2 \times \cdots \times A_n$ in τ.

A.5 A Language *l* for a Theory of Physics and Its Representation in a Topos τ

We will now try to construct a physics theory for a system S. The construction of such a theory is defined by an interplay between a language $l(S)$, associated to the system S, a topos and the representation of the theory in the topos. In particular, we can say that a theory of the system S is defined by choosing a representation/model, M, of the language $l(S)$ in a topos τ_M. The choice of both topos and representation depend on the kind of theory being used, i.e. if it is classical or quantum theory.

As we have seen above, since each topos τ has an internal language $l(\tau)$ associated to it, constructing a theory of physics consists in translating the language, $l(S)$, of the system to the local language $l(\tau)$ of the topos.

As a first step in constructing a theory of physics we need to specify, exactly, what $l(S)$ is. In particular we need to analyse which primitive type terms and formulas should be present in $l(S)$ for it to be a language that will enable us to talk about the physical system S.

A.5.1 The Language $l(S)$ of a System S

The minimum set of type symbols and formulas, which are needed for a language to be used as a language to talk about a physical system S, are the following:

1. *The state space object and the quantity value object*. These objects are represented in $l(S)$ by the ground type symbols Σ and $\mathcal{R}$.
2. *Physical quantities*. Given a physical quantity A, it is standard practice to represent such a quantity in terms of a function from the state space to the quantity value object. Thus, we require $l(S)$ to contain the set function symbols $F_{l(S)}(\Sigma, \mathcal{R})$ of signature $\Sigma \rightarrow \mathcal{R}$, such that the physical quantity is $A : \Sigma \rightarrow \mathcal{R}$.

3. *Values*. We would like to have values of physical quantities. These are defined in $l(S)$ as terms of type $\mathcal{R}$ with free variables s of type Σ, i.e. they are the terms $A(s) \in \mathcal{R}$, where $A : \Sigma \to \mathcal{R} \in F_{l(S)}(\Sigma, \mathcal{R})$.
4. *Propositions*. Imagine we would like to talk about collections of states of the system with a particular property. Such a collection is represented in terms of sub-objects of the state space, which comprises the states with that particular property in question. Thus we have terms $Q = \{s|A(s) \in \Delta\}$ which are of type $P\Sigma$ with a free variable Δ of type $P\mathcal{R}$.
5. *Truth values*. We generally would like to talk about values of physical quantities for a given state of the system, thus we require the presence of formulas of the type $A(s) \in \Delta$, where Δ is a variable of type $P\mathcal{R}$ and s is a variable of type Σ. Such a formula is a term of type Ω.

   A formula w with no free variables, called a sentence, is a special element of Ω which is represented, in a topos, by a global element of Ω, i.e.

$$[w] : 1 \to \Omega. \tag{A.36}$$

   These, as we will see later on, will represent truth values for propositions about the system.
6. *States*. There are three options for describing a state, which we will analyse separately.
   (i) *Microstate option*. The microstate option is the one used in classical physics where a state is identified with an element of the state space. Hence in the context of the language $l(S)$, a micro-state is a term t of type Σ, i.e. $t \in \Sigma$. To understand how the micro-state option is utilised to evaluate proposition consider the term $A(s) \in \Delta$, this is a term of type $\underline{\Omega}$ with free variables s and Δ of type $\underline{\Sigma}$ and $P(\mathcal{R})$, respectively. On the other hand $\{s|A(s) \in \Delta\}$ is a term of type $P(\underline{\Sigma})$ with free variable Δ of type $P(\mathcal{R})$. Given a state $t \in \underline{\Sigma}$ we can then form a term of type $\underline{\Omega}$ as follows: $t \in \{s|A(s) \in \Delta\}$. This term has free variables t and Δ of type Σ and $P\mathcal{R}$, respectively.

   Intuitively, $A(t) \in \Delta$ represents the proposition stating: "the value of A, given the state t, lies in the range Δ". However semantically[4] we have the following equivalence

$$t \in \left\{s|A(s) \in \Delta\right\} \quad \Leftrightarrow \quad A(t) \in \Delta. \tag{A.37}$$

   Therefore the proposition "the value of A, given the state t, lies in the range Δ" becomes the term $A(t) \in \Delta$ of type Ω with free variable $t \in \Sigma$ and $\Delta \subseteq \mathbb{R}$.
   (ii) *Pseudo-state option*. This method consists in defining a term $\mathfrak{w}$ of type $P(\Sigma)$. Then, given the term $\{s|A(s) \in \Delta\}$, which is of type $P\Sigma$ with a

[4]Note that there are two distinct notions of equivalence: (i) syntactical (ii) semantical. The first one is defined in terms of inference rules as discussed in Sect. A.6, while two propositions are semantically equivalence whenever, in each topos τ, they are represented by the same element in $\underline{\Omega}_{\tau}$.

free variable Δ of type $P\mathcal{R}$, we want to know whether the elements in $\mathfrak{w}$ have the property $A(s) \in \Delta$, i.e. we want to know whether the proposition $A(s) \in \Delta$ is true, given the pseudo-state $\mathfrak{w}$. To this end we need to check the assertion

$$\mathfrak{w} \subseteq \{s|A(s) \in \Delta\}. \tag{A.38}$$

This is a term of type Ω.

(iii) *Truth object option*. This method consists in defining a term $\mathbb{T}$ of type $P(P(\Sigma))$. The simplest choice is a variable of type $P(P(\Sigma))$ defined as

$$\mathbb{T} : P\big(P(\Sigma)\big) \to P\big(P(\Sigma)\big). \tag{A.39}$$

A term of type Ω is then obtained by

$$\{s|A(s) \in \Delta\} \in \mathbb{T} \tag{A.40}$$

which has as free variable $\Delta \in P(\mathcal{R})$ and whatever free variables are contained in $\mathbb{T}$.

Intuitively we can think of $\mathbb{T}$ as a collection of subsets of the state space that have a particular property which we know to be true. Then we consider another subset of the state space, namely $Q \subseteq S$ and we would like to know if the collection of states in Q have the property $A(s) \in \Delta$. Since we know that there is a $\mathbb{T}$ to which all collection of sets of states with the property $A(s) \in \Delta$ belong, we simply check if $\{s|A(s) \in \Delta\} \in \mathbb{T}$.

7. Any axioms added to the language have to be represented by the arrow *true* : $1 \to \Omega$.

A.5.2 Representation of $l(S)$ in a Topos

Given a topos τ with representation M, we now want to know how $l(S)$ is represented in τ.

1. *State space and quantity value object*. The objects Σ and $\mathcal{R}$ are represented by the objects $\Sigma^{\tau M}$ and $\mathcal{R}^{\tau M}$ in τ, which take the role of the state object and the quantity value object.
2. *Physical quantities*. Physical quantities are defined in terms of τ-arrows between the τ-objects $\Sigma^{\tau M}$ and $\mathcal{R}^{\tau M}$.

   We will generally require the representation to be faithful, i.e. the map $A \mapsto A^{\tau M}$ is one-to-one.
3. *Values*. Values are represented in τ by terms of type $\mathcal{R}^{\tau M}$, i.e. $A^{\tau M}(s) \in \mathcal{R}^{\tau M}$ where $A^{\tau M} : \Sigma^{\tau M} \to \mathcal{R}^{\tau M}$.
4. *Truth values*. A formula $A(s) \in \Delta$ is a term of type Ω, thus it is represented in a topos τ by an arrow

$$\big[A(s) \in \Delta\big]^{\tau M} : \Sigma^{\tau M} \times P\mathcal{R}^{\tau M} \to \Omega^{\tau M}.$$

Such an arrow gets factored as follows:

$$[A(s) \in \Delta]^{\tau_M} = e_{\mathcal{R}^{\tau_M}} \circ \langle [A(s)]^{\tau_M}, [\Delta]^{\tau_M} \rangle \tag{A.41}$$

where $e_{\mathcal{R}^{\tau_M}} : \mathcal{R}^{\tau_M} \times P\mathcal{R}^{\tau_M} \to \Omega^{\tau_M}$ is the evaluation map, $[A(s)]^{\tau_M} : \Sigma^{\tau_M} \to \mathcal{R}^{\tau_M}$ is the arrow representing the physical quantity A and $[\Delta]^{\tau_M} : P\mathcal{R}^{\tau_M} \to P\mathcal{R}^{\tau_M}$ is simply the identity arrow. Putting the two results together we have

$$\Sigma^{\tau_M} \times P\mathcal{R}^{\tau_M} \xrightarrow{[A(s)]^{\tau_M} \times [\Delta]^{\tau_M}} \mathcal{R}^{\tau_M} \times P\mathcal{R}^{\tau_M} \xrightarrow{e_{\mathcal{R}^{\tau_M}}} \Omega^{\tau_M}. \tag{A.42}$$

Truth values are terms of type Ω with no free variables. Hence in the topos τ, they will be represented by elements of the sub-object classifier Ω^{τ_M}, i.e. global elements $\gamma^{\tau_M} : 1^{\tau_M} \to \Omega^{\tau_M}$, $\gamma^{\tau_M} \in \Gamma\Omega^{\tau_M}$.

5. *Propositions*. A proposition is a term of type $P(\Sigma)$, hence in a topos it will be defined as an element in $P(\Sigma^{\tau_M})$. In particular, consider a term of type $P(\Sigma^{\tau_M})$ with free variable Δ of type $P(\mathcal{R}^{\tau_M})$. In a topos this is represented by an arrow

$$\big[\{s|A(s) \in \Delta\}\big]^{\tau_M} : P\mathcal{R}^{\tau_M} \to P\Sigma^{\tau_M}. \tag{A.43}$$

Using this term of type $P(\Sigma^{\tau_M})$, which is represented in τ by the arrow $[\Xi]^{\tau_M} : 1^{\tau_M} \to P(\mathcal{R}^{\tau_M})$, a proposition $A \in \Delta$ is represented as:

$$\big[\{s|A(s) \in \Delta\}\big]^{\tau_M} \circ [\Xi]^{\tau_M} : 1^{\tau_M} \to P\big(\Sigma^{\tau_M}\big). \tag{A.44}$$

6. *States*. We will now analyse how the different 'types' of states described above are represented in a topos.
   (a) *Micro-state option*. We have seen that a micro-state is essentially a term of type Σ, hence in a topos it is represented by a global element (if it exists) of Σ^{τ_M}, i.e.

$$s : 1^{\tau_M} \to \Sigma^{\tau_M}. \tag{A.45}$$

   Moreover, given a term of type $P(\mathcal{R})$, which is represented in τ by an arrow $[\Xi]_{\tau_M} : 1^{\tau_M} \to P(\mathcal{R}^{\tau_M})$ it is possible to define a map $\langle s, [\Xi]\rangle : 1^{\tau_M} \to \Sigma^{\tau_M} \times P(\mathcal{R}^{\tau_M})$, which, if combined with the arrow $[A(s) \in \Delta]^{\tau_M} : \Sigma^{\tau_M} \times P(\mathcal{R}^{\tau_M}) \to \Omega^{\tau_M}$ gives

$$1_{\tau_M} \xrightarrow{\langle s, [\Xi]_{\tau_M}\rangle} \Sigma^{\tau_M} \times P\big(\mathcal{R}^{\tau_M}\big) \xrightarrow{[A(s)\in\Delta]^{\tau_M}} \Omega^{\tau_M}. \tag{A.46}$$

   This is the global element of Ω^{τ_M} representing the truth value of the proposition $(A(s) \in \Delta)$.
   (b) *Psuedo-state object*. Pseudo-states are identified with terms of type $P(\Sigma)$, so in a topos they are represented by elements

$$\mathfrak{w}^{\tau_M} : 1^{\tau_M} \to P\big(\Sigma^{\tau_M}\big). \tag{A.47}$$

Given a proposition

$$\left[\left\{s|A(s)\in\Delta\right\}\right]^{\tau_M}\circ[\Xi]^{\tau_M}:1\to P\left(\Sigma^{\tau_M}\right) \tag{A.48}$$

we combine the two maps to give

$$\left(\mathfrak{w}^{\tau_M},\left[\left\{s|A(s)\in\Xi\right\}\right]^{\tau_M}\circ[\Delta]^{\tau_M}\right):1^{\tau_M}\to P\left(\Sigma^{\tau_M}\right)\times P\left(\Sigma^{\tau_M}\right). \tag{A.49}$$

Considering the arrow $[\mathfrak{w}\underline{\subseteq}[\{s|A(s)\in\Delta\}]^{\tau_M}]:P(\Sigma^{\tau_M})\times P(\Sigma^{\tau_M})\to\Omega^{\tau_M}$, which represents the term $(\mathfrak{w}\underline{\subseteq}\{s|A(s)\in\Delta\})$ of type Ω, we can define the truth value of the proposition (A.48) given the pseudo-state (A.47) as

$$1^{\tau_M}\xrightarrow{(\mathfrak{w}^{\tau_M},[\{s|A(s)\in\Delta\}]^{\tau_M}\circ[\Xi]^{\tau_M})}P\left(\Sigma^{\tau_M}\right)\times P\left(\Sigma^{\tau_M}\right)\xrightarrow{[\mathfrak{w}\underline{\subseteq}[\{s|A(s)\in\Delta\}]^{\tau_M}]}\Omega^{\tau_M}. \tag{A.50}$$

(c) *Truth object*. A truth object is a term $\mathbb{T}$ of type $P(P(\Sigma))$ such that, given the proposition $\{s|A(s)\in\Delta\}$, the term $(\{s|A(s)\in\Delta\}\in\mathbb{T})$ is of type Ω. Such a term has free variables Δ of type $P(\mathcal{R})$ and $\mathbb{T}$ of type $P(P(\Sigma))$. Therefore, its representation in a topos τ is

$$\left[\left\{s|A(s)\in\Delta\right\}\in\mathbb{T}\right]^{\tau_M}:P\left(\mathcal{R}^{\tau_M}\right)\times P\left(P\left(\Sigma^{\tau_M}\right)\right)\to\Omega^{\tau_M} \tag{A.51}$$

which can be factored as follows:

$$\left[\left\{s|A(s)\in\Delta\right\}\in\mathbb{T}\right]^{\tau_M}=e^{P(\Sigma_{\tau_M})}\circ\left(\left[\left\{s|A(s)\in\Delta\right\}\right]^{\tau_M}\times[\mathbb{T}]^{\tau_M}\right). \tag{A.52}$$

Here $e_{P(\Sigma^{\tau_M})}:P(\Sigma^{\tau_M})\times P(P(\Sigma^{\tau_M}))\to\Omega^{\tau_M}$ is the evaluation map and

$$\left[\left\{s|A(s)\in\Delta\right\}\right]^{\tau_M}:P\left(\mathcal{R}^{\tau_M}\right)\to P\left(\Sigma^{\tau_M}\right) \tag{A.53}$$

$$[\mathbb{T}]^{\tau_M}:P\left(P\left(\Sigma^{\tau_M}\right)\right)\xrightarrow{id}P\left(P\left(\Sigma^{\tau_M}\right)\right). \tag{A.54}$$

Given the above, the truth value of the proposition $A\in\Delta$ is represented as

$$v(A\in\Delta;\mathbb{T}):\left[\left\{s|A(s)\in\Delta\right\}\in\mathbb{T}\right]^{\tau_M}\circ\left\langle[\Delta]^{\tau_M},[\mathbb{T}]^{\tau_M}\right\rangle \tag{A.55}$$

where $\langle[\Delta^{\tau_M}],[\mathbb{T}^{\tau_M}]\rangle:1^{\tau_M}\to P(\mathcal{R}^{\tau_M})\times P(P(\Sigma^{\tau_M}))$.

For example in classical physics the topos in which we represent the language $l(S)$ is **Sets**, thus the truth object Ω is simply the set $\{0,1\}$. In this context we have

$$v(A\in\Delta;\mathbb{T}):=\left[\left\{s|A(s)\in\Delta\right\}\in\mathbb{T}^s\right]:P(\mathcal{R})\times P\left(P(\Sigma)\right)\to\{0,1\} \tag{A.56}$$

such that

$$v(A\in\Delta;\mathbb{T})(\Delta,\mathbb{T})=\begin{cases}1 & \text{if } \{s\in\Sigma|A(s)\in\Delta\}\in\mathbb{T}\\ 0 & \text{otherwise}\end{cases} \tag{A.57}$$

$$=\begin{cases}1 & \text{if } A^{-1}(\Delta)\in\mathbb{T}\\ 0 & \text{otherwise.}\end{cases} \tag{A.58}$$

.

A.6 Deductive System of Reasoning for First Order Logic

So far, we have defined the symbols and formation rules for the first order language l. However, in order to actually use l as a language that enables us to talk about things, we also require rules of inference. Such rules will allow us to derive true statements from other true statements.

In order to describe this better we need to introduce the notion of a sequent.

Definition A.6 Given two formulae ψ and ϕ a sequent is an expression $\psi \vdash_{\vec{x}} \phi$ which indicates that ϕ is a logical consequence of ψ in the context $\vec{x}$.[5]

What this means is that any assignment of values of the variables in $\vec{x}$ which makes ψ true, will also make ϕ true.

The deduction system will then be defined as a sequent calculus, i.e. a set of inference rules which will allow us to infer a sequent from other sequents. Symbolically, a rule of inference is written as follows:

$$\frac{\Gamma}{\psi \vdash_{\vec{x}} \phi} \tag{A.59}$$

which means that the sequent $\psi \vdash_{\vec{x}} \phi$ can be inferred by the collection of sequents Γ. We can also have a double inference as follows:

$$\frac{\Gamma}{\overline{\psi \vdash_{\vec{x}} \phi}}.$$

This can be read in both directions, thus it means that $\psi \vdash_{\vec{x}} \phi$ can be inferred from the collection of sequents Γ, but also that the collection of sequents Γ can be inferred from $\psi \vdash_{\vec{x}} \phi$.

We will now define a list of inference rules. In the following, the symbol Γ will represent a collection of sequents, the letters γ, β, α will represent formulae, the letters $\sigma, \tau, \ldots$ will represent terms of some type and $\alpha \cup \Gamma$ represent the collections of formulas in both Γ and the formula α.

Variables in a term can be either *free* or *bounded*. We say that a variable α of a term σ is *bounded* if it appears within a context of the form $\{\alpha \vdash_{\vec{x}} \sigma\}$, otherwise it will be called *free*. The inference rules are:

- *Thinning*

$$\frac{\beta \cap \Gamma \vdash_{\vec{x}} \alpha}{\overline{\Gamma \vdash_{\vec{x}} \alpha}}.$$

- *Cut*

$$\frac{\Gamma \vdash_{\vec{x}} \alpha, \alpha \cup \Gamma \vdash_{\vec{x}} \beta}{\overline{\Gamma \vdash_{\vec{x}} \beta}}.$$

[5]A context $\vec{x}$ is a list of distinct variables. When applied to a formula α it indicates that α has variables only within that context.

For any free variable of α free in Γ or β.

- *Substitution*

$$\begin{array}{c} \Gamma \vdash_{\vec{x}} \alpha \\ \hline\hline \Gamma(x/\sigma) \vdash_{\vec{x}} \alpha(x/\sigma) \end{array}$$

when σ is free in Γ and α. The term $\Gamma(x/\sigma)$ indicates the term obtained from Γ by substituting σ (which is a term of some type) for each free occurrence of x.

- *Extensionality*

$$\begin{array}{c} \Gamma \vdash_{\vec{x}} x \in \sigma \Leftrightarrow x \in \rho \\ \hline\hline \Gamma \vdash_{\vec{x}} \sigma = \rho \end{array}$$

where x is not free in either Γ, σ or ρ.

- *Equivalence*

$$\begin{array}{c} \alpha \cup \Gamma \vdash_{\vec{x}} \beta \quad \beta \cup \Gamma \vdash_{\vec{x}} \alpha \\ \hline\hline \Gamma \vdash_{\vec{x}} \alpha \Leftrightarrow \beta \end{array}.$$

- *Finite Conjunction*

The rules for finite conjunction are the following:

$$\alpha \vdash_{\vec{x}} \alpha = \top, \qquad \alpha \wedge \beta \vdash_{\vec{x}} \beta, \qquad \alpha \wedge \beta \vdash_{\vec{x}} \alpha. \tag{A.60}$$

Note that we have used part of the definition of the logical connective '*if then*'.

A consequence of these rules is

$$\frac{\alpha \vdash_{\vec{x}} \beta \;\; \alpha \vdash_{\vec{x}} \gamma}{\alpha \vdash_{\vec{x}} \gamma \wedge \beta}. \tag{A.61}$$

Proof

$$\dfrac{\alpha \vdash_{\vec{x}} \beta \qquad \dfrac{\alpha \vdash_{\vec{x}} \gamma \qquad \dfrac{\dfrac{}{\beta \vdash_{\vec{x}} \beta = \top}\,(3) \qquad \dfrac{\dfrac{}{\gamma \vdash_{\vec{x}} \gamma = \top}\,(1) \qquad \dfrac{}{\gamma = \top \cup \beta = \top \vdash_{\vec{x}} \gamma \wedge \beta}\,(2)}{\gamma \cup \beta = \top \vdash_{\vec{x}} \gamma \wedge \beta}\,(4)}{\gamma \cup \beta \vdash_{\vec{x}} \gamma \wedge \beta}\,(5)}{\alpha \cup \beta \vdash_{\vec{x}} \alpha \wedge \beta}\,(6)}{\alpha \vdash_{\vec{x}} \gamma \wedge \beta}\,(7)$$

$\square$

This proof should be read from top to bottom and consists, as one can see, of a finite collection of sequents called a finite *tree*, in which the bottom vertex represents the *conclusion* of the proof. All the sequents of the proof are correlated to each other in the following way:

1. A sequent belonging to a node[6] which has nodes above it is derived by applying a rule of inference to the sequents belonging to the above nodes.
2. Every top most node is either a basic axiom or a premise of the proof.

[6] A node is an inference step: $\frac{\Gamma_1}{\Gamma_2}$.

In the proof above we have that

$$\overline{\gamma \vdash_{\vec{x}} \gamma = \top}$$

is derived by the thinning axiom, the equivalence axiom and the axioms $\gamma \vdash_{\vec{x}} \gamma$ and $\vdash_{\vec{x}}$ true as follows:

Proof

$$\dfrac{\dfrac{\dfrac{\vdash_{\vec{x}} \text{true}}{\gamma \vdash_{\vec{x}} \top}\,(1)}{\gamma \cup \gamma \vdash_{\vec{x}} \top}\,(3) \qquad \dfrac{\overline{\gamma \vdash_{\vec{x}} \gamma}}{\top \cup \gamma \vdash_{\vec{x}} \gamma}\,(2)}{\gamma \vdash_{\vec{x}} \gamma = \top}\,(4)$$

□

Where the lines (1), (2) and (3) are an application of the thinning axiom, while line (4) is the application of the equivalence axiom where the equivalence $\alpha \Leftrightarrow \beta := \alpha = \beta$ was used.

Going back to the proof of the *conjunction axiom* the remaining lines are derived as follows:

(i) Line (2) is the definition of the logical connective $\wedge$.
(ii) All the other lines are derived from applications of the *cut axiom*.

It should be noted that it is also possible to form a more general version of the *conjunction axiom* by replacing the single sequent α by a collection of sequents Γ, thus the *conjunction Axiom* becomes:

$$\frac{\Gamma \vdash_{\vec{x}} \beta \quad \Gamma \vdash_{\vec{x}} \gamma}{\Gamma \vdash_{\vec{x}} \beta \wedge \gamma}\,.$$

- *Finite Disjunction*
  The rules for finite disjunction consist of the following axioms:

$$\bot \vdash_{\vec{x}} \alpha \;\; \alpha \vdash_{\vec{x}} \alpha \vee \beta \;\; \beta \vdash_{\vec{x}} \alpha \vee \beta \tag{A.62}$$

  and the following rule of inference:

$$\frac{\alpha \vdash_{\vec{x}} \gamma \quad \beta \vdash_{\vec{x}} \gamma}{\alpha \vee \beta \vdash_{\vec{x}} \gamma} \tag{A.63}$$

  whose generalisation is

$$\frac{\alpha \cup \Gamma \vdash_{\vec{x}} \gamma \quad \beta \cup \Gamma \vdash_{\vec{x}} \gamma}{\alpha \vee \beta \cup \Gamma \vdash_{\vec{x}} \gamma}\,.$$

- *Implication*
  For implication we have the double inference rule

$$\begin{array}{c} \beta \wedge \alpha \vdash_{\vec{x}} \gamma \\ \hline\hline \alpha \vdash_{\vec{x}} \beta \Rightarrow \gamma \end{array}\,.$$

Again the general form of which the above is a specification is

$$\frac{\beta \cup \Gamma \vdash_{\vec{x}} \gamma}{\overline{\Gamma \vdash_{\vec{x}} \beta \Rightarrow \gamma}}.$$

To see why that is the case we will prove the above generalisation, but only one way:

Proof

$$\dfrac{\dfrac{\dfrac{\beta \vdash_{\vec{x}} \beta}{\beta \cup \Gamma \vdash_{\vec{x}} \beta}\,(1)}{\beta \wedge \gamma \cup \Gamma \vdash_{\vec{x}} \beta}\,(2) \qquad \dfrac{\beta \cup \Gamma \vdash_{\vec{x}} \beta \quad \beta \cup \Gamma \vdash_{\vec{x}} \gamma}{\beta \cup \Gamma \vdash_{\vec{x}} \beta \wedge \gamma}\,(3)}{\Gamma \vdash_{\vec{x}} \beta \Rightarrow \gamma}\,(4).$$

where in (4) we used the definition of implication given in Sect. A.3: $\alpha \Rightarrow \beta := (\alpha \wedge \beta) \Leftrightarrow \alpha$. □

- *Negation*
  For negation we only have one axiom

$$\bot \vdash_{\vec{x}} \alpha \tag{A.64}$$

  while the inference rules are

$$\frac{(\alpha \cup \Gamma) \vdash_{\vec{x}} \bot}{\Gamma \vdash_{\vec{x}} \neg\alpha}$$

  and

$$\frac{\Gamma \vdash_{\vec{x}} \alpha}{(\neg\alpha \cup \Gamma) \vdash_{\vec{x}} \bot}.$$

- *Universal Quantification*
  We have the following double inference rule

$$\frac{\alpha \vdash_{\vec{x}y} \beta}{\overline{\alpha \vdash_{\vec{x}} \forall y\beta}}$$

  where y is not free in either β or α.
  Again the generalisation is

$$\frac{\Gamma \vdash_{\vec{x}y} \beta}{\overline{\Gamma \vdash_{\vec{x}} \forall y\beta}}.$$

- *Existential Quantifier*
  We have the double inference rule

$$\frac{\alpha \vdash_{\vec{x}y} \beta}{\overline{(\exists y)\alpha \vdash_{\vec{x}} \beta}}$$

where y is a free variable in β.

Again the generalization would be

$$\frac{\alpha \cup \Gamma \vdash_{\vec{x}y} \beta}{\overline{(\exists y)\alpha \cup \Gamma \vdash_{\vec{x}} \beta}}\,.$$

- *Distributive Axiom*

$$\big(\alpha \wedge (\beta \vee \gamma)\big) \vdash_{\vec{x}} \big((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)\big). \tag{A.65}$$

- *Frobenius Axiom*

$$\big(\alpha \wedge (\exists y)\beta\big) \vdash_{\vec{x}} (\exists y)(\alpha \wedge \beta) \tag{A.66}$$

where $y \notin \vec{x}$.

- *Law of Excluded Middle*

$$\top \vdash_{\vec{x}} \alpha \vee \neg\alpha. \tag{A.67}$$

It should be noted that, for intuitionistic type of higer order languages the law of excluded middle does not hold. All the rest does.

With this we end our definition of higher order languages $\mathcal{L}$, which are comprised of a set of term types, a set of logical connectives and a set of rules of inference which determine the logic.

Appendix B
Worked out Examples

B.1 Category Theory

Example B.1 An iso arrow is always epic. In fact, consider an iso f such that $g \circ f = h \circ f$ ($f : a \to b$ and $g, h : b \to c$)

$$g = g \circ id_b = g \circ \left(f \circ f^{-1}\right) = (g \circ f) \circ f^{-1} = (h \circ f) \circ f^{-1}$$
$$= h \circ \left(f \circ f^{-1}\right) = h$$

therefore f is right cancellable.

Example B.1 Given a category $\mathcal{C}$, isomorphism is an equivalence relation on $\mathcal{C}$. This is defined as follows:

Given any two objects $A, B \in \mathcal{C}$ we say that they are equivalent iff there exists an iso $i : A \to B$. This is a well defined equivalence relation since it satisfies the following properties:

1. Reflexivity. $\forall A \in \mathcal{C}$, $id_A : A \to A$ is an iso since $id_A \circ id_A = id_A$.
2. Transitive. Assume that we have two isos $i_1 : A \to B$ and $i_2 : B \to C$. We define the composite $h := i_2 \circ i_1$. For this to be an iso it has to have an inverse.

$$\begin{aligned}(i_2 \circ i_1) \circ (i_2 \circ i_1)^{-1} = i_2 \circ i_1 \circ i_1^{-1} \circ i_2^{-1} &\overset{\text{associativity}}{=} i_2 \circ id_B \circ i_2^{-1} \\ &\overset{\text{identity}}{=} i_2 \circ i_2^{-1} = id_C. \end{aligned} \tag{B.1}$$

   Similarly we can show that $(i_1 \circ i_2)^{-1} \circ (i_1 \circ i_2) = id_B$
3. Symmetry. If we have an iso $i : A \to B$ then there exists a unique inverse $i^{-1} : B \to A$. Since $i^{-1} : B \to A$ is unique with inverse i then i^{-1} is the desired iso.

Example B.2 In any category the following are true:

1. $g \circ f$ is monic if both g and f are monic.
2. If $g \circ f$ is monic then so is f.

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In fact we have that

(1) If $f : A \to B$ and $g : B \to C$ are monic then, given two arrows $h, j : D \to A$ such that $(g \circ f) \circ h = (g \circ f) \circ j$ we have:

$$\begin{aligned} &(g \circ f) \circ h = (g \circ f) \circ j \\ &\xrightarrow{\text{associativity}} g \circ (f \circ h) = g \circ (f \circ j) \\ &\xrightarrow{\text{monic}} f \circ h = f \circ j \\ &\xrightarrow{\text{monic}} h = j. \end{aligned} \tag{B.2}$$

It follows that $(g \circ f)$ is monic.

(2) Given $f : A \to B$, and $h, k : D \to A$, assume that $f \circ h = f \circ k$. Then consider $g : B \to C$ such that $(g \circ f)$ is monic. It follows that

$$(g \circ f) \circ h = g \circ (f \circ h) = g \circ (f \circ k) = (g \circ f) \circ k \tag{B.3}$$

implies that $h = k$.

Example B.3 Given any category $\mathcal{C}^{op}$, a map $f \in \mathcal{C}^{op}$ is monic in $\mathcal{C}^{op}$ if and only if it is epic in $\mathcal{C}$. In fact $f : B \to A$ is monic in $\mathcal{C}^{op}$ iff for all $g, h : C \to B$ in $\mathcal{C}^{op}$ then

$$f \circ g = f \circ h \quad \text{implies that} \quad g = h. \tag{B.4}$$

However from the definition of dual category the above holds iff for all $g, h : B \to C$ in $\mathcal{C}$ we have

$$g \circ f = h \circ f \quad \text{implies} \quad g = h \tag{B.5}$$

where now $f : A \to B$. However, this is true iff f is epic in $\mathcal{C}$. We then say that monic and epic are dual notions.

Example B.4 The category $(\mathbb{R}, \leq)^{op}$ has as:

1. Objects: $r \in \mathbb{R}$.
2. Morphisms: given any two elements $x, y \in (\mathbb{R}, \leq)$ such that $x \leq y$ then there exists a unique arrow $f : y \to x$ in $(\mathbb{R}, \leq)^{op}$.

   These morphisms undergo the following properties:

   (i) Composition: if $x \leq y$ and $y \leq z$ then in $(\mathbb{R}, \leq)$, $x \leq z$. This implies that there exist three maps in $(\mathbb{R}, \leq)^{op}$, namely $f : y \to x$, $g : z \to y$ and $h : z \to x$ such that $h := f \circ g$.
   (ii) Associativity: given that[1] $f' : x \leq y$, $g' : y \leq z$ and $k' : z \leq w$ in $(\mathbb{R}, \leq)$ we then have that $k' \circ (g' \circ f') = (k' \circ g') \circ f'$, therefore in $(\mathbb{R}, \leq)^{op}$ we have $f' \circ (g' \circ k') = (f' \circ g') \circ k'$.

[1] Here $x \leq y$ means that there is a map $f' : x \to y$ in $(\mathbb{R}, \leq)^{op}$ or equivalently a map $f : y \to x$ in $(\mathbb{R}, \leq)$.

(iii) Identity element: in $\mathbb{R}$ we have that $x \leq x$, by duality $i_x : x \to x$ is the identity morphism on x in $(\mathbb{R}, \leq)^{op}$.

Example B.5 Given a category $\mathcal{C}$ with initial object 0, the following hold:

(i) If $A \simeq 0$ (i.e. there is an iso map between them) then A is an initial object.
(ii) If there exists a monic arrow $f : A \to 0$, then f is an iso.

Proof (i) An initial object 0 is such that, given any other object $A \in \mathcal{C}$ there exists one and only one map $i : 0 \to A$. If such an arrow i is iso, then $i^{-1} : A \to 0$ exists and is the unique inverse.

Now consider any other object $C \in \mathcal{C}$, we know that there exists a unique arrow $f : 0 \to C$. We then assume that we have two maps $h, g : A \to C$, we then obtain the following diagram

$$\begin{tikzcd} A \arrow[d, shift right, "g"'] \arrow[d, shift left, "h"] & 0 \arrow[l, "i"] \arrow[dl, "f"] \\ C & \end{tikzcd}$$

From the property of the initial object we have that

$$h \circ i = g \circ i = f. \tag{B.6}$$

However since i is iso, it is right cancellable, therefore $h = g$. This implies that given any object $C \in \mathcal{C}$ there exists one and only one map $f : A \to C$. Thus A is an initial object.

(ii) $f : A \to 0$ is monic. Since 0 is the initial object we have the unique arrow $i : 0 \to A$. From the property of 0 being an initial object it follows that $f \circ i = id_0$ (since there is one and only one arrow from any object to 0, including from 0 to itself). On the other hand

$$f \circ (i \circ f) = f \circ id_C \quad \text{implies} \quad i \circ f = id_C \quad \text{(monic property).} \tag{B.7}$$

□

Example B.6 Given a category $\mathcal{C}$ with terminal object 1, if there exists an arrow $g : 1 \to A$ with domain the terminal object, then g must be monic. In fact, if we assume that there exists an arrow $g : 1 \to A$, since 1 is the terminal object then, given any other object $A \in \mathcal{C}$, $h : A \to 1$ is unique, including $1 \to 1$. It follows that $h \circ g = id_1$. Now consider two maps $f, k : B \to 1$ such that $g \circ k = g \circ f$ we then have

$$f = id_1 \circ f = (h \circ g) \circ f = h \circ (g \circ f) = h \circ (g \circ k) = (h \circ g) \circ k = id_1 \circ k = k \tag{B.8}$$

which implies that g is monic.

Example B.7 Given two objects A, B in some category $\mathcal{C}$, consider the category $\mathbf{Pair}(A, B)$, whose objects are triplets (P, p_1, p_2) where $p_1 : P \to A$ and $p_2 : P \to B$. The morphisms in $\mathbf{Pair}(A, B)$ are maps

$$f : (P, p_1, p_2) \to (Q, q_1, q_2) \tag{B.9}$$

such that $f : P \to Q$ is a morphism in $\mathcal{C}$ and $q_1 \circ f = p_1$; $q_2 \circ f = p_2$, i.e. the following diagram commutes

$$\begin{tikzcd}
 & P \arrow[dl, "p_2"'] \arrow[d, "f"] \arrow[dr, "p_1"] & \\
A & Q \arrow[l, "q_1"] \arrow[r, "q_2"'] & B
\end{tikzcd}$$

We then have that

1. $\mathbf{Pair}(A, B)$ is a category.
2. $(A \times B, \pi_1, \pi_2)$ is a product if it is a terminal object in $\mathbf{Pair}(A, B)$.

Proof 1. To show that $\mathbf{Pair}(A, B)$ is a category, we need to show that the following properties hold:

(a) *Composition*. Given $f : (P, p_1, p_2) \to (Q, q_1, q_2)$ and $g : (Q, q_1, q_2) \to (S, s_1, s_2)$ we need to define composition. Thus we say that $g \circ f$ is the map which makes the following diagram commute:

$$\begin{tikzcd}
 & P \arrow[dl, "p_2"'] \arrow[d, "f"] \arrow[dr, "p_1"] & \\
A & Q \arrow[l, "q_1"] \arrow[r, "q_2"'] \arrow[d, "g"] & B \\
 & S \arrow[ul, "s_1"] \arrow[ur, "s_2"'] &
\end{tikzcd}$$

(b) *Identity morphism.* $Id_{(P,p_1,p_2)} : (P, p_1, p_2) \to (P, p_1, p_2)$

$$
\begin{tikzcd}
& P \arrow[dl, "p_2"'] \arrow[d, "id_P"] \arrow[dr, "p_1"] & \\
A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}
$$

(c) *Associativity*

$$
\begin{tikzcd}
& P \arrow[dl, "p_2"'] \arrow[d, "f"] \arrow[dr, "p_1"] & \\
A & Q \arrow[l, "q_1"] \arrow[r, "q_2"'] \arrow[d, "g"] & B \\
& S \arrow[ul, "s_1"] \arrow[ur, "s_2"'] &
\end{tikzcd}
$$

composed with

$$
\begin{tikzcd}
& S \arrow[dl, "s_2"'] \arrow[d, "h"] \arrow[dr, "s_1"] & \\
A & R \arrow[l, "r_1"] \arrow[r, "r_2"'] & B
\end{tikzcd}
$$

gives

$$\begin{tikzcd}
& P \arrow[dl, "p_2"'] \arrow[d, "f"] \arrow[dr, "p_1"] & \\
A & Q \arrow[l, "q_1"] \arrow[r, "q_2"'] \arrow[d, "g"] & B \\
& S \arrow[ul, "s_1"] \arrow[ur, "s_2"'] \arrow[d, "h"] & \\
& R \arrow[uul, "s_2"] \arrow[uur, "s_1"'] &
\end{tikzcd}$$

which is the same as

$$\begin{tikzcd}
& P \arrow[dl, "p_2"'] \arrow[d, "f"] \arrow[dr, "p_1"] & \\
A & Q \arrow[l, "q_1"] \arrow[r, "q_2"'] & B
\end{tikzcd}$$

composed with

$$\begin{tikzcd}
& Q \arrow[dl, "q_2"'] \arrow[d, "g"] \arrow[dr, "q_1"] & \\
A & S \arrow[l, "s_1"] \arrow[r, "s_2"'] \arrow[d, "h"] & B \\
& R \arrow[ul, "r_1"] \arrow[ur, "r_2"'] &
\end{tikzcd}$$

which gives

$$
\begin{tikzcd}
 & P \arrow[d, "f"] \arrow[ddl, "p_2"'] \arrow[ddr, "p_1"] & \\
 & Q \arrow[dl, "q_2"'] \arrow[d, "g"] \arrow[dr, "q_1"] & \\
A & S \arrow[l, "s_1"] \arrow[r, "s_2"'] \arrow[d, "h"] & B \\
 & R \arrow[ul, "r_1"] \arrow[ur, "r_2"'] &
\end{tikzcd}
$$

2. If $(A \times B, \pi_1, \pi_2)$ is a terminal object then, for all elements $(P, p_1, p_2) \in$ **Pair**(A, B) there exists one and only one arrow

$$(P, p_1, p_2) \to (A \times B, \pi_1, \pi_2) \tag{B.10}$$

which, by definition implies that the following diagram commutes:

$$
\begin{tikzcd}
 & P \arrow[dl, "p_2"'] \arrow[d, "f"] \arrow[dr, "p_1"] & \\
A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}
$$

But this is precisely the condition satisfied by the product. □

Example B.8 For any triple $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$ the following propositions are equivalent.

1. Given any triple $A \xleftarrow{f} C \xrightarrow{g} B$, there exists a unique morphism $\langle f, g \rangle : C \to A \times B$, such that

$$\pi_1 \circ \langle f, g \rangle = f \quad \text{and} \quad \pi_2 \circ \langle f, g \rangle = g. \tag{B.11}$$

2. For any triple $A \stackrel{f}{\leftarrow} C \stackrel{g}{\rightarrow} B$ there exists a morphism $\langle f, g \rangle : C \to A \times B$, such that

$$\pi_1 \circ \langle f, g \rangle = f \quad \text{and} \quad \pi_2 \circ \langle f, g \rangle = g \tag{B.12}$$

and, moreover, for any $h : C \to A \times B$,

$$h = \langle \pi_1 \circ h, \pi_2 \circ h \rangle. \tag{B.13}$$

Proof (a) (1) $\Rightarrow$ (2). Given any $h : C \to A \times B$, we need to show that $h = \langle \pi_1 \circ h, \pi_2 \circ h \rangle$. From (1) we have that

$$A \xleftarrow{\pi_1 \circ h} C \xrightarrow{\pi_2 \circ h} B. \tag{B.14}$$

Thus there exists a unique arrow $k : C \to A \times B$ such that

$$\pi_1 \circ k = \pi_1 \circ h \quad \text{and} \quad \pi_1 \circ k = \pi_1 \circ h. \tag{B.15}$$

Since this equation holds for $k := h$ and from (1) $k := \langle \pi_1 \circ h, \pi_2 \circ h \rangle$, it follows that $h = \langle \pi_1 \circ h, \pi_2 \circ h \rangle$.

(b) (2) $\Rightarrow$ (1). For any triple $A \stackrel{f}{\leftarrow} C \stackrel{g}{\rightarrow} B$, (2) implies that there exists a map $\langle f, g \rangle : C \to A \times B$ such that

$$\pi_1 \circ \langle f, g \rangle = f \quad \text{and} \quad \pi_2 \circ \langle f, g \rangle = g. \tag{B.16}$$

What remains to show is that $\langle f, g \rangle$ is unique. To this end let $k : C \to A \times B$ be such that

$$\pi_1 \circ k = f \quad \text{and} \quad \pi_2 \circ k = g. \tag{B.17}$$

Then, from (2) it follows that

$$k = \langle \pi_1 \circ k, \pi_2 \circ k \rangle = \langle f, g \rangle. \tag{B.18}$$

$\square$

Example B.9 Given the category $\mathcal{C}$:

$$0 \to 1, \qquad 1' \to 2 \tag{B.19}$$

and the category $\mathcal{D}$:

$$\begin{tikzcd}
0 \arrow[rr] \arrow[dr] & & 1 \arrow[dl] \\
& 2 &
\end{tikzcd}$$

we define the functor $F : \mathcal{C} \to \mathcal{D}$ as

$$\begin{aligned} F : 2+2 &\to 3 \\ 0 &\mapsto 0 \\ 1 &\mapsto 1 \\ 1' &\mapsto 1 \\ 2 &\mapsto 2. \end{aligned} \tag{B.20}$$

It follows that

$$F(f : 0 \to 1) := 0 \to 1; \qquad F\big(g : 1' \to 2\big) := 1 \to 2. \tag{B.21}$$

The only arrow which does not lie in the image of the F functor is $0 \to 2$ which is $0 \to 2$ which is the composite $F(g) \circ F(f)$. Thus the image of F is not a subcategory.

Example B.10 We will now define the *bi-variant Hom functor* $\mathcal{C}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to$ *Set* and show that it is a functor. A possible definition would be:

$$\begin{aligned} \mathcal{C}(-,-) : \mathcal{C}^{op} \times C &\to \mathit{Sets} \\ (A, B) &\mapsto \mathcal{C}(A, B) \\ (f, g) &\mapsto \mathcal{C}(f, g) \end{aligned} \tag{B.22}$$

where $f : C \to D$ and $g : E \to F$ are such that

$$(f, g) : (D, E) \to (C, F) \tag{B.23}$$

and

$$\begin{aligned} \mathcal{C}(f, g) : \mathcal{C}(D, E) &\to \mathcal{C}(C, F) \\ h &\to g \circ h \circ f. \end{aligned} \tag{B.24}$$

The requirements for $\mathcal{C}(-,-)$ to be a well defined functor are

$$\mathcal{C}(id_A, id_A) = id_{\mathcal{C}(A,A)} \tag{B.25}$$

which is trivially satisfied and

$$\mathcal{C}(g \circ f, g \circ f) = \mathcal{C}(f, g) \circ \mathcal{C}(f, g) \tag{B.26}$$

where

$$\begin{aligned} \mathcal{C}(g \circ f, g \circ f) : \mathcal{C}(C, A) &\to \mathcal{C}(A, C) \\ h &\mapsto g \circ f \circ h \circ g \circ f \end{aligned} \tag{B.27}$$

while

$$\begin{aligned}\mathcal{C}(f,g)\circ\mathcal{C}(f,g) : \mathcal{C}(C,A) \to \mathcal{C}(B,B) \to \mathcal{C}(A,C)\\ h \mapsto f\circ h\circ g \mapsto g\circ f\circ h\circ g\circ f.\end{aligned} \tag{B.28}$$

Thus (B.22) is a well defined functor.

Example B.11 The diagram

$$\begin{tikzcd}
A \arrow[r, "id_A"'] \arrow[d, "id_A"] & A \arrow[d, "f"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

is a pullback iff f is monic. We start by assuming that f is monic. Given any pair of maps $h, g : C \to A$, such that $f \circ id_A \circ g = f \circ id_A \circ h$ then $g = h$, thus g will be the only map making the following diagram commute:

$$\begin{tikzcd}
C \arrow[dr, dashed, "g"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "g"'] & & \\
& A \arrow[r, "id_A"] \arrow[d, "id_A"] & A \arrow[d, "f"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

Hence the square is a pullback.

On the other hand, if the diagram is a pullback then for any other $h : C \to A$, such that $f \circ id_A \circ g = f \circ id_A \circ h$, by uniqueness of g it follows that $g = h$. Thus f is monic.

Example B.12 Given any category, if

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "i"'] & B \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

is a pullback, then i is an equaliser of f and g. In fact, let us assume that the diagram is a pullback,

$$\begin{tikzcd}
E \arrow[drr, bend left, "h"] \arrow[dr, dashed, "l"] \arrow[ddr, bend right, "h"'] & & \\
& A \arrow[r, "i"] \arrow[d, "i"'] & B \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}$$

then $f \circ i = g \circ i$. Moreover for any arrow $h : E \to B$, such that $f \circ h = g \circ h$, there exists a unique l such that $h = i \circ l$, thus i is an equaliser.

On the other hand, if i is an equaliser, then $f \circ i = g \circ i$. Moreover from the universal property of an equaliser, given any arrow $h : E \to B$ there exists a unique arrow $l : E \to A$ which makes the outer square of the above diagram commute. It follows that the inner square is a pullback.

Example B.13 Given a category $\mathcal{C}$ with products and terminal object 1. For any two objects $A, B \in \mathcal{C}$ the pullback of $A \to 1 \leftarrow B$ is the product of A and B. To see this we start by assuming that

$$
\begin{tikzcd}
D \arrow[r, "g"] \arrow[d, "f"'] & A \arrow[d, "!"] \\
B \arrow[r, "!"'] & 1
\end{tikzcd}
$$

is indeed a pullback. Therefore, for each pair of map $h : E \to A$ and $k : E \to S$ such that $! \circ h = ! \circ k$, there exists a unique $l : E \to D$ such that the following diagram commutes:

$$
\begin{tikzcd}
E \arrow[drr, bend left, "h"] \arrow[dr, dashed, "l"] \arrow[ddr, bend right, "k"'] & & \\
& D \arrow[r, "g"] \arrow[d, "f"'] & A \arrow[d, "!"] \\
& B \arrow[r, "!"'] & 1
\end{tikzcd}
$$

Thus $h = g \circ l$, $k = f \circ l$. Moreover since ! is the unique arrow to the terminal object, the condition $! \circ h = ! \circ k$ is trivially satisfied (always satisfied), thus we end up with the following diagram:

$$
\begin{tikzcd}
& E \arrow[dl, "k"'] \arrow[d, dashed, "l"] \arrow[dr, "h"] & \\
B & D \arrow[l, "f"] \arrow[r, "g"'] & A
\end{tikzcd}
$$

But this is precisely the definition of a product.

Example B.14 If A is an object in a category with a terminal object 1, then

$$\begin{tikzcd} 1 & A \arrow[l, "!"'] \arrow[r, "id_A"] & A \end{tikzcd}$$

is a product diagram. In fact, given the maps $! : B \to 1$ and $f : B \to A$ we construct

$$\begin{tikzcd} & B \arrow[dl, "!"'] \arrow[d, "h"] \arrow[dr, "f"] & \\ 1 & A \arrow[l, "!"] \arrow[r, "id_A"'] & A \end{tikzcd}$$

Obviously, the only arrow making the above diagram commute would be $h := f$.

Example B.15 Consider a pair of morphisms $A \xrightarrow{f} C \xleftarrow{g} B$. We then define a category $Con(f, g)$ whose objects are (f, g)-cone defined as a triple (D, p, q), such that the following diagram commutes:

$$\begin{tikzcd} D \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

Given two (f, g)-cones (D, p, q) (D_1, p_1, q_1) a morphism $h : (D, p, q) \to (D_1, p_1, q_1)$ between them is a map $h : D \to D_1$, such that the following diagram commutes:

$$\begin{tikzcd} & D \arrow[dl, "p"'] \arrow[d, "h"] \arrow[dr, "q"] & \\ A & D_1 \arrow[l, "p_1"] \arrow[r, "q_1"'] & B \end{tikzcd}$$

A pullback of f along g is defined via the diagram

$$\begin{tikzcd}
E \arrow[dr, dashed, "q_1"] \arrow[drr, bend left, "l"] \arrow[ddr, bend right, "p_1"'] & & \\
& D \arrow[r, "q"] \arrow[d, "p"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

where the map l is unique for a given E. However the top part of the diagram is simply an object (E, p_1, q_1) in $Con(f, g)$. Thus the pullback property tells us that for each object $(E_i, p_i, q_i) \in Con(f, g)$ there exists a unique map $l : (E_i, p_i, q_i) \to (D, p, q)$. This means precisely that (D, p, q) is a terminal object

Example B.16 Given any map $f : A \to B$ then the characteristic functions of the identities id_A and id_B are such that $\chi_{id_B} \circ f = \chi_{id_A}$. To see this, consider the diagram

$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[d, "f"] & B \arrow[d, "id_B"] \\
B \arrow[r, "id_B"] \arrow[d] & B \arrow[d, "\chi_{id_B}"] \\
1 \arrow[r, "true"'] & \Omega
\end{tikzcd}$$

where $!_B : B \to 1$ and $!A : A \to 1$.

Since pullbackness implies commutativity, it follows that $\chi_{id_B} \circ f = true \circ !_B \circ f = true \circ !_A = \chi_{id_A}$.

B.2 Topos Quantum Theory

Example B.17 Let us consider a 4 dimensional Hilbert space $\mathbb{C}^4$, with basis $\psi_1 = (1,0,0,0)$, $\psi_2 = (0,1,0,0)$, $\psi_3 = (0,0,1,0)$, $\psi_4 = (0,0,0,1)$. We would like to define the proposition $S_z \in [-3,1] \wedge S_z \in [1,3]$, where S_z represents the value of the spin in the z direction of a two particle system. Total spin in the z direction can only have values -2, 0, 2 since the self-adjoint operator representing S_z is

$$\hat{S}_z = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

Thus the only value that S_z can take in the interval $[-3,1]$ is -2 while the only value it can take in the interval $S_z \in [1,3]$ is 2.

In this setting the proposition $\hat{S}_z \in [1,3]$ is represented by the projection operator $\hat{P}_1 = \text{diag}(1,0,0,0)$ (see Sect. 10.3). On the other hand the proposition $\hat{S}_z \in [-3,-1]$ is represented by the projection operator $\hat{P}_4 = \text{diag}(0,0,0,1)$ (see Sec. 12.3). Therefore all that remains to compute is $\underline{\delta(\hat{P}_1)} \wedge \underline{\delta(\hat{P}_4)}$. Let us compute this for each context $V \in \mathcal{V}(\mathcal{H})$. For the maximal algebra V and for $V_{\hat{P}_1,\hat{P}_4}$ we have

$$\underline{\delta(\hat{P}_1)}_{V_{\hat{P}_1,\hat{P}_4}} \wedge \underline{\delta(\hat{P}_4)}_{V_{\hat{P}_1,\hat{P}_4}} = \underline{\delta(\hat{P}_1)}_V \wedge \underline{\delta(\hat{P}_4)}_V = \{\lambda_1\} \cap \{\lambda_4\} = \emptyset. \tag{B.29}$$

For $V_{\hat{P}_1}$ we have

$$\underline{\delta(\hat{P}_1)}_{V_{\hat{P}_1}} \wedge \underline{\delta(\hat{P}_4)}_{V_{\hat{P}_1}} = \{\lambda_1\} \cap \{\lambda_{123}\} = \emptyset. \tag{B.30}$$

For $V_{\hat{P}_4}$ we have

$$\underline{\delta(\hat{P}_1)}_{V_{\hat{P}_4}} \wedge \underline{\delta(\hat{P}_4)}_{V_{\hat{P}_4}} = \{\lambda_{123}\} \cap \{\lambda_4\} = \emptyset. \tag{B.31}$$

For $V_{\hat{P}_2,\hat{P}_3}$ we have

$$\underline{\delta(\hat{P}_1)}_{V_{\hat{P}_2,\hat{P}_3}} \wedge \underline{\delta(\hat{P}_4)}_{V_{\hat{P}_2,\hat{P}_3}} = \{\lambda_{14}\} \cap \{\lambda_{14}\} = \{\lambda_{14}\}. \tag{B.32}$$

For $V_{\hat{P}_1,\hat{P}_j}$, $j \in \{2,3\}$ we have

$$\underline{\delta(\hat{P}_1)}_{V_{\hat{P}_1,\hat{P}_j}} \wedge \underline{\delta(\hat{P}_4)}_{V_{\hat{P}_1,\hat{P}_j}} = \{\lambda_1\} \cap \{\lambda_{4i}\} = \emptyset. \tag{B.33}$$

For $V_{\hat{P}_i,\hat{P}_4}$, $i \in \{2,3\}$ we have

$$\underline{\delta(\hat{P}_1)}_{V_{\hat{P}_i,\hat{P}_4}} \wedge \underline{\delta(\hat{P}_4)}_{V_{\hat{P}_i,\hat{P}_4}} = \{\lambda_{1j}\} \cap \{\lambda_4\} = \emptyset. \tag{B.34}$$

It is interesting to compare such a proposition with the proposition $S_z \in ([-3.1] \cap [1,3])$ which is represented by $\delta^o(\hat{P}_1 \wedge \hat{P}_4)$. This is clearly equal to $\hat{0}$ for all contexts, hence $\delta^o(\hat{P}_1 \wedge \hat{P}_4) \leq \delta^o(\hat{P}_1) \wedge \delta^o \hat{P}_4)$.

Example B.18 Given the same setting as above we would like to give the topos analogue of the proposition $S_z \in [-3,3]$. This is represented by the projection operators $\hat{1}$. Therefore, for each context $V \in \mathcal{V}(\mathcal{H})$, we have that

$$\delta^o(\hat{1})_V = \hat{1} \tag{B.35}$$

which implies that

$$S_{\delta^o(\hat{1})_V} = \underline{\Sigma}_V. \tag{B.36}$$

Example B.19 Consider the algebra of bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. We then define the category $\mathcal{V}(\mathcal{B})$ of abelian sub-algebras of $\mathcal{B}(\mathcal{H})$. This can be easily seen to be a category under sub-algebra inclusion. We now would like to define a covariant functor $F : \mathcal{V}(\mathcal{B}) \to \mathcal{V}(\mathcal{H})$. A first guess would be

$$\begin{aligned} F : \mathcal{V}(\mathcal{B}) &\to \mathcal{V}(\mathcal{H}) \\ B &\mapsto F(B) := B'' \end{aligned} \tag{B.37}$$

where B'' represents the double commutant of B. Is this a functor? First we need to show that, given two sub-algebras $i : B_i \subseteq B$, the following diagram commutes:

$$\begin{tikzcd} B_i \arrow[r, "F"] \arrow[d, "i"'] & F(B_i) \arrow[d, "F(i)"] \\ B \arrow[r, "F"'] & F(B) \end{tikzcd}$$

The fact that the above diagram commutes follows trivially from the fact that if $B_i \subseteq B$ then $B_i'' \subseteq B''$. Given the commutativity of the above diagram it follows at once that $F(i \circ j) = F(i) \circ F(j)$ for $j : B_j \subseteq B_i$.

Moreover

$$F(id_B) := \big(F(B) \to F(B)\big) = id_{F(B)}. \tag{B.38}$$

Example B.20 Consider a set of classical observables $\mathcal{O}$ which you want to quantized. Such a set forms a Lie algebra with respect to an appropriately defined commutator. For example the Poisson algebra of a set of functions on phase space such that two elements $A, B \in \mathcal{O}$ are considered to be non-commuting when $\{A, B\} = 0$.

We now want to define a possible quantisation funtor. Usually quantization of $\mathcal{O}$ is defined via an irreducible map $v : A \to \hat{A}$ for all $A \in \mathcal{O}$. The assignment of this map is such that the Lie-non-commutativity in $\mathcal{O}$ is reflected by the non-commutativity in the operator algebra on $\mathcal{H}$. Moreover v has to be faithful, i.e. $A \neq B \Rightarrow \hat{A} \neq \hat{B}$.

To ensure that the operator associated to each observable is bounded one refines the quantization map as follows

$$A \to \hat{A} \to e^{i\hat{A}}. \tag{B.39}$$

Next we define the category $\mathbf{C}_0$ of Lie-abelian sub-algebras of the classical observables in $\mathcal{O}$ This category forma a posetunder sub-algebra inclusion. Moreover, we will assume that $\mathbf{C}_0$ is invariant under any symplectic covariance transformation. Given such a category it is possible to define a possible quantisation functor. In particular, given any classical observable (a) we can define a possible quantisation of (a) through the faithful map [37]

$$\tilde{v} : a \mapsto e^{i\hat{a}}. \tag{B.40}$$

If we then consider the collection of all Lie-abelian sub-algebras of classical observables,the above map translates to:

$$\begin{aligned} \phi : \mathbf{C}_0 &\to \mathcal{V}(\mathcal{H}) \\ C &\mapsto \phi(C) := \Upsilon(C)'' \end{aligned} \tag{B.41}$$

where $\Upsilon(C)'' = (\tilde{v}(C) \cup \tilde{v}(C)^*)''$ (here $''$ represents the double commutant operator) and $\mathcal{V}(\mathcal{H})$ is the category of abelian von Neumann sub-algebras in $\mathcal{H}$. Thus $\Upsilon(C)''$ is the smallest abelian von Neumann algebra containing $\Upsilon(C) := \tilde{v}(C) \cup \tilde{v}(C)^*$. However we are not only interested in a single quantisation, but in all possible unitary equivalent quantisations, thus we need to define the action of $G \subseteq U(\mathcal{H})$ on ϕ. In this way we define the notion of unitary equivalent quantisations implementing Dirac covariance[2] of quantum theory. So, for each $g \in G$ and $C \in \mathbf{C}_0$ we define

$$l_g\phi(C) := l_g\big(\phi(C)\big) = \hat{U}_g\phi(C)\hat{U}_g^{-1}. \tag{B.42}$$

Having defined the action of G on the quantisation functor we can define the *quantisation presheaf* over $\mathbf{C}_0$ as follows:

Definition B.7 The quantisation presheaf $\underline{Q} : \mathbf{C}_0 \to \mathit{Sets}$ is defined on

1. Objects: for each $C \in \mathbf{C}_0$ we assign the collection of unitary equivalent quantisation maps, i.e. $\underline{Q}(C) := \{l_g\phi : \downarrow C \to \mathcal{V}(\mathcal{H}) | g \in G\}$ where $l_g\phi(C) := l_g(\phi(C))$. We assume that there is no group action on $\mathbf{C}_0$.

[2]By Dirac covariance we mean the fact that given a unitary group G then considering an operator $\hat{A}$ and a state $|\psi\rangle$ is equivalent to considering $\hat{U}_g\hat{A}\hat{U}_g^{-1}$ and $\hat{U}_g|\psi\rangle$.

2. Morphisms: given a map $i_{C_1,C_2} : C_1 \subseteq C_2$ the corresponding presheaf map is

$$\begin{aligned} \underline{Q}(i_{C_1,C_2}) : \underline{Q}(C_2) &\to \underline{Q}(C_1) \\ \phi &\mapsto \phi_{|C_1}. \end{aligned} \tag{B.43}$$

Example B.21 Given a sheaf $\overline{\underline{A}}$ over a topological space with Alexandroff topology, we want to define the bundle of germs of $\overline{\underline{A}}$. This concept simplifies for our Alexandroff base spaces as, given any point $V \in \mathcal{V}(\mathcal{H})$, there is a unique smallest open set, namely $\downarrow V$, to which V belongs.

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be open neighbourhoods of $V \in \mathcal{V}(\mathcal{H})$ with $s_1 \in \bar{\underline{A}}(\mathcal{O}_1)$ and $s_2 \in \bar{\underline{A}}(\mathcal{O}_2)$. Then s_1 and s_2 have the same germ at V, if there is some open $\mathcal{O} \subseteq \mathcal{O}_1 \cap \mathcal{O}_2$ such that $s_1|_{\mathcal{O}} = s_2|_{\mathcal{O}}$. Since $\mathcal{V}(\mathcal{H})$ has the Alexandroff topology, we can see at once that s_1 and s_2 have the same germ at V iff

$$s_1|_{\downarrow V} = s_2|_{\downarrow V}. \tag{B.44}$$

It follows that if $V \in \mathcal{O}$, $s \in \bar{\underline{A}}(\mathcal{O})$, then $germ_V s = s_{|\downarrow V}$. Hence

$$(\Lambda\bar{\underline{A}})_V = \bar{\underline{A}}(\downarrow V). \tag{B.45}$$

Example B.22 Given the presheaf $\underline{\Sigma}$ we define the set $\Sigma := \coprod_{V\in\mathcal{V}(\mathcal{H})} \underline{\Sigma}_V = \bigcup_{V\in\mathcal{V}(\mathcal{H})}\{V\} \times \underline{\Sigma}_V$. Associated with this is the map $p_\Sigma : \Sigma \to \mathcal{V}(\mathcal{H})$ defined by $p_\Sigma(\lambda) = V$, where V is the context such that $\lambda \in \underline{\Sigma}_V$, therefore $p_\Sigma^{-1}(V) = \underline{\Sigma}_V$.

We now prove the following theorem:

Theorem B.1 *Given $\Sigma_{|\downarrow V} := \coprod_{V'\in\downarrow V} \underline{\Sigma}_V$, a local section $\sigma : \downarrow V \to \Sigma_{|\downarrow V}$ of the bundle $p_\Sigma : \Sigma \to \mathcal{V}(\mathcal{H})$ is continuous with respect to the spectral topology on Σ, if and only if it is a local section of the presheaf $\underline{\Sigma}_{|\downarrow V}$. Therefore a continuous local section of the bundle $\Lambda\underline{\breve{\Sigma}}$ equipped with the étale topology.*

Proof Given a section σ of the presheaf $\underline{\Sigma}_{\downarrow V}$, let us consider $\sigma^{-1}(S \cap \Sigma_{\downarrow V})$ for any basis set, S, for the spectral topology of Σ. From the definition of S it is clear that if $\sigma(V_1) \in S_{V_1}$ for some $V_1 \in \mathcal{V}(\mathcal{H})$ then, $\sigma(V_2) \in S_{V_2}$ for all $V_2 \subseteq V_1$, i.e., $\sigma^{-1}(S_{|\downarrow V_1}) = \downarrow V_1$. It follows that:

$$\sigma^{-1}(S \cap \sigma_{\downarrow V}) = \bigcup\left\{\downarrow V_1 | V_1 \subseteq V, \sigma(V_1) \in S_{V_1}\right\}. \tag{B.46}$$

This is a union of lower sets and is, hence, open in the Alexandroff topology on $\mathcal{V}(\mathcal{H})$. It follows that σ is a continuous section of the bundle $\Sigma_{|\downarrow V}$.

Conversely, if σ is a continuous section of the bundle $\Sigma_{|\downarrow V}$, then, for any basis element $S := \coprod_{V\in\mathcal{V}(\mathcal{H})} \underline{S}_V$, $\sigma^{-1}(S \cap \Sigma_{\downarrow V})$ is a lower set. In particular, consider any topological base set $\underline{S}$ such that $\sigma(V) \in \underline{S}_V$. Then in order for $\sigma^{-1}(\underline{S} \cap \underline{\Sigma}_{\downarrow V})$ to be open we must have $\sigma(V_1) \in \underline{S}_{V_1}$ for all $V_1 \subseteq V$. Thus σ is 'collared' for any $\underline{S}$, such that $\sigma(V) \in \underline{S}_V$. Now the induced topology on each $\underline{\Sigma}_V$ is just the spectral topology on $\underline{\Sigma}_V$, and this is extremely disconnected. It follows that for any $\lambda \in \underline{\Sigma}_V$, $\lambda = \bigcap\{\underline{C} | \underline{C}$ is clopen and $\lambda \in \underline{C}\}$. Thus taking the intersection of the basis set, $\underline{S}$, that contain $\lambda \in \underline{\Sigma}_V$ implies that σ is a section of the presheaf $\underline{\Sigma}_{|\downarrow V}$. □

Example B.23 The space $\mathcal{V}(\mathcal{H})/G$ can be given the structure of a poset by defining, for each pair of orbits w_1, w_2 in $\mathcal{V}(\mathcal{H})/G$ the relation

$$w_1 \leq w_2 \quad \text{iff there exists} \quad V_1 \in O_{w_1} \text{ and } V_2 \in O_{w_2} \quad \text{such that} \quad V_1 \subseteq V_2 \tag{B.47}$$

where $O_{w_1} \subset \mathcal{V}(\mathcal{H})$ is the orbit associated to w_1.

Proof 1. Reflexivity: it is trivial that $w \leq w$ for all $w \in \mathcal{V}(\mathcal{H})/G$.

2. Transitivity: if $w_1 \leq w_2$ and $w_2 \leq w_3$ then there exists (i) $V_1 \in O_{w_1}$ and $V_2 \in O_{w_2}$ such that $V_1 \subseteq V_2$ and (ii) $V_3 \in O_{w_2}$ and $V_4 \in O_{w_3}$ such that $V_3 \subseteq V_4$. Now, since G acts transitively on the orbit w_2 there exists $g \in G$ such that $V_3 = l_g(V_2)$, and hence $l_g(V_2) \subseteq V_4$. Therefore, $V_2 \subseteq l_{g^{-1}}(V_4)$ and thus $V_1 \subseteq l_{g^{-1}}(V_4)$, and so $w_1 \leq w_3$, as required.

3. Antisymmetry: suppose $w_1 \leq w_2$ and $w_2 \leq w_1$. Then there exists $V_1 \in O_{w_1}$, $V_2 \in O_{w_2}$, $V_3 \in O_{w_2}$, $V_4 \in O_{w_1}$ such that $V_1 \subseteq V_2$ and $V_3 \subseteq V_4$. Since G acts transitively on w_2 there exists $g_1 \in G$ such that $V_3 = l_{g_1}(V_2)$ and hence $V_2 = l_{g_1^{-1}}(V_3) \subseteq l_{g_1^{-1}}(V_4)$. Hence we have $V_1, V_4 \in O_{w_1}$ such that $V_1 \subseteq l_{g_1^{-1}}(V_4)$. Now, because G acts transitively on the orbit w_1 there exists $g_2 \in G$ such that $V_4 = l_{g_2}(V_1)$ and hence $V_1 \subseteq l_g(V_1)$ where $g := g_1^{-1} g_2$. Now, the orbits of G form an anti-chain, since the algebra-map $V \to l_g(V) := \hat{U}(g) V \hat{U}(g)^{-1}$ is an isomorphism, then V cannot be a proper subset of $l_g(V)$. Therefore, $V_1 \subseteq l_g(V_1)$ implies that $V_1 = l_g(V_1) = l_{g_1^{-1}}(l_{g_2}(V_1)) = l_{g_1^{-1}}(V_4)$, and hence $V_2 \subseteq V_1$. Thus $V_1 = V_2$ and $w_1 = w_2$. □

Since $\mathcal{V}(\mathcal{H})/G$ is a poset, the Alexandroff topology is simply the topology whose basis are the lower sets $\downarrow w$ with the ordering defined above.

We now want to define the map

$$\begin{aligned} \pi : \mathcal{V}(\mathcal{H}) &\to \mathcal{V}(\mathcal{H})/G \\ V &\mapsto w_V \end{aligned} \tag{B.48}$$

which maps each algebra to its equivalence class. In order for this map to give rise to a geometric morphism we need to show that it is continuous. By placing the Alexandroff topology on $\mathcal{V}(\mathcal{H})/G$, then by Lemma 14.2, π is indeed continuous. However, the map $\pi : \mathcal{V}(\mathcal{H}) \to \mathcal{V}(\mathcal{H})/G$ although continuous is not étalé, since this would require π to be a local homeomorphism, and this cannot be true at those points in the base space where there is a 'sudden' change of stability group for the corresponding G-orbit. In fact even if $V' \subseteq V$ there is no obvious relation between the respective stability groups.

The fact that π is continuous implies that we can define a geometric morphism (see Theorem 14.4) whose direct and inverse image are, respectively:

$$\pi_* : Sh\mathcal{V}(\mathcal{H}) \to Sh\big(\mathcal{V}(\mathcal{H})/G\big) \tag{B.49}$$

$$\pi^* : Sh\big(\mathcal{V}(\mathcal{H})/G\big) \to Sh\mathcal{V}(\mathcal{H}) \tag{B.50}$$

such that $\pi^*(\underline{A})(\downarrow V) := \underline{A}(\pi(\downarrow V)) = \underline{A}\downarrow w$ where $V \in O_w$, while $\pi_*((\underline{S}))(\downarrow w) := \underline{S}(\pi^{-1}(\downarrow W))$. We now would like to define the sheaf of (continuous) section of π, which we denote as $\gamma(\pi) \in Sh(\mathcal{V}(\mathcal{H})/G)$. In order to do so we need the following lemma.

Lemma B.1 *Let $\alpha : P_1 \to P_2$ be a map between posets P_1 and P_2. Then α is order preserving, if and only if, for each lower set $L \in P_2$, we have that $\alpha^{-1}(L)$ is a lower subset of P_1.*

Proof Suppose α is order preserving and let $L \in P_2$ be lower. Now let $z \in \alpha^{-1}(L) \subseteq P_1$, i.e., $\alpha(z) = l$ for some $l \in L$, and suppose $y \in P_1$ is such that $y \leq z$. Since α is order preserving we have $\alpha(y) \leq \alpha(z) = l \in L$, which, since L is lower, means that $\alpha(y) \in L$ i.e., $y \in \alpha^{-1}(L)$. Hence $\alpha^{-1}(L)$.

Conversely, suppose that for any lower set $L \subseteq P_2$ we have that $\alpha^{-1}(L) \subseteq P_1$ is lower and consider a pair $x, y \in P_1$ such that $x \leq y$. Now $\downarrow\alpha(y)$ is lower in P_2 and hence $\alpha^{-1}(\downarrow\alpha(y))$ is a lower subset of P_1. However $\alpha(y) \in \downarrow\alpha(y)$ and hence $y \in \alpha^{-1}(\downarrow\alpha(y))$. Therefore the fact that $x \leq y$ implies that $x \in \alpha^{-1}(\downarrow\alpha(y))$, i.e., $\alpha(x) \in \downarrow\alpha(y)$, which means that $\alpha(x) \leq \alpha(y)$. Thus α is order preserving. □

We now would like to define the sheaf of (continuous) section of π, which we denote as $\gamma(\pi) \in Sh(\mathcal{V}(\mathcal{H})/G)$. In this context we recall that a section of a bundle $p_Y : Y \to X$ over an open subset $U \subseteq X$ is a map $s : U \to Y$ such that $p_Y \circ s = id_U$. In our case, using the Alexandroff topology on both $\mathcal{V}(\mathcal{H})$ and $\mathcal{V}(\mathcal{H})/G$ we know from Lemma 14.2 that $s : U \subseteq \mathcal{V}(\mathcal{H})/G \to \mathcal{V}(\mathcal{H})$ is continuous, if and only if, it is order preserving. Thus $\gamma(\pi)$ is the sheaf of order preserving local sections of the bundle $\pi : \mathcal{V}(\mathcal{H}) \to \mathcal{V}(\mathcal{H})/G$. In particular, for each open set $U \subseteq \mathcal{V}(\mathcal{H})/G$ we obtain $\gamma(\pi)(U) := \{s : U \to \mathcal{V}(\mathcal{H}) | s \text{ is order preserving}\}$, while the morphisms are given by restriction.

We can now define the bundle

$$\begin{aligned} p_{\gamma(\pi)} : \Lambda\big(\gamma(\pi)\big) &\to \mathcal{V}(\mathcal{H})/G \\ s &\mapsto w^s \end{aligned} \tag{B.51}$$

where $s : \downarrow w \to \mathcal{V}(\mathcal{H})$. It follows that the stalk is

$$\Lambda\big(\gamma(\pi)\big)_w := \gamma(\pi)(\downarrow w) \tag{B.52}$$

which is the set of all local sections $s : \downarrow w \subseteq \mathcal{V}(\mathcal{H})/G \to \mathcal{V}(\mathcal{H})$.

We now want to show that $p_{\gamma(\pi)}$ is étale also with respect to the Alexandroff topology (yet to be defined), i.e. we want to show that $p_{\gamma(\pi)}$ is a local homeomorphism with respect to the Alexandroff topology. To this end we first of all define the Alexandroff topology on $\Lambda(\gamma(\pi))$. This is easy since $\Lambda(\gamma(\pi))$ is actually a poset, thus it comes with an Alexandroff topology. The poset ordering is given as follows:

$$s_1 \leq s_2 \quad \text{iff} \quad p_{\gamma(\pi)}(s_1) \leq p_{\gamma(\pi)}(s_2) \quad \text{and} \quad s_1 = (s_2)_{|p_{\gamma(\pi)}(s_1)}. \tag{B.53}$$

That this is indeed a partial ordering is easy to prove so we will not report the proof here.

Given such an ordering, then a basis open in $\Lambda(\gamma(\pi))$ will be $\downarrow s$. We want to show that $p_{\gamma(\pi)}$, restricted to such an open, is a homeomorphism. In particular we obtain

$$(p_{\gamma(\pi)})_{|\downarrow s} : \downarrow s \to \mathcal{V}(\mathcal{H})/G \tag{B.54}$$

such that $(p_{\gamma(\pi)})_{|\downarrow s}(\downarrow s) = \downarrow (p_{\gamma(\pi)})_{|\downarrow s}(s) = \downarrow w^s$ is an open set in the Alexandroff topology. That this is a local homeomorphism is then easy to see. The only difficult part is showing injectivity. In particular given two section s_1 and s_2 in $\downarrow s$ assume that $w^{s_1} = w^{s_2}$. However, $s_1 \in \downarrow s$ then $s_1 \leq s$ therefore $s_1 = s_{|w^{s_1}} = s_{|w^{s_2}} = s_2$ since $s_2 \leq s$. It follows that the bundle $\gamma(\pi)$ in (B.51) is étale with corresponding sheaf $\gamma(\pi)$.

We know from Sect. 14.2 that $\Lambda : Sh(\mathcal{V}(\mathcal{H})/G) \to Bund(\mathcal{V}(\mathcal{H})/G)$ and $\gamma : Bund(\mathcal{V}(\mathcal{H})/G) \to Sh(\mathcal{V}(\mathcal{H})/G)$ are adjoints. We thus want to compute the co-unit ϵ as applied to the bundle $\pi : \mathcal{V}(\mathcal{H}) \to \mathcal{V}(\mathcal{H})/G$. This is the map

$$\epsilon_Y : \Lambda\gamma(p_Y) \to Y \tag{B.55}$$

where $p_Y : Y \to \mathcal{V}(\mathcal{H})/G$ is any bundle over $\mathcal{V}(\mathcal{H})/G$. Thus, considering the bundle $\pi : \mathcal{V}(\mathcal{H}) \to \mathcal{V}(\mathcal{H})/G$ we obtain the commutative triangle

$$\begin{tikzcd}
\Lambda\gamma(\pi) \arrow[rr, "\epsilon_{\mathcal{V}(\mathcal{H})}"] \arrow[dr, "p_{\gamma(\pi)}"'] & & \mathcal{V}(\mathcal{H}) \arrow[dl, "\pi"] \\
& \mathcal{V}(\mathcal{H})/G &
\end{tikzcd}$$

such that

$$\begin{aligned} \epsilon_{\mathcal{V}(\mathcal{H})}(w) : \Lambda\big(\gamma(\pi)\big) &\to \mathcal{V}(\mathcal{H}) \\ s &\mapsto s(w) \end{aligned} \tag{B.56}$$

for $s : \downarrow w \to \mathcal{V}(\mathcal{H})$, i.e. $s \in \Lambda\gamma(\pi)_w$.

We can now pullback the bundle $\Lambda\underline{\Sigma} \to \mathcal{V}(\mathcal{H})$ via the co-unit to obtain

$$\begin{tikzcd}
\Lambda(\gamma(\pi)) \times_{\mathcal{V}(\mathcal{H})} \Lambda(\underline{\Sigma}) \arrow[rr, "pr_2"] \arrow[d, "pr_1"] & & \Lambda\underline{\Sigma} \arrow[d, "p_E"] \\
\Lambda(\gamma(\pi)) \arrow[rr, "\epsilon_{\mathcal{V}(\mathcal{H})}"] \arrow[dr, "p_{\gamma(\pi)}"'] & & \mathcal{V}(\mathcal{H}) \arrow[dl, "\pi"] \\
& \mathcal{V}(\mathcal{H})/G &
\end{tikzcd}$$

where

$$\Lambda\big(\gamma(\pi)\big) \times_{\mathcal{V}(\mathcal{H})} \Lambda(\underline{\Sigma}) := \big\{(s, \lambda) \subseteq \Lambda\big(\gamma(\pi)\big) \times \Lambda(\underline{\Sigma})\big|\epsilon_{\mathcal{V}(\mathcal{H})}(s) = p_E(\lambda)\big\}. \tag{B.57}$$

Now, since p_E is étale and, since a pullback of an étale bundle by a continuous map (in this case $\epsilon_{\mathcal{V}(\mathcal{H})}$) is étale, it follows that $\Lambda(\gamma(\pi)) \times_{\mathcal{V}(\mathcal{H})} \Lambda(\underline{\Sigma}) \to \Lambda(\gamma(\pi))$ is étale. Moreover, since $p_{\gamma(\pi)}$ is étale, the combination $\Lambda(\gamma(\pi)) \times_{\mathcal{V}(\mathcal{H})} \Lambda(\underline{\Sigma}) \xrightarrow{pr_1} \Lambda(\gamma(\pi)) \xrightarrow{p_{\gamma(\pi)}} \mathcal{V}(\mathcal{H})/G$ is étale.

From the above reasoning it follows that we can define a functor $F : Sh(\mathcal{V}(\mathcal{H})) \to Sh(\mathcal{V}(\mathcal{H})/G)$ as follows

$$\begin{aligned} F : Sh\big(\mathcal{V}(\mathcal{H})\big) &\to Sh\big(\mathcal{V}(\mathcal{H})/G\big) \\ \underline{\Sigma} &\mapsto \gamma\big(\Lambda\underline{\Sigma} \xrightarrow{p_E} \mathcal{V}(\mathcal{H}) \xrightarrow{\pi} \mathcal{V}(\mathcal{H})/G\big) \end{aligned} \tag{B.58}$$

where $\gamma(\Lambda\underline{\Sigma} \to \mathcal{V}(\mathcal{H}) \to \mathcal{V}(\mathcal{H})/G)$ is the sheaf of sections of the composite bundle $\Lambda\underline{\Sigma} \to \mathcal{V}(\mathcal{H}) \to \mathcal{V}(\mathcal{H})/G$. We then obtain the following proposition:

Proposition B.1

$$\Lambda\big(\gamma(p_E \circ \pi)\big) \simeq \Lambda\big(\gamma(\pi)\big) \times_{\mathcal{V}(\mathcal{H})} \Lambda(\underline{\Sigma}) \tag{B.59}$$

as bundles over $\mathcal{V}(\mathcal{H})/G$.

Proof Given any $w \in \mathcal{V}(\mathcal{H})/G$ we have

$$\Lambda\big(\gamma(p_E \circ \pi)\big)_w = \{\sigma : \downarrow w \to \underline{\Sigma}\} \tag{B.60}$$

where σ is a local section of $\pi \circ p_E : \underline{\Sigma} \to \mathcal{V}(\mathcal{H})/G$ defined on $\downarrow w$, while

$$\Lambda\big(\gamma(\pi)\big) \times_{\mathcal{V}(\mathcal{H})} \Lambda(\underline{\Sigma})_w = \big\{(s,\lambda)|s(w) = p_E(\lambda)\big\}. \tag{B.61}$$

Now since $p_E \circ \sigma$ is a local section of $\pi : \mathcal{V}(\mathcal{H}) \to \mathcal{V}(\mathcal{H})/G$ on $\downarrow w$, we can define the map

$$\begin{aligned} i_w : \Lambda\big(\gamma(p_E \circ \pi)\big)_w &\to \Lambda\big(\gamma(\pi)\big) \times_{\mathcal{V}(\mathcal{H})} \Lambda(\underline{\Sigma})_w \\ \sigma &\mapsto \big(p_E \circ \sigma, \sigma(w)\big). \end{aligned} \tag{B.62}$$

To show that the pair $(p_E \circ \sigma, \sigma(w))$ satisfies the condition for a pullback over $\mathcal{V}(\mathcal{H})$ we simply note that

$$\epsilon_{\mathcal{V}(\mathcal{H})}(p_E \circ \sigma) = p_E\big(\sigma(w)\big) \tag{B.63}$$

as required. Moreover, for the étalce bundle $\Lambda\underline{\Sigma}$ there is an inverse in which $(s,\lambda) \in \Lambda(\gamma(\pi)) \times_{\mathcal{V}(\mathcal{H})} \Lambda(\underline{\Sigma})_w$ is taken to the unique lift of the local section $s : \downarrow w \to \mathcal{V}(\mathcal{H})$ that passes through λ, where $p_E(\lambda) = s(w)$. □

Now that we have defined the functor F we would like to analyse its action on certain constructs. In particular we will analyse its effect on the terminal object $\underline{1}_{Sh(\mathcal{V}(\mathcal{H}))}$. Our claim is that

$$F(\underline{1}_{Sh(\mathcal{V}(\mathcal{H}))}) = \gamma(\pi). \tag{B.64}$$

Applying the definition to the left hand side we obtain

$$F(\underline{1}_{Sh(\mathcal{V}(\mathcal{H}))}) = \gamma(\Lambda\underline{1} \xrightarrow{p_1} \mathcal{V}(\mathcal{H}) \xrightarrow{\pi} \mathcal{V}(\mathcal{H})/G). \tag{B.65}$$

We then recall that $\Lambda\underline{1}$ is simply a collection of singletons, one for each $V \in \mathcal{V}(\mathcal{H})$, thus $\Lambda\underline{1} \to \mathcal{V}(\mathcal{H})$ is simply $\mathcal{V}(\mathcal{H})$. Hence the result follows.

As a last step in our example we will construct a right adjoint to F, i.e. we will define a functor

$$G : Sh\big(\mathcal{V}(\mathcal{H})/G\big) \to Sh\big(\mathcal{V}(\mathcal{H})\big) \tag{B.66}$$

such that

$$F \dashv G. \tag{B.67}$$

There are two ways of approaching this. One is by explicitly constructing the G functor and then showing that it is indeed a right adjoint of F, while the second is to show that F is defined as a combination of left adjoint, such that G will be defined as the combination of the respective right adjoints. We will use the second method. To this end we note that

$$F = p_{\gamma(\pi)}! \circ \epsilon^* \tag{B.68}$$

where $\epsilon^*_{\mathcal{V}(\mathcal{H})} : Sh(\mathcal{V}(\mathcal{H})) \to Sh(\Lambda(\gamma(\pi)))$ is the inverse image part (hence left adjoint of ϵ_*) of the geometric morphism induced by the continuous map $\epsilon_{\mathcal{V}(\mathcal{H})}$:

$\Lambda(\gamma(\pi)) \to \mathcal{V}(\mathcal{H})$. On the other hand $p_{\gamma(\pi)}! : Sh(\Lambda(\gamma(\pi))) \to Sh(\mathcal{V}(\mathcal{H})/G)$ is the left adjoint of $p^*_{\gamma(\pi)}$. It follows that:

$$G := \epsilon_* \circ p^*_{\gamma(\pi)} \tag{B.69}$$

and $F \dashv G$.

References

1. J.S. Bell, *Against 'measurement'* (Physics World, Institute of Physics Publishing, Bristol and Philadelphia, 1990)
2. H. Everett, Relative state formulation of quantum mechanics. Rev. Mod. Phys. **29**, 141–149 (1957)
3. B. DeWitt, N. Graham, *The Many-Worlds Interpretation of Quantum Mechanics* (Princeton University Press, Princeton, 1973)
4. A. Einstein, B. Podolsky, N. Rosen, Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. **47**(10), 777–780 (1935)
5. N. Bohr, Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. **48**, 700 (1935)
6. D. Bohm, A suggested interpretation of the quantum theory in terms of "hidden" variables. I. Phys. Rev. **85**, 166–179 (1952)
7. D. Bohm, A suggested interpretation of the quantum theory in terms of "hidden" variables. II. Phys. Rev. **85**, 180–193 (1952)
8. M. Lockwood, *Mind, Brain and the Quantum: The Compound I* (Blackwell, Oxford, 1992)
9. N.G. van Kampen, The scandal of quantum mechanics. Am. J. Phys. **76**, 989 (2008)
10. R. Peierls, Selected Scientific Papers of Sir Rudolf Peierls. World Scientific Series in 20th Century Physics, vol. 19 (1997)
11. K. Gottfried, *Quantum Mechanics* (Benjamin, Elmsford, 1966)
12. V.F. Lenzen, in *Philosophical Problems of the Statistical Interpretation of Quantum Mechanics Proceedings*. Second Berkeley Symp. on Math. Statist. and Prob. (University of California Press, Berkeley, 1951), pp. 567–579
13. J.S. Bell, *Speakable and Unspeakable in Quantum Mechanics (Collected Papers on Quantum Philosophy)* (Cambridge University Press, Cambridge, 2004)
14. C.J. Isham, J. Butterfield, A topos perspective on the Kochen-Specker theorem: I. Quantum states as generalized valuations (1998). arXiv:quant-ph/9803055
15. J. Butterfield, C.J. Isham, A topos perspective on the Kochen-Specker theorem: II. Conceptual aspects, and classical analogues (1998). arXiv:quant-ph/9808067
16. J. Butterfield J. Hamilton, C.J. Isham, A topos perspective on the Kochen-Specker theorem: III. Von Neumann algebras as the base category (1999). arXiv:quant-ph/9911020
17. R. Goldblatt, *Topoi the Categorial Analysis of Logic* (North-Holland, London, 1984)
18. S. MacLane, I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory* (Springer, London, 1968)
19. J.L. Bell, *Toposes and Local Set Theories* (Clarendon Press, Oxford, 1988)
20. S. MacLane, *Categories for the Working Mathematician* (Springer, London, 1997)
21. C.J. Isham, *Lectures on Quantum Theory, Mathematical and Structural Foundations* (Imperial College Press, London, 1995)

C. Flori, *A First Course in Topos Quantum Theory*, Lecture Notes in Physics 868,
DOI 10.1007/978-3-642-35713-8,

22. J. Bub, *Interpreting the Quantum World* (Cambridge University Press, Cambridge, 1997)
23. M. Kernaghan, Bell-Kochen-Specker theorem for 20 vectors. J. Phys. A **27** (1994)
24. A. Doering, C.J. Isham, *Classical and Quantum Probabilities as Truth Values* (2011). arXiv:1102.2213v1
25. A. Doering, Quantum states and measures on the spectral presheaf (2008). arXiv:0809.4847v1 [quant-ph]
26. C.J. Isham, J. Butterfield, Some possible roles for topos theory in quantum theory and quantum gravity. Found. Phys. **30**, 1707 (2000). arXiv:gr-qc/9910005
27. A. Doring, C.J. Isham, A topos foundation for theories of physics. I. Formal languages for physics. J. Math. Phys. **49**, 053515 (2008). arXiv:quant-ph/0703060 [quant-ph]
28. J.B. Geloun, C. Flori, Topos analogues of the KMS state. arXiv:1207.0227v1 [math-ph]
29. A. Doring, C.J. Isham, A topos foundation for theories of physics. II. Daseinisation and the liberation of quantum theory. J. Math. Phys. **49**, 053516 (2008). arXiv:quant-ph/0703062 [quant-ph]
30. A. Doring, C.J. Isham, A topos foundation for theories of physics. III. The representation of physical quantities with arrows. J. Math. Phys. **49**, 053517 (2008). arXiv:quant-ph/0703064 [quant-ph]
31. A. Doring, C.J. Isham, A topos foundation for theories of physics. IV. Categories of systems. J. Math. Phys. **49**, 053518 (2008). arXiv:quant-ph/0703066 [quant-ph]
32. A. Doring, C. Isham, 'What is a thing?': topos theory in the foundations of physics. arXiv:0803.0417 [quant-ph]
33. A. Doring, Topos theory and 'neo-realist' quantum theory. arXiv:0712.4003 [quant-ph]
34. J. Harding, A. Doering, Abelian subalgebras and the Jordan structure of a von Neumann algebra. arXiv:1009.4945 [math-ph]
35. C.J. Isham, Is it true or is it false; or somewhere in between? The logic of quantum theory. Contemp. Phys. **46**, 207 (2005)
36. C.J. Isham, A topos perspective on state-vector reduction. arXiv:quant-ph/0508225
37. K. Nakayama, Sheaves in quantum topos induced by quantization. arXiv:1109.1192 [math-ph]
38. C.J. Isham, Topos theory and consistent histories: the internal logic of the set of all consistent sets. Int. J. Theor. Phys. **36**, 785 (1997). arXiv:gr-qc/9607069
39. A. Doering, Topos quantum logic and mixed states. arXiv:1004.3561 [quant-ph]
40. A. Doering, The physical interpretation of daseinisation. arXiv:1004.3573 [quant-ph]
41. C. Flori, Concept of quantization in a topos. In preparation
42. C. Flori, Review of the topos approach to quantum theory. arXiv:1106.5660 [math-ph]
43. C. Flori, A topos formulation of history quantum theory. J. Math. Phys. **51**, 053527 (2010). arXiv:0812.1290 [quant-ph]
44. C.J. Isham, Topos methods in the foundations of physics. arXiv:1004.3564 [quant-ph]
45. C. Flori, Group action in topos quantum physics. arXiv:1110.1650 [quant-ph]
46. S. Wolters, A comparison of two topos-theoretic approaches to quantum theory. arXiv:1010.2031 [math-ph]
47. C. Heunen, N.P. Landsman, B. Spitters, A topos for algebraic quantum theory. Commun. Math. Phys. **291**, 63 (2009). arXiv:0709.4364 [quant-ph]
48. C. Heunen, N.P. Landsman, B. Spitters, S. Wolters, The Gelfand spectrum of a noncommutative C*-algebra: a topos-theoretic approach. J. Aust. Math. Soc. **90**, 39 (2011). arXiv:1010.2050 [math-ph]
49. W. Brenna, C. Flori, Complex numbers and normal operators in topos quantum theory. arXiv:1206.0809 [quant-ph]
50. C. Flori, Approaches to quantum gravity. arXiv:0911.2135 [gr-qc]
51. F. Dowker, A. Kent, On the consistent histories approach to quantum mechanics. J. Stat. Phys. **82**, 1575 (1996). arXiv:gr-qc/9412067
52. C.J. Isham, Topos theory and consistent histories: the internal logic of the set of all consistent sets. Int. J. Theor. Phys. **36**, 785 (1997). arXiv:gr-qc/9607069

53. C.J. Isham, Quantum logic and the histories approach to quantum theory. J. Math. Phys. **35**, 2157 (1994). arXiv:gr-qc/9308006
54. M. Gell-Mann, J.B. Hartle, in *Complexity, Entropy and the Physics of Information*, ed. by W. Zurek. SFI Studies in the Sciences of Complexity, vol. VIII (Addison Wesley, Reading, 1990)
55. M. Gell-Mann, J.B. Hartle, in *Proceedings of the 3rd International Symposium on the Foundations of Quantum Mechanics in the Light of New Technologies*, ed. by S. Kobayashi, H. Ezawa, Y. Murayama, S. Nomura (Physical Society of Japan, Tokyo, 1990)
56. M. Gell-Mann, J.B. Hartle, in Proceedings of the 25th International Conference on High Energy Physics, ed. by K.K. Phua, Y. Yamaguchi, Singapore, August 2–8 1990 (South East Asia Theoretical Physics Association and Physical Society of Japan) (Worlds Scientific, Singapore, 1990)
57. R.B. Griffiths, Consistent histories and the interpretation of quantum mechanics. J. Stat. Phys. **36**, 219 (1984)
58. R.B. Griffiths, Logical reformulation of quantum mechanics. I. Foundations. J. Stat. Phys. **53**, 893 (1988)
59. R.B. Griffiths, Logical reformulation of quantum mechanics. II. Interferences and the Einstein-Podolsky-Rosen experiment. J. Stat. Phys. **53**, 933 (1988)
60. R.B. Griffiths, Logical reformulation of quantum mechanics. III. Classical limit and irreversibility. J. Stat. Phys. **53**, 957 (1988)
61. R.B. Griffiths, Logical reformulation of quantum mechanics. IV. Projectors in semiclassical physics. J. Stat. Phys. **57**, 357 (1989)
62. R.B. Griffiths, The consistency of consistent histories: a reply to d'Espagnat. Found. Phys. **23**, 1601 (1993)
63. R. Omnes, Consistent interpretations of quantum mechanics. Rev. Mod. Phys. **64**, 339–382 (1992)
64. J.J. Halliwell, A review of the decoherent histories approach to quantum mechanics. Ann. N.Y. Acad. Sci. **755**, 726 (1995). arXiv:gr-qc/9407040
65. C.J. Isham, N. Linden, Quantum temporal logic and decoherence functionals in the histories approach to generalized quantum theory. J. Math. Phys. **35**, 5452 (1994). arXiv:gr-qc/9405029
66. P. Mittelstaedt, Quantum logic and decoherence. Int. J. Theor. Phys. **43**(6) (2004)
67. P. Mittelstaedt, Time dependent propositions and quantum logic. J. Philos. Log. **6**, 463–472 (1977)
68. E.W. Stachow, Logical foundations of quantum mechanics. Int. J. Theor. Phys. **19**(4) (1980)
69. E.W. Stachow, A Model Theoretic Semantics for Quantum Logic, in *Proceedings of the Biennial Meeting of the Philosophy of Science Association*, vol. 1 (1980), pp. 272–280
70. K. Savvidou, C. Anastopoulos, Histories quantisation of parameterised systems: I. Development of a general algorithm. Class. Quantum Gravity **17**, 2463 (2000). arXiv:gr-qc/9912077
71. N.K. Savvidou, Continuous time in consistent histories (1999). arXiv:gr-qc/9912076v1
72. S. Vickers, *Topology Via Logic* (Cambridge University Press, Cambridge, 1989)
73. A.M. Gleason, Measures on the closed subspaces of a Hilbert space. J. Math. Phys. **6**, 885 (1957)
74. G. Helmberg, *Introduction to Spectral Theory in Hilbert Space* (Dover Publications, New York, 2008)
75. R. Whitley, The spectral theorem for a normal operator. Am. Math. Mon. **75**, 856–861 (1968)
76. H.F. de Groote, On a canonical lattice structure on the effect algebra of a von Neumann algebra (2007). arXiv:math-ph/0410018
77. R. Kubo, Statistical mechanical theory of irreversible processes. 1. General theory and simple applications in magnetic and conduction problems. J. Phys. Soc. Jpn. **12**, 570 (1957)
78. P.C. Martin, J.S. Schwinger, Theory of many particle systems. 1. Phys. Rev. **115**, 1342 (1959)
79. H. Araki, P.D.F. Ion, On the equivalence of KMS and Gibbs conditions for states of quantum lattice systems. Commun. Math. Phys. **35**, 1 (1974)

80. T. Matsui, Purification and uniqueness of quantum Gibbs states. Commun. Math. Phys. **162**, 321 (1994)
81. G.L. Sewell, *Quantum Mechanics and Its Emergent Macrophysics* (Princeton University Press, Princeton, 2002)
82. S. Stratila, *Modular Theory in Operator Algebras* (Abacus Press, Tunbridge Wells, 1981)
83. M. Takesaki, *Tomita's Theory of Modular Hilbert Algebras and Its Applications* (Springer, New York, 1970)
84. M. Takesaki, *Theory of Operator Algebras I*. Encyclopaedia of Mathematical Sciences, vol. 124 (Springer, Berlin, 2002)
85. S. Maeda, Probability measures on projections in von Neumann algebras. Rev. Math. Phys. **1**, 235 (1989)
86. H.J. Borchers, On revolutionizing quantum field theory with Tomita's modular theory. J. Math. Phys. **41**, 3604 (2000)
87. S.J. Summers, Tomita-Takesaki modular theory. arXiv:math-ph/0511034
88. A. Connes, *Noncommutative Geometry* (Academic Press, San Diego, 1994)
89. B. Banaschewski, C.J. Mulvey, The spectral theory of commutative C^*-algebras: the constructive spectrum. Quaest. Math. **23**, 4 (2009)
90. B. Banaschewski, C.J. Mulvey, A globalisation of the Gelfand duality theorem. Ann. Pure Appl. Log., vol. 137 (2006)
91. C. Rousseau, Topos theory and complex analysis, in Applications of Sheaves. Lecture Notes in Mathematics, vol. 753 (1979)
92. A. Doering, B. Dewitt, Self-adjoint operators as functions I (2012). arXiv:1208.4724v1 [math-ph]
93. A. Doering, R.S. Barbosa, Unsharp values, domains and topoi (2011). arXiv:1107.1083v1 [quant-ph]
94. P.T. Johnstone, *Sketches of an Elephant A Topos Theory Compendium*, vols. 1, 2 (Oxford University Press, London, 2002)
95. J. Dugundji, *Topology* (Allyn and Bacon, Needham Heights, 1976)
96. C.J. Isham, *Modern Differential Geometry for Physicists*. Lecture Notes in Physics (World Scientific, Singapore, 1999)
97. A. Doering, Generalised gelfand spectra of Nonabelian unital C^*-algebras II: flows and time evolution of quantum systems (2012). arXiv:1212.4882 [math.OA]
98. A. Doering, Generalised gelfand spectra of Nonabelian unital C^*-algebras I: categorical aspects, automorphisms and Jordan structure (2012). arXiv:1212.2613 [math.OA]
99. A. Doering, Topos-based logic for quantum systems and bi-Heyting algebras (2012). arXiv:1202.2750 [quant-ph]

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DOI 10.1007/978-3-642-35713-8,