nLab Home Page All Pages Latest Revisions Discuss this page Feeds Search

Isbell duality

higher geometry← ***Isbell duality*** → higher algebra

Context

Higher algebra

Higher geometry

Duality

Contents

1. Idea

A general abstract adjunction

$$(\mathcal{O} \dashv \mathrm{Spec}) : \mathrm{CoPresheaves} \underset{\mathrm{Spec}}{\overset{\mathcal{O}}{\leftrightarrows}} \mathrm{Presheaves}$$

relates (higher) presheaves with (higher) copresheaves on a given (higher) category C: this is called ***Isbell conjugation*** or ***Isbell duality*** (after John Isbell).

To the extent that this adjunction descends to presheaves that are (higher) sheaves and copresheaves that are (higher) algebras this duality relates higher geometry with higher algebra.

Objects preserved by the monad of this adjunction are called ***Isbell self-dual***.

Under the interpretation of presheaves as generalized spaces and copresheaves as generalized quantities modeled on C (Lawvere 86, see at *space and quantity*), Isbell duality is the archetype of the duality between geometry and algebra that permeates mathematics (such as Gelfand duality, Stone duality, or the embedding of smooth manifolds into formal duals of R-algebras).

2. Definition

Let $\mathcal{V}$ be a good enriching category (a cosmos, i.e. a complete and cocomplete closed symmetric monoidal category).

Let $\mathcal{C}$ be a small $\mathcal{V}$-enriched category.

Write $[\mathcal{C}^{op}, \mathcal{V}]$ and $[\mathcal{C}, \mathcal{V}]$ for the enriched functor categories.

Proposition 2.1. *There is a V-adjunction*

$$(\mathcal{O} \dashv \mathrm{Spec}) : [C, \mathcal{V}]^{op} \underset{\mathrm{Spec}}{\overset{\mathcal{O}}{\leftrightarrows}} [C^{op}, \mathcal{V}]$$

where

$$\mathcal{O}(X) : c \mapsto [C^{op}, \mathcal{V}](X, \mathcal{V}(-, c)) \,,$$

and

$$\mathrm{Spec}(A) : c \mapsto [C, \mathcal{V}]^{op}(\mathcal{V}(c, -), A) \;.$$

Remark 2.2. This is also called Isbell duality. Objects which are preserved by $\mathcal{O} \circ \mathrm{Spec}$ or $\mathrm{Spec}\, \mathcal{O}$ are called ***Isbell self-dual***.

The proof is mostly a tautology after the notation is unwound. The mechanism of the proof may still be of interest and be relevant for generalizations and for less tautological variations of the setup. We therefore spell out several proofs.

Proof A. Use the end-expression for the hom-objects of the enriched functor categories to compute

$$\begin{aligned}
[C, \mathcal{V}]^{op}(\mathcal{O}(X), A) &:= \int_{c \in C} \mathcal{V}(A(c), \mathcal{O}(X)(c)) \\
&:= \int_{c \in C} \mathcal{V}(A(c), [C^{op}, \mathcal{V}](X, \mathcal{V}(-, c))) \\
&:= \int_{c \in C} \int_{d \in C} \mathcal{V}(A(c), \mathcal{V}(X(d), \mathcal{V}(d, c))) \\
&\simeq \int_{d \in C} \int_{c \in C} \mathcal{V}(X(d), \mathcal{V}(A(c), \mathcal{V}(d, c))) \\
&=: \int_{d \in C} \mathcal{V}(X(d), [C, \mathcal{V}]^{op}(\mathcal{V}(d, -), A)) \\
&=: \int_{d \in C} \mathcal{V}(X(d), \mathrm{Spec}(A)(d)) \\
&=: [C^{op}, \mathcal{V}](X, \mathrm{Spec}(A))
\end{aligned} \;.$$

∎

Remark 2.3. Here apart from writing out or hiding the ends, the only step that is not a definition is precisely the middle one, where we used that $\mathcal{V}$ is a symmetric closed monoidal category.

The following proof does not use ends and needs instead slightly more preparation, but has then the advantage that its structure goes through also in great generality in higher category theory.

Proof B. Notice that

Lemma 1: $\mathrm{Spec}(\mathcal{V}(c,-)) \simeq \mathcal{V}(-,c)$

because we have a natural isomorphism

$$\begin{aligned} \mathrm{Spec}(\mathcal{V}(c,-))(d) &:= [C,\mathcal{V}](\mathcal{V}(c,-),\mathcal{V}(d,-)) \\ &\simeq \mathcal{V}(d,c) \end{aligned}$$

by the Yoneda lemma.

From this we get

Lemma 2: $[C^{\mathrm{op}},\mathcal{V}](\mathrm{Spec}\,\mathcal{V}(c,-),\mathrm{Spec}\,A) \simeq [C,\mathcal{V}](A,\mathcal{V}(c,-))$

by the sequence of natural isomorphisms

$$\begin{aligned} [C^{\mathrm{op}},\mathcal{V}](\mathrm{Spec}\,\mathcal{V}(c,-),\mathrm{Spec}\,A) &\simeq [C^{\mathrm{op}},\mathcal{V}](\mathcal{V}(-,c),\mathrm{Spec}\,A) \\ &\simeq (\mathrm{Spec}\,A)(c) \\ &:= [C,\mathcal{V}](A,\mathcal{V}(c,-)) \end{aligned} \,,$$

where the first is Lemma 1 and the second the Yoneda lemma.

Since (by what is sometimes called the co-Yoneda lemma) every object $X \in [C^{\mathrm{op}},\mathcal{V}]$ may be written as a colimit

$$X \simeq \underset{\longrightarrow_i}{\lim}\, \mathcal{V}(-,c_i)$$

over representables $\mathcal{V}(-,c_i)$ we have

$$X \simeq \underset{\longrightarrow_i}{\lim}\, \mathrm{Spec}(\mathcal{V}(c_i,-)) \,.$$

In terms of the same diagram of representables it then follows that

Lemma 3:

$$\mathcal{O}(X) \simeq \underset{\longleftarrow_i}{\lim}\, \mathcal{V}(c_i,-)$$

because using the above colimit representation and the Yoneda lemma we have natural isomorphisms

$$
\begin{aligned}
\mathcal{O}(X)(d) &= [C^{op}, \mathcal{V}](X, \mathcal{V}(-,c)) \\
&\simeq [C^{op}, \mathcal{V}](\underset{\longrightarrow_i}{\lim} \mathcal{V}(-,c_i), \mathcal{V}(-,c)) \\
&\simeq \underset{\longleftarrow_i}{\lim} [C^{op}, \mathcal{V}](\mathcal{V}(-,c_i), \mathcal{V}(-,c)) \\
&\simeq \underset{\longleftarrow_i}{\lim} \mathcal{V}(c_i, c)
\end{aligned}
.
$$

Using all this we can finally obtain the adjunction in question by the following sequence of natural isomorphisms

$$
\begin{aligned}
[C, \mathcal{V}]^{op}(\mathcal{O}(X), A) &\simeq [C, \mathcal{V}](A, \underset{\longleftarrow_i}{\lim} \mathcal{V}(c_i, -),) \\
&\simeq \underset{\longleftarrow_i}{\lim} [C, \mathcal{V}](A, \mathcal{V}(c_i, -)) \\
&\simeq \underset{\longleftarrow_i}{\lim} [C^{op}, \mathcal{V}](\mathrm{Spec}\, \mathcal{V}(c_i, -), \mathrm{Spec}\, A) \\
&\simeq [C^{op}, \mathcal{V}](\underset{\longrightarrow_i}{\lim} \mathrm{Spec}\, \mathcal{V}(c_i, -), \mathrm{Spec}\, A) \\
&\simeq [C^{op}, \mathcal{V}](X, \mathrm{Spec}\, A)
\end{aligned}
.
$$

▮

The pattern of this proof has the advantage that it goes through in great generality also on higher category theory without reference to a higher notion of enriched category theory.

Definition 2.4. An object X or A is ***Isbell-self-dual*** if

- $A \to \mathcal{O}\,\mathrm{Spec}(A)$ is an isomorphism in $[C, \mathcal{V}]$;
- $X \to \mathrm{Spec}\,\mathcal{O}X$ is an isomorphism in $[C^{op}, \mathcal{V}]$, respectively.

Remark 2.5. Under certain circumstances, Isbell duality can be extended to large $\mathcal{V}$-enriched categories C. For example, if C has a small generating subcategory S and a small cogenerating subcategory T, then for each $F: C^{op} \to \mathcal{V}$ and $G: C \to \mathcal{V}$, one may construct $\mathcal{O}(F)$ and $\mathrm{Spec}(G)$ objectwise as appropriate subobjects in $\mathcal{V}$:

$$
\begin{aligned}
\mathcal{O}(F)(c) = [C^{op}, \mathcal{V}](F, C(-,c)) &\hookrightarrow \int_{s:S} \mathcal{V}(Fs, \hom(s,c)) \\
\mathrm{Spec}(G)(c) = [C, \mathcal{V}](G, C(c,-)) &\hookrightarrow \int_{t:T} \mathcal{V}(Gt, \hom(c,t))
\end{aligned}
$$

3. Example

In the simplest case, namely for an ordinary category $\mathcal{C}$, the adjunction between presheaves and copresheaves arises as follows.

The category of presheaves $[\mathcal{C}^{op}, \mathrm{Set}]$ is the free cocompletion of $\mathcal{C}$. This means that any functor

$$
f : \mathcal{C} \to \mathcal{D}
$$

to a cocomplete category $\mathcal{D}$ extends along the Yoneda embedding $y : \mathcal{C} \to [\mathcal{C}^{op}, \mathrm{Set}]$ to a cocontinuous functor

$$F : [\mathcal{C}^{op}, \mathrm{Set}] \to \mathcal{D}$$

in a manner unique up to natural isomorphism.

Dually, the category of copresheaves $[\mathcal{C}, \mathrm{Set}]^{op}$ is the free completion of $\mathcal{C}$. This means that any functor

$$g : \mathcal{C} \to \mathcal{D}$$

to a complete category $\mathcal{D}$ extends along the co-Yoneda embedding $z : \mathcal{C} \to [\mathcal{C}, \mathrm{Set}]^{op}$ to a continuous functor.

$$G : [\mathcal{C}, \mathrm{Set}]^{op} \to \mathcal{D}$$

in a manner unique up to natural isomorphism.

We can apply these ideas to get the functors involved in Isbell duality. The presheaf category $[\mathcal{C}^{op}, \mathrm{Set}]$ has all limits, so we can extend the Yoneda embedding to a continuous functor

$$Y : [\mathcal{C}, \mathrm{Set}]^{op} \to [\mathcal{C}^{op}, \mathrm{Set}]$$

from copresheaves to presheaves. Dually, the copresheaf category $[\mathcal{C}, \mathrm{Set}]^{op}$ has all colimits, so we can extend the co-Yoneda embedding to a cocontinuous functor

$$Z : [\mathcal{C}^{op}, \mathrm{Set}] \to [\mathcal{C}, \mathrm{Set}]^{op}$$

from presheaves to copresheaves.

Isbell duality says that these are adjoint functors: Y is right adjoint to Z.

4. Properties

Relation to Yoneda embedding

Spec is the left Kan extension of the Yoneda embedding along the contravariant Yoneda embedding, while $\mathcal{O}$ is the left Kan extension of the contravariant Yoneda embedding along the Yoneda embedding.

The codensity monad of the Yoneda embedding is isomorphic to the monad induced by the Isbell adjunction, $\mathrm{Spec}\,\mathcal{O}$ (Di Liberti 19, Thrm 2.7).

Respect for limits

Choose any class L of limits in C and write $[C, \mathcal{V}]_{\times} \subset [C, \mathcal{V}]$ for the full subcategory consisting of those functors preserving these limits.

Proposition 4.1. *The $(\mathcal{O} \dashv \mathrm{Spec})$-adjunction does descend to this inclusion, in that we have an adjunction*

$$(\mathcal{O} \dashv \mathrm{Spec}) : [C, \mathcal{V}]_{\times}^{op} \underset{\mathrm{Spec}}{\overset{\mathcal{O}}{\leftrightarrows}} [C^{op}, \mathcal{V}]$$

Proof. Because the hom-functors preserves all limits:

$$
\begin{aligned}
\mathcal{O}(X)(\underset{\longleftarrow_j}{\lim} c_j) &:= [C^{op}, \mathcal{V}](X, \mathcal{V}(-, \underset{\longleftarrow_j}{\lim} c_j)) \\
&\simeq [C^{op}, \mathcal{V}](X, \underset{\longleftarrow_j}{\lim} \mathcal{V}(-, c_j)) \\
&\simeq \underset{\longleftarrow_j}{\lim} [C^{op}, \mathcal{V}](X, \mathcal{V}(-, c_j)) \\
&=: \underset{\longleftarrow_j}{\lim} \mathcal{O}(X)(c_j)
\end{aligned}
\,.
$$

▮

Isbell self-dual objects

***Proposition 4.2**. All representables are Isbell self-dual.*

Proof. By Proof B , lemma 1 we have a natural isomorphisms in $c \in C$

$$
\mathrm{Spec}(\mathcal{V}(c, -)) \simeq \mathcal{V}(-, c) \,.
$$

Therefore we have also the natural isomorphism

$$
\begin{aligned}
\mathcal{O}\,\mathrm{Spec}\,\mathcal{V}(c, -)(d) &\simeq \mathcal{O}\mathcal{V}(-, c)(d) \\
&:= [C^{op}, \mathcal{V}](\mathcal{V}(-, c), \mathcal{V}(-, d))\,, \\
&\simeq \mathcal{V}(c, d)
\end{aligned}
$$

where the second step is the Yoneda lemma. Similarly the other way round. ▮

Isbell envelope

See Isbell envelope.

5. Examples and similar dualities

Isbell duality is a template for many other space/algebra-dualities in mathematics.

Function T-Algebras on presheaves

Let $\mathcal{V}$ be any cartesian closed category.

Let $C := T$ be the syntactic category of a $\mathcal{V}$-enriched Lawvere theory, that is a $\mathcal{V}$-category with finite products such that all objects are generated under products from a single object 1.

Then write $T\,\mathrm{Alg} := [C, \mathcal{V}]_\times$ for category of product-preserving functors: the category of T-algebras. This comes with the canonical forgetful functor

$$
U_T : T\,\mathrm{Alg} \to \mathcal{V} : A \mapsto A(1)
$$

Write

$$
F_T : T^{op} \hookrightarrow T\,\mathrm{Alg}
$$

for the Yoneda embedding.

Definition 5.1. Call

$$\mathbb{A}_T := \mathrm{Spec}(F_T(1)) \in [C^{\mathrm{op}}, \mathcal{V}]$$

the T***-line object***.

Observation 5.2. *For all $X \in [C^{\mathrm{op}}, \mathcal{V}]$ we have*

$$\mathcal{O}(X) \simeq [C^{\mathrm{op}}, \mathcal{V}](X, \mathrm{Spec}(F_T(-))) \,.$$

In particular

$$U_T(\mathcal{O}(X)) \simeq [C^{\mathrm{op}}, \mathcal{V}](X, \mathbb{A}_T) \,.$$

Proof. We have isomorphisms natural in $k \in T$

$$\begin{aligned} [C^{\mathrm{op}}, \mathcal{V}](X, \mathrm{Spec}(F_T(k))) & \simeq T\,\mathrm{Alg}(F_T(k), \mathcal{O}(X)) \\ & \simeq \mathcal{O}(X)(k) \end{aligned}$$

by the above adjunction and then by the Yoneda lemma. ▮

All this generalizes to the following case:

instead of setting $C := T$ let more generally

$$T \subset C \subset T\,\mathrm{Alg}^{\mathrm{op}}$$

be a small full subcategory of T-algebras, containing all the free T-algebras.

Then the original construction of $\mathcal{O} \dashv \mathrm{Spec}$ no longer makes sense, but that in terms of the line object still does

Proposition 5.3. *Set*

$$\mathrm{Spec}\, A : B \mapsto T\,\mathrm{Alg}(A, B)$$

and

$$\mathcal{O}(X) : k \mapsto [C^{\mathrm{op}}, \mathcal{V}](X, \mathrm{Spec}(F_T(k))) \,.$$

Then we still have an adjunction

$$(\mathcal{O} \dashv \mathrm{Spec}) : T\,\mathrm{Alg}^{\mathrm{op}} \underset{\mathrm{Spec}}{\overset{\mathcal{O}}{\leftrightarrows}} [C^{\mathrm{op}}, \mathcal{V}] \,.$$

$$
\begin{aligned}
T\,\mathrm{Alg}^{\mathrm{op}}(\mathcal{O}(X), A) &:= \int_{k \in T} \mathcal{V}(A(k), \mathcal{O}(X)(k)) \\
&:= \int_{k \in T} \mathcal{V}(A(k), [C^{\mathrm{op}}, \mathcal{V}](X, \mathrm{Spec}(F_T(k)))) \\
&:= \int_{k \in T} \int_{B \in C} \mathcal{V}(A(k), \mathcal{V}(X(B), T\,\mathrm{Alg}(F_T(k), B))) \\
&\simeq \int_{k \in T} \int_{B \in C} \mathcal{V}(A(k), \mathcal{V}(X(B), B(k))) \\
&\simeq \int_{k \in T} \int_{B \in C} \mathcal{V}(X(B), \mathcal{V}(A(k), B(k))) \\
&=: \int_{B \in C} \mathcal{V}(X(B), T\,\mathrm{Alg}(A, B)) \\
&=: \int_{B \in C} \mathcal{V}(X(B), \mathrm{Spec}(A)(B)) \\
&=: [C^{\mathrm{op}}, \mathrm{Set}](X, \mathrm{Spec}(A))
\end{aligned}
.
$$

Proof. The first step that is not a definition is the Yoneda lemma. The step after that is the symmetric-closed-monoidal structure of $\mathcal{V}$. ▮

Function k-algebras on derived ∞-stacks

The structure of our Proof B above goes through in higher category theory.

Formulated in terms of derived stacks over the (∞,1)-category of dg-algebras, this is essentially the argument appearing on page 23 of (Ben-ZviNadler).

Function T-algebras on ∞-stacks

for the moment see at *function algebras on ∞-stacks*.

Function 2-algebras on algebraic stacks

see Tannaka duality for geometric stacks

Gelfand duality

Gelfand duality is the equivalence of categories between (nonunital) commutative C*-algebras and (locally) compact topological spaces. See there for more details.

Serre-Swan theorem

The Serre-Swan theorem says that suitable modules over an commutative C*-algebra are equivalently modules of sections of vector bundles over the Gelfand-dual topological space.

6. Related concepts

- function algebra

- function algebras on ∞-stacks
- nucleus of a profunctor

Isbell duality* between *algebra* and *geometry

geometry	**category**	**dual category**	**algebra**
topology	$\mathrm{TopSpaces}_{H,cpt}$	$\overset{\text{Gelfand-Kolmogorov}}{\hookrightarrow} \mathrm{Alg}^{op}_{\mathbb{R}}$	commutative algebra
topology	$\mathrm{TopSpaces}_{H,cpt}$	$\overset{\text{Gelfand duality}}{\simeq} \mathrm{TopAlg}^{op}_{C^\ast, comm}$	comm. C-star-algebra
noncomm. topology	$\mathrm{NCTopSpaces}_{H,cpt}$	$\coloneqq \mathrm{TopAlg}^{op}_{C^\ast}$	general C-star-algebra
algebraic geometry	$\mathrm{Schemes}_{Aff}$	$\overset{\text{almost by def.}}{\hookrightarrow} \mathrm{Alg}^{op}_{fin}$	fin. gen. commutative algebra
noncomm. algebraic geometry	$\mathrm{NCSchemes}_{Aff}$	$\coloneqq \mathrm{Alg}^{op}_{fin, red}$	fin. gen. associative algebra
differential geometry	SmoothManifolds	$\overset{\text{Milnor's exercise}}{\hookrightarrow} \mathrm{Alg}^{op}_{comm}$	commutative algebra
supergeometry	$\mathrm{SuperSpaces}_{Cart}$ $\mathbb{R}^{n\vert q}$	$\hookrightarrow \mathrm{Alg}^{op}_{\mathbb{Z}_2}$ $\mapsto C^\infty(\mathbb{R}^n) \otimes \wedge^\bullet \mathbb{R}^q$	supercommutative superalgebra
formal higher supergeometry (super Lie theory)	$\mathrm{Super}\, L_\infty \,\mathrm{Alg}_{fin}$ $\mathfrak{g}$	$\overset{\text{Lada-Markl}}{\hookrightarrow} \mathrm{sdgcAlg}^{op}$ $\mapsto \mathrm{CE}(\mathfrak{g})$	differential graded-commutative superalgebra ("FDAs")

***in physics*:**

algebra	**geometry**
Poisson algebra	Poisson manifold
deformation quantization	geometric quantization
algebra of observables	space of states
Heisenberg picture	Schrödinger picture
AQFT	FQFT
higher algebra	***higher geometry***
Poisson n-algebra	n-plectic manifold

algebra	geometry
En-algebras	higher symplectic geometry
BD-BV quantization	higher geometric quantization
factorization algebra of observables	extended quantum field theory
factorization homology	cobordism representation

7. References

The original articles on Isbell duality and the Isbell envelope are

- John Isbell, *Structure of categories*, Bulletin of the American Mathematical Society 72 (1966), 619– 655. (project euclid)
- John Isbell, *Normal completions of categories*, Reports of the Midwest Category Seminar, vol. 47, Springer, 1967, 110–155.

More recent discussion is in

- William Lawvere, p. 17 of *Taking categories seriously*, Revista Colombiana de Matematicas, XX (1986) 147-178, reprinted as: Reprints in Theory and Applications of Categories, No. 8 (2005) pp. 1-24 (web)
- Michael Barr, John Kennison, R. Raphael, Isbell Duality, Theory and Applications of Categories, Vol. 20, 2008, No. 15, pp 504-542. (web)
- Richard Garner, *The Isbell monad*, Advances in Mathematics ***274*** (2015) pp.516-537. (draft)
- Vaughan Pratt, *Communes via Yoneda, from an elementary perspective*, Fundamenta Informaticae 103 (2010), 203–218.
- Ivan Di Liberti, Fosco Loregian, *On the Unicity of Formal Category Theories*, arXiv:1901.01594 (2019). (abstract)
- Ivan Di Liberti, *Codensity: Isbell duality, pro-objects, compactness and accessibility*, (arXiv:1910.01014)

Isbell conjugacy for (∞,1)-presheaves over the (∞,1)-category of duals of dg-algebras is discussed around page 32 of

- David Ben-Zvi, David Nadler, *Loop spaces and connections* (arXiv:1002.3636)

in

- Bertrand Toën, *Champs affines* (arXiv:math/0012219)

Isbell self-dual ∞-stacks over duals of commutative associative algebras are called *affine stacks*. They are characterized as those objects that are *small* in a sense and local with respect to the cohomology with coefficients in the canonical line object.

A generalization of this latter to ∞-stacks over duals of algebras over arbitrary abelian Lawvere theories is the content of

- Herman Stel, *∞-Stacks and their function algebras – with applications to ∞-Lie theory*, master thesis (2010) (web)

See also

- MathOverflow: theme-of-isbell-duality
- R.J. Wood, *Some remarks on total categories*, J. Algebra ***75_:2, 1982, 538–545 doi***

Last revised on May 31, 2020 at 20:43:51. See the history of this page for a list of all contributions to it.

Edit Back in time (36 revisions) See changes History Cite Print TeX Source