

Functor

In mathematics, specifically category theory, a **functor** is a map between categories. Functors were first considered in algebraic topology, where algebraic objects (such as the fundamental group) are associated to topological spaces, and maps between these algebraic objects are associated to continuous maps between spaces. Nowadays, functors are used throughout modern mathematics to relate various categories. Thus, functors are important in all areas within mathematics to which category theory is applied.

The word *functor* was borrowed by mathematicians from the philosopher Rudolf Carnap,^[1] who used the term in a linguistic context;^[2] see function word.

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Definition

Let *C* and *D* be categories. A **functor** *F* from *C* to *D* is a mapping that^[3]

- associates to each object *X* in *C* an object *F*(*X*) in *D*,
- associates to each morphism *f*: *X* → *Y* in *C* a morphism *F*(*f*): *F*(*X*) → *F*(*Y*) in *D* such that the following two conditions hold:
 - *F*(*id*_{*X*}) = *id*_{*F*(*X*)} for every object *X* in *C*,
 - *F*(*g* ∘ *f*) = *F*(*g*) ∘ *F*(*f*) for all morphisms *f*: *X* → *Y* and *g*: *Y* → *Z* in *C*.

That is, functors must preserve identity morphisms and composition of morphisms.

Covariance and contravariance

There are many constructions in mathematics that would be functors but for the fact that they "turn morphisms around" and "reverse composition". We then define a **contravariant functor** *F* from *C* to *D* as a mapping that

- associates to each object *X* in *C* an object *F*(*X*) in *D*,

- associates to each morphism $f: X \rightarrow Y$ in C a morphism $F(f): F(Y) \rightarrow F(X)$ in D such that the following two conditions hold:
 - $F(\text{id}_X) = \text{id}_{F(X)}$ for every object X in C ,
 - $F(g \circ f) = F(f) \circ F(g)$ for all morphisms $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ in C .

Note that contravariant functors reverse the direction of composition.

Ordinary functors are also called **covariant functors** in order to distinguish them from contravariant ones. Note that one can also define a contravariant functor as a *covariant* functor on the opposite category C^{op} .^[4] Some authors prefer to write all expressions covariantly. That is, instead of saying $F: C \rightarrow D$ is a contravariant functor, they simply write $F: C^{\text{op}} \rightarrow D$ (or sometimes $F: C \rightarrow D^{\text{op}}$) and call it a functor.

Contravariant functors are also occasionally called *cofunctors*.^[5]

There is a convention which refers to "vectors"—i.e., vector fields, elements of the space of sections $\Gamma(TM)$ of a tangent bundle TM —as "contravariant" and to "covectors"—i.e., 1-forms, elements of the space of sections $\Gamma(T^*M)$ of a cotangent bundle T^*M —as "covariant". This terminology originates in physics, and its rationale has to do with the position of the indices ("upstairs" and "downstairs") in expressions such as $x'^i = \Lambda^i_j x^j$ for $\mathbf{x}' = \mathbf{\Lambda} \mathbf{x}$ or $\omega'_i = \Lambda^j_i \omega_j$ for $\boldsymbol{\omega}' = \boldsymbol{\omega} \mathbf{\Lambda}^T$. In this formalism it is observed that the coordinate transformation symbol Λ^j_i (representing the matrix $\mathbf{\Lambda}^T$) acts on the basis vectors "in the same way" as on the "covector coordinates": $\mathbf{e}_i = \Lambda^j_i \mathbf{e}_j$ —whereas it acts "in the opposite way" on the "vector coordinates" (but "in the same way" as on the basis covectors: $\mathbf{e}^i = \Lambda^i_j \mathbf{e}^j$). This terminology is contrary to the one used in category theory because it is the covectors that have *pullbacks* in general and are thus *contravariant*, whereas vectors in general are *covariant* since they can be *pushed forward*. See also Covariance and contravariance of vectors.

Opposite functor

Every functor $F: C \rightarrow D$ induces the **opposite functor** $F^{\text{op}}: C^{\text{op}} \rightarrow D^{\text{op}}$, where C^{op} and D^{op} are the opposite categories to C and D .^[6] By definition, F^{op} maps objects and morphisms identically to F . Since C^{op} does not coincide with C as a category, and similarly for D , F^{op} is distinguished from F . For example, when composing $F: C_0 \rightarrow C_1$ with $G: C_1^{\text{op}} \rightarrow C_2$, one should use either $G \circ F^{\text{op}}$ or $G^{\text{op}} \circ F$. Note that, following the property of opposite category, $(F^{\text{op}})^{\text{op}} = F$.

Bifunctors and multifunctors

A **bifunctor** (also known as a **binary functor**) is a functor whose domain is a product category. For example, the Hom functor is of the type $C^{\text{op}} \times C \rightarrow \mathbf{Set}$. It can be seen as a functor in *two* arguments. The Hom functor is a natural example; it is contravariant in one argument, covariant in the other.

A **multifunctor** is a generalization of the functor concept to n variables. So, for example, a bifunctor is a multifunctor with $n = 2$.

Examples

Diagram: For categories C and J , a diagram of type J in C is a covariant functor $D: J \rightarrow C$.

(Category theoretical) presheaf: For categories C and J , a J -presheaf on C is a contravariant functor $D: C \rightarrow J$.

Presheaves: If X is a topological space, then the open sets in X form a partially ordered set $\text{Open}(X)$ under inclusion. Like every partially ordered set, $\text{Open}(X)$ forms a small category by adding a single arrow $U \rightarrow V$ if and only if $U \subseteq V$. Contravariant functors on $\text{Open}(X)$ are called presheaves on X . For instance, by assigning to every open set U the associative algebra of real-valued continuous functions on U , one obtains a presheaf of algebras on X .

Constant functor: The functor $C \rightarrow D$ which maps every object of C to a fixed object X in D and every morphism in C to the identity morphism on X . Such a functor is called a *constant* or *selection* functor.

Endofunctor: A functor that maps a category to that same category; e.g., polynomial functor.

Identity functor: in category C , written 1_C or id_C , maps an object to itself and a morphism to itself. The identity functor is an endofunctor.

Diagonal functor: The diagonal functor is defined as the functor from D to the functor category D^C which sends each object in D to the constant functor at that object.

Limit functor: For a fixed index category J , if every functor $J \rightarrow C$ has a limit (for instance if C is complete), then the limit functor $C^J \rightarrow C$ assigns to each functor its limit. The existence of this functor can be proved by realizing that it is the right-adjoint to the diagonal functor and invoking the Freyd adjoint functor theorem. This requires a suitable version of the axiom of choice. Similar remarks apply to the colimit functor (which assigns to every functor its colimit, and is covariant).

Power sets: The power set functor $P : \mathbf{Set} \rightarrow \mathbf{Set}$ maps each set to its power set and each function $f : X \rightarrow Y$ to the map which sends $U \subseteq X$ to its image $f(U) \subseteq Y$. One can also consider the contravariant power set functor which sends $f : X \rightarrow Y$ to the map which sends $V \subseteq Y$ to its inverse image $f^{-1}(V) \subseteq X$.

For example, if $X = \{0, 1\}$ then $F(X) = \mathcal{P}(X) = \{\{\}, \{0\}, \{1\}, X\}$. Suppose $f(0) = \{\}$ and $f(1) = X$. Then $F(f)$ is the function which sends any subset U of X to its image $f(U)$, which in this case means $\{\} \mapsto f(\{\}) = \{\}$, where \mapsto denotes the mapping under $F(f)$, so this could also be written as $(F(f))(\{\}) = \{\}$. For the other values, $\{0\} \mapsto f(\{0\}) = \{f(0)\} = \{\{\}\}$, $\{1\} \mapsto f(\{1\}) = \{f(1)\} = \{X\}$, $\{0, 1\} \mapsto f(\{0, 1\}) = \{f(0), f(1)\} = \{\{\}, X\}$. Note that $f(\{0, 1\})$ consequently generates the trivial topology on X . Also note that although the function f in this example mapped to the power set of X , that need not be the case in general.

Dual vector space: The map which assigns to every vector space its dual space and to every linear map its dual or transpose is a contravariant functor from the category of all vector spaces over a fixed field to itself.

Fundamental group: Consider the category of pointed topological spaces, i.e. topological spaces with distinguished points. The objects are pairs (X, x_0) , where X is a topological space and x_0 is a point in X . A morphism from (X, x_0) to (Y, y_0) is given by a continuous map $f : X \rightarrow Y$ with $f(x_0) = y_0$.

To every topological space X with distinguished point x_0 , one can define the fundamental group based at x_0 , denoted $\pi_1(X, x_0)$. This is the group of homotopy classes of loops based at x_0 . If $f : X \rightarrow Y$ is a morphism of pointed spaces, then every loop in X with base point x_0 can be composed with f to yield a loop in Y with base point y_0 . This operation is compatible with the homotopy equivalence relation and the composition of loops, and we get a group homomorphism from $\pi(X, x_0)$ to $\pi(Y, y_0)$. We thus obtain a functor from the category of pointed topological spaces to the category of groups.

In the category of topological spaces (without distinguished point), one considers homotopy classes of generic curves, but they cannot be composed unless they share an endpoint. Thus one has the **fundamental groupoid** instead of the fundamental group, and this construction is functorial.

Algebra of continuous functions: a contravariant functor from the category of topological spaces (with continuous maps as morphisms) to the category of real associative algebras is given by assigning to every topological space X the algebra $C(X)$ of all real-valued continuous functions on that space. Every continuous map $f : X \rightarrow Y$ induces an algebra homomorphism $C(f) : C(Y) \rightarrow C(X)$ by the rule $C(f)(\varphi) = \varphi \circ f$ for every φ in $C(Y)$.

Tangent and cotangent bundles: The map which sends every differentiable manifold to its tangent bundle and every smooth map to its derivative is a covariant functor from the category of differentiable manifolds to the category of vector bundles.

Doing this constructions pointwise gives the tangent space, a covariant functor from the category of pointed differentiable manifolds to the category of real vector spaces. Likewise, cotangent space is a contravariant functor, essentially the composition of the tangent space with the dual space above.

Group actions/representations: Every group G can be considered as a category with a single object whose morphisms are the elements of G . A functor from G to **Set** is then nothing but a group action of G on a particular set, i.e. a G -set. Likewise, a functor from G to the category of vector spaces, **Vect** $_K$, is a linear representation of G . In general, a functor $G \rightarrow C$ can be considered as an "action" of G on an object in the category C . If C is a group, then this action is a group homomorphism.

Lie algebras: Assigning to every real (complex) Lie group its real (complex) Lie algebra defines a functor.

Tensor products: If C denotes the category of vector spaces over a fixed field, with linear maps as morphisms, then the tensor product $V \otimes W$ defines a functor $C \times C \rightarrow C$ which is covariant in both arguments.^[7]

Forgetful functors: The functor $U : \mathbf{Grp} \rightarrow \mathbf{Set}$ which maps a group to its underlying set and a group homomorphism to its underlying function of sets is a functor.^[8] Functors like these, which "forget" some structure, are termed *forgetful functors*. Another example is the functor $\mathbf{Rng} \rightarrow \mathbf{Ab}$ which maps a ring to its underlying additive abelian group. Morphisms in **Rng** (ring homomorphisms) become morphisms in **Ab** (abelian group homomorphisms).

Free functors: Going in the opposite direction of forgetful functors are free functors. The free functor $F : \mathbf{Set} \rightarrow \mathbf{Grp}$ sends every set X to the free group generated by X . Functions get mapped to group homomorphisms between free groups. Free constructions exist for many categories based on structured sets. See free object.

Homomorphism groups: To every pair A, B of abelian groups one can assign the abelian group $\text{Hom}(A, B)$ consisting of all group homomorphisms from A to B . This is a functor which is contravariant in the first and covariant in the second argument, i.e. it is a functor $\mathbf{Ab}^{\text{op}} \times \mathbf{Ab} \rightarrow \mathbf{Ab}$ (where **Ab** denotes the category of abelian groups with group homomorphisms). If $f : A_1 \rightarrow A_2$ and $g : B_1 \rightarrow B_2$ are morphisms in **Ab**, then the group homomorphism $\text{Hom}(f, g) : \text{Hom}(A_2, B_1) \rightarrow \text{Hom}(A_1, B_2)$ is given by $\varphi \mapsto g \circ \varphi \circ f$. See Hom functor.

Representable functors: We can generalize the previous example to any category C . To every pair X, Y of objects in C one can assign the set $\text{Hom}(X, Y)$ of morphisms from X to Y . This defines a functor to **Set** which is contravariant in the first argument and covariant in the second, i.e. it is a functor $C^{\text{op}} \times C \rightarrow \mathbf{Set}$. If $f : X_1 \rightarrow X_2$ and $g : Y_1 \rightarrow Y_2$ are morphisms in C , then the group homomorphism $\text{Hom}(f, g) : \text{Hom}(X_2, Y_1) \rightarrow \text{Hom}(X_1, Y_2)$ is given by $\varphi \mapsto g \circ \varphi \circ f$.

Functors like these are called representable functors. An important goal in many settings is to determine whether a given functor is representable.

Properties

Two important consequences of the functor axioms are:

- F transforms each commutative diagram in C into a commutative diagram in D ;
- if f is an isomorphism in C , then $F(f)$ is an isomorphism in D .

One can compose functors, i.e. if F is a functor from A to B and G is a functor from B to C then one can form the composite functor $G \circ F$ from A to C . Composition of functors is associative where defined. Identity of composition of functors is the identity functor. This shows that functors can be considered as morphisms in categories of categories, for example in the category of small categories.

A small category with a single object is the same thing as a monoid: the morphisms of a one-object category can be thought of as elements of the monoid, and composition in the category is thought of as the monoid operation. Functors between one-object categories correspond to monoid homomorphisms. So in a sense, functors between arbitrary categories are a kind of generalization of monoid homomorphisms to categories with more than one object.

Relation to other categorical concepts

Let C and D be categories. The collection of all functors from C to D forms the objects of a category: the functor category. Morphisms in this category are natural transformations between functors.

Functors are often defined by universal properties; examples are the tensor product, the direct sum and direct product of groups or vector spaces, construction of free groups and modules, direct and inverse limits. The concepts of limit and colimit generalize several of the above.

Universal constructions often give rise to pairs of adjoint functors.

Computer implementations

Functors sometimes appear in functional programming. For instance, the programming language Haskell has a class `Functor` where `fmap` is a polytypic function used to map functions (*morphisms* on *Hask*, the category of Haskell types)^[9] between existing types to functions between some new types.^[10]

See also

- Functor category
- Kan extension
- Pseudofunctor

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- [The catsters \(https://www.youtube.com/user/TheCatsters\)](https://www.youtube.com/user/TheCatsters), a YouTube channel about category theory.
- "[Category Theory](https://planetmath.org/?op=getobj&from=objects&id=5622)" (<https://planetmath.org/?op=getobj&from=objects&id=5622>). *PlanetMath*.
- [Video archive \(http://categorieslogicphysics.wikidot.com/events\)](http://categorieslogicphysics.wikidot.com/events) of recorded talks relevant to categories, logic and the foundations of physics.
- [Interactive Web page \(https://web.archive.org/web/20080916162345/http://www.j-paine.org/cgi-bin/webcats/webcats.php\)](https://web.archive.org/web/20080916162345/http://www.j-paine.org/cgi-bin/webcats/webcats.php) which generates examples of categorical constructions in the category of finite sets.

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