Secret Blogging Seminar

Concrete Categories

In many introductions to category theory, you first learn the notion of a concrete category: A concrete category is a collection of sets, called the objects of the category and, for each pair (X, Y) of objects, a subset of the maps $X \to Y$. (There are, of course, axioms that these things must obey.) In a concrete category, the objects are sets, and the morphisms are maps that obey certain conditions. So the category of groups is concrete: a map of groups is just a map of the underlying sets such that multiplication is preserved. So are the category of vector spaces, topologicial spaces, smooth manifolds and most of the other most intuitive examples of categories.

Using terminology from a discussion (http://mathoverflow.net/questions/2015/can-the-category-of-schemes-be-concretized) at MO, I'll call a category *concretizable* if it is isomorphic to a concrete category. For example, Set^{op} can be concretized by the functor which sends a set X to the set 2^X of subsets of X, and sends a map of sets $f : X \to Y$ to the preimage map $f^{-1} : 2^Y \to 2^X$.

At one point, I learned of a result of Freyd: The category of topological spaces, with maps up to homotopy, is not concretizable. I thought this was an amazing reflection of how subtle homotopy is. But now I think this result is sort of a cheat. As I'll explain in this post, **if you are the sort of person who ignores details of set theory, then you might as well treat all categories as concrete.** My view now is that specific concretizations are very interesting; but the question of whether a category has a concretization is not. I'll also say a few words about small concretizations, and Freyd's proof.

Let me start by saying exactly what you need to check to see whether a functor is a concretization. Let C be a category and F a functor from C to Set. Then F is a concretization if, for any objects X and Y, and any morphisms f and g from X to Y, we have $F(f) = F(g)$ only if $f = g$.

Now, Yoneda's lemma almost gives us such a functor in every case. Define

$F(X) := \bigsqcup_{S \in \mathrm{Ob}(C)} \mathrm{Hom}_C(S, X).$

Yoneda's lemma tells us that, if f and g induce the same map from $\mathrm{Hom}_C(S, X)$ to $\mathrm{Hom}_C(S, Y)$ for every S, then $f = g$. The proof is simply to take $S = X$.

So, why doesn't Yoneda's lemma tell us that all categories are concretizable? Because the collection of objects of our category may not be a set. I assume that you have at some point been introduced to Russell's paradox (http://en.wikipedia.org/wiki/Russel%27s_paradox), which is resolved by declaring that the collections of all sets is "too big" to be a set. Similarly, you run into trouble if you try to talk about "the set of all vector spaces" or "the set of all groups." And since people do consider the category of all vector spaces; category theorists have learned to be careful with expressions like $\bigsqcup_{S \in \mathrm{Ob}(C)}$, which act as if the objects of C are a set.

If, like me, you don't care about this sort of set theoretic issue, then you might as well think that all categories are concretizable. But you should still object to the Yoneda method of concretization. When will you ever be able to check something for all the objects of a category? What you want is some reasonable collection $T \subset \mathrm{Ob}(C)$ of test objects, so that it is enough to see whether $f_* = g_*$ on $X(S)$ for $S \in T$.

There are several great theorems of this form: In the category of varieties of finite type over $\mathbb{C}$, the Nullstellansatz tells us that a map is determined by its values on $X(\mathbb{C})$ — so we can think of a variety as made up of its points. ~~In the category of finite CW complexes, with maps up to homotopy, Whitehead's theorem tells us that it is enough to study $\mathrm{Hom}(S^n, X)$, for all spheres S^n.~~ The preceeding statement is completely false; thanks to Eric Wofsey (http://mathoverflow.net/questions/2672/whitehead-for-maps) for a counter-example. I'll try to find a better, and more true, example.

My mathematical aesthetic would be to adopt a subjective standard here: the goal of concretization is to find a "nice" set of test objects, and the term "nice" is defined by the judgement of the mathematical community. The choice of a single point, in the Nullstellansatz example, is very nice. The choice of, for example, all Artinian rings, would still be nice, but less so. (**PARAGRAPH REWRITTEN** due to error in the preceding paragraph.)

For those who seek an objective criterion, Tom Leinster proposes (http://mathoverflow.net/questions/2015/can-the-category-of-schemes-be-concretized/2050#2050) saying that C has a *small concretization* if there is a set T of objects of C such that $X \mapsto \bigsqcup_{S \in T} \mathrm{Hom}_C(S, X)$ is a concretization. **Cautionary exercise:** The category whose objects are sets, and whose morphisms are surjections, is a concrete category but has no small concretization.

I don't want to close the post without saying something about how it is proved that the category of topological spaces, with maps modulo homotopy, is not concretizable. Even though I don't find concretization interesting, the idea that it can be proved impossible is interesting to me. This is a result of Peter Freyd, whose explanation of the technical points (http://www.tac.mta.ca/tac/reprints/articles/6/tr6abs.html) is about as good as it could be, so I'll leave the details to him.

Suppose that F were a concretization. For simplicity, I'll assume that $F(\{\mathrm{point}\})$ is the set with a single element. For any connected space X, let $z(X)$ be the element of $F(X)$ corresponding to the unique homotopy class of map $\{\mathrm{point}\} \to X$.

Freyd constructs a totally ordered set P, with cardinality greater than $2^{F(S^2)}$, two sequences of connected spaces A_p and B_p indexed by P, and maps $A_p \to S^2 \to B_p$. These have the property that the composite $A_p \to B_p$ does not factor through $\{\mathrm{point}\}$ but $A_p \to B_q$, with $q < p$, does. Since P is so big, there must be some $q < p$ such that $F(A_p)$ and $F(A_q)$ map to the same subset of $F(S^2)$. Call this subset I.

Then $A_p \to S^2 \to B_q$ factors through $\{\mathrm{point}\}$, so the map $S^2 \to B_q$ sends I to $z(B_q)$. But then $A_q \to S^2 \to B_q$ also sends $F(A_q)$ to $z(B_q)$, contradicting that $A_q \to B_q$ does not factor through a point.

To give a little more detail, one first constructs a sequence of groups G_p, with nonzero maps $\mathbb{Z} \to G_p$, such that, for any map $G_p \to G_q$ with $q < p$, the composite $\mathbb{Z} \to G_q$ would be zero. Let C_p be a $K(G_p, 1)$

Note that S^1 is a $K(\mathbb{Z}, 1)$, so we have functorial maps $S^1 \to C_p$. Then B_p is the suspension $\Sigma(C_p)$, the map $S^2 \to B_p$ is the suspension of $S^1 \to C_p$, and A_p is the mapping cone of $S^1 \to C_p$. If you want more detail than this, you should read Freyd's paper. (http://www.tac.mta.ca/tac/reprints/articles/6/tr6abs.html)

October 26, 2009 David Speyer

12 thoughts on "Concrete Categories"

1. *range* says:
   October 26, 2009 at 10:30 am
   Interesting. I've been reading for a while, but I actually understood the whole post. Helps that I just started a graduate degree in pure mathematics.

2. Pingback: Concretizable Categories « memoirs on a rainy day ~
3. *Andrew Stacey* says:
   October 27, 2009 at 12:08 am
   Here's some further reading on the idea of "test objects":

   The Isbell envelope in the n-lab. Here one turns the idea on its head and says, "Suppose I have a category of test objects, what can it reasonably detect?".

   Leading on from that, one should look at the examples of Generalised smooth spaces and in particular, Frolicher spaces (well, I would say that, wouldn't I).

   There's also some more on this in my preprint On Category of Smooth Objects. Although the motivation is primarily the various categories of smooth objects, I found a formulation of the idea of having a "test category" for probing objects which would work more generally than just with smooth spaces.

4. *Mike Shulman* says:
   October 28, 2009 at 5:08 pm
   I'm not sure I agree that it is legit to say that modulo details of set theory, all categories are concrete. You may, of course, reply that I'm not the sort of person who ignores details of set theory. And you'd be right; but I think I would say that maybe this is one of the details of set theory that no one is justified in ignoring. (There do, of course, *exist* details of set theory that no one is justified in ignoring!)

   I *would* say that the Yoneda lemma means that many of the properties we might intuitively attribute only to "concrete" categories do, in fact, apply to all categories. However, I also think there may be a real difference between concrete and nonconcrete categories *relative to the same size of universe*.

   Consider the homotopy category, which as Freyd proved is not concrete. I imagine someone who ignores details of set theory as saying "sure, it's sort of not concrete, but it'll become concrete if you move up a universe." In other words, it doesn't have a faithful functor to the category Set of small sets, but it does have a faithful functor to the category SET of large sets. However, once you move up a universe to allow large sets, then the category you're looking at is not really "the homotopy category" any more! Relative to the new universe, "the homotopy category" would also include "large homotopy types," and thus would not be concretizable over SET.

Here's an analogy: every small category with split idempotents is accessible. Therefore, *every* category with split idempotents becomes "accessible" relative to a larger universe SET. Does that mean that modulo details of set theory, all categories are accessible? I don't think so.

5. *David Speyer* says:
October 29, 2009 at 6:27 am
"the Yoneda lemma means that many of the properties we might intuitively attribute only to "concrete" categories do, in fact, apply to all categories"

I would certainly agree with this. My tentative thought is that I would be willing to go farther and say "all of the properties that we care about and might intuitively attribute only to concrete categories do, in fact, apply to all categories." But I'd be interested to hear counter-examples.

I have not heard of accessible categories before, and am trying to digest the wikipedia article now. Any help is appreciated.

David Speyer says:
October 29, 2009 at 8:31 am
Whether or not I wind up agreeing with you, it probably is worth everyone learning the idea that sometimes you have to move up a universe to make the object that you want.

7. *Mike Shulman* says:
October 29, 2009 at 6:37 pm
Well, I would definitely argue that it's *not* true that "modulo details of set theory, all categories are accessible." There are certainly important properties possessed by accessible categories, such as the possibility of transfinite arguments to generate free constructions, which are lacking in non-accessible categories. And since it seems that your *argument* for why "all categories might as well be concrete" would apply equally well to show that "all categories might as well be accessible," I am skeptical of the argument.

That doesn't mean it isn't *true*, though, that all categories "might as well be concrete", at least as far as "the properties that we care about and might intuitively attribute only to concrete categories" go. I don't think I can evaluate that statement, because I don't really have much intuition myself that's restricted to concrete categories. Perhaps that just means I've internalized the Yoneda lemma enough by now that most of my intuition applies directly to all categories. (-:

8. *Tom Leinster* says:
November 2, 2009 at 7:48 pm
David, here is one way to understand accessible categories.

There is, as you may know, a notion of "flat functor" on a category, very closely analogous to the notion of flat module. (A functor A –> Set can be called a "left A-module".) Definition: for a (small) category A, a functor X: A –> Set is flat iff the functor

preserves finite limits.

For any small category A, we may form the category Flat(A, Set) of flat functors A –> Set and all natural transformations between them. A category is accessible if and only if it is of this form.

So, accessible categories are categories of flat modules.

9. *Tom Leinster* says:
November 2, 2009 at 7:54 pm
Oh, oops. What I gave was a characterization of *finitely* accessible categories, which are just a special kind of accessible category.

I don't know much about this stuff, but I think the following is correct. A category is accessible iff it is k-accessible for some infinite regular cardinal k. But for any such cardinal k there's an accompanying notion of "k-flat", meaning that (in the notation of my last comment) preserves limits of cardinality less than k. And I guess a category is k-accessible iff it's the category of k-flat functors on some small category.

That's not so clean an idea, though.

10. *Mike Shulman* says:
November 2, 2009 at 8:38 pm
That definition (in Tom's comment #9) is correct. (It doesn't seem any less clean to me than the finitary one. I've never really understood why some people seem to find cardinal numbers greater than aleph-0 somehow distasteful. (-:)

But anyway, although that definition is correct, I like it better as a theorem than a definition. I feel like defining an accessible category in that way is like defining a manifold as a certain subset of Euclidean space. Yes, it's correct, but the "real" definition is more intrinsic. The "embedding" definition makes the notion sound very limited, while the "intrinsic" definition makes you realize how general it is.

Here's a definition that's sort of in between. A *sketch* is a small category equipped with certain chosen cones and cocones. A *model* of a sketch S is a functor from S to Set which takes the chosen cones and cocones to limiting cones and colimiting cocones, respectively. A category is *accessible* if it is equivalent to the category of models for some sketch. Intuitively, that means the objects of an accessible category can be considered to be "families of sets equipped with structure that can be defined in terms of limits and colimits."

11. *Tom Leinster* says:
November 2, 2009 at 9:59 pm
Mike, it's not that I find cardinals larger than aleph_0 distasteful. It's that going from my first (incorrect) description to my second (correct) description involved making the description more complicated: we had to quantify over all infinite regular cardinals. We (or I) had to say "A category is accessible if it is k-accessible for some infinite regular cardinal k" and *then* "A category is k-accessible if…" It's simply an extra layer of stuff.

Anyway, your characterization by sketches eliminates that extra layer.

12. *Mike Shulman* says:
November 3, 2009 at 8:37 am
Okay, I see your point; sorry that I leaped to the wrong conclusion. (I do know other people who seem to have that sort of distaste.)

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