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Categories in Context: Historical, Foundational, and Philosophical†

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The aim of this paper is to put into context the historical, foundational and philosophical significance of category theory. We use our historical investigation to inform the various category-theoretic foundational debates and to point to some common elements found among those who advocate adopting a foundational stance. We then use these elements to argue for the philosophical position that category theory provides a framework for an algebraic *in re* interpretation of mathematical structuralism. In each context, what we aim to show is that, whatever the significance of category theory, it need not rely upon any set-theoretic underpinning.

1. History

Any (rational) reconstruction of a history, if it is not merely to consist in a list of dates and 'facts', requires a perspective. Noting this, the perspective taken in our detailing the history of category theory will be bounded by our investigation of category theorists' top-down approach towards analyzing mathematical concepts in a category-theoretic context. Any perspective too has an agenda: ours is that, contrary to popular belief, whatever the worth (mathematical, foundational, logical, and philosophical) of category theory, its significance need not rely on any set-theoretical underpinning.

1.1 *Categories as a Useful Language*

In 1942, Eilenberg and Mac Lane started their collaboration by applying methods of computations of groups, developed by Mac Lane, to a problem in algebraic topology formulated earlier by Borsuk and Eilenberg. The problem was to compute certain homology groups of specific spaces.[1]

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[1] Here is the problem: given a solenoid Σ in the sphere S^3, how many homotopy classes of continuous mappings $f(S^3 - \Sigma) \subset S^2$ are there? As its name indicates, a solenoid

The methods employed were those of the theory of group extensions, which were then used to compute homology groups. In the process, it became apparent that many group homomorphisms were 'natural'. While the expression 'natural isomorphism' was already in use, because Eilenberg and Mac Lane relied on its use more heavily and specifically, a more exact definition was needed; they state: 'We are now in a position to give a precise meaning to the fact that the isomorphisms established in Chapter V are all "natural".' (Eilenberg and Mac Lane [1942b], p. 815) It was clear from their joint work, and from other results known to them, that the phenomenon which they refer to as 'naturality' was a common one and appeared in different contexts. They therefore decided to write a short note in which they set up the 'basis for an appropriate general theory' wherein they restricted themselves to the natural isomorphisms of group theory. (See Eilenberg and Mac Lane [1942a], p. 537.) In this note, they introduce the notion of a functor, in general, and the notion of natural isomorphisms, in particular. These two notions were used to give a precise meaning to 'what is shared' by all cases of natural isomorphisms. At the end of the note, Eilenberg and Mac Lane announced that the general axiomatic framework required to present natural isomorphisms in other areas, *e.g.*, in the areas of topological spaces and continuous mappings, simplical complexes and simplical transformations, Banach spaces and linear transformations, would be studied in a subsequent paper.

This next paper, appearing in 1945 under the title 'General theory of natural equivalences', marks the official birth of category theory. Again, the objective is to give a general axiomatic framework in which the notion of natural isomorphism could be both defined and used to capture what structure is shared in various areas of inquiry. In order to accomplish the former, they had to define functors in full generality, and, in order to do this, they had to define categories. Here is how Mac Lane details the order of discovery: 'we had to discover the notion of a natural transformation. That in turn forced us to look at functors, which in turn made us look at categories' (Mac Lane [1996c], p. 136). Having made this finding, 'the conceptual development of algebraic topology inevitably uncovered the three basic notions: *category, functor* and *natural transformation*' (Mac Lane [1996c], p. 130) .

is an infinitely coiled thread. Thus, the complement of a solenoid in the sphere S^3 is infinitely tangled around it. Eilenberg showed that these homotopy classes were in one-to-one correspondence with the elements of a specific homology group, which he could not, however, compute. Although it seems to be a purely technical problem, its feasibility leads to a better understanding and control of (co-)homology. Using a different method, Steenrod discovered a way to compute various relevant groups, but the computations were quite intricate. What prompted the collaboration between Eilenberg and Mac Lane was the discovery that Steenrod's groups were isomorphic to extensions of groups, which were much easier to compute.

It should be noted that, at this point, Eilenberg and Mac Lane thought that the concept of a category was required only to satisfy a certain constraint on the definition of functors. Indeed, they took functors to be (set-theoretical) *functions*, and therefore as needing well-defined domains and codomains, *i.e.*, as needing sets. They were immediately aware, too, that the category of *all* groups, or the category of *all* topological spaces, was an illegitimate construction from such a set-theoretic point of view. One way around this problem, as they explicitly suggested, was to use the concept of a category as a heuristic device, so that

> . . . the whole concept of a category is essentially an auxiliary one; our basic concepts are essentially those of a *functor* and of a natural transformation . . . The idea of a category is required only by the precept that every function should have a definite class as domain and a definite class as range, for the categories are provided as the domains and ranges of functors. Thus one could drop the category concept altogether and adopt an even more intuitive standpoint, in which a functor such as 'Hom' is not defined over the category of 'all' groups, but for each particular pair of groups which may be given. The standpoint would suffice for the applications, inasmuch as none of our developments will involve elaborate constructions on the categories themselves. (Eilenberg and Mac Lane [1945], p. 247)

This heuristic stance was basically the position underlying the status of categories from 1945 until 1957–1958. Eilenberg and Mac Lane did, however, examine alternatives to their 'intuitive standpoint', including the idea of adopting NBG (with its distinction between sets and classes) as a set-theoretical framework, so that one could say that the category of all groups is a class and not a set. Of course, one has to be careful with the operations performed on these classes and make sure that they are legitimate. But, as Eilenberg and Mac Lane mention in the passage quoted above, these operations were, during the first ten years or so, rather simple, which meant that their 'legitimacy' did not pose much of a problem for using the NBG strategy. The view that such 'large' categories are best taken as classes is adopted, for instance, in Eilenberg and Steenrod's very influential book on the foundations of algebraic topology, and also in all other books on category theory that appeared in the sixties. (See, for example, Eilenberg and Steenrod [1952], Freyd [1964], Mitchell [1965], Ehresmann [1965], Bucur and Deleanu [1968], Pareigis [1970].) Side-stepping the issue of what categories are, Cartan and Eilenberg's equally influential book on homological algebra, which is about the role of certain functors, does not even attempt to define categories! (See Cartan and Eilenberg [1956].)

The books by Eilenberg and Steenrod and by Cartan and Eilenberg contained the seeds for the next developments of category theory in three important aspects. First, they introduced categories and functors into mathematical practice and were the source by which many students learned algebraic topology and homological algebra. This allowed for the assimilation of the language and notions as a matter of course. Second, they used categories, functors, and diagrams throughout and suggested that these were the right tools for both setting the problems and defining the concepts in these fields. Third, they employed various other tools and techniques that proved to be essential in the development of category theory itself. As such, these two books undoubtedly offered up the seeds that revolutionized the mathematics of the second half of the twentieth century and allowed category theory to blossom into its own.

1.2 *Categories as Mathematically Autonomous*

The [1945] introduction of the notions of category, functor, and natural transformation led Mac Lane and Eilenberg to conclude that category theory 'provided a handy language to be used by topologists and others, and it offered a conceptual view of parts of mathematics'; however, they 'did not then regard it as a field for further research effort, but just as a language of orientation' (Mac Lane [1988], pp. 334–335). The recognition that category theory was more than 'a handy language' came with the work of Grothendieck and Kan in the mid-fifties and published in 1957 and 1958, respectively.[2]

Cartan and Eilenberg had limited their work to functors defined on the category of modules. At about the same time, Leray, Cartan, Serre, Godement, and others were developing sheaf theory. From the start, it was clear to Cartan and Eilenberg that there was more than an analogy between the cohomology of sheaves and their work. In 1948 Mac Lane initiated the search for a general and appropriate setting to develop homological algebra, and, in 1950, Buchsbaum's dissertation set out to continue this development (a summary of this was published as an appendix in Cartan and Eilenberg's book). However, it was Grothendieck's Tôhoku paper, published in 1957, that really launched categories into the field. Not only did Grothendieck define abelian categories in that now classic paper, he also introduced a hierarchy of axioms that may or may not be satisfied by abelian categories and yet allow one to determine what can be constructed and/or proved in such contexts. Within this framework, Grothendieck generalized not only Cartan and Eilenberg's work, something which Buchsbaum had similarly done, but also generalized various special

[2] Mac Lane's work on group duality was certainly important with hindsight, but was not initially recognized as such by the mathematical community. In contrast, even at the outset, it was clear that Grothendieck and Kan's work was to have a profound impact on the mathematical community.

results on spectral sequences, in particular Leray's spectral sequences on sheaves.

In the context of abelian categories, as defined by Grothendieck, it came to matter not what the system under study is about (what groups or modules are 'made of'), but only that one can, by moving to a common level of description, *e.g.*, the level of abelian categories and their properties, cash out the claim, via the use of functors, that 'the Xs relate to each other the way the Ys relate to each other', where X and Y are now category-theoretic 'objects'. Providing the axioms of abelian categories[3] thus allowed for talk about the shared structural features of its constitutive systems, *qua* category-theoretic objects, without having to rely on what 'gives rise' to those features. In category-theoretic terminology, it allows one to characterize a type of structure in terms of the (patterns of) functors that exist between objects without our having to specify what such objects or morphisms are 'made of'. As McLarty points out:

> [c]onceptually this [the axiomatization of abelian categories] is not like the axioms for a abelian groups. This is an axiomatic description of the whole category of abelian groups and other similar categories. We pay no attention to what the objects and arrows are, only to what patterns of arrows exist between the objects. (McLarty [1990], p. 356)

More generally, since in characterizing a particular category, we need not concern ourselves with what the objects and morphisms are 'made of', there is no need to rely on set theory or NGB to tell us what the objects and morphisms of categories 'really are'. In the case of abelian categories, for example, we note that 'the basic [categorical] axioms let you perform the basic constructions of homological algebra and prove the basic theorems with no use of set theory at all' (McLarty [1990], p. 356).

At about the same time, *i.e.*, in the spring of 1956, Kan introduced the notion of adjoint functor. Kan was working in homotopy theory, developing what is now called combinatorial homotopy theory. He soon realized that he could use the notion of adjoint functor to unify various results that he had obtained in previous years. He published the unified version of these results, together with new homotopical results, in 1958 in a paper entitled 'Functors involving c.s.s. complexes'. For this paper to make sense to the reader, Kan had to write a paper on adjoint functors themselves. It was simply called 'Adjoint functors' and was published just before the paper on homotopy theory in the *AMS Transactions*. It was while writing the paper on adjoint functors that Kan discovered how general the notion was; specifically, he

[3] Mac Lane [1950], did not completely succeed in his attempt to axiomatize abelian categories. This was first done by Grothendieck [1957].

noted the connection to other fundamental categorical notions, *e.g.*, to the notions of limit and colimit. As Mac Lane himself observed, it took quite a while before the notion of adjoint functor was itself seen as a fundamental concept of category theory,[4] *i.e.*, before it was taken as the concept upon which a whole and autonomous theory could be built and developed. (See Mac Lane [1971a], p. 103.)

According to Mac Lane, category theory became an independent field of mathematical research between 1962 and 1967. (See Mac Lane [1988].) From the above, it is clear that abelian categories and adjoint functors played a key role in that development. One also has to mention the work done by Grothendieck and his school on the foundations of algebraic geometry, which appeared in 1963 and 1964; the work done by Ehresmann and his school on 'structured categories' and differential geometry in 1963; Lawvere's doctoral dissertation [1963]; and the work done on triples by Barr, Beck, Kleisli, and others in the mid-sixties. Perhaps more telling of its rising independence is the fact that the first textbooks on category theory appeared during this period, these starting with Freyd [1964], Mitchell [1965], and Bucur and Deleanu [1968]. The ground-breaking work of Quillen [1967], although not concerned with 'pure' category theory, but using categories in an indispensable way, should also be mentioned.

One can thus summarize the shifts required to recognize category theory as mathematically autonomous as follows:

1. In the first period, that is, from 1945 until about 1963, mathematicians started with kinds of set-structured systems, *e.g.*, abelian groups, vector spaces, modules, rings, topological spaces, Banach spaces, *etc.*, moved to the categories of such structured systems as specified by the morphisms between them, and then moved to functors between the now defined categories (these functors usually going in one direction only). Insofar as kinds of set-structured systems preceded the formation of a category, one could say that categories themselves were taken as types of set-structured systems (or class-structured systems, depending on the choice of the foundational framework) just as any other algebraic system.
2. In the sixties, it became possible to start directly with the categorical language and use the notions of object, morphism, category, and functor to define and develop mathematical concepts and theories in terms of cat-structured systems. In other words, one need not first define the types of structured systems one is interested in as kinds of

[4] It is interesting to note that even in Freyd's book, *Abelian Categories*, published in 1964, adjoint functors are introduced in the exercises, although Freyd himself was probably one of the very first mathematicians to recognize the importance of the concept.

set-structured systems and then move to the category of these kinds. Instead one defines a category with specific properties, the objects of which are the very kinds of structured systems that one is interested in. Thus the objects and their properties are characterized by the 'structure' of the category in which they are considered; this structure as presented by the (patterns of) morphisms that exist between the objects. The 'nature' of both the objects and morphisms is left unspecified and is considered as entirely irrelevant. Set-structured systems and functions may, of course, then be used to illustrate, exemplify, or represent (even in the technical, mathematical, sense of that expression) such 'abstract' categories, but they are not constitutive of what categories are.

3. The category-theoretic way of working and thinking points to a reversal of the traditional presentation of mathematical concepts and theories, *i.e.*, points to a top-down approach. This approach is best characterized by an adherence to a category-theoretic 'context principle' according to which one never asks for the meaning of a mathematical concept in isolation from, but always in the context of, a category.

An analogy with the concepts of spaces and points of spaces can be used to further illuminate this last shift.[5] It is well-known that two irreconcilable claims can be made about points and spaces: first, one can claim that points pre-exist spaces—that the latter are 'made of' points; second, one can claim that spaces pre-exist points—that the latter are 'extracted' or 'boundaries' of spaces, *e.g.*, line segments. Bringing this situation to bear on the category-theoretic case, the first claim corresponds to an 'atomistic' approach to mathematics, or, in the terminology of Awodey [2004], to a bottom-up approach. This approach is clearly expressed in Russell's philosophical and logical work, and, more generally, in most (if not all) set-theoretical accounts. The second claim, in contrast, corresponds to an 'algebraic' approach to mathematics, or, again in the terminology of Awodey [2004], to a top-down approach. It is this latter, top-down, approach that finds clear expression in category theory as it has developed since the mid-sixties. More to the point, the idea that this approach can be (and should be) extended and applied to logic and, more generally, to the foundations of mathematics itself is to be attributed to Lawvere, who first made such attempts in his Ph.D. thesis [1963].

[5] This, in fact, can be seen as much more than an analogy, since a categorical approach to points of spaces has been developed in this manner, mostly under the influence of Grothendieck, in a topos-theoretical framework. See Cartier [2001] for a survey.

1.3 *Categories and the Foundation of Mathematics*

In the late fifties and early sixties, it seemed possible to define various mathematical concepts and characterize many mathematical branches directly in the language of category theory and, in some cases, it appeared to provide the most appropriate setting for such analyses. As we have seen, the concepts of functor and the branches of algebraic topology, homological algebra, and algebraic geometry were prime examples. Lawvere took the next step and suggested that even logic and set theory, and whatever else could be defined set-theoretically, should be defined by categorical means. And so, in a more substantial way, he advanced the claim that category theory provided *the* setting for a conceptual analysis of the logical/foundational aspects of mathematics.

This bold step was initially considered, even by the founders of category theory, to be almost absurd. Here is how Mac Lane expresses his first reaction to Lawvere's attempts:

> [h]e [Lawvere] then moved to Columbia University. There he learned more category theory from Samuel Eilenberg, Albrecht Dold, and Peter Freyd, and then conceived of the idea of giving a direct axiomatic description of the category of all categories. In particular, he proposed to do set theory without using the elements of a set. His attempt to explain this idea to Eilenberg did not succeed; I happened to be spending a semester in New York (at Rockefeller University), so Sammy asked me to listen to Lawvere's idea. I did listen, and at the end I told him 'Bill, you can't do that. Elements are absolutely essential to set theory.' After that year, Lawvere went to California. (Mac Lane [1988], p. 342)

More precisely, Lawvere went to Berkeley in 1961–62 to learn more about logic and the foundations of mathematics from Tarski, his collaborators, and their students. One should note, however, that Lawvere's goal was to find an alternative, more appropriate, foundation for continuum mechanics; he thought that the standard set-theoretical foundations were inadequate insofar as they introduced irrelevant, and problematic, properties into the picture. In his own words:

> [t]he foundation of the continuum physics of general materials, in the spirit of Truesdell, Noll, and others, involves powerful and clear physical ideas which unfortunately have been submerged under a mathematical apparatus including not only Cauchy sequences and countably additive measures, but also *ad hoc* choices of charts for manifolds and of inverse limits of Sobolev Hilbert spaces, to get at the simple nuclear spaces of intensively and extensively variable quantities. But as Fichera

> lamented, all this apparatus gives often a very uncertain fit to the phenomena. This apparatus may well be helpful in the solution of certain problems, but can the problems themselves and the needed axioms be stated in a direct and clear manner? And might this not lead to a simpler, equally rigorous account? These were the questions to which I began to apply the topos method in my 1967 Chicago lectures. It was clear that work on the notion of topos itself would be needed to achieve the goal. I had spent 1961–62 with the Berkeley logicians, believing that listening to experts on foundations might be a road to clarifying foundational questions. (Perhaps my first teacher Truesdell had a similar conviction 20 years earlier when he spent a year with the Princeton logicians.) Though my belief became tempered, I learned about constructions such as Cohen forcing which also seemed in need of simplification if large numbers of people were to understand them well enough to advance further. (Lawvere [2000], p. 726)

With an eye toward presenting a 'simpler, equally rigorous account', Lawvere, in his Ph.D. thesis submitted at Columbia under Eilenberg's supervision, started working on the foundations of universal algebra and, in so doing, ended by presenting a new and innovative account of mathematics itself. In particular, he proposed to develop the whole theory in the category of categories instead of using a set-theoretical framework. The thesis contained the seeds of Lawvere's subsequent ideas and, indeed, had an immediate and profound impact on the development of category theory. As Mac Lane notes:

> Lawvere's imaginative thesis at Columbia University, 1963, contained his categorical description of algebraic theories, his proposal to treat sets without elements and a number of other ideas. I was stunned when I first saw it; in the spring of 1963, Sammy and I happened to get on the same airplane from Washington to New York. He handed me the just completed thesis, told me that I was the *reader*, and went to sleep. I didn't. (Mac Lane [1988], p. 346)

One of the key features of Lawvere's thesis is the use of adjoint functors; they are precisely defined, their properties are developed, and they are used systematically in the development of results. In fact, they constitute the main methodological tool of this work. More generally, the results themselves use categories and functors in an original way. As McLarty explains:

> [h]e [Lawvere] showed how to treat an algebraic theory itself as a category so that its models are functors. For example the

> theory of groups can be described as a category so that a group is a suitable functor from that category to the category of sets (and a Lie group is a suitable functor to the category of smooth spaces, and so on). (McLarty [1990], p. 358)

By both adopting a top-down approach and undertaking our analyses in a category-theoretic context, we can claim that an algebraic theory *is* a category and its mathematical models *are* functors.[6] Thus, our analysis of the very notion of an algebraic theory is itself characterized by purely categorical means, that is, by categorical properties in the category of categories. The category of models of an algebraic theory is amenable to the same analysis and, moreover, Lawvere showed how to 'recover' the theory from the category of models.

In 1964, Lawvere went on to axiomatize the category of sets and, in the same spirit, axiomatized the category of categories in 1966. It is important to emphasize that Lawvere did not, contrary to what Mac Lane had initially thought, try to 'get rid of' sets and their elements. Rather, he conceived of sets as being, like any other mathematical entity, part of the categorical universe. Such an analysis of the concept of category, in general, and of the concept of set, in particular, can thus be seen as an example of the use of the context principle: we are to ask about the meaning of these concepts only in the context of the universe of categories. Sets do play a role in mathematics, but this role should be analyzed, revealed, and clarified in the category-theoretic context.[7] More generally, this suggests that a mathematical concept, no matter what it is, is always meaningful (should be analyzed) in a context and that the universe of categories provides the proper context. Thus, the concept set ought to be analyzed by first considering categories of sets. One ought not start with sets and functions, rather, one should begin by looking for a purely category-theoretic context in which the characterization of set-structured categories can be given; this in the same way that abelian, algebraic, and other categories had been characterized. (See Blanc and Preller [1975], Blanc and Donnadieu [1976], and McLarty [1991] for more on using, in the spirit of Lawvere, the category of categories as such a context, and McLarty [2004] for more on using the elementary theory of the category of sets (ETCS) in a like manner.)

As is well known, Lawvere's foundational research did not stop there. Not long after completing the preceding work, Lawvere, inspired by

[6] For more detail on how a group can be described as a functor, see Adamek and Rosicky [1994], p. 138.

[7] In particular, Lawvere tried to tackle the issue of 'small' and 'large' categories in a categorical context. Joyal and Moerdijk [1995] provide a different but revealing illustration of the way a categorical approach can handle questions of size in algebraic set theory.

Grothendieck's use of toposes in algebraic geometry, formulated, in collaboration with Miles Tierney, the axioms of an elementary topos. As we have previously remarked, Lawvere's motivation was to find the appropriate setting for, or proper foundation of, continuum mechanics. (See Kock [1981], Lavendhomme [1996], and Bell [1998] for various aspects of this development.) More specifically, Lawvere was attempting to analyze the notion of 'variable set' as it arises in sheaf theory. He thus saw the theory of elementary toposes as the proper context for such an analysis and, indeed, as providing for a 'generalization' of set theory; this as analogous to the generalization from integers or reals to rings and R-algebras. As things turned out, the concept of an elementary topos was to have more far-reaching results, *e.g.*, it turned out to be adequate for conceptual analyses of forcing and independence results in set theory. (See Tierney [1972], Bunge [1974] for early applications. See also Freyd [1980], Scedrov [1984], Blass and Scedrov [1989], [1992].)

Perhaps even more significantly, it was then shown that an arbitrary elementary topos is equivalent, in a precise sense, to an intuitionistic higher-order type theory. Furthermore, the axioms of an elementary topos, when written as a higher-order type theory, were shown to be algebraic, i.*e*., they were shown to express basic 'equalities'. In this sense, categorical logic *is* algebraic logic. (See, for instance, Boileau and Joyal [1981], Lambek and Scott [1986].) As a further example, a category of sets was shown *to be* an elementary topos. Thus, in Lawvere's sense of the term, one can say that topos theory is a 'generalization' of set theory.[8] Speaking then to Lawvere's aims, it seems entirely possible to perform foundational research in a topos-theoretical setting, or, more generally, in a category-theoretic setting. But one must guard against a possible ambiguity concerning what is meant by the term 'foundational', for it turns out to mean different things to different mathematicians. However, despite these variations, it seems possible to state what is shared amongst category theorists interested in foundational research. It is to these variations, and to their common basis, that we now turn.

2. Categorical Foundations

We will admittedly be rather sketchy here and seek to give only an overview of the different 'foundational' positions found in the category-theory literature. We believe that five different positions can be identified: these are characterized by the works of Lawvere, Lambek, Mac Lane, Bell and Makkai. We will first detail these positions and then describe what we take to be the common standpoint of the categorical community.

[8] One has to be very cautious about what this claim entails. For an excellent account of the misuses of topos theory in foundational work, see McLarty [1990].

2.1 *Lawvere*

Lawvere's views on mathematical knowledge, the foundations of mathematics, and the role of category theory have evolved through the years. But, as we have seen, from his Ph.D. thesis onwards we find the conviction that category theory provides the proper setting for foundational studies. What Lawvere has in mind when considering foundational questions should be emphasized at the outset, for his considerations presume a creative mixture of philosophical and mathematical preoccupations. In his 1966 paper 'The category of categories as a foundation of mathematics', Lawvere claims that 'here by "foundation" we mean a single system of first-order axioms in which all usual mathematical objects can be defined and all their usual properties proved' (Lawvere [1966], p. 1). It is to this very conservative view of what a foundation ought to be that the axioms for a theory of the category of categories, which would be strong enough to develop most of mathematics (including set theory), are herein proposed.

It is important to note that, although Lawvere himself is aware of using the term 'foundations' differently at different times, his purpose is already both clear and steadfast: to provide the context in which a mathematical domain may be characterized categorically so that a top-down approach to the analysis of its concepts may be undertaken, *e.g.*, in the same way that abelian categories, algebraic categories, *etc.*, are characterized, namely by those categorical properties expressed by adjoint functors and/or by additional constraints (*e.g.*, by exactness conditions, by the existence of specific objects, *etc.*). Thus although Lawvere's [1966] explicit foundational goal is to develop a first-order theory,[9] his underlying motivation is perhaps more clearly expressed in another paper that was published in 1969, entitled 'Adjointness in foundations'. There we read that '[f]oundations will mean here the study of what is universal in mathematics' (Lawvere [1969], p. 281), the assumption being that what is universal is to be revealed by adjoint functors. Speaking then to his preference for top-down analyses in a categorical context, Lawvere here asserts that

> [t]hus Foundations in this sense cannot be identified with any 'starting-point' or 'justification' for mathematics, though partial results in these directions may be among its fruits. But among the other fruits of Foundations so defined would presumably be guide-lines for passing from one branch of mathematics to another and for gauging to some extent which directions of research are likely to be relevant. (Lawvere [1969], p. 281)

9 As was shown by Isbell [1967], Lawvere's original attempt was technically flawed, but not irrevocably. Isbell himself suggested a correction in his review, and Blanc and Preller [1975], Blanc and Donnadieu [1976], and McLarty [1991] all have made different proposals to circumvent the difficulty.

Examples of such 'other fruits' provided by category theory were already numerous when Lawvere expressed the foregoing sentiment: Eilenberg and Steenrod's work in algebraic topology; Cartan and Eilenberg's and Grothendieck's results in homological algebra; Grothendieck's writings in algebraic geometry; and, finally, Lawvere's work in universal algebra and, as he hoped, continuum mechanics.[10]

It should be clear from the above quote that Lawvere does not have an 'atomistic', or bottom-up, conception of the foundations of mathematics; there is no point in looking for an 'absolute' starting-point, a portion of mathematical ontology and/or knowledge that would constitute its bedrock and upon which everything else would be developed. In fact, Lawvere's position, far more than being top-down, is deeply historical and dialectical. (See Lawvere and Schanuel [1998].) This belief in the underlying foundational value of the historical/dialectical origins of mathematical knowledge has been explicitly expressed in a recent collaboration with Robert Rosebrugh:

> [a] foundation makes explicit the essential features, ingredients, and operations of a science as well as *its origins* and *general laws of development*. The purpose of making these explicit is to provide a guide to the learning, use, and further development of the science. A 'pure' foundation that forgets this purpose and pursues a speculative 'foundation' for its own sake is clearly a nonfoundation. (Lawvere and Rosebrugh [2003], p. 235; italics added)

It is clear that, for Lawvere, the proper setting for any foundational study ought to be a category (and in some cases, a category of categories). For most purposes, this background framework need not, for practical purposes, be made explicit, nor need it be used to any great depth, but since the underlying foundational goal is to state the universal/essential features of the science of mathematics by taking a top-down approach to the characterization of mathematical concepts in terms of category-theoretic concepts and properties thereof, it needs to be presumed. Notice, too, that there is no such thing as *the* foundation for mathematics; the overall framework itself is assumed as evolving. This assumption, in combination with the historical/dialectical nature of mathematical knowledge, means that rather than being prescriptive about what constitutes mathematics, 'foundations' are to be descriptive about both the 'origins' and the 'essential features' of mathematics.[11] (See Lawvere [2003], where the dialectical approach is explicitly adopted.) In the spirit of the aforementioned use of the context

[10] Other similar examples of this kind of 'foundations' not involving categories abound; Weyl's work on Riemann surfaces is but one remarkable case.

[11] We should also point out that Lawvere has recently launched a different foundational program: he has presented, and is still developing, a general classification of categories and

principle, Lawvere's descriptive account of foundations allows us to see how the universe of categories is taken as providing the context for both analyzing concepts in terms of their 'essential features' and, indeed, for understanding mathematics as a science of what is 'universal'.

2.2 *Lambek*

Lambek's work in the foundations of mathematics is radically different from Lawvere's. Although he is also clearly concerned with the history of mathematics, *e.g.*, Anglin and Lambek [1995], this interest does not seem to be reflected in his more philosophically motivated work.[12] Lambek has focused on investigating how the standard philosophical positions in the foundations of mathematics, namely, logicism, intuitionism, formalism, and Platonism, square with a categorical, or more specifically, a topos-theoretical approach to mathematics. In this light, he adopts a thoroughly logical standpoint toward foundational analyses, a point of view that he takes as being consistent with the standard conception of foundational work. Identifying toposes with higher-order type theories, Lambek has tried to show that:

1. The position framed by the so-called free topos, or more precisely, by pure higher-order intuitionistic type theory, is compatible with that of the logicist[13] and might be acceptable to what he calls moderate intuitionists, moderate formalists, and moderate Platonists. Lambek justifies this claim as follows:

> the free topos is a suitable candidate for the real (meaning ideal) world of mathematics. It should satisfy a moderate formalist because it exhibits the correspondence between truth and provability. It should satisfy a moderate Platonist because it is distinguished by being initial among all models and because truth in this model suffices to ensure provability. It should satisfy a moderate intuitionist, who insists that 'true' means 'knowable', not only because it has been constructed from pure intuitionistic type theory, but also because it illustrates all kinds of intuitionistic

toposes that is clearly philosophically motivated. For instance, one can talk of intensive categories and extensive categories, the distinction resting on simple categorical properties which themselves are meant to capture the difference between intensive qualities and extensive qualities. Similarly, some toposes are categories of spaces. The goal is to provide a categorical characterization of those toposes that are categories of spaces and, in so doing, yield a characterization of the notion of space itself.

[12] See, however, Lambek [1981].

[13] His position on this issue has evolved somewhat. In 1991, for example, he did not believe that a logicist could accept such a position.

> principles. The free topos would also satisfy a logicist who accepts pure intuitionistic type theory as an updated version of symbolic logic and is willing to overlook the objection that the natural numbers have been postulated rather than defined. (Lambek [1994], p. 58)

2. There is no ‘absolute’ topos that could satisfy the classical Platonist, although Lambek and Scott [1986] suggest that the moderate Platonist might accept any Boolean topos (with a natural-number object) in which the terminal object is a non-trivial indecomposable projective.[14] (See Lambek [2004].)

Some, such as Mac Lane in his review of Lambek and Scott, have objected to Lambek’s approach. However, the motivation for Mac Lane’s objection is not entirely clear; it may stem from his belief that there is more than one adequate foundational system for mathematics. The resulting nominalism[15] and the underlying assumption that a type theory is *the* fundamental system that one has to adopt[16] might also be the culprit. To have to make this assumption in the first place is taken by some as being unnecessarily complex and as not reflecting the ways in which mathematicians think and work. Lambek too has noted its more formal limitations, *viz.*, that ‘[t]ype theory as presented here suffices for arithmetic and analysis, although not for category theory and modern metamathematics’.[17] Yet despite this acknowledgment Lambek maintains that type theory can be a foundation at least to the degree that set theory can, and moreover, that it can provide for a philosophy more agreeable than those inspired by set-theoretical investigations.

[14] Lambek and Scott did not realize at that point that such a topos could be described more simply by saying that the terminal object is a generator. It is not clear who was the first to make this latter characterization.

[15] Lambek has called his position ‘constructive nominalism’. (See Lambek [1994], [1995]; Couture and Lambek [1991]; Lambek and Scott [1980], [1981], [1986].)

[16] The nominalism referred to here is a consequence of the fact that the free topos, which is taken as the ideal world of mathematics, is the topos generated by pure type theory. Hence all the entities involved are (equivalence classes) of linguistic entities. It should be noted that these linguistic entities may be transfinite and that the type formation may also be transfinite. As such it need not be feasible to either inscribe or utter these expressions of these entities. It is indeed a very moderate form of both nominalism and intuitionism.

[17] This is less straight-forward than it might seem. In fact it is a quite delicate issue. It is clear, however, that a substantial amount of category theory can be done internally, *i.e.*, within a topos. See, for instance, McLarty [1992], Chapter 20. One of the subtler issues that is left to be dealt with has to do, again, with large categories, *e.g.*, the category of *all* groups.

2.3 *Mac Lane*

Mac Lane's position on foundations is somewhat ambiguous and has evolved over the years. As a founder of category theory, he did not at first see category theory as providing a general foundational framework. As we have seen, he and Eilenberg thought of category theory as a 'useful language' for algebraic topology and homological algebra. In the sixties, under the influence of Lawvere, he reconsidered foundational issues and published several papers on set-theoretical foundations for category theory. (See Mac Lane [1969a], [1969b], [1971].) Although clearly enthusiastic about Lawvere's work on the category of categories, he never fully endorsed that position himself. After the advent of topos theory in the seventies, he advanced the idea that a well-pointed topos with choice and a natural-number object might offer a legitimate alternative to standard ZFC, thus going back to Lawvere's ETCS programme but in a topos-theoretical setting. The point underlying this proposal was to convince mathematicians of the *possibility* of alternative foundations, and so was not aimed at showing that category theory was a definite or 'true' framework. This proposal, together with Mac Lane's other pronouncements against set theory as *the* foundational framework, led to a debate with the set-theorist Mathias, and ended with the publication of Mathias's 2001 paper which sought to prove some of the mathematical limitations of Mac Lane's proposal. (See Mac Lane [1992], [2000] and Mathias [1992], [2000], [2001].)

Mac Lane's views on foundations follow from his convictions about the nature of mathematical knowledge itself, which we cannot possibly hope to address in detail here. In a nutshell, as set out in his book *Mathematics Form and Function*, mathematics is presented as arising from a formal network based on (mostly informal but objective) ideas and concepts that evolve through time according to their function. It is in this light, of seeing mathematics as form and function, that we are to understand why Mac Lane has stated, on various occasions, his opinion concerning the inadequacy of both foundations and 'standard' philosophical positions about mathematical ontology and knowledge. Thus, when we read his repeated calls for new research in these areas (See Mac Lane [1981] and [1986].) we are to understand that these appeals do not arise from a preference for either a set-theoretic or category-theoretic perspective, but rather are to note that, in their attempts to deal with mathematics as form and function, 'none of the usual systematic foundations or philosophies . . . seem . . . satisfactory' (Mac Lane [1986], p. 455).

2.4 *Bell*

Bell's position is somewhat akin to Lambek's, but with certain important differences. Like Lambek, Bell has an interest in the history of mathematics.

(See Bell [2001].) While in 1981 Bell argued explicitly *against* category theory as a foundational framework, he also recommended the development of a topos-theoretical 'outlook'. Later, like Lawvere, he adopted a distinctly dialectical attitude towards foundations, asserting, for example, that 'the genesis of category theory is an instance of the dialectical process of replacing the constant by the variable' and 'the [dialectical process] of negating negation . . . underlies two key developments in the foundations of mathematics: Robinson's nonstandard analysis and Cohen's independence proofs in set theory' (Bell [1986], pp. 410, 421).

By 1986 he had also begun to attach more significance to the foundational role of category theory, coming to view toposes and their associated higher-order intuitionistic type theories, or in his terminology 'local set theories', as providing a network of 'co-ordinate systems' within which one could both fix and analyze, albeit only *locally*, the meanings of mathematical concepts. It should be pointed out, too, that Bell suggests that the types in such a context be thought of as 'natural kinds', and so sets can only be subsets of these natural kinds, whence the term 'local'. In this respect, these 'local frameworks of interpretation' came to be seen as serving a role analogous to frames of reference of relativity theory. (See Bell [1981], [1986], [1988].) It is precisely for this reason that, in contrast to Lambek, Bell does not argue in favor of one specific topos, or kind of topos, as a 'candidate for the real world of mathematics'. Rather, he endorses a pluralist top-down approach towards the foundations of mathematics. As he explains:

> the topos-theoretical viewpoint suggests that the absolute universe of sets be replaced by a plurality of 'toposes of discourse', each of which may be regarded as a possible 'world' in which mathematical activity may (figuratively) take place. The mathematical activity that takes place within such 'worlds' is codified within local set theories; it seems appropriate, therefore, to call this codification *local mathematics*, to contrast it with the *absolute* (*i.e.*, classical) mathematics associated with the absolute universe of sets. *Constructive provability* of a mathematical assertion now means that it is *invariant*, *i.e.*, valid in *every* local mathematics. (Bell [1988], p. 245)

As in the case of Lambek's proposal, it is recognized that category theory itself cannot be developed fully in this framework, but it nonetheless remains foundationally significant. This is because it speaks to the value of taking a top-down approach to the analysis of mathematical concepts from within a category-theoretic context, albeit a local one. And more so because it speaks to the 'algebraic' structuralists' attempt to overlook the 'concrete' (atomistic) nature of kinds of mathematical systems in favour of abstractly characterizing the shared structure of such kinds in terms of the morphisms between them. Again, as Bell explains

> . . .with the rise of abstract algebra. . . the attitude gradually emerged that the crucial characteristic of mathematical structures is not their internal constitution as set-theoretical entities but rather the relationship among them as embodied in the network of morphisms. . . However, although the account of mathematics they [Bourbaki] gave in their *Éléments* was manifestly structuralist in intention, in actuality they still defined structures as sets of a certain kind, thereby failing to make them truly independent of their 'internal constitution'. (Bell [1981], p. 351)

2.5 *Makkai*

Makkai's motivation is both philosophical and technical. Technically, he takes very seriously the fact that a topos-theoretical perspective cannot provide an adequate foundation for category theory itself. Thus, on Makkai's view, one has to face the question of the foundations of category theory, *i.e.*, the question of what is to be an appropriate metatheory. To this end, and following Lawvere, Makkai's aim is to provide a metatheoretic description of a category of categories. From a logician's point of view, this means:

- providing a proper syntax for the theory, which is, according to Makkai, provided by FOLDS, that is, first-order logic with dependent sorts. (See Makkai [1997a], [1997b], [1997c], [1998].)
- providing a proper background universe for the interpretation of the theory, *e.g.*, a universe that would play an analogous role to the one played by the cumulative hierarchy in set theory, and which is, according to Makkai's account, the universe of higher-dimensional categories, or weak n-categories. (See Hermida, Makkai, and Power [2000], [2001], [2002].)
- providing a theory as such that would be adequate for category theory and, perhaps, a large part of 'abstract' mathematics. (See Makkai [1998] for this and a short and very clear synthesis of his foregoing papers.)

Philosophically, Makkai has explored how these issues are related to mathematical structuralism, which he characterizes as follows:

> I take it to be a tenet of structuralism that everything accessible to rational inquiry is a structure; the conceptual world consists of structures. (Makkai [1998], p. 155)

Makkai's fundamental contribution to a category-theoretically framed structuralism is the idea that, in formal languages, the relation of identity for entities is not given *a priori* by first-order axioms. The relation of identity is *derived* from within a context. This position, then, is a natural and coherent extension of a structurally interpreted context principle: one has first to determine a context for talking about shared structure; then a criterion of identity for objects having that structure is given by the context itself. The simplest example of this is the suggestion that the notion of isomorphism is the proper criterion of identity for objects in a category and that it is defined by categorical means. In this sense a category acts as a context for analyzing kinds of systems in terms of their shared structure.

The systematic development of this idea, *i.e.*, the consideration that types of (higher-level) categories can act as a context for analyzing the shared structure of kinds of categories, may be seen as naturally leading to higher-dimensional categories, also known as weak n-categories. (See Leinster [2002] for a review of the various definitions in the literature.) Although it is not yet clear whether such structuralism can be made systematic, Makkai's work points to the belief, common among categorists, that the category-theoretic methods of analysis that mathematicians use to talk about kinds of structured systems in terms of their shared structure (methods that have perhaps proved far more powerful in proving theorems than older methods) also speak to the power of such methods to provide a more adequate framework for a conceptual account of mathematics itself.

2.6 *Some Common Elements*

The first, and probably most important, common element present in all the previous developments, and shared by all category theories and categorical logicians, is the assumption that by adopting a top-down approach to analyzing mathematical concepts the 'shared structure' between abstract mathematical systems can be accounted for in terms of the morphisms between them. For example, as we have seen in Lawvere's [1969] work, adjoint functors are taken to reveal fundamental structural connections between kinds of abstract mathematical systems. Second, it is fair to say that category theorists and categorical logicians believe that mathematics does not require a unique, absolute, or definitive foundation and that, for most purposes, frameworks logically weaker than ZF are satisfactory. Categorical logic, for instance, is taken to provide the tools required to perform an analysis of the shared logical structure, in a categorical sense of that expression, involved in any mathematical discipline. Third, the categorical perspective shows that it is not necessary to assume that mathematics is 'about' sets. Although sets may in some contexts be descriptive, *e.g.*, some types of categories might have a set structure, they are not constitutive of the structure of categories themselves, *i.e.*, types of categories need not be ‘built up from’ kinds of set-structured systems. In accordance with (or perhaps as a consequence of) the previous claim, there is, from a categorical perspective, no unique conception of a set, although the notion of topos, in the categorical context, captures the fundamental structural characteristics of the concept. Finally, category theorists and categorical logicians endorse, either implicitly or explicitly, the aforementioned context principle: the top-down approach to characterizing mathematical concepts in a category-theoretic context is taken to be the means by which we should analyze the ‘shared structure’ of mathematical concepts (presented as objects and categories) in terms of the morphisms that exist between them.

3. Philosophical Implications

It should be obvious by now that category theory ought to have an impact on current discussions of mathematical structuralism. In fact, we can point to a common philosophical position that threads itself through the foundational positions here considered, namely, the structuralist belief that ‘mathematics studies structure and that mathematical objects are nothing but positions in structures . . .’ (Resnik [1996], p. 83). On the one hand, however, it is far from clear that all category theorists, even those with a foundationalist orientation, would call themselves structuralists.[18] On the other hand, perhaps the reason for this is that it is far from clear, unfortunately, what structuralism amounts to. In this closing section we will attempt to clarify the various interpretations and versions of philosophically positioned mathematical structuralism, and consider the extent to which category theory can be used to frame a structuralist philosophy of mathematics.

3.1 *Mathematical Structuralism as a Philosophical Position*

The slogan that mathematics studies structure is itself interpreted in at least two different ways. On the first interpretation, the slogan amounts to the claim that mathematics is about *structures* (Bourbaki [1950], [1968]; Resnik [1996], [1999]; and Shapiro [1996], [1997]). On the second interpretation, it amounts to the claim that mathematics is about systems that ‘have’ a structure, or that mathematics is about *structured systems* (Mac Lane [1996b]; Awodey [1996], [2004]; and Hellman [1996], [2001], [2003]). Setting aside this difference for the moment, mathematical

[18] See, for example, Taylor [1999] who claims that his position is closer to a form of logicism than to anything else. And, as we have seen, Lambek [1994] considers himself a nominalised platonist.

structuralism is further found at two distinct levels: the concrete and the abstract.[19]

Before attending in detail to these interpretations and levels, and to situate our claims with respect to the current philosophical literature, we note Hale's [1996] distinction between what he calls model-structuralism, abstract-structuralism, and pure-structuralism. What Hale (and Hellman [1996]) calls model-structuralism, we will characterize as structuralism at the concrete level; and what he calls abstract-structuralism, will characterize as bottom-up, 'set theoretic',[20] *ante rem* structuralism at the abstract level. Structuralists of this stripe seek to define what an abstract structure is, as an independently existing entity, by abstractly considering concrete kinds of set/place-structured systems and by considering those abstractly considered kinds as constitutive of what an abstract structure is. In Hale's words:

> [according to the abstract-structuralist] . . . structures . . . are entities in their own right, akin in some respects to model-structures, but distinguished from them by the fact that their elements have no non-structural properties, but are to be conceived

[19] At the abstract level of structural analysis, this difference of interpretation can be taken as corresponding to Shapiro's ([1996], [1997]) *ante rem*/*in re* distinction or Dummett's [1991] mystical/hard-headed distinction. In this paper, at both the concrete and abstract levels of analysis, we speak of systems that have a structure; this is a means of indicating that we take the aim of the structuralist to provide an account of the shared structure of mathematical systems in terms of their being an instance of the same kind or type, as opposed to having to answer the questions: 'What is a structure?', or 'What are the kinds or types that are constitutive of what a structure is?'. For example, Bourbaki held *both* that there are types of structures *and* that sets are constitutive of these types and so that a 'structure' is an abstractly considered type of set-structured system. Speaking to this analysis of structures in terms of types, we note Shapiro's claim that 'according to Bourbaki, there are three great types of structures, or "mother structures": algebraic structures, such as group, ring, field; order structures, such as partial order, linear order, and well order; and topological structures [which provide a formalization of the concepts of limit, neighbourhood, and continuity] . . .' (Shapiro [1997], p. 176). What Shapiro leaves out is the manner in which set-theory is taken as constitutive of these types: that '[e]ach [type of] structure carries with it its own language, freighted with special intuitive references derived from the theories which the axiomatic analysis . . . has derived the structure. . . Mathematics . . . possesses the powerful tools furnished by the theory [i.e., set theory] of the great type of structures; in a single view, it sweeps over immense domains, now unified by the axiomatic method. . . ' (Bourbaki [1950], pp. 227–228) so that '. . . whereas in the past it was thought that every branch of mathematics depended on its own particular intuitions which provided its concepts and primary truths, nowadays it is known to be possible, logically speaking, to derive practically the whole of mathematics from a single source, the theory of sets' (Bourbaki [1968], p. 9).

[20] To appreciate why we consider both Bourbaki and Shapiro's notion of 'structure' as a bottom-up 'set-theoretic' conception we point the reader to Shapiro's ([1997], p. 96) claim that 'set theory and the envisioned structure theory are notational variants of each other. In particular, structure theory without the reflection principle is a variant of second-order ZFC'.

> as no more than 'bare positions'[21] in the structure . . . On this approach, an abstract-structure is just is what is left when, beginning with a model-structure, we abstract away from all that is inessential, leaving behind only what is common to all other model-structures isomorphic to it. (Hale [1996], p. 125)

Finally, what he calls pure-structuralism, we will characterize as top-down, 'algebraic', *in re*[22] structuralism at the abstract level:

> . . . [it] has no truck with abstract-structures as entities at all . . . the theory tells us what holds true of any collection of objects satisfying a certain structural description, but speaks of no one such collection of objects. The terms of the theory . . . are not to be understood as genuine singular terms . . . not even bare positions in an abstract-structure—but rather are to be interpreted as purely schematic or variable. (Hale [1996], p. 125)

We pause here to point out that while Hellman is typically read (see Hale [1996]) as advancing a modalized version of pure, *in re*, structuralism, we prefer to interpret him as arguing for a top-down, '*non*-algebraic', *in re* structuralism at the abstract level. While, as an *in re* structuralist, he does not claim that 'structures' exist or that they are made up of 'objects', he does hold that statements about possible types of abstract structured systems are determined by assertions about possible systems, so that the terms of any 'theory' of structured systems are not purely schematic or variable but rather are terms of 'modalized assertions'. This with the result that

> [c]ategorical axioms of logical possibility of various types of structures replace ordinary existence axioms of MT [model theory] or CT [category theory] and typical mathematical theorems are represented as modal universal conditionals asserting what would necessarily hold in any structure of the appropriate type that there might be. (Hellman [1996], p. 102)

Having noted the way in which we intend our terminology to be understood, we now turn to consider these distinctions in greater detail. At the *concrete* level, mathematical structuralism (or model-structuralism)[23] is

[21] Shapiro [1997] refers to such an abstractly considered 'bare positions' account as the *ante rem* 'places-are-objects perspective', and distinguishes this from the *in re* 'places-are-offices perspective'.

[22] Another term for *in re* structuralism is *eliminative* structuralism, since the *in re* structuralist eliminates talk of 'structures' in favor of talk of systems that 'have' a structure.

[23] Dummett [1991], as we shall see, argues that the term 'structuralism' ought to only be applied to structuralism at the abstract level, *i.e.*, that model-structuralism ought not be labeled structuralism.

the philosophical position that the subject matter of a particular mathematical theory is concrete kinds of structured systems (models) and their morphology. A particular kind of mathematical object, then, is nothing but 'a position in a concrete system' that has a kind of structure; and a particular mathematical theory aims to characterize such kinds of objects 'up to isomorphism', that is, in terms of the shared structure of those concrete systems in which they are positions. For example, the theory of natural numbers, as characterized by the Peano axioms, may be seen as providing a framework[24] for presenting those concrete kinds of structured systems (models) that have the same natural-number structure (that are isomorphic). Its objects, *i.e.*, natural numbers, may then be presented as nothing but positions in a concrete system that is structured by the axioms[25] that characterize that kind, *e.g.*, may be presented as von Neumann ordinals, Zermelo numerals, or, indeed, as any other object which shares the same structure. If all concrete systems that exemplify this structure are isomorphic, we say that the natural-number structure and its morphology determine its objects 'up to isomorphism'.[26]

[24] At the concrete level, to say that the axioms provide a framework is not intended to be read as the 'formalist' claim that a theory ought to be viewed syntactically as an empty form or uninterpreted calculus devoid of content. One could equally interpret this semantically; one could say that, though framed by its axioms, a theory is the collection of all its isomorphic models, *i.e.*, is the collection of all its concrete systems that have the same kind of structure. One must also distinguish, then, the formalist claim that a mathematical theory is about contentless form from the essentialist/Fregean claim that a mathematical theory is about independently existing 'objects', and so, too, from the structuralist claim that a mathematical theory is about concrete systems that satisfy the axioms that are claimed to characterize a kind of structure, so that a *kind of* object is characterized by being a position in any concrete system that satisfies the axioms. (For a similar point, see also Benacerraf's ([1965], pp. 285–294) distinction between the formalist, the Fregean, and what he calls the 'formist'.) Such a structuralist view of particular mathematical theories as axiom systems is characterized by Weyl ([1949], pp. 25–27): '. . . an axiom system is a logical mold of possible sciences . . . . A science can determine its domain of investigation up to an isomorphic mapping. In particular it remains quite indifferent as to the "essence" of its objects . . .'. Weyl then goes on to distinguish further between what we call the concrete and the abstract levels of structuralism. That is, while a particular mathematical theory, what he calls 'a science', can be presented in terms of all those concrete systems (models) that share the same kind of structure, '[p]ure mathematics . . . develops the theory of logical "molds" without binding itself to one or the other among possible concrete interpretations . . .' We will show, in the next paragraph, how this aim of 'pure mathematics' corresponds to top-down, 'algebraic', structuralism at an abstract level.

[25] To get the isomorphism results needed, *i.e.*, to guarantee categoricity in the logical sense of the term, we assume that the axioms are second-order.

[26] In reference, then, to Benacerraf [1965], structuralism at the concrete level implies that there are no numbers as 'objects' *qua* things whose essence can be individuated independently of the role they play in a concrete system of a given kind. There are, in our terminology, only *kinds of* objects *qua* positions in concrete systems that have the same kind of structure, *i.e.*, that can be individuated only up to isomorphism. We note, however,

So, at the concrete level, the structuralist has three possible replies to the question: 'What are natural numbers?'. First, she may reply, in Hilbertian style,[27] that they are positions in *any* concrete system that has the appropriate kind of structure, *i.e.*, in any interpretation that satisfies the axioms that are taken to characterize the natural-number structure, *e.g.*, that satisfies the Peano axioms. That is, in reply to the question: 'What allows us to talk about particular objects as instances of the same kind of structure?', the Hilbert-inspired structuralist replies: 'The axioms that are claimed to characterize the kind of structure in question provide us with a *framework* that in turn allows us to characterize as objects "up to isomorphism" all those positions that "have" the same kind of structure.' Second, likewise in Hilbertian-style, the structuralist may simply reject the question 'What are natural numbers?' and argue that such a framework does not licence us to talk about natural numbers as objects at all; rather one ought to eliminate talk of (reference to) natural numbers as objects. All such seeming reference is to be understood as a convenient device for 'filling-in' the following schema: 'Let a structure of a kind (*e.g.*, a natural-number structure) be given (based on the assumption of possibility), then ...', where the '...' introduces constants that are only schematic, *i.e.*, that are allowed by the axioms *qua* defining conditions, and so spell out the given kind of structure, but are not thought of as genuinely referring to natural numbers as objects at all.

In either case, as a Hilbert-inspired structuralist, one eschews the Fregean demand that, before we turn to talking about natural numbers (as, for example, objects that saturate concepts in the context of a sentence), one must first provide a *background theory* for talking about natural numbers as "objects"[28] *qua* independently existing things. This Frege-inspired position

that Benacerraf moves, bottom-up, from the concrete to the abstract level of analysis by abstractly considering all such positions in concrete systems of a given kind as 'elements' of an abstract kind, *i.e.*, as elements of an ω-sequence. (See also Benacerraf [1996] for a re-evaluation of this.)

[27] In what follows we are not attempting to read either Hilbert or Frege as structuralists. There are many reasons for avoiding such claims; for example, even if one could so interpret Hilbert's *Grundlagen der Geometrie* in this light, his subsequent neo-Kantian aim of founding all of mathematics on a finitary/intuitive arithmetic certainly precludes any structural analysis of arithmetic and so too of mathematics. In addition, Frege's assumption that there is a fixed universe of discourse certainly precludes the structuralist/model-theoretic picture we have presented. However, the interested reader is strongly encouraged to see Shapiro [1996], pp. 161–170, and [1997], pp. 152–170, for a reconstruction of Hilbert as an *in re* structuralist and Frege as an *ante rem* structuralist.

[28] To assist the reader, when we use the term 'object' or 'structure' in an ontological sense, *i.e.*, in the sense of existing independently of us and of language, we write "object" and "structure".

represents the third possible reply: natural numbers are “objects” that exist, as Shapiro ([1997], p. 168) explains, ‘. . . in exactly the same way as any other objects, including horses, planets, Caesars, and pocket watches.’[29] On this view, then, axioms are truths or assertions about “objects” that are in the background theory. It is to this Hilbert/Frege distinction between viewing an axiom system as a framework, or scaffolding or schemata,[30] and viewing axioms as truths or assertions of some background theory[31] that a category-theoretic account of mathematical structuralism has much to say.

[29] This is where the difference between the Fregean and Shapiro’s *ante rem* structuralism becomes apparent. The Fregean holds that there is a fixed and absolute domain of “objects”, while the *ante rem* structuralist holds that ‘[w]hen it comes to mathematics, the Fregean all inclusive domain gives way to the ontological relativity urged here. Each mathematical object *is a place* in a particular structure . . .’ (Shapiro [1997], p. 169; italics added). We will see, however, that, once we move to structuralism at the abstract level, Shapiro retains the Frege-style demand for a background theory that fixes the meaning of the term ‘structure’; by taking up his objects-are-places view we can conclude that a “structure” *is* a place-structured system.

[30] Our use of the term ‘framework’, then, is intended to accord with Hilbert’s claim that ‘. . . it is certainly obvious that every theory is only a scaffolding or schema of concepts together with their necessary relations to one another, and that the basic elements can be thought of in any way one likes . . . . One only needs to apply a reversible one-one transformation and lay it down that the axioms shall be correspondingly the same for all transformed things’ (Hilbert [1899], pp. 40–41). And so it is to be understood in light of Hilbert’s use of the axiomatic method, as implemented in his *Grundlagen*. Bernays best sums up this use as follows:

> [a] main feature of Hilbert’s axiomatization of geometry is that the axiomatic method is presented and practiced in the spirit of the abstract conception of mathematics that arose at the end of the nineteenth century and which has been adopted in modern mathematics. It consists in. . . understanding the assertions (theorems) of the axiomatized theory in a hypothetical sense, that is, as holding true for any interpretation. . . for which the axioms are satisfied. Thus, an axiom system is regarded not as a system of statements about a subject matter but as a system of conditions for what might be called a relational structure. (Bernays [1967], p. 497)

For an informative overview of the Frege/Hilbert debate and its relation to the *ante rem*/*in re* debate see Shapiro [1996], [1997].

[31] This second position, as exemplified by Hellman’s *in re* modal-structuralism, is thus Hilbertian to the extent that the ‘standard’ axioms encountered in ordinary mathematics are treated as schematic, *i.e.*, are treated as ‘defining conditions’ on systems said to have a given kind of structure. Yet external, Fregean, modal-existence axioms as assertions also are needed to speak to the ‘assumption of possibility’ of their being a system having the structure in question. This means that the ‘standard’ axioms show up in the antecedents of modalized, universally quantified conditionals in which the primitive constants of the axiom system in question have been replaced by variables.

3.2 *Interpretations and Varieties of Mathematical Structuralism*

At the next level, the *abstract* level, mathematical structuralism may be characterized as the philosophical position that the subject matter of mathematics itself is abstract kinds of structured systems (or what others have called abstract structures) and their morphology. Viewed from this level, an abstract kind of mathematical object is nothing but 'a position in an abstract system' that itself has an abstract kind (or type[32]) of structure, and an abstract mathematical theory aims to characterize such types of abstract systems in terms of their shared structure. It is at this abstract level of inquiry, then, that one encounters the question: 'What are (abstract) structures?'. In response to this question one finds, in the philosophical literature, two interpretations and three varieties of philosophically positioned mathematical structuralism. The two interpretations, already touched upon in brief, are: *ante rem* (realist) and *in re* (nominalist) structuralism. Shapiro explicates these as follows: the *ante rem* structuralist believes

> that structures exist as legitimate objects of study *in their own right*. According to this view, a given structure exists *independently of any system that exemplifies it*... Mathematical objects, such as natural numbers, are places in these structures. So numerals, for example, are *genuine singular terms denoting genuine objects*, the objects being *places* [as opposed to placeholders] in a structure. (Shapiro [1996], pp. 149–150; italics added)

The *in re* structuralist, by contrast, believes that

> [a] statement of arithmetic is *not taken at face value* as a statement about a particular collection of objects. Instead, a statement of arithmetic *is a generalization over all systems of a certain type*... Thus, [*in re* structuralism] does not countenance mathematical objects, or structures for that matter, as *bona fide* objects. Talk of numbers is convenient shorthand for talk about all systems that exemplify the structure. Talk of structure generally is convenient shorthand for talk about systems of objects. (Shapiro [1996], p. 150; italics added)

Foregoing, for the moment, Shapiro's conflation here of structuralism at the concrete and abstract levels, these two interpretations correspond, in a rough and ready way, to the Hilbert/Frege distinction at the concrete level.

[32] We have used the term 'type' here to express the claim that abstract systems can be taken as instances of the same type of structure by satisfying those axioms that are taken to characterize that type. This is intended in a way analogous to the claim that concrete systems can be taken as instances of the same kind of structure by satisfying those axioms that are taken to characterize that kind.

That is, in response to the question: 'What are abstract structures?', the *in re* structuralist says, in Hilbertian or 'algebraic' style: 'They are anything that satisfy the axioms that are taken to characterize the abstract kind, or type, of structure under consideration'. This is because, given our Hilbertian stance, the question can be re-phrased as: 'What framework allows us to talk about abstract kinds of structured systems as instances of the same type?' That is, for the *in re* 'algebraic'[33] structuralist, an abstract kind of structured system is an object only if it can be considered as a position in another type of structured system.[34] One foregoes talking about abstract structures as "objects" in favour of talking about abstract kinds of systems that 'have' a type of structure. Thus one eschews, once again, the Fregean demand that, before talking about abstract structured systems as objects *qua* positions in a type of structured system, one must first provide a background theory for talking about (making assertions about) types of structures, or "structures", themselves as "objects" *qua* independently existing things.

Failing, then, to heed Resnik's counsel that abstract structuralism is not committed to asserting the independent existence of "structures", yet,

[33] In contrast to Shapiro, who focuses his abstract structural analysis on what he calls 'concrete, non-algebraic, theories', we intend our top-down 'algebraic' analysis to apply to both of what he calls 'algebraic' and 'non-algebraic' theories. In a similar vein, while Hale [1996] sees pure-structuralism as applying only to algebraic theories, we see it as applying to both. For more on this distinction, see Shapiro [1997], especially pp. 40–41, 50, 73n, and 133. See Resnik [1996] and Hellman [1996], [2003], for views that, like ours, consider both algebraic and non-algebraic theories. Note, however, that Resnik takes axioms to be assertions and Hellman takes certain modal-existence axioms to be assertions.

[34] While Resnik is often characterized as an *ante rem* structuralist in Shapiro's sense, he does not believe, as Shapiro does, that realism about structures must be guaranteed by an ontology of "structures" that exist both independently of us and of language. In fact (see next footnote) he would agree with the *in re* structuralist that an abstract kind of structured system is an object only if it can be considered as a position in another type of structured system. For Resnik, we affirm realism about structures, neither by adopting an ontology nor by accepting an axiom system, but rather by committing ourselves to a language of interpretation. As Resnik explains:

> [t]his . . . is how first-order structuralists can affirm their realism. In other words, they can make the language of mathematics their own, if they have not already done so, and simply assert the appropriate axioms. Until we [interpret the language of, say, number theorists in terms of ours or make it ours by learning it] their sentences might as well be schemata. Yet once we make their language part of our own we can apply our truth-predicate to it, and attribute truth-values to its sentences. (Resnik [1996], p. 98)

So Resnik's realism, or *ante rem* structuralism, is born out of linguistic considerations, not ontological ones. The *in re* structuralist, in contrast, does not countenance objects or structures as real "objects", linguistic or otherwise, and so only needs to *accept* the axioms as schemata, as opposed to needing to *assert* them.

in response to this worry,[35] three varieties of mathematical structuralism have been proposed. These are: the *set-theoretic*, the *sui generis*, and the *modal*.[36] In essence, these varieties seek to speak to the Fregean demand for pre-conditions for the independent existence of abstract structures; they suggest set-theory, structure-theory, or modal-logic as background theories[37] (or meta-languages) that allow us to talk about "structures" as either actually or possibly existing "objects". As such they allow us to say that either set-theory, structure-theory, or modal logic, provides the conditions for our *asserting* the actuality or possibility of an abstract system's *being* a structure of the appropriate type. In any case, in taking structures to be "objects", we either run into the problems of having to assume a foundational 'background ontology' and/or of the 'reification of structure', or we make mathematics dependent on a primitive notion/logic of possibility. The end result being that structuralism provides no improvement, either ontological or epistemological, over platonism.[38]

Where does category theory fit in among all these interpretations and varieties of mathematical structuralism? One could claim that categories are, after all, types of structured sets, and thus that a structuralism framed by category theory falls under the set-theoretic variety of structuralism.[39] As we indicated in the opening paragraph of this paper, we believe that

[35] Typically, the worry of the abstract structuralist is, as Resnik explains: '. . . whether first-order structuralism is strong enough for this realism'. Yet Resnik successfully dissolves this worry by pointing out that,

> structuralism even when combined with realism, is *not* committed to holding that number theory, analysis and set theory *assert the existence of structures*. Indeed, a general mathematical theory of structures asserts the existence of structures *only by representing them as positions in other structures,* so structuralism in not committed to viewing structures as entities at all". (Resnik [1996], p. 96; italics added)

Yet, while not committed to asserting the independent existence of "structures", Resnik, as an *ante rem* structuralist, is committed to asserting the truth of the axioms.

[36] See Hellman [2001], for an excellent overview of these varieties and the problems associated with each.

[37] That Shapiro himself sees structuralism in this Fregean light is evidenced by his claim that '. . . on *any* structuralist programme, some background theory is needed. The present options are set theory, modal model theory, and *ante rem* structure theory' (Shapiro [1997], p. 96; italics added).

[38] See Hale [1996] for an informative discussion of why what he calls abstract-structuralism, be it set or place inspired, can do little to overcome the ontological problems traditionally associated with mathematical platonism. And more so for a thoughtful criticism of why a modalized version of pure-structuralism faces serious epistemological problems.

[39] There is no doubt, too, that the proponents of the *sui generis* and the modal approaches have made similar claims, *i.e.*, that category theory falls under the general strategy they have put forward; see, in particular, Shapiro [1997], pp. 87–88, and Hellman [1996], p. 104.

this claim fails to do justice to the actual practice of category theory and, more importantly, fails to recognize the fact that category theory *is* both a foundational and philosophical alternative.

Our claim is that, in taking abstract kinds of structured systems, categories included, as “objects” (either possible or actual), all *ante rem* varieties of philosophically interpreted mathematical structuralism have failed us. Underlying this mistaken stance is the aforementioned conflation of concrete and abstract levels of structuralism, which derives from the assumption that abstract kinds of structured systems or “structures” *qua* “objects” are ‘made up’ of *abstractly considered* concrete kinds of structured systems. Such a bottom-up structuralist holds that one moves to “structures” at the abstract level by abstractly considering a kind of concrete system. The set-theoretic structuralist, for example Bourbaki, construes an abstract kind of object as an ‘element’ in an abstractly considered concrete kind of set, so that a type of structure is ‘made up’ of appropriately related abstractly considered set-structured systems.[40] And, likewise, the place-theoretic structuralist, for example Shapiro, construes an abstract kind of object as a place, *i.e.*, as an abstractly considered concrete kind of position, so that a ‘structure’ is ‘made up’ of an appropriately related, abstractly considered, system consisting of places-as-objects.

We again pause to note that we do not intend here to read Hellman’s modal approach as a bottom-up *ante rem* variety of structuralism; on the contrary, it is obviously meant as an *in re* interpretation. As indicated by the title of his 1996 article ‘Structuralism without structures’, Hellman clearly does not see “structures” as *actually* existing independently of any system that has a given type of structure. However, his modal aim appears nonetheless to be founded on the (external) Fregean assumption that one requires pre-conditions for the possibility that there is a system of the appropriate type. It is in this sense that we characterize Hellman’s view as a top-down, ‘*non*-algebraic’, *in re* structuralist position. While, for Hellman, the axioms for any type of structured system need not be assertory/true in the robust ontological Fregean sense, certain modal-existence axioms need to be assertory to guarantee the assumption of possibility that such a type can be given (Hellman [2003], p. 7). This with the result that ‘Fregean axioms only appear externally, as it were, in the form of modal-existence axioms (mainly of infinity) and comprehension principles governing wholes and pluralities.’[41] Modal structuralism, then, is intended to apply to this assertory requirement, yet it is this same requirement that leaves Hellman vulnerable to the objection that even his modalized version of structuralism must be concerned with whether there are enough possible objects to

[40] See footnote 18 for an explanation of this claim.

[41] Hellman, e-mail correspondence.

'make up' his possible types of structured systems.[42] Thus, for all varieties it is assumed that certain conditions, either truth conditions or modal conditions, for the assertion of the actual existence of "structures" or possible existence of types of structured systems must be provided *before* we seek to give a framework for what we can say about the shared structure of abstract kinds of structured systems.

In all cases, in concerning ourselves with background theories and/or pre-conditions of existence or assertion, we seem inevitably returned to Hale's abstract-structuralism, with little room left for pure-structuralism. Witnessing the confusion that this engenders is Dummett's remark that:

> [t]here is an unfortunate ambiguity in the standard use of the word 'structure', which is often applied to an algebraic or relational system—a set with certain operations or relations defined on it, perhaps with some designated elements; that is to say, a model considered independently of any theory which it satisfies. This terminology hinders a more abstract use of the word 'structure'; if, instead we use 'system' for the foregoing purpose, we may speak of two systems as having an identical structure, in this more abstract sense, just in case they are isomorphic. The dictum that mathematics is the study of structure is ambiguous between these two senses of 'structure'. If it is meant in the less abstract sense, the dictum is hardly disputable, since any model of a mathematical theory will be a structure in this sense. It is probably usually intended in accordance with the more abstract sense of 'structure'; in this case, it expresses a philosophical doctrine that may be labeled 'structuralism'. (Dummett [1991], p. 295)

While Dummett's analysis is in some sense helpful, in that it separates the concrete level from the abstract level, it, too, confuses top-down (pure in Hale's sense) accounts with those that are bottom-up (abstract in Hale's sense), *i.e.*, it confuses accounts that presume that abstract structures as objects must be presented as positions in types of abstract structured systems with accounts that presume either that abstract structures as types of "objects" must be 'made up' of abstractly considered concrete kinds of objects, like sets or places, or that assertions about types of systems must be modalized. These latter presumptions, however, are merely a residue of the Fregean assumption that axioms are assertions, as opposed to schemata. How, then, does bottom-up structuralism differ from top-down structuralism? In the case of bottom-up structuralism, one must first provide

[42] See Hale [1996] and Shapiro [1997] for a discussion of the problems associated with having to justify this assumption. Note too that Shapiro [1997], p. 93, foregoes any demand for justifying this assumption and simply adds the claim that 'there is at least one structure that has an infinite number of places' as an axiom of his structure theory.

a Frege-inspired background theory. In the case of top-down structuralism, this requirement is simply dropped in favour of providing a Hilbert-inspired framework.

3.3 *Structures* versus *Schemata*

In light of this difference, and in line with our Hilbertian path, we will focus on clarifying, and providing a framework for, the notion of an abstract system as a *schema*, instead of focusing on clarifying, and providing background theories for, the notion of an abstract structure as an “object”. Thus, our aim as an *in re*, yet ‘algebraic’, structuralist is not the analysis of the constitutive character of “structures” or the modal status of assertions about types of structured systems, but rather the analysis of the shared structure of abstract systems in terms of types of structured systems.[43] And, as category theorists, in addition to taking such a top-down approach, we heed our adherence to the aforementioned context principle and so consider this analysis from within a category-theoretic context.

As explained, the problem with standard structural approaches is that they cleave to the residual Fregean assumption that there is one unique context that provides us with the pre-conditions for the actual existence of “structures” or for the possible existence of types of structured systems. As we previously tried to illustrate, in a categorical framework the context, though systematized by the category-theoretic axioms, varies, and so a mathematical concept has to be thought of in a context that can be varied in a systematic fashion. It is our claim that, in this sense, a categorical framework provides us with the conditions a *context* has to satisfy in order for us to talk about or do mathematics. Such a framework allows us to attend to how abstract kinds of structured systems may be seen as instances of the same type, and further provides us with the proper means to understand how such structural contexts may vary and yet are, nonetheless, still related to one another.

Consider, by way of illustration, Hale’s [1996] example of group theory as a purely structural theory. Group theory must be presented in a certain

[43] It should be underscored, however, that Hellman [2003], does appreciate the distinction between the algebraic-schematic use of categories (what he calls the ‘algebraico-structuralist’ perspective, p. 9), but his suggestion that the ‘problem of the “home address” remains’ (pp. 8, 15) clearly indicates that he is still thinking of structures (be they categories or toposes) as “objects” requiring conditions for the possibility of existence. In fact, if, on the algebraic approach, the aim of structuralism is to account for the shared structure of abstract kinds of structured systems in terms of schematic types, as opposed to answering ‘What is (or where is!) a structure?’, then why should we be troubled by the fact that ‘[b]y themselves they [the category-theoretic axioms] assert nothing’ that ‘They merely tell us what it is to be a structure of a certain kind’ and thus are ‘unlike the axioms of set theory, [in that] its axioms are not assertory?’ (p. 7).

language, and categories can be used to carve out contexts for that purpose. The models of the theory can be represented as functors from the theory considered as a category to another category with the right properties, which can themselves be abstractly represented in the language of category theory, *i.e.*, in a Cartesian category. At this stage, we are already in *a* category of categories (notice that we are in not in *the* category of categories). One can then investigate groups in a specific context: in a category of differential manifolds, an internal group is a *Lie* group; in a category of groups, an internal group is an *Abelian* group, *etc*. In other words, what the terms of the theory refer to depends on an underlying category-theoretic context and the latter can vary and yet be expressed in a systematic way so as to reveal its type of structure, *e.g.*, to reveal its group structure.

Against Hale and Shapiro, the same 'algebraic' analysis applies equally well to non-algebraic theories, *e.g.*, to theories of natural numbers or real numbers or sets. One can write down the usual axioms for such structured systems and interpret them in various contexts; and what an appropriate context is can be precisely specified using a categorical framework. Thus, we do not have to say that

> [non-algebraic theories] ... go against the thesis [of pure-structuralism]. Such theories are replete with what appear to be singular terms for particular mathematical objects. . . . which form their ostensible subject matter. The pure structuralist must hold that the surface syntax of such theories presents an entirely misleading appearance, to be dispelled by some suitably eliminative paraphrase [like that provided by modal-structuralism]. (Hale [1996], p. 125)

What is misleading here is the reintroduction of the idea that there is a unique context for all such theories, *i.e.*, that the singular terms to be organized according to their type have to be uniquely interpreted. The terms of the theory are variable precisely because the contexts of interpretation are variable, but they are nonetheless related to one another systematically, *i.e.*, are related in another specifiable context.

We believe, then, that the real difference between abstract-structuralism and pure-structuralism, and the reason why the terminology turns out from our point of view to be misleading, is that the distinction relies upon the process of abstraction itself. This point is left entirely open in the literature (with, of course, the notable exception of Awodey [2004]). The question at hand is: 'What, for the mathematical structuralist, is the direction of abstraction?'. More specifically, is abstraction top-down or bottom-up? Do we *begin with* or *arrive at* the notion of an abstract system? Does this notion depend on the 'things' upon which the abstraction process is carried out? As we mentioned, for the abstract, bottom-up, structuralist an abstract kind of structured system is *arrived at* by abstractly considering a concrete system *qua* a system of a specific kind. The 'details' of the underlying system might be forgotten, but the abstract system depends directly on the 'structure' of these concrete systems. Our claim is that, as the history of mathematics and the history of category theory show, the abstraction process, once it has at its disposal an appropriate language that allows one to express identity conditions adequately, yields an *autonomous level of description*, which does not depend on an underlying system and/or any background theory. Moreover, it is in this respect that we claim that category theory provides a framework that allows one to *begin with* the notion of an abstract system.

Once the abstract level has reached this autonomy, one begins with abstract kinds of structured systems: this autonomy becomes ontologically significant once criteria of identity are given and used systematically throughout. In other words, the identification of the various abstract kinds of structured systems depends solely on the criterion provided by the new level of description. For instance, groups were long considered to be groups *of* permutations (in algebra) or groups *of* transformations (in geometry). As such, the identity of a group was thought to be determined by, or depend upon, a previously given entity with its own criterion of identity, *e.g.*, a geometry presented axiomatically or algebraically. We claim that for groups to be considered purely abstractly, mathematicians had to have an axiomatic presentation of the concept together with a criterion of identity for the entities that fall under that concept, *i.e.*, had to have the correct notion of group isomorphism, which was then used intrinsically to determine which groups there are. As sketched above, category theory provides just such an axiomatization and so provides us with an autonomous language together with criteria of identity for such contexts.

Pure-structuralism as characterized by a top-down account of an abstract system is now truly pure in the following sense: the axioms for a category provide the framework, or scaffolding, for what we can say about abstract kinds of structured systems independently of what those kinds are 'made of'. Taking our top-down approach we *begin with* the notion of an *abstract system*; we do not seek to *arrive at* this notion by abstractly considering a kind of concrete system. In its most general sense, then, an abstract system is considered in a Hilbertian, 'algebraic', sense, as a *schema* for our talk of the shared structure of an abstract kind of structured system: it allows us to talk about such abstract kinds of structured systems as instances of the same type without our having to consider what these types are types of. Considered then from within a category-theoretic context, a cat-structured abstract system has 'objects' and 'morphisms' as its abstract kinds, which are structured by the category-theoretic axioms. This means that a type of cat-structured system, *i.e.*, an abstract kind of structured system *qua* a type of category

> . . . is *anything* satisfying these axioms. The objects need not have 'elements', nor need the morphisms be 'functions'. . . We do not really care what non-categorical properties the objects and morphisms of a given category may have; that is to say, we view it 'abstractly' by restricting to the language of objects and morphisms, domains and codomains, composition, and identity morphisms. (Awodey [1996], p. 213)

That is, the axioms for a category provide the context from within which we can analyze the shared structure of abstract kinds of structured systems in terms of the morphisms that exist between them.

For example, suppose one wanted to give a characterization of abstract kinds of set-structured systems in terms of *the* category of sets. Consider what this would mean: *the* category of sets would be a category satisfying additional axioms expressed in the language of category theory, *e.g.*, having finite limits and colimits, exponentials, a subobject classifier, *etc*. (See, for instance, Lawvere's [1964] original proposal or Lawvere and Rosebrugh [2003], Chapters 6 and 9.) One would then try to show that any two categories satisfying these axioms have to be categorically *equivalent* (not isomorphic). It is only in this way that one could talk about *the* category of sets. It is not that it is *unique*, but that it is unique *up to a specified criterion of identity*. However, we are simplifying the situation somewhat: the notion of equivalence used would, itself, also be part of a context. Be that as it may, the category of sets would be specified without our having to consider what the abstract kind 'set' is 'made of' (*e.g.*, without our having to consider it as being made up of elements or consider it as being an element of a class) and so, too, without our having to specify the axioms that 'give rise' to systems having this structure (*e.g.*, ZF, ZFC, GB).

Returning to the philosophical implications, at once we see important differences from the standard bottom-up accounts of mathematical structuralism: on the category-theoretic view, not only are there are no abstract kinds of "objects", *e.g.*, either sets-with-structure (Dummett [1991], p. 295) or places-with-structure (Shapiro [1997]), there are no abstract kinds of "structures", *e.g.*, either '(equivalence types of) systems-with-structure' or 'the abstract form of a system, highlighting the interrelationships among the objects' (Shapiro [1997], p. 74). This means that the bottom-up conception of an abstract kind of system (of an abstractly considered concrete kind of system whose abstract "objects" are 'positions in a set-structure', or 'positions in a place-structure') is to be considered as *a* type of abstract kind of structured system: it is not, however, *the* archetype of either the concept 'system' or the concept 'structure'. As Corry explains of Mac Lane [1980]:

> Bourbaki's concepts defined 'mathematical structures' by taking an abstract set and appending to it an additional construct,

> in category theory there is no subordination of 'mathematical structures' to sets, and this is the source of the supremacy of this theory over Bourbaki. (Corry [1996], p. 382)

A category, too, is neither a privileged abstract kind of system nor is it an abstract Fregean "structure" *qua* an "object": it is a Hilbertian style abstract structure *qua* a schematic type, to be used as a framework for expressing what we can say about the shared structure of the various abstract kinds of structured systems in terms of 'having' the same type of structure. Again, as Mac Lane makes clear:

> ... a structure is essentially a list of operations and relations and their required properties, commonly given as axioms, and often so formulated as to be properties shared by a number of possibly quite different specific mathematical objects... [A] mathematical object 'has' a particular structure when specified aspects of the objects satisfy the (standard) list of axioms for the structure. This notion of 'structure' is clearly an outgrowth of the widespread use of the axiomatic method in mathematics [as exemplified by Hilbert's *Grundlagen*]. (Mac Lane [1996b], pp. 174 and 176)

3.4 *An Algebraic* in re *Interpretation*

The value of this top-down, 'algebraic', or schematic, notion of structure is not that it provides a 'constitutive foundation' for mathematics, but rather that it can be used to provide a 'descriptive foundation' for what we mean by the structuralist claim that (pure) mathematics studies structure, where we interpret this as the claim that mathematics is about systems that 'have' a structure, or is about structured systems.[44] In this sense, category theory provides a framework for a top-down *in re* interpretation of mathematical structuralism; a category provides a context from within which we can analyze the shared structure of abstract kinds of structured systems, *independently* of any abstractly considered concrete kind of structured system, *e.g.*, independently of its set-structure or place-structure or, more generally, independently of what its abstract kinds are 'made of'.

For example, in the type of category called **Top**, we present the shared topological-structure of any abstract kind of structured system by taking

[44] If one sees, as Mac Lane seems to, the task of a philosophy of mathematics to be, in addition to adopting a structuralist stance, that of providing for an epistemologically tractable account of both the form and function of mathematics, then one might agree with him that the notion of structure as here presented 'seems at best just one possible aspect of an adequate philosophy of mathematics. Such an adequate philosophy is not now available' (Mac Lane [1996b], p. 183). But if one sees a structuralist philosophy of mathematics as adequate, then this notion of structure appears to be the best one possible.

'objects' as abstract kinds of topological spaces and 'morphisms' as abstract kinds of continuous mappings, independently of what those abstract kinds are kinds of.[45] Awodey nicely describes the situation as follows:

> ... suppose we have somehow specified a particular kind of structure in terms of objects and morphisms... Then that category characterizes that kind of mathematical structure, independently of the initial means of specification. For example, the topology of a given space is determined by its continuous mappings to and from the other spaces, regardless of whether it was initially specified in terms of open sets, limit points, a closure operator, or whatever. The category **Top** thus serves the purpose of characterizing the notion of 'topological structure'. (Awodey [1996], p. 213)

Similarly, as detailed above, we can present the shared abstract kind of structure of any set-structured system as a type of cat-structured system, by taking our 'objects' to be sets and our 'morphisms' to be functions. The result is the type of category called **Set**. In this context the type of category, **Set**, allows us to talk about the shared structure of all abstract kinds of set-structured systems[46] as instances of the same abstract kind, without our having to ask what these kinds are kinds of. In any case, we may say that the result of our so framing the abstract kind of set-structured system is the type of category called **Set**. But this does not mean that the 'objects' *are* sets and that 'morphisms' *are* functions; it means rather, that in this type of system, propositions that talk about 'objects' and 'morphisms' can be interpreted as being about abstract kinds of sets and functions. So, Shapiro is simply mistaken to claim that

[45] That Hellman has not appreciated the distinction between constitutive and descriptive aims is witnessed by his claim that 'the categorical foundationalist cannot take these [topological] notions for granted. The very notions of "open set", "collection of open sets closed under finite intersections and arbitrary union", "inverse image of an open set", and "continuous function" must be *built up* somehow from categorical primitives' (Hellman [2003], p. 8; italics added). If the category theorist's aim is to give a constitutive account of, say, 'topological space', then he might, as Hellman suggests, need to consider the 'preconditions of intelligibility' of this concept. If, however, his aim is descriptive then he can, as Awodey suggests, begin by assuming such preconditions have be laid down.

[46] For instance, in his recent book with Rosebrugh, Lawvere [2003] presents axioms for categories of sets, which include what he calls variable sets and constant sets. One of the goals of this book is to present a purely axiomatic conception of *abstract* sets, which have properties, according to Lawvere, different from variable or cohesive sets. Those properties are, mainly: abstract sets form a category with finite limits and colimits and exponentiation; they have a Boolean two-valued subobject classifier; and they satisfy the categorical version of the axiom of choice and an axiom of infinity. Thus, sets, be they variable or constant, form a topos, but abstract sets form a special kind of topos, *i.e.*, they satisfy additional properties.

> [t]he category theorist characterizes a structure or type of structure in terms of structure-preserving functions, called 'morphisms', between systems that exemplify the structures.[47] (Shapiro [1997], p. 93)

As explained above, any category-theoretic context is not intended as a characterization of what a "structure", or type of structure, is; what such a category theorist, who takes structure-preserving 'morphisms' as functions, characterizes is a type of category, *viz*., **Set**, but this is not intended as an archetype for a "structure" *qua* a category.

We are now in a position to see how Shapiro's mistake undermines his argument for the necessity of 'structure theory'[48] over and above category theory: to wit, not every structuralist programme requires a background theory that tells us what "structures" *qua* "objects" are, because a top-down, 'algebraic', approach, framed in the language of category theory, does not require structures as "objects", either possible or actual, nor does it require axioms as truths or assertions. Thus, contra Shapiro, yet in line with Hellman and Resnik, we need *not* claim that categories exist as "objects" independently of any abstract system that exemplifies them: categories too are only claimed to exist in virtue of their being an 'object' in some (possibly higher-level) type of cat-structured system. The frameworks for such systems could be provided, as McLarty [2004] points out, via the category-of-categories approach (by CCAF axioms) or via the elementary theory of the category-of-sets approach (by ETCS axioms).[49]

Foregoing, then, Fregean concerns of pre-conditions for the existence of categories, categories can, as schematic types, act as Hilbertian

[47] We should point out that, as early as 1945, Eilenberg and Mac Lane had already given two examples of categories that were not 'made up' of set-structured systems and structure-preserving functions; specifically they showed that: i) any group G can be considered to be a one-object category with the morphisms being the elements of the group, and ii) any poset P can be considered as a category with the objects being the elements of the P and the morphisms given by the ordering relation. Interestingly enough, Eilenberg and Mac Lane did not mention these cases in their original list of examples of categories, but they are introduced and used as heuristic devices.

[48] See Shapiro [1997], pp. 93–97, for the characterization and justification of his structure theory.

[49] As McLarty [2004], p. 43, points out, the question of the existence of categories is not a question of whether its axioms are assertory or not:

> Indeed category theory *per se* has no such [assertory] axioms, but that is no lack, since category theory *per se* is a general theory applicable to many structures. Each specific categorical foundation offers various quite strong existence axioms.

frameworks; the category-theoretic axioms need not be truths (as Shapiro requires) nor need they make assertions (as Resnik or Hellman require) to be useful for our analysis of the shared structure of abstract kinds of structured systems in terms of their having the same type of structure. Thus, as Bernays says of Hilbert's conception of an axiom system,[50] a category, as a schematic type, is to be regarded not as a system of statements about a subject matter, *i.e.*, about either "structures" or about possible types of structured systems, but rather is taken as a context specifying those conditions for what might be called a relational structure, *i.e.*, for what might be called a type of structure in Mac Lane's sense of the term.

Simply put, to talk about the shared structure of abstract kinds of structured systems in terms of types of cat-structured systems, there is no need for either set theory or structure theory or modal logic. A category can (and does) act as a schematic type that is used to frame what we say about the shared structure of abstract kinds of mathematical systems, in terms of types of cat-structured systems, and for types of cat-structured systems, in terms of the abstract types of categories. And, speaking against the need for any background theory, it does so *without* our having to specify what either kinds or types are 'made of'. Thus, to be an 'algebraic' *in re* structuralist about abstract kinds of mathematical systems, we need not state what a structure *is*, nor need we say what a category *qua* a structure *is*, in the ontological or modal sense of 'is'. All we need to do is provide the appropriate context from within which we can talk about the shared structure of these abstract kinds in terms of schematic types, *i.e.*, in terms of types of cat-structured systems.

We have shown that if category theory is taken as the framework for what we say about the shared structure of abstract kinds of mathematical systems, then we can account for an interpretation of mathematical structuralism that respects both the category theorist's top-down approach and his use of the category-theoretic context principle. More significantly, we can use this framework to provide an interpretation that, against both the standard *ante rem* and *in re* interpretations, neither requires a 'theory of structures' nor demands the elimination of types of structured systems as objects, and that, against the various varieties, does not require us to replace, or even reconstruct, talk of the shared structure between abstract kinds of structured systems with talk of either set-structures, place-structures, or modalities.

[50] See footnote 30.

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