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GROUP EXTENSIONS AND HOMOLOGY*

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CONTENTS

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INTRODUCTION

In 1937 the following problem was formulated by Borsuk and Eilenberg: Given a solenoid[1] Σ in the three sphere S^3, how many homotopy classes of continuous mappings $f(S^3 - \Sigma) \subset S^2$ are there? In 1939 Eilenberg proved ([4], p. 251) that the homotopy classes in question are in a 1-1-correspondence with the elements of the one-dimensional homology group $H^1(K, I) = Z^1(K, I)/B^1(K, I)$, where K is any representation of $S^3 - \Sigma$ as a complex, $Z^1(K, I)$ is the group of infinite 1-cycles in K with the additive group I of integers as coefficients and $B^1(K, I)$ is the subgroup of bounding cycles. This homology group is generally much "larger" than the conventional homology group $H_t^1(K, I) = Z^1/\bar{B}^1$ where $\bar{B}^1(K, I)$ is the group of cycles that bound on every finite portion of K; with an appropriate topology in the group Z^1, $\bar{B}^1$ turns out to be exactly the closure of B^1.

At this point the investigation was taken up by Steenrod [10]. By using "regular cycles" he computed the groups $H^1(S^3 - \Sigma)$ for the various solenoids Σ. The groups are uncountable and of a rather complicated nature.[2]

This paper originated from an accidental observation that the groups obtained by Steenrod were identical with some groups that occur in the purely algebraic theory of *extensions of groups*. An abelian group E is called an ex-

[1] For the definition see Appendix B below.

[2] A popular exposition of Steenrod's results can be found in his article in Lectures in Topology, Ann Arbor, University of Michigan Press, 1941, pp. 43–55.

tension of the group G by the group H if $G \subset E$ and $H = E/G$. With a proper definition of equivalence and addition, the extensions of G by H themselves form an abelian group Ext $\{G, H\}$. It turns out that $H^1(S^3 - \Sigma, I)$ is isomorphic with Ext $\{I, \Sigma^*\}$ where Σ^* is a properly chosen subgroup of the group of rational numbers.[3]

The thesis of this paper is that the theory of group extensions forms a natural and powerful tool in the study of homologies in infinite complexes and topological spaces. Even in the simple and familiar case of finite complexes the results obtained are finer than the existing ones.

Our fundamental theorem concerns the homology groups of a star finite complex K. Let $H^q(G)$ denote the homology group of infinite cycles with coefficients in an arbitrary topological group G. We obtain an explicit expression for $H^q(G)$ in terms of G and the cohomology groups $\mathcal{H}_q$ of *finite* cocycles with integral coefficients. ($\mathcal{H}_q$ is the factor group $\mathcal{Z}_q/\mathcal{B}_q$ of cocycles modulo coboundaries). This expression is

$$H^q(G) = \text{Hom}\ \{\mathcal{H}_q, G\} \times \text{Hom}\ \{\mathcal{B}_{q+1}, G\}/\text{Hom}\ \{\mathcal{Z}_{q+1} \mid \mathcal{B}_{q+1}, G\}.$$

Here Hom $\{H, G\}$ stands for the (topological) group of all homomorphisms of H into G, while Hom $\{\mathcal{Z}_{q+1} \mid \mathcal{B}_{q+1}, G\}$ denotes the group of those homomorphisms of $\mathcal{B}_{q+1}$ into G which can be extended to homomorphisms of $\mathcal{Z}_{q+1}$ into G. The factor group on the right in this expression appears to depend on the groups $\mathcal{B}_{q+1}$ and $\mathcal{Z}_{q+1}$, but actually depends only on the cohomology group $\mathcal{H}_{q+1} = \mathcal{Z}_{q+1}/\mathcal{B}_{q+1}$. In fact this factor group can best be interpreted as the group "Ext" of group extensions of G by $\mathcal{H}_{q+1}$. The fundamental theorem then has the form

$$H^q(G) = \text{Hom}\ \{\mathcal{H}_q, G\} \times \text{Ext}\ \{G, \mathcal{H}_{q+1}\}.$$

The paper is self contained as far as possible, both in algebraic and topological respects. The first four chapters below develop the requisite group-theoretical notions. Chapter I discusses the groups of homomorphisms involved in the above formula, while Chapter II introduces the group of group extensions, and proves the fundamental theorem relating this group to groups of homomorphisms. This fundamental theorem is essentially a formulation of the known fact that a group extension of G by H can be described either by generators of H (and hence by homomorphisms) or by certain "factor sets." Chapter III analyzes the group Ext $\{G, H\}$ for some special cases of G. Chapter IV introduces some additional groups, closely related to Ext, which arise as inverse limit groups in the treatment of homologies of topological spaces.

The last two chapters analyze homology groups. Chapter V treats the case of a complex, and proves the fundamental theorem quoted above, as well as parallel theorems for some of the other homology groups of a complex. Chapter

[3] More precisely Σ^* is the character group of Σ. The detailed treatment appears in Appendix B below.

VI obtains analogous theorems for the Čech homology groups of a topological space.

Appendix A discusses the case when G is a group with operators. Appendix B contains a computation of the group Ext $\{I, \Sigma^*\}$ mentioned above.

Each chapter is preceded by a brief outline. The chapters are related as in the following diagram:

$$\begin{array}{ccccccc} \mathrm{I} & \rightarrow & \mathrm{II} & \rightarrow & \mathrm{III} & \rightarrow & \mathrm{V} \\ & & & & & \searrow & & \searrow \\ & & & & & & \mathrm{IV} & \rightarrow & \mathrm{VI} \end{array}$$

Almost all of V can be read directly after I and II, and a major portion after I alone.

Chapters V and VI are strongly influenced by S. Lefschetz's recent book "Algebraic Topology" [7], that the authors had the privilege of reading in manuscript.

Chapter I. Topological Groups and Homomorphisms

After a certain preliminary definitions, this chapter introduces the basic group Hom $\{R, G\}$ of homomorphisms. In the case when R is a subgroup of a free group, we require two subgroups of "extendable" homomorphisms. The topology of these subgroups is investigated when the "coefficient group" G is itself topological.

1. Topological spaces

A set X is called a *space* if there is given a family of subsets of X, called *open* sets, such that

(1.1) *X and the void set are open,*

(1.2) *the union of any number of open sets is open,*

(1.3) *the intersection of two open sets is open.*

Complements of open sets are called *closed.* X is called a *Hausdorff space* if in addition

(1.4) *every two distinct points are contained respectively in two disjoint open sets.*

X is called a *compact* (= bicompact) space if

(1.5) *every covering of X by open sets contains a finite subcovering.*

A space X is *discrete* if every set in X is open.

The intersection of an open set of a space X with a subset A of X will be called *open in A*. With this convention A becomes a space.

Let X and Y be spaces and $x \rightarrow f(x) = y$ a mapping of X into a subset of Y. The mapping f is *continuous* if for every open set $U \subset Y$ the set $f^{-1}(U)$ is open (in X). The mapping f is *open* if for every open set $U \subset X$ the set $f(U)$ is open (in Y). A well known result is

Lemma 1.1. *If f is a continuous mapping of a compact space X into a Hausdorff space Y, then $f(X)$ is closed in Y.*

A *product space* $\prod_\alpha X_\alpha$ of a given collection $\{X_\alpha\}$ of spaces X_α is defined as the space whose points are all collections $\{x_\alpha\}$, $x_\alpha \in X_\alpha$ and in which open sets are unions of sets of the form $\prod_\alpha U_\alpha$, where U_α is an open subset of X_α and $U_\alpha = X_\alpha$ except for a finite number of indices α.[4] It is known that $\prod X_\alpha$ is a Hausdorff or compact space if and only if for every α the space X_α is a Hausdorff or compact space.[5]

Let Λ be a set of elements and X be a space. We consider the set X^Λ of all functions with arguments in Λ and values in X. The set X^Λ is clearly in a 1-1 correspondence with the product $\prod X_\lambda$ where $\lambda \in \Lambda$ and $X_\lambda = X$. Hence we may consider X^Λ as a space.

2. Topological groups

Only abelian groups (written additively) will be considered.

A group G will be called a *generalized topological group* if G is a space in which the group composition (as a mapping $G \times G \to G$) and the group inverse (as a mapping $G \to G$) are continuous.

If G, considered as a space, is a Hausdorff space, then G will be called a *topological group*.[6] Similarly, if G is compact as a space we shall say that G is a *compact group*.

A subgroup of a (generalized) topological group is a (generalized) topological group. A closed subgroup of a compact group is compact.

LEMMA 2.1. *In a generalized topological group G the following properties are equivalent*:

(a) *every point of G is a closed set,*

(b) *the zero element of G is a closed set,*

(c) *G is a topological group.*[7]

The *factor group* $H = G/G_1$ of a generalized topological group G modulo a subgroup G_1 is the group of all cosets $g + G_1$ of G_1 in G. The correspondence $\varphi(G) = H$ carrying each $g \in G$ into its coset $\varphi g = g + G_1$ in H is the "natural" mapping of G on H. We introduce a topology in H by calling a set $U \subset H$ open if and only if $\varphi^{-1}(U)$ is open in G. It can be shown that this topology is the only one under which φ will be both open and continuous.

LEMMA 2.2. *If G is a generalized topological group and G_1 is an arbitrary subgroup of G, then the factor group $H = G/G_1$ is a generalized topological group; it is a topological group if and only if G_1 is a closed subgroup of G. If G is compact, then G/G_1 is compact.*

LEMMA 2.3. *The closure $\bar{0}$ of the zero element of a generalized topological group is a closed subgroup of G. Its factor group $G/\bar{0}$ is the "largest" factor group of G which is a topological group.*

The preceding two statements show the utility of the study of generalized

[4] If $\{\alpha\} = 1, 2, \cdots, n$ we also use the symbol $X_1 \times X_2 \times \cdots \times X_n$ for the product space.

[5] See C. Chevalley and O. Frink, Bulletin Amer. Math. Soc. 47 (1941), pp. 612–614.

[6] G is then a topological group in the sense of Pontrjagin [8].

[7] To prove that a) implies c) one first proves that each neighborhood of g contains the closure of a neighborhood of g, as in Pontrjagin [8], p. 43, proposition F.

topological groups. Several times in the sequel we need to consider an isomorphism

$$G_1/H_1 \cong G_2/H_2 \tag{2.1}$$

where the G_i are topological groups, while the H_i are not closed, so that G_i/H_i are only generalized topological groups. However, if we are able to prove that the isomorphism (2.1) is continuous in both directions in the "generalized" topology of the groups G_i/H_i, we obtain as a corollary the bicontinuous isomorphism of the topological groups $G_i/\bar{H}_i$.

If $\{G_\alpha\}$ is a collection of generalized topological groups the direct product $\prod_\alpha G_\alpha$ is a generalized topological group, provided we define the sum $\{g_\alpha\} = \{g'_\alpha\} + \{g''_\alpha\}$ by setting $g_\alpha = g'_\alpha + g''_\alpha$ for every α. Similarly, if Λ is any set and G is a generalized topological group, then the set G^Λ of all mappings of Λ into G is a generalized topological group. It follows from the results quoted in §1 that $\prod_\alpha G_\alpha$ and G^Λ are topological or compact groups if and only if the groups G_α and G are all topological or compact, respectively.

3. The group of homomorphisms

Let G and H be generalized topological groups. A *homomorphism* θ of H into G is a continuous function $\theta(h)$ defined for all $h \in H$ with values in G, such that $\theta(h_1 + h_2) = \theta(h_1) + \theta(h_2)$. For instance, the natural mapping of a group into one of its factor groups is a homomorphism. If θ_1 and θ_2 are two homomorphisms their sum $\theta_1 + \theta_2$, defined by

$$(\theta_1 + \theta_2)(h) = \theta_1(h) + \theta_2(h), \qquad \text{(all } h \text{ in } H)$$

is also a homomorphism. Under this addition, the set of all homomorphisms θ of H into G constitutes a group, which we denote by Hom $\{H, G\}$:

$$\text{Hom } \{H, G\} = [\text{all homomorphisms } \theta \text{ of } H \text{ into } G]. \tag{3.1}$$

To introduce a (generalized) topology in Hom $\{H, G\}$, take any compact subset X of H and any open subset V of G with $0 \in V$ and consider the set $U(X, V)$ of all θ with $\theta(X) \subset V$. In the usual sense ([8], p. 55) these sets $U(X, V)$ constitute a complete set of neighborhoods of 0 in Hom $\{H, G\}$, and are used to define the topology of Hom $\{H, G\}$.[8]

If H is discrete, the compact subsets X of H are just the finite ones. In this case Hom $\{H, G\}$ is a subgroup of the group G^H with the topology as defined in §2.

LEMMA 3.1. *If G is a topological group and H is discrete, then Hom $\{H, G\}$ is a closed subgroup of the group G^H of all mappings of H into G.*

PROOF. Let $\phi_0 \in G^H$ be a mapping of H into G that is not a homomorphism. There are then elements h_1, h_2, h_3 in H such that $h_1 + h_2 = h_3$ and $\phi_0(h_1) + \phi_0(h_2) \neq \phi_0(h_3)$. Since G is a Hausdorff space and the group composi-

[8] This is the general definition stated by Weil [11], p. 99, and Lefschetz [7], Ch. II.

tion is continuous there are in G three open sets U_1, U_2, U_3 containing $\phi_0(h_1)$, $\phi_0(h_2)$, and $\phi_0(h_3)$, respectively, such that[9] $(U_1 + U_2) \cap U_3 = 0$. Consequently the open subset U of G^H consisting of the mappings ϕ such that $\phi(h_1) \in U_1$, $\phi(h_2) \in U_2$, and $\phi(h_3) \in U_3$ has no elements in common with Hom $\{H, G\}$. Hence Hom $\{H, G\}$ is closed.

COROLLARY 3.2. *If H is discrete and G is a topological (and compact) group, then* Hom $\{H, G\}$ *is a topological (and compact) group.*

Note that the topology of Hom $\{H, G\}$ may not be discrete even though H and G both have discrete topologies. Observe also that if H is discrete, an alteration in the topology of G may alter the topology of Hom $\{H, G\}$ but not its algebraic structure. However, if H carries a non-discrete topology, an alteration in the topology of either H or G may alter the algebraic structure of Hom $\{H, G\}$, in that continuous homomorphisms may cease to be continuous, or vice versa.

If H is compact, we can take H itself to be the compact set X used in the definition of the topology in Hom $\{H, G\}$. Consequently, given any open set V in G containing 0, the homomorphisms θ, such that $\theta(H) \subset V$, constitute an open set. Hence if V can be picked so as not to contain any subgroups but 0, we see that Hom $\{H, G\}$ is discrete.

Subgroups and factor groups of H will correspond respectively to factor groups and subgroups of Hom $\{H, G\}$, as stated in the following lemmas.

LEMMA 3.3. *If H/H_1 is a factor group of the discrete group H, then* Hom $\{H/H_1, G\}$ *is (bicontinuously) isomorphic to that subgroup of* Hom $\{H, G\}$ *which consists of the homomorphisms θ mapping every element of H_1 into zero.*

The proof is readily given by observing that each homomorphism θ with $\theta(H_1) = 0$ maps each coset of H_1 into a single element of G, so induces a homomorphism θ' of H/H_1. The continuity of the isomorphism $\theta \to \theta'$ can be established, as always for isomorphisms between groups, by showing continuity at $\theta = 0$. ([8], p. 63).

LEMMA 3.4. *If L is a subgroup of H, then each homomorphism θ of H into G induces a homomorphism $\theta' = \theta \mid L$ of L into G. The correspondence $\theta \to \theta'$ is a (continuous) homomorphism of* Hom $\{H, G\}$ *into* Hom $\{L, G\}$. *If L is a direct factor of H, this correspondence maps* Hom $\{H, G\}$ *onto* Hom $\{L, G\}$.

4. Free groups and their factor groups

The homology groups will be interpreted later as certain groups of homomorphisms of "free" groups, which we now define. If the elements z_α of a discrete group F are such that every element of F can be represented uniquely as a finite sum $\sum n_\alpha z_\alpha$ with integral coefficients n_α, F is said to be a *free abelian group* with generators (or basis elements) $\{z_\alpha\}$. The number of generators may be infinite. A free group can be constructed with any assigned set of symbols as basis elements.

[9] $U_1 + U_2$ is the set of all sums $g_1 + g_2$, with $g_i \in U_i$. The symbol $\cap$ stands for the set-theoretic intersection.

LEMMA 4.1. *Every proper subgroup of a free group is free.*

For the denumerable case, this is proved by Čech [3]; a general proof is given in Lefschetz [7] (II, (10.1)).

Any discrete group H can be represented as a homomorphic image of a free group. Specifically, if we choose any set of elements t_α in H which together generate all of H, and if we then construct a free group F with generators z_α in 1-1 correspondence $z_\alpha \leftrightarrow t_\alpha$ with the given t's, the correspondence $\sum n_\alpha z_\alpha \rightarrow \sum n_\alpha t_\alpha$ will map the free group F homomorphically onto the given group H. If the kernel of this homomorphism[10] is R, H may be represented as the factor group $H = F/R$. R is essentially the group of "relations" on the generators t_α of H.

Given $R \subset F$, each homomorphism ϕ of F into G induces a homomorphism $\theta = \phi \mid R$ of the subgroup R into G, and the homomorphisms so induced form a subgroup of Hom $\{R, G\}$, denoted as

$$\text{(4.1)} \qquad \operatorname{Hom}\{F \mid R, G\} = [\text{all } \theta = \phi \mid R, \text{ for } \phi \in \operatorname{Hom}\{F, G\}].$$

Alternatively, the elements of this subgroup can be described as those homomorphisms θ of R into G which can be extended (in at least one way) to homomorphisms of F into G.

A similar, but lighter, restriction may be imposed as follows: Given $\theta \in \operatorname{Hom}\{R, G\}$, require that for every subgroup $F_0 \supset R$ of F for which F_0/R is finite there exist an extension of θ to a homomorphism of F_0 into G. The θ's meeting this requirement also constitute a subgroup,

$$\text{(4.2)} \qquad \operatorname{Hom}_f\{R, G; F\} = [\text{all } \theta \in \operatorname{Hom}\{F_0 \mid R, G\} \text{ for every finite } F_0/R].$$

These two subgroups,

$$\operatorname{Hom}\{F \mid R, G\} \subset \operatorname{Hom}_f\{R, G; F\} \subset \operatorname{Hom}\{R, G\},$$

are important because the corresponding factor groups in Hom $\{R, G\}$ are invariants of the group $H = F/R$, in that they do not depend on the particular free group F chosen to represent H. This fact may be stated as follows.

THEOREM 4.2. *If H is isomorphic to two factor groups F/R and F'/R' of free groups F and F', then*

$$\text{(4.3)} \qquad \operatorname{Hom}\{R, G\}/\operatorname{Hom}\{F \mid R, G\} \cong \operatorname{Hom}\{R', G\}/\operatorname{Hom}\{F' \mid R', G\},$$

the isomorphism being both algebraic and topological. The same result holds for the factor groups

$$\text{(4.4)} \qquad \operatorname{Hom}\{R, G\}/\operatorname{Hom}_f\{R, G; F\}, \qquad \operatorname{Hom}_f\{R, G; F\}/\operatorname{Hom}\{F \mid R, G\}.$$

This theorem is a corollary of a result to be established in Chapter II, as Theorem 10.1. It can also be proved directly, by appeal to the following lemma, which we state without proof.

[10] The *kernel* of a homomorphism θ of a group H is the set of all elements $h \in H$ with $\theta(h) = 0$.

LEMMA 4.3. *Let $F/R = E/G$, where $F \supset R$ is a free group and $E \supset G$ is any other group. There exists a homomorphism ϕ of F into E such that, in the given identification of cosets of G with cosets of R,*

$$\phi(x) + G = x + R, \qquad \text{for all } x \in F. \tag{4.5}$$

Any other $\phi^* \in$ Hom $\{F, E\}$ with this property (4.5) *has the form $\phi^* = \phi + \beta$, for some $\beta \in$ Hom $\{F, G\}$. Conversely, given ϕ with the property* (4.5) *any such $\phi^* = \phi + \beta$ has the same property.*

Although a given group H can be represented in many ways as a factor group $H = F/R$ of a gree group, there is a "natural" such representation, in which F is the additive group F_H of the (integral) group ring of H. Specifically, given H, we choose for each $h \in H$ a symbol z_h and construct a free group F_H generated by the symbols z_h. The correspondence $z_h \to h$ induces a homomorphism of F_H on H. Let R_H denote the kernel of this homomorphism. The factor group (4.3) of the Theorem can then be described invariantly in terms of H and G as the group

$$\text{Hom }\{R_H, G\}/\text{Hom }\{F_H \mid R_H, G\}.$$

The same remark applies to the factor groups of (4.4). It would be possible to use the groups so described as substitutes for the group of group extensions to be introduced in Chapter II.

5. Closures and extendable homomorphisms

If G is topological, we wish to examine the closures of the groups Hom $\{F \mid R, G\}$ and Hom_f in the topological group Hom $\{R, G\}$. A preliminary is a characterization of the subgroup Hom_f.

LEMMA 5.1. *A homomorphism θ of* Hom $\{R, G\}$ *lies in* Hom_f $\{R, G; F\}$ *if and only if for each element t in F with a multiple mt in R there exists $h \in G$ with $\theta(mt) = mh$.*

PROOF. Let F_t be the subgroup of F generated by t and R. If $mt \in R$ for $m \neq 0$, F_t/R is finite and cyclic, so that $\theta \in \text{Hom}_f$ is extendable to F_t. Hence the condition stated on $\theta(mt)$ is necessary. Conversely, for any given group $F_0 \subset F$ with F_0/R finite we can write F_0/R as a direct product of cyclic groups. By applying the given condition on θ to each of these cyclic groups, we find an extension of θ to F_0, as required.

Another characterization of Hom_f can be found; the proof is similar:

LEMMA 5.2. *A homomorphism θ of* Hom $\{R, G\}$ *lies in* Hom_f $\{R, G; F\}$ *if and only if θ can be extended to a homomorphism (into G) of each subgroup F_0 of F which contains R and for which the factor group F_0/R has a finite number of generators.*

We now consider the topology on Hom $\{R, G\}$.

LEMMA 5.3. *If G and hence* Hom $\{R, G\}$ *are generalized topological groups,* Hom_f $\{R, G; F\}$ *is contained in the closure of* Hom $\{F \mid R, G\}$, or

$$\mathrm{Hom}\,\{F \mid R, G\} \subset \mathrm{Hom}_f\,\{R, G; F\} \subset \overline{\mathrm{Hom}}\,\{F \mid R, G\} \subset \mathrm{Hom}\,\{R, G\}.$$

PROOF. Let θ_0 be in $\mathrm{Hom}_f\,\{R, G; F\}$, while U is any open set of $\mathrm{Hom}\,\{R, G\}$ containing θ_0. Since F is discrete, the definition of the topology in $\mathrm{Hom}\,\{R, G\}$ implies that there is a finite set of elements $r_1, \cdots, r_n$ of R such that U contains all θ for which each $\theta(r_i) = \theta_0(r_i)$. The elements r_i are all contained in a subgroup F_0 of F generated by a finite number of the given independent generators of the free group F. Since $\theta_0 \in \mathrm{Hom}_f$, θ_0 has an extension θ' to the group generated by F_0 and R (Lemma 5.2). Introduce a new homomorphism θ^* of F by setting $\theta^*(z_\alpha) = \theta'(z_\alpha)$ for each generator z_α of F_0, $\theta^*(z_\alpha) = 0$ otherwise. This θ^* induces a homomorphism θ of R, which agrees with θ_0 on the original elements $r_1, \cdots, r_n$ and which is by construction an element of $\mathrm{Hom}\,\{F \mid R, G\}$. In other words, the arbitrary neighborhood U of θ_0 does contain a homomorphism $\theta \in \mathrm{Hom}\,\{F \mid R, G\}$. This proves the lemma.

LEMMA 5.4. *If G is a compact topological group,* $\mathrm{Hom}\,\{F \mid R, G\}$ *is a closed sub-group of* $\mathrm{Hom}\,\{R, G\}$, *and hence* $\mathrm{Hom}\,\{F \mid R, G\} = \mathrm{Hom}_f\,\{R, G; F\}$.

PROOF. By Corollary 3.2, both the groups $\mathrm{Hom}\,\{R, G\}$ and $\mathrm{Hom}\,\{F, G\}$ are compact and topological. The second of these groups is mapped homomorphically onto $\mathrm{Hom}\,\{F \mid R, G\}$ by the continuous correspondence $\theta \to \theta \mid R$ of Lemma 3.4. Therefore, by Lemma 1.1, the image $\mathrm{Hom}\,\{F \mid R, G\}$ is closed.

For any integer m, let mG be the subgroup of all elements of the form mg, with g in G. A condition for the closure of Hom_f may be stated in terms of these subgroups.

LEMMA 5.5. *If G is a generalized topological group, then* $\mathrm{Hom}_f\,\{R, G; F\}$ *is closed in* $\mathrm{Hom}\,\{R, G\}$ *whenever every subgroup mG of G is closed in G, for* $m = 2, 3, \cdots$.[11]

PROOF. Let θ be a homomorphism in the closure of $\mathrm{Hom}_f\,\{R, G; F\}$. Consider an arbitrary t in F such that $mt \in R$. By Lemma 5.1 and the given condition on G it will suffice to prove that $\theta(mt) \in \overline{mG}$. Let V be any open set containing 0 in G. By the definition of the topology in $\mathrm{Hom}\,\{R, G\}$, there exists for θ in the closure of Hom_f an element θ' in Hom_f itself, such that $\theta'(mt) - \theta(mt) \in V$. But $\theta'(mt)$ is in mG, so that the arbitrary open set $V + \theta(mt)$ does contain an element of mG. This proves $\theta(mt)$ in $\overline{mG}$, as required.

An examination of this proof shows that the given condition on G can be somewhat weakened. It suffices to require that the subgroup mG be closed in G for every integer m which is the order of an element of F/R. The same remark will apply in various subsequent cases when this condition on G is used.

CHAPTER II. GROUP EXTENSIONS

This chapter introduces the basic group $\mathrm{Ext}\,\{G, H\}$ of all group extensions of G by H, and its subgroup $\mathrm{Ext}_f\,\{G, H\}$ of all extensions which are "finitely trivial"

[11] If every subgroup mG is closed in G, Steenrod [9] and Lefschetz [7] say that G has the "division closure property."

(§8). Each individual group extension can be described either by a suitable "factor set" (§7) or by a certain homomorphism. The equivalence of these two representations is the fundamental theorem of this chapter (Theorem 10.1); it gives an expression of Ext $\{G, H\}$ as one of the factor-homomorphism groups already considered in Chapter I. This fundamental theorem, which is implicit in previous algebraic work on group extensions, is of independent algebraic interest. The chapter closes with a proof that the representation of Ext $\{G, H\}$ by homomorphisms is a "natural" one (§12). This conclusion is needed for the subsequent limiting process, which is used in defining the Čech homology groups.

6. Definition of extensions

A group E having G as subgroup and $H = E/G$ as the corresponding factor group is said to be an "extension" of G by H. More explicitly, if the groups G and H are given, a *group extension* of G by H is a pair (E, β), where E is a group containing G and β is a homomorphism of E onto H under which exactly the elements of G are mapped into $0 \in H$.[12] Such a β induces an isomorphism of E/G to H. For given G and H, two extensions (E_1, β_1) and (E_2, β_2) are regarded as *equivalent* if and only if there is an isomorphism ω of E_1 to E_2 which leaves elements of G and cosets of H fixed. In other words, the isomorphism ω of E_1 to E_2 must have $\omega g = g$ for $g \in G$ and $\beta_2 \omega x = \beta_1 x$ for $x \in E_1$. We regard equivalent extensions as identical, and so study the equivalence classes of extensions of G by H. It will appear that these equivalence classes are themselves the elements of a group.

For given G and H, the direct product $G \times H$ has the "natural" homomorphism $(g, h) \rightarrow h$ onto H, and so can be regarded as an extension of G by H. Any extension (E, β) equivalent to this direct product (with its natural homomorphism) is said to be a *trivial* extension of G by H.

7. Factor sets for extensions

A given extension (E, β) of G by H can be described in terms of representatives for elements of H. To each h in H select in E a representative $u(h)$, such that $\beta(u(h)) = h$. Every element of E lies in some coset h, so has the form $g + u(h)$ for g in G. The sum of any two representatives $u(h)$ and $u(k)$ will lie in the same coset, modulo G, as does the representative of the sum $h + k$. Hence there is an addition table of the form

$$u(h) + u(k) = u(h + k) + f(h, k), \tag{7.1}$$

where $f(h, k)$ lies in G for each pair of elements h, k in H. The commutative and associative laws in the group E imply two corresponding identities for f,

$$f(h, k) = f(k, h), \tag{7.2}$$

[12] Group extensions are discussed by Baer [2], Hall [6], Turing [11], Zassenhaus [15], and elsewhere. Much of the discussion in the literature treats the more general case in which G but not H is assumed to be abelian and in which G is not necessarily in the center of H.

$$f(h, k) + f(h + k, l) = f(h, k + l) + f(k, l). \tag{7.3}$$

The sum of any two elements $g_1 + u(h)$ and $g_2 + u(k)$ of E is determined by the addition table (7.1) and the addition given within G and H.

The extension E does not uniquely determine the corresponding function f. An arbitrary set of representatives $u'(h)$ for the elements of H can be expressed in terms of the given representatives as

$$u'(h) = u(h) + g(h), \qquad \text{each } g(h) \in G;$$

they will have an addition table like that of (7.1) with a function f' given by

$$f'(h, k) = f(h, k) + [g(h) + g(k) - g(h + k)]. \tag{7.4}$$

Conversely, a *factor set* of H in G is any function $f(h, k)$, with values in G for h, k in H which satisfies the "commutative" and "associative" conditions (7.2) and (7.3) for all h, k, and l in H. A *transformation set* is any function of h and k like the term in brackets in (7.4); thus for any function $g(h)$ defined for each $h \in H$ and taking on values in G, the function

$$t(h, k) = g(h) + g(k) - g(h + k) \tag{7.5}$$

is a transformation set. Such a set automatically satisfies the conditions (7.2) and (7.3), hence is always a factor set. Two factor sets f and f' are said to be *associate* if their difference is, as in (7.4), a transformation set. The correspondence between group extensions and factor sets may now be formulated as follows.

THEOREM 7.1. *For given groups G and H, there is a many-one correspondence $f \to (E, \beta)$ between the factor sets f of H in G and the group extensions (E, β) of G by H, where $f \to (E, \beta)$ holds if and only if f is the factor set which appears in one of the possible "addition tables" (7.1) for E. Two factor sets f and f' of H in G determine equivalent group extensions of G by H if and only if they are associate. In particular, the group extension determined by f is trivial if and only if f is a transformation set.*

PROOF. As a preliminary, observe that the associative relations (7.3) for f show (with $k = l = 0$, $h = k = 0$) that $f(0, 0) = f(h, 0) = f(0, l)$. Now, given f, we construct E_f as the group of all pairs (g, h) with addition given by the rule

$$(g_1, h) + (g_2, k) = (g_1 + g_2 + f(h, k), h + k),$$

and the homomorphism β_f defined by $\beta_f(g, h) = h$. Since $f(0, 0) = f(0, l)$, each element $(g, 0)$ may be identified with the corresponding element $g + f(0, 0)$ in G; the pair (E_f, β_f) is then indeed an extension of G by H. As a representative of h in E_f, we may choose $u(h) = (0, h)$; the addition table (7.1) then involves exactly the original factor set f. If E is an arbitrary group extension of G by H in which f appears as the factor set of E, the correspondence $g + u(h) \leftrightarrow (g, h)$ shows that the extension E is in fact equivalent to the extension E_f just constructed. Therefore $f \to (E_f, \beta_f)$ is a many-one correspondence with the defining property stated in the theorem.

If f and f' are associate, as in (7.4), the correspondence

$$(g, h) \to (g - g(h), h)'$$

shows that the corresponding extensions E_f and $E_{f'}$ are equivalent. Conversely, the argument leading to (7.4) shows in effect that E_f is equivalent to $E_{f'}$ only if f is associate to f'.

We turn now to two special applications of transformation sets. In the first place, the representative for the zero element of H may always be chosen as the zero in E. This means that $u'(0) = 0$, $u'(0) + u'(h) = u'(h)$, so that

$$f'(0, h) = f'(h, 0) = 0 \qquad (\text{all } h \in H). \tag{7.6}$$

A factor set f' with the property (7.6) may be called *normalized*; we have proved that every factor set f is associate to a normalized factor set.

Free groups may be characterized in terms of group extensions as follows:

THEOREM 7.2. *A group with more than one element H is free if and only if every extension of any group by H is the trivial extension.*

PROOF. Suppose first that H satisfies the condition that every extension of every G is trivial. Represent H as F/R, where F is free. Then F is a trivial extension of R by H, hence is a direct sum of R and H. Therefore H, as a subgroup of the free group F, is itself free. The other half of the theorem is stated in more detail in the following Lemma.

LEMMA 7.3. *Every factor set f' of a free group F in a group G is a transformation set, so that*

$$f'(x, y) = \phi(x + y) - \phi(x) - \phi(y), \qquad \phi(x) \in G, \tag{7.7}$$

holds for all $x, y \in F$. If F has generators z_α, the function ϕ may be chosen so that $\phi(0) = -f'(0, 0)$, $\phi(z_\alpha) = 0$ for each generator z_α.

PROOF. In the extension $E_{f'}$ of G by F we have an addition table

$$u'(x) + u'(y) = u'(x + y) + f'(x, y) \qquad (x, y \in F).$$

In E we introduce a new set of representatives $u(\sum e_\alpha z_\alpha) = \sum e_\alpha u'(z_\alpha)$ for the elements $\sum e_\alpha z_\alpha$ of F. These are related to the original representatives by an equation $u(z) = u'(z) + \phi(z)$, where $\phi(z)$ has values in G. Because F is a free group, $z \to u(z)$ as defined is a homomorphism of F into E, so that $u(x + y) = u(x) + u(y)$, and the factor set belonging to u is identically zero. But the given f' is associate to this zero factor set, as in (7.4). Setting $f = 0$, $\phi = -g$ in (7.4) gives (7.7), as desired. By construction, $u(z_\alpha) = u'(z_\alpha)$, so $\phi(z_\alpha) = 0$. Also $u'(0) + u'(0) = u'(0) + f'(0, 0)$, so that $u'(0) = f'(0, 0)$, $u(0) = 0$, and therefore $\phi(0) = -f'(0, 0)$. This completes the proof.

8. The group of extensions

For fixed H and G the sum of two factor sets f_1 and f_2 is a third factor set, defined as

$$(f_1 + f_2)(h, k) = f_1(h, k) + f_2(h, k) \qquad (h, k \in H).$$

Under this addition, the factor sets and the transformation sets form groups, denoted respectively by

(8.1) Fact $\{G, H\}$ = group of all factor sets of H in G,

(8.2) Trans $\{G, H\}$ = group of all transformation sets of H in G.

The factor sets belonging to a given group extension E constitute a coset of the subgroup Trans $\{G, H\}$, as in (7.4). Hence the correspondence of factor sets to extensions is a one-one correspondence between cosets of Fact/Trans and equivalence classes of extensions. This correspondence carries the addition of factor sets into an addition of group extensions. We are thus led to define the *group of group extensions* of G by H as[13]

(8.3) $$\text{Ext } \{G, H\} = \text{Fact } \{G, H\}/\text{Trans } \{G, H\}.$$

If H is discrete while G is a (generalized) topological group, there will be a corresponding induced topology on Ext $\{G, H\}$. For each factor set f is a function on $H \times H$ with values in G, so that Fact $\{G, H\}$ is a subgroup of the generalized topological group $G^{H \times H}$ of all such functions. The subgroup "Trans" and the factor group "Ext" also carry topologies. Much as in §3 one can prove that if H is discrete and G topological, then Fact $\{G, H\}$ is a closed subgroup of $G^{H \times H}$. This proves

LEMMA 8.1. *If H is discrete and G is a topological (and compact) group, then* Fact $\{G, H\}$ *is a topological (and compact) group.*

In general, however, Trans $\{G, H\}$ will not be closed in Fact $\{G, H\}$, even when G is topological. In such cases Ext $\{G, H\}$ is necessarily a generalized topological group.

If (E, β) is an extension of G by H, each subgroup $S \subset H$ determines a corresponding subgroup $E_S \subset E$, consisting of all $e \in E$ with $\beta(e) \in S$. Since $E_S \supset G$, we may thus say that E "induces" an extension (E_S, β) of G by S. We call an extension E *finitely trivial* if E_S is trivial for every finite subgroup $S \subset H$.

Similarly, any factor set f of H in G determines for each subgroup $S \subset H$ a factor set f_S of S in G, where $f_S(h, k) = f(h, k)$ for h, k in S (i.e., f_S is obtained by "cutting off" f at S). The correspondence between factor sets and group extensions readily gives

LEMMA 8.2. *A factor set f of H in G determines a finitely trivial extension of*

[13] It is possible to define the sum of two group extensions directly, without using the factor sets (see Baer [2] p. 394); it also is possible to give an analogous definition of the topology introduced below in Ext $\{G, H\}$.

G by H if and only if, for every finite subgroup $S \subset H$, the factor set f_S "cut off" at S is a transformation set of S in G. Hence the finitely trivial extensions of G by H constitute a subgroup $\mathrm{Ext}_f \{G, H\}$ *of* $\mathrm{Ext} \{G, H\}$.

9. Group extensions and generators

A group extension can be described not only by factor sets, but also by certain homomorphisms related to the generators of the extending group H. For let (E, β) be a given extension of G by H, and $H = F/R$ a representation of H as a factor group of a free group F. Let F have the generators z_α, as in §4; the corresponding elements (or cosets) t_α of H will then be a set of generators of H. For each generator t_α choose a corresponding representative u_α in the given group extension E, so that $\beta u_\alpha = t_\alpha$. Then $\beta(\sum e_\alpha u_\alpha) = \sum e_\alpha t_\alpha$, so that any element $\sum e_\alpha t_\alpha \in H$ has a representative of the form $\sum e_\alpha u_\alpha$. This means that each element of E can be written in the form

$$x = g + \sum e_\alpha u_\alpha, \qquad g \in G, \qquad e_\alpha \text{ integers}.$$

From this representation one can at once determine how to add the elements of E. However, this representation is not in general unique, for $(\sum e_\alpha u_\alpha) \in G$ is equivalent to $\sum e_\alpha t_\alpha = 0$, which in turn is equivalent to $(\sum e_\alpha z_\alpha) \in R$. Thus to each $r = \sum e_\alpha z_\alpha$ in the group R of "relations" there is assigned an element $\theta(r) \in G$, defined as

$$\theta(r) = \theta(\sum e_\alpha z_\alpha) = \sum e_\alpha u_\alpha$$

These assignments $\theta(r)$ completely determine the extension E.

The function θ hereby defined[14] is a homomorphism of R into G. Conversely every such homomorphism θ can be used to construct a corresponding group extension of G by $H = F/R$; it suffices to construct E by reducing the direct product $F \times G$ modulo the subgroup of all elements of the form $(r, \theta(r))$, for $r \in R$. There is thus a correspondence between homomorphisms of R into G and extensions of G by $H = F/R$.[15]

10. The connection between homomorphisms and factor sets

Given G and $H = F/R$, an extension E of G by H may be given either by a factor set or by a homomorphism of R into G. There must therefore be a relation between factor sets and homomorphisms of this type. We now propose to establish this relation directly, without using extensions explicitly. (Actually, the correspondence which we obtain is identical with that obtained by going from a homomorphism first to the corresponding group extension and then to its factor set.)

THEOREM 10.1. *If $H = F/R$ is a factor group of a free group F, while G is any other group, then*

[14] Actually θ may be obtained by "cutting off" one of the homomorphisms ϕ as described in Lemma 4.3.

[15] This correspondence has been stated by Baer ([2], p. 395) and used by Hall [6].

(10.1) $$\text{Ext}\ \{G, H\} \cong \text{Hom}\ \{R, G\}/\text{Hom}\ \{F \mid R, G\}.$$

Under the correspondence which gives this isomorphism

(10.2) $$\text{Ext}_f\ \{G, H\} \cong \text{Hom}_f\ \{R, G; F\}/\text{Hom}\ \{F \mid R, G\},$$

(10.3) $$\text{Ext}\ \{G, H\}/\text{Ext}_f\ \{G, H\} \cong \text{Hom}\ \{R, G\}/\text{Hom}_f\ \{R, G; F\}.$$

If G is a generalized topological group while F and H are discrete, all these isomorphisms are bicontinuous.

PROOF. As a preliminary, observe that the representation $H = F/R$ means that the free group F is a group extension of R by H. In this extension choose a representative $u_0(h)$ in F for each $h \in H$. F is then described, as in (7.1), by an addition table

(10.4) $$u_0(h) + u_0(k) = u_0(h + k) + f_0(h, k),$$

where f_0 is a factor set of H in R. This factor set will be fixed throughout the proof.

Since Ext $\{G, H\}$ is defined as Fact/Trans, the required isomorphism (10.1) could be established by a suitable correspondence of homomorphisms to factor sets. Let $\theta \in \text{Hom}\ \{R, G\}$ be given, and define f_θ by

(10.5) $$f_\theta(h, k) = \theta[f_0(h, k)] \qquad (h, k \in H).$$

The requisite commutative and associative laws (7.2) and (7.3) for f_θ follow from those for f_0, and the correspondence $\theta \to f_\theta$ is a homomorphism of Hom $\{R, G\}$ into Fact $\{G, H\}$, and therefore into Ext $\{G, H\}$.

Suppose next that θ can be extended to a homomorphism θ^* of F into G. This homomorphism applied to (10.4) gives

$$\theta^*[f_0(h, k)] = \theta^*[u_0(h)] + \theta^*[u_0(k)] - \theta^*[u_0(h + k)].$$

If we set $g(h) = \theta^*[u_0(h)]$, the result asserts that $\theta^* f_0 = \theta f_0 = f_\theta$ is a transformation set.

Conversely, suppose that f_θ is a transformation set, so that $f_\theta(h, k) = g(h) + g(k) - g(h + k)$ for some function g. Now any element in F can be written, in only one way, in the form $r + u_0(h)$, with r in R, h in H. We define $\theta^*(r + u_0(h))$ as $\theta(r) + g(h)$. Clearly θ^* is an extension of θ; a straightforward computation with (10.4) shows that θ^* is actually a homomorphism. In this case, then, θ is extendable to F.

We know now that the correspondence $\theta \to f_\theta$ is an isomorphism of Hom $\{R, G\}$/Hom $\{F \mid R, G\}$ into a subgroup of Ext $\{G, H\}$. It remains to prove that it is a homomorphism onto. At this juncture we use for the first time the assumption that F is a free group. Let E be a given extension of G by H, with a factor set f which we can assume is normalized, as in (7.6). Let β_0 be the given homomorphism of F on H. Use f to define a factor set f' of F in G by the equation

(10.6) $$f'(x, y) = f(\beta_0 x, \beta_0 y), \qquad x, y \in F.$$

Since F is free, f' is a transformation set, so we can find, as in Lemma 7.3, a function $\phi(z)$ on F to G with

(10.7) $$\phi(x + y) = \phi(x) + \phi(y) + f'(x, y).$$

In particular, if x and y lie in R, $\beta_0 x = \beta_0 y = 0$, and $f'(x, y) = f(0, 0) = 0$, because f is normalized. Thus ϕ, restricted to R, is a homomorphism $\theta = \phi \mid R$ of R into G. Furthermore, if ϕ is applied to the addition table (10.4) for F, the property (10.7) gives

$$\phi[u_0(h) + u_0(k)] = \phi[u_0(h + k)] + \phi[f_0(h, k)],$$

where a term $f'(u_0(h + k), f_0(h, k))$, which would have entered by (10.7), is zero because f is normalized, $f_0(h, k) \in R$, and $\beta_0 f_0(h, k) = 0$. Now compute $f(h, k)$ for h, k in H. By (10.6),

$$\begin{aligned} f(h, k) &= f'(u_0(h), u_0(k)) \\ &= \phi[u_0(h) + u_0(k)] - \phi[u_0(h)] - \phi[u_0(k)] \\ &= \phi[u_0(h + k)] - \phi[u_0(h)] - \phi[u_0(k)] + \phi[f_0(h, k)], \end{aligned}$$

in virtue of the equation displayed just above. This equation asserts that f is associate to the factor set $\phi f_0 = \theta f_0$. In other words, given the normalized factor set f, we have constructed a homomorphism θ for which f is essentially θf_0. This completes the proof of (10.1).

It is desirable to find a more explicit expression for this dependence of θ on f. A simple induction applied to (10.7) will show that, for z_i in F,

$$\phi\left(\sum_{i=1}^{n} z_i\right) = \sum_{i=1}^{n} \phi(z_i) + \sum_{k=1}^{n-1} f'\left(\sum_{i=1}^{k} z_i,\ z_{k+1}\right).$$

If z_i is one of the generators z_α of F, then $\phi(z_i) = 0$, by Lemma 7.2. If $z_i = -z_\alpha$ is the negative of a generator, then by (10.7)

$$\phi(0) = \phi(z_\alpha + (-z_\alpha)) = \phi(z_\alpha) + \phi(-z_\alpha) + f'(z_\alpha, -z_\alpha),$$

so that $\phi(-z_\alpha) = -f'(z_\alpha, -z_\alpha)$. Now any element of F can be written as a finite linear combinations of generators and hence as a sum $\sum x_i$, where each x_i is either a generator or the negative of a generator z_α, and where any given generator may appear several times in this sum. In particular, for any element $r = \sum x_i$ in the subgroup R, the previous formula for ϕ becomes a formula for $\theta = \phi \mid R$,

(10.8) $$\theta\left(\sum_{i=1}^{n} x_i\right) = -\sum{}' f(\beta_0 x_i, -\beta_0 x_i) + \sum_{k=1}^{n-1} f\left(\sum_{i=1}^{k} \beta_0 x_i,\ \beta_0 x_{k+1}\right),$$

where β_0 is the given homomorphism of F into H, and where the sum $\sum'$ is taken over those elements x_i which are the negatives of generators. The essential feature of this formula is the fact that it expresses $\theta(r)$ for $r \epsilon R$ as a sum of a finite number of values of the given factor set f of H in G.

Now consider the continuity of the correspondence $\theta \to f_\theta$ used to establish (10.1). It suffices to establish the continuity at 0. If U is any open set, containing zero, in Hom $\{R, G\}$/Hom $\{F \mid R, G\}$, there will be an open set V containing 0 in G and a finite set of elements $r_1, \cdots, r_s \epsilon R$ such that U contains the cosets of all homomorphisms θ with $\theta(r_i) \epsilon V$, $i = 1, \cdots, s$.

For a given f, the expressions $\theta(r_i)$ of (10.8) for these elements r will involve but a finite number of elements of the factor set f. Because of the continuity of addition in G, we can construct an open set U' in Fact $\{G, H\}$ such that each $\theta(r_i)$ does in fact lie in the given V. This establishes the continuity of the correspondence $f \to \theta$. The continuity of the inverse correspondence is obtained by a similar argument on the definition (10.5) of this correspondence.

It remains only to consider the formulas (10.2) and (10.3) on finitely trivial extensions. Let θ and its correspondent f_θ be given, and let $F_0 \supset R$ be any subgroup of F for which F_0/R is finite. A previous argument, applied to F_0 instead of F, shows that θ can be extended to a homomorphism of F_0 into G if and only if f_θ, regarded as a factor set for F_0/R in G, is a transformation set. But the subgroup Hom_f $\{R, G; F\}$ by definition consists of all those θ which are extendable to every such F_0, while Ext_f by Lemma 8.2 is obtained from those factor sets which are transformation sets on every such subgroup F_0. Hom_f $\{R, G; F\}$/Hom $\{F/R, G\}$ is the subgroup corresponding to Ext_f $\{G, H\}$ under $\theta \to f_\theta$. This proves (10.2) and with it (10.3). The continuity of the isomorphisms in this case follows from the continuity of the isomorphism (10.1).

For subsequent purposes we observe that the correspondence $\theta \to f_\theta$ obtained in this proof is essentially independent of the choice of the fixed factor set f_0 for H in R. Specifically, if f_0 is replaced by an associate factor set f_0', f_θ will be replaced also by an associate factor set, so that the corresponding element of Ext $\{G, H\}$ is not altered.

11. Applications

The representation of Ext $\{G, H\}$ as Hom $\{R, G\}$/Hom $\{F \mid R, G\}$ gives an immediate proof of the invariance of the latter group, as stated in Theorem 4.2 of Chapter I. There are a number of other simple corollaries.

COROLLARY 11.1. *For a direct product $H \times H'$,*

$$\text{Ext } \{G, H \times H'\} \cong \text{Ext } \{G, H\} \times \text{Ext } \{G, H'\}. \tag{11.1}$$

If G is a generalized topological group, the isomorphism is bicontinuous.

PROOF. If $H = F/R$ and $H' = F'/R'$, we may write $H \times H' = (F \times F')/(R \times R')$, where $F \times F'$, like F and F', is free. Each homomorphism of $R \times R'$ into G determines homomorphisms θ and θ' of the subgroups R and R' into G, and this correspondence yields a (bicontinuous) isomorphism

$$\text{Hom } \{R \times R', G\} \cong \text{Hom } \{R, G\} \times \text{Hom } \{R', G\}.$$

Furthermore, under the same correspondence

$$\text{Hom}\,\{(F \times F') \mid (R \times R'), G\} \cong \text{Hom}\,\{F \mid R, G\} \times \text{Hom}\,\{F' \mid R', G\}.$$

These two relations yield a corresponding isomorphism between the respective factor groups such as Hom $\{R, G\}$/Hom $\{F \mid R, G\}$. By the fundamental theorem, the latter isomorphism is the one asserted in (11.1).

This conclusion can also be established without using homomorphisms, by a direct argument like that of Lemma 7.2. (Choose new representatives in E for elements of $H \times H'$ by setting $u'(hh') = u(h)u(h')$). Another simple argument directly with the factor sets will give a companion "direct product" representation,

$$\text{Ext}\,\{G \times G', H\} \cong \text{Ext}\,\{G, H\} \times \text{Ext}\,\{G', H\}; \tag{11.2}$$

this isomorphism is also bicontinuous.

Corollary 11.2. *If H is a cyclic group of order m, then*

$$\text{Ext}\,\{G, H\} \cong G/mG, \qquad (mG = \text{all } mg, \text{for } g \in G). \tag{11.3}$$

This isomorphism is also bicontinuous.

This is a well known result, which can be derived directly from our main theorem. The cyclic group H can be written as $H = F/R$, where F is an infinite cyclic group with generator z, R the subgroup generated by mz. Then any $\theta \in \text{Hom}\,\{R, G\}$ is uniquely determined by the image $\theta(mz) = h$ of the generator mz. This correspondence $\theta \to h$ (mod mG) gives the isomorphism (11.3).

A similar representation can be found for any finite abelian group H, simply by representing H as a direct product of cyclic groups of orders m_i, $i = 1, \cdots, t$. By Corollary 11.1, Ext $\{G, H\}$ is then isomorphic to the direct product of the groups G/m_iG. A similar decomposition applies if the abelian group H has a finite number of generators. The result may be stated as follows.

Corollary 11.3. *If H has a finite number of generators, and T is the subgroup of all elements of finite order in H, then* Ext $\{G, H\} \cong$ Ext $\{G, T\}$, *algebraically and topologically. The latter group is a direct product of groups of the form G/mG.*

Theorem 7.2 (extensions by a free group are trivial) has an analogue for infinitely divisible groups. Recall that G is *infinitely divisible* if for each $g \in G$ and each integer $m \neq 0$ the equation $mx = g$ has a solution $x \in G$.

Corollary 11.4. *A group G is infinitely divisible if and only if every extension of G by any group is the trivial extension.*

Proof. If G is not infinitely divisible, some $G/mG \neq 0$, so that there will be a non-trivial extension of G by a cyclic group, as in Corollary 11.2. Conversely, suppose G is infinitely divisible. If $R \subset F$ are groups, a transfinite induction will show that every homomorphism of R into G can be extended to a homomorphism into G of the larger group F. Therefore the subgroup Hom $\{F \mid R, G\}$ exhausts the group Hom $\{R, G\}$, and Ext $\{G, F/R\} = 0$.

Corollary 11.5. *If T is the subgroup of all elements of finite order in H, then*

(11.4) $\quad \text{Ext}\,\{G,\ H\}/\text{Ext}_f\,\{G,\ H\} \cong \text{Ext}\,\{G,\ T\}/\text{Ext}_f\,\{G,\ T\}.$

This isomorphism is bicontinuous (if G is a generalized topological group).

PROOF. In the representation $H = F/R$, let F_T denote the set of all elements of F of finite order modulo R. The group T then has the representation $T = F_T/R$, while F_T, as a subgroup of a free group, is itself free. Now the group $\text{Hom}_f\,\{R, G; F\}$ by definition consists of all homomorphisms extendable to subgroups of F finite over R; as these subgroups are all contained in F_T, the group Hom_f is identical with $\text{Hom}_f\,\{R, G; F_T\}$. If both factor groups in (11.4) are now represented by groups of homomorphisms, as in (10.3), the result is immediate.

Observe that when T has only elements of finite order, the group $\text{Ext}_f\,\{G, T\}$, though it consists of extensions E of G by T trivial on every finite subgroup of T, can contain non-trivial extensions. This is illustrated by the following example. Let p be a prime, and G a group with generators $g, h_1, h_2, \cdots$ and relations $p^i h_i = g$, for $i = 1, 2, \cdots$. In this group G the intersection of all the subgroups p^iG is the group generated by g alone. Let T be the group of all rational numbers of the form a/p^i, reduced modulo 1. Then all elements of T have finite order, and T may be written as $T = F/R$, where F is a free group with generators $z_1, z_2, \cdots$, and R the free subgroup generated by pz_1, $pz_2 - z_1$, $pz_3 - z_2, \cdots$. (The homomorphism $F \to T$ maps z_i into $1/p^i$.)

To prove $\text{Ext}_f\,\{G, T\} \neq 0$ it suffices to find a $\theta \in \text{Hom}_f\,\{R, G; F\}$ which is not in $\text{Hom}\,\{F \mid R, G\}$. Such a θ is determined by setting $\theta(pz_1) = g$, $\theta(pz_{i+1} - z_i) = 0$, $i = 1, 2, \cdots$. The definition $\theta^*(z_{n-i}) = p^ih_n$ will provide an extension θ^* of θ to the finite subgroup of F generated by $z_1, \cdots, z_n$. However, suppose that θ had an extension ϕ to F. Then $\phi(pz_{i+1}) = \phi(z_i)$, so that $\phi(z_1) = p^n\phi(z_{n+1})$ for every n. This means that $\phi(z_1)$ is in every subgroup p^nG, hence has the form eg for an integer e. But then $g = \theta(pz_1) = p\phi(z_1) = epg$ gives a contradiction. Therefore $\text{Ext}_f\,\{G, T\} \neq 0$ in this case. However, if G has no elements of finite order, one can prove easily that $\text{Ext}_f\,\{G, T\} = 0$, using Lemma 5.1 (see §17 below).

For several types of topological groups G, §5 gives information on the topology of the various relevant subgroups of $\text{Hom}\,\{R, G\}$. By the main theorem, the conclusions of Lemmas 5.3, 5.4, and 5.5 can now be rewritten as conclusions about the topology of $\text{Ext}\,\{G, H\}$, as follows.

COROLLARY 11.6. *If H is discrete and G a generalized topological group, the closure of the zero element in the generalized topological group* $\text{Ext}\,\{G, H\}$ *contains* $\text{Ext}_f\,\{G, H\}$. *If, in addition, every subgroup mG is closed in G, for $m = 2, 3, \cdots$, then* $\text{Ext}_f\,\{G, H\}$ *is closed in* $\text{Ext}\,\{G, H\}$.

In particular, if H has no elements of finite order, then every extension of G by H is trivial on (the non-existent) finite subgroups of H, consequently $\text{Ext}_f\,\{G, H\} = \text{Ext}\,\{G, H\}$ and the closure of 0 is the whole group $\text{Ext}\,\{G, H\}$. This means that $\text{Ext}\,\{G, H\}$ carries the "trivial" (generalized) topology in which the only open sets are the whole group and the empty set.

COROLLARY 11.7. *If H is discrete and G compact and topological, then* $\mathrm{Ext}_f \{G, H\} = 0$ *and* $\mathrm{Ext} \{G, H\}$ *is itself a compact topological group.*

This conclusion is obtained from Lemma 8.1 and from Lemma 5.4.

12. Natural homomorphisms

The basic homomorphism $\eta(\theta) = f_\theta$ mapping elements θ of Hom $\{R, G\}$ into factor sets f, as in Theorem 10.1, is a "natural" one. Specifically, this means that the application of η "commutes" with the application of any homomorphism T to the free group F and its subgroup R. To state this more precisely, we need to consider first the homomorphisms which T induces on the groups Hom $\{R, G\}$ and Ext $\{G, H\}$.

Let F' be a free group with subgroup R', T a homomorphism $z' \to Tz'$ of F' into the free group F such that $T(R') \subset R$. T induces a homomorphism of $H' = F'/R'$ into $H = F/R$. This induced homomorphism will be written with the same letter T, so that $T(g + R') = Tg + R$, for any coset $g + R'$.

Now consider $\theta \in \mathrm{Hom}\,\{R, G\}$. Clearly the product $\theta' = \theta T$ is an element of Hom $\{R', G\}$, and the correspondence $\theta \to \theta'$ is a homomorphism T_h^* of Hom $\{R, G\}$ into Hom $\{R', G\}$. Furthermore $\theta \in \mathrm{Hom}\,\{F \mid R, G\}$ implies $\theta T \in \mathrm{Hom}\,\{F' \mid R', G\}$, so that T_h^* also induces a homomorphism T_h^*,

(12.1) $T_h^* : \mathrm{Hom}\,\{R, G\}/\mathrm{Hom}\,\{F \mid R, G\} \to \mathrm{Hom}\,\{R', G\}/\mathrm{Hom}\,\{F' \mid R', G\}$.

Similarly, consider $f \in \mathrm{Fact}\,\{G, H\}$. The function f' defined by

$$f'(h', k') = f(Th', Tk') \qquad (h', k' \in H')$$

is a factor set of H' in G, and the correspondence $f \to f'$ is a homomorphism T_e^* of Fact $\{G, H\}$ into Fact $\{G, H'\}$. Furthermore, $f \in \mathrm{Trans}\,\{G, H\}$ implies $f' \in \mathrm{Trans}\,\{G, H'\}$, so that T_e^* also induces a homomorphism T_e^* for the corresponding factor groups Ext = Fact/Trans,

$$T_e^* : \mathrm{Ext}\,\{G, H\} \to \mathrm{Ext}\,\{G, H'\}. \tag{12.2}$$

By the (dual) homomorphisms induced on Ext or Hom by T we always mean these homomorphisms T_h^* and T_e^*.

THEOREM 12.1. *Let T be a homomorphism of F' into F with $T(R') \subset R$, where $F \supset R$ and $F' \supset R'$ are free groups, while η (or η') is the homomorphism of* Hom $\{R, G\}$ *onto* Ext $\{G, F/R\}$ *established in the proof of Theorem* 10.1. *Then*

$$\eta' T_h^* = T_e^* \eta, \tag{12.3}$$

where T_h^*, T_e^* are the appropriate homomorphisms induced by T on Hom *and* Ext, *respectively.*

PROOF. The figure involved is

$$\begin{tikzcd} \mathrm{Hom}\,\{R, G\} \arrow[r, "\eta"'] \arrow[d, "T_h^*"] & \mathrm{Ext}\,\{G, F/R\} \arrow[d, "T_e^*"] \\ \mathrm{Hom}\,\{R', G\} \arrow[r, "\eta'"'] & \mathrm{Ext}\,\{G, F'/R'\} \end{tikzcd}$$

The correspondence η was constructed from a factor set f_0 for F as an extension of R; similarly, η' is based on a factor set f_0' for H' in R', such that

$$u_0'(h') + u_0'(k') = u_0'(h' + k') + f_0'(h', k'), \tag{12.4}$$

where $u_0'(h')$ is a representative of $h' \in H'$ in F'. First we determine the relation between f_0 and f_0'. The given homomorphism T carries F' into F, H' into H and thus $u_0'(h')$ into $Tu_0'(h')$, a representative in F of Th' in H. This representative will differ from the given representative $u_0(Th')$ by an element of R, so that

$$Tu_0'(h') = u_0(Th') + \rho(h') \qquad \text{(all } h' \text{ in } H').$$

where each $\rho(h')$ lies in R. Now the representatives $Tu_0'(h')$ will add with a factor set $Tf_0'(h', k')$, as may be seen by applying T to both sides of (12.4). This factor set in associate (in the group TH') to the originally given factor set f_0 of $H \supset TH'$; explicitly we have, by the argument leading to (7.4), that

$$Tf_0'(h', k') = f_0(Th', Tk') + [\rho(h') + \rho(k') - \rho(h' + k')].$$

Suppose now that $\theta \in \text{Hom}\,\{R, G\}$ is given. Application of η and then T_e^* will give, by the definitions of these correspondences, a factor set f', with

$$\begin{aligned} f'(h', k') &= \theta[f_0(Th', Tk')] \\ &= \theta T[f_0'(h', k')] + [\theta\rho(h' + k') - \theta\rho(h') - \theta\rho(k')]. \end{aligned}$$

On the other hand, application of T_h^* and then η' will give, again by the appropriate definitions, a factor set f^* with

$$f^*(h', k') = \theta'[f_0'(h', k')] = \theta T[f_0'(h', k')].$$

Since $\theta\rho(h')$ is an element in G for each $h' \in H'$, these two equations show that f^* and f' are associate, hence that $f' = T_e^*\eta\theta$ and $f^* = \eta' T_h^* \theta$ do determine the same element of Ext $\{G, H\}$, as asserted in the theorem.

CHAPTER III. EXTENSIONS OF SPECIAL GROUPS

In this chapter we shall determine Ext $\{G, H\}$ more explicitly for various special groups G and H. We begin with a brief review of the theory of characters, which will be used extensively in this chapter and also in Chapters V and VI.

13. Characters[16]

Let G, H, and J be three generalized topological groups. G and H are said to be *paired* to J if a continuous function[17] $\phi(g, h)$ with values in J is given

[16] The character theory was discovered by Pontrjagin (see [8]), generalized by van Kampen (see Weil [12], Ch. VI and Lefschetz [7] Ch. II).

[17] As a mapping $G \times H \to J$; for discussion of pairing, cf. [8], [14].

such that for any fixed g_0, $\phi(g_0, h)$ is a homomorphism of H into J and for any fixed h_0, $\phi(g, h_0)$ is a homomorphism of G into J.

Each subset $A \subset G$ determines a corresponding subset Annih $A \subset H$, called the *annihilator* of A, such that $h \in$ Annih A if and only if $\phi(g, h) = 0$ for all $g \in A$. Annihilators of subsets of H are defined similarly. It is clear that the annihilators are subgroups.

LEMMA 13.1. *If G and H are paired to a topological group J, then for each $A \subset G$,* Annih A *is a closed subgroup of H.*

This is an immediate consequence of the continuity of ϕ for fixed g.

G and H are said to be *dually paired* to J if they are so paired that

$$\text{Annih } G = 0 \quad \text{and} \quad \text{Annih } H = 0.$$

LEMMA 13.2. *If G and H are paired to J then $G/$*Annih H *and $H/$*Annih G *are dually paired to J.*

The most important group pairings arise when $J = P$ is the additive group of reals reduced modulo 1. A homomorphism of a group G into P will be called a *character* and the group Hom $\{G, P\}$ will be written as Char G. Since P has no "arbitrarily small" subgroups, it follows from a remark in §3 that if G is compact, Char G is discrete. Vice versa, by Corollary 3.2, if G is discrete, Char G is compact and topological.

The basic lemma of the theory of characters is

LEMMA 13.3. *Let G be a discrete or compact topological group and let $g \neq 0$ be an element of G. There is then a character $\theta \in$* Char G *such that $\theta(g) \neq 0$.*

In the case of discrete G the lemma follows easily from the proof of Corollary 11.4, since P is infinitely divisible. In the compact case the proof is much less elementary and uses the theory of invariant integration in compact groups.

The lemma can be equivalently formulated as follows:

LEMMA 13.4. *Let G be a discrete or compact topological group. G and* Char G *are dually paired to P with the multiplitation*

$$\phi(g, \theta) = \theta(g), \qquad g \in G,\ \theta \in \text{Char } G.$$

Now let G and H be paired to P with $\phi(g, h)$ as multiplication. Since, for a fixed g, $\phi(g, h)$ is a character of H and, for fixed h, $\phi(g, h)$ is a character of G, we obtain induced mappings

$$G \to \text{Char } H, \qquad H \to \text{Char } G. \tag{13.1}$$

A basic result of the character theory is

THEOREM 13.5. *Let the compact topological group G and the discrete group H be paired to P. The pairing is dual if and only if the induced mappings* (13.1) *are isomorphisms:*

$$G \cong \text{Char } H \quad \textit{and} \quad H \cong \text{Char } G.$$

The following two theorems are consequences of the previous results:

THEOREM 13.6. *If G is a discrete or a compact topological group, then*

$$\text{Char Char } G \cong G.$$

THEOREM 13.7. *If the compact topological group G and the discrete group H are dually paired to P, then for every closed subgroup G_1 of G and every subgroup H_1 of H we have*

$$\text{Annih [Annih } G_1] = G_1, \qquad \text{Annih [Annih } H_1] = H_1.$$

14. Modular traces

To study Ext $\{G, H\}$ for compact G we need a certain modification of the "trace" of an endomorphism of a free group. The simplest case of this modification refers to a correspondence which is not a homomorphism, but is a homomorphism, modulo m-folds of elements. It may be stated as follows.

LEMMA 14.1. *Let m be an integer, and let $r \to S(r)$ be a correspondence carrying the free group R into a finite subset of itself in such manner that*

$$S(r_1 + r_2) \equiv S(r_1) + S(r_2) \quad (\text{mod } mR), \tag{14.1}$$

for all $r_1, r_2 \in R$. Let the elements y_α be any independent basis for R, and write $S(y_\alpha) = \sum_\beta c_{\alpha\beta} y_\beta$, with integral coefficients $c_{\alpha\beta}$. Then the "trace"

$$t_m(S) \equiv \sum_\alpha c_{\alpha\alpha} \quad (\text{mod } m) \tag{14.2}$$

is a well defined finite integer, modulo m, independent of the choice of the basis y_α for R.

The proof is exactly parallel to the standard one (e.g. [1], p. 569) for an actual homomorphism of R to itself, using the "modular" homomorphism condition (14.1) at the appropriate junctures in place of the full homomorphism condition. A similar analogue of a special case of the "additivity" of traces will give the following conclusion.

LEMMA 14.2. *If in Lemma 14.1 the elements $w_1, \cdots, w_t$ are any independent elements of R such that $S(R)$ lies in the group generated by $w_1, \cdots, w_t$, and if $S(w_i) = \sum_j d_{ij} w_j$, then $t_m(S) \equiv \sum_i d_{ii}$ (mod m).*

Now let R be a subgroup of the free group F, σ a homomorphism of R into a finite subgroup of F/R. There will then be at least one integer m for which $m\sigma(R) = 0$. Choose for each coset u of F/R a representative $\rho(u)$ in F; then $\rho(u + v) \equiv \rho(u) + \rho(v)$ (mod R). For each $r \in R$, $m(\rho\sigma r)$ is also an element of R, and $S(r) = m(\rho\sigma r)$, where

$$R \xrightarrow[\sigma]{} F/R \xrightarrow[\rho]{} F \xrightarrow[m]{} R,$$

is a correspondence of R to R with the modular homomorphism property (14.1).[18] The trace of the original homomorphism σ is now defined as

$$t(\sigma) \equiv t_m(S)/m \equiv t_m(m\rho\sigma)/m \quad (\text{mod } 1). \tag{14.3}$$

[18] S could also be described in terms of m and σ as follows: S is the essentially unique correspondence of R to a finite subset of $mF \cap R$ with property (14.1) and such that each $\sigma(r)$ is the coset modulo R of $S(r)/m$.

THEOREM 14.3. *If $R \subset F$, F a free group, and if σ is any homomorphism of R into a finite subgroup of F/R, then the trace $t(\sigma)$ defined by* (14.3) *is a unique real number, modulo* 1, *independent of the choices of m and ρ made in its definition. If σ_1 and σ_2 are two such homomorphisms of R to F/R,*

$$t(\sigma_1 + \sigma_2) \equiv t(\sigma_1) + t(\sigma_2) \quad (mod\ 1). \tag{14.4}$$

In particular, $t(_0) \equiv 0$ (mod 1). Furthermore, if T_0 is a fixed finite subgroup of F/R, the correspondence $\sigma \rightarrow t(\sigma)$ is a continuous homomorphism of Hom $\{R, T_0\}$ *into the reals modulo* 1.

We are to prove the invariance of the definition of t. First, hold ρ fixed and replace m by a proper multiple $m' = km$. Then S and $t_m(S)$ are both multiplied by k, hence $t'(\sigma) \equiv t_{km}(kS)/km \equiv kt_m(S)/km \equiv t(\sigma)$ is unaltered, mod 1. Now hold m fixed and let ρ' be any second set of representatives $\rho'(u)$ for cosets $u \in F/R$. Then $\rho'(u) \equiv \rho(u)$ (mod R), so $S'(r) \equiv S(r)$ (mod mR), which implies that $t_m(S') \equiv t_m(S)$ (mod m). This shows that the trace is independent of ρ and m.

The additive property (14.4) is readily established; it is only necessary to choose a single integer in such a way that both $m\sigma_1 R$ and $m\sigma_2 R$ are zero.

Before establishing the continuity of $t(\sigma)$, we propose a more explicit representation of the finiteness of $t(\sigma)$. Let T_0 be a fixed finite subgroup of F/R, and choose a direct summand F_0 of F with a finite number of generators such that $F_0/(F_0 \cap R)$ contains T_0. We can choose simultaneously ([1], p. 566) a basis $z_1, \cdots, z_n$ for F_0 and a basis $y_1, \cdots, y_s$ for $F_0 \cap R$ so that $y_i = d_i z_i$, for integers d_i, $i = 1, \cdots, s \leqq n$. Furthermore, one can prove $F_0 \cap R$ a direct summand of R; there is then a (not necessarily denumberable) basis for R of the form $y_1, \cdots, y_s, y_\alpha, y_\beta, \cdots$. In particular, if $\sigma(R) \subset T_0$, we may choose $\rho(0) = 0$, $\rho(T_0) \subset F_0$, hence $S(R) = m\rho\sigma(R) \subset F_0 \cap R$. The equations for S and its trace then take the form

$$S(y_\gamma) = \sum_{i=1}^{s} c_{\gamma i} y_i, \qquad t_m(S) \equiv \sum_{i=1}^{s} c_{ii} \quad (\text{mod } m), \tag{14.5}$$

where $\gamma = 1, 2, \cdots, s, \alpha, \beta, \cdots$.

To prove $t(\sigma)$ continuous it suffices to establish the continuity at $\sigma = 0$, and hence to prove that $t(\sigma) \equiv 0$ for σ in a suitable neighborhood U of 0 in Hom $\{R, T_0\}$. Let U be the open set in Hom $\{R, T_0\}$ consisting of all σ with $\sigma(y_1) = \cdots = \sigma(y_s) = 0$, where y_i is the special basis constructed from F_0 above. Then, because $\rho(0) = 0$, we have $S(y_i) = 0$, $t_m(S) \equiv 0$ (mod m), and therefore $t(\sigma) \equiv 0$ (mod 1) for σ in U.

We next consider circumstances under which the traces will vanish.

LEMMA 14.4. *If $\sigma \in$ Hom $\{R, F/R\}$ has an extension σ^* which carries F homomorphically into a finite subgroup T_0 of F/R, then $t(\sigma) \equiv 0$ (mod 1).*

PROOF. For T_0 we choose $y_i = d_i z_i$ as above, and then select ρ with $\rho(T_0) \subset F_0$ and m with $mT_0 = 0$ and each $d_i \equiv 0$ (mod m). Then, for suitable integers e_{ij},

$$\rho\sigma^*(z_i) = \sum_{j=1}^{s} e_{ij} z_j, \qquad i = 1, \cdots, n;$$

furthermore $\rho\sigma^*(kz_i) \equiv k\rho\sigma^*(z_i)$ (mod R_0), for any integer k. But $S(y_i) = m\rho\sigma(y_i) = m\rho\sigma^*(d_iz_i) \equiv md_i\rho\sigma^*(z_i)$ (mod mR_0). Then computing $t_m(S)$ by (14.5) and using the fact that $m \equiv 0$ (mod d_j) for each j, we find that $t_m(S) \equiv m \sum e_{ii} \equiv 0$ (mod m), as asserted.

Conversely, we can find certain circumstances in which the trace will assuredly not vanish.

LEMMA 14.5. *If $z \in F$ has order n, modulo R, and if σ is a homomorphism of R into the subgroup of F/R generated by the coset of z, then $\sigma(nz) \neq 0$ implies $t(\sigma) \not\equiv 0$ (mod 1).*

PROOF. Let u denote the coset of z, modulo R. Choose the system of representatives so that $\rho(iu) = iz$, for $i = 0, \cdots, n-1$, and use n as the integer m in the definition of the trace. Then $S = m\rho\sigma$ carries R into the cyclic subgroup generated by mz. Since $\sigma(nz) = ku$, where $k \not\equiv 0$ (mod m), $S(nz) \equiv knz$, and the trace, as computed by Lemma 14.2, is $t_m(S) \equiv k \not\equiv 0$ (mod m), as asserted.

15. Extensions of compact groups

The group of extensions of a compact topological group G can be expressed as an appropriate character group.

THEOREM 15.1. *If G is compact and topological, H discrete, then* $\operatorname{Ext}_f \{G, H\} = 0$ *and there is a (bicontinuous) isomorphism:*

$$\text{Ext } \{G, H\} \cong \text{Char Hom } \{G, H\}. \tag{15.1}$$

If G_0 is the component of 0 in G and T the subgroup of all elements of finite order in H, then also

$$\text{Ext } \{G, H\} \cong \text{Char Hom } \{G, T\} \cong \text{Char Hom } \{G/G_0, T\}.$$

The last conclusion follows at once from the first, for Hom $\{G, H\}$ includes only continuous homomorphisms ϕ of the compact group G; every such homomorphism must map the connected subgroup G_0 into 0. Furthermore each ϕ carries G into a finite subgroup of the discrete group H, hence into a subgroup of T. Observe also that H is discrete, hence has no arbitrarily small subgroups; therefore (cf. §3) Hom $\{G, H\}$ is discrete, as should be the case for a character group of the compact group Ext $\{G, H\}$.

It remains to prove (15.1). Represent H as F/R; then, according to the fundamental theorem of Chapter II, (15.1) is equivalent to

$$\text{Hom } \{R, G\}/\text{Hom } \{F \mid R, G\} \cong \text{Char Hom } \{G, F/R\}. \tag{15.2}$$

According to Theorem 13.5 it will thus suffice to provide a suitable pairing of the compact group Hom $\{R, G\}$ and the discrete group Hom $\{G, F/R\}$ to the reals modulo 1. To this end, take any $\theta \in$ Hom $\{R, G\}$ and $\phi \in$ Hom $\{G, F/R\}$. As just above, $\phi(G)$ is a finite subgroup of F/R. Therefore $\sigma = \phi\theta$ is a homomorphism of R into a finite subgroup of F/R, so that the trace introduced in the previous section can be used to define

$$t(\theta, \phi) \equiv t(\phi\theta) \pmod{1}. \tag{15.3}$$

We propose to show that this is the requisite pairing.

In the first place, this product is additive, for

$$t(\theta + \theta', \phi) \equiv t(\theta, \phi) + t(\theta', \phi) \pmod{1},$$

$$t(\theta, \phi + \phi') \equiv t(\theta, \phi) + t(\theta, \phi') \pmod{1}$$

follow from the corresponding property (14.4) for $\sigma = \phi\theta$. Secondly, if ϕ is fixed, the product $t(\theta, \phi)$ is continuous in θ. For when ϕ is fixed, $\sigma = \phi\theta$ maps R into a fixed finite subgroup of F/R. Since $\theta \to \phi\theta = \sigma$ is continuous, and since $\sigma \to t(\sigma)$ is continuous, by Theorem 14.3, the continuity of $t(\theta, \phi)$ follows.

As to the annihilators under this pairing, we assert that

$$\text{Annih Hom } \{G, F/R\} = \text{Hom } \{F \mid R, G\}. \tag{15.4}$$

For suppose first that $\theta \in \text{Hom } \{F \mid R, G\}$ and let θ^* be an extension of θ to F. Then $\sigma^* = \phi\theta^*$ is an extension of $\sigma = \phi\theta$ to F, and σ^* still carries F into (the same) finite subgroup of F/R. Therefore, by Lemma 14.4, $t(\theta, \phi) \equiv t(\sigma) \equiv 0 \pmod{1}$. Hence θ is in the annihilator in question.

Conversely, let θ be fixed, and suppose that $t(\theta, \phi) \equiv 0 \pmod{1}$ for every ϕ; then $\theta \in \text{Hom } \{F \mid R, G\}$. Since G is compact and topological, it will suffice by Lemma 5.4 to prove that $\theta \in \text{Hom}_f \{R, G; F\}$. If this were not the case, there would be in F an element z of some order n, modulo R, such that $\theta(nz) = g_0$ is not an element of nG. But nG is a continuous image (under $g \to ng$) of the compact group G, hence (Lemma 1.1) is a closed subgroup of G; therefore G/nG is compact and topological. By Lemma 13.3 there is then character χ of G/nG with $\chi(g_0') \neq 0$, where g_0' is the coset of g_0 modulo nG. Since every coset of G/nG has as order some divisor of n, this character χ carries G/nG into the group generated by the fraction $1/n$, modulo 1. This is a cyclic group of order n, and so can be replaced by the isomorphic cyclic group of order n generated by the coset z' of z in F/R. The so-modified character X of G/nG then induces a continuous homomorphism ϕ of G into F/R, where

$$\phi(g_0) \neq 0, \qquad \phi(G) \subset [0, z', z'^2, \cdots, z'^{n-1}].$$

For this particular ϕ, the homomorphism $\sigma = \phi\theta$ carries nz into $\phi\theta(nz) = \phi(g_0) \neq 0$. Lemma 14.5 of the previous section then shows that $t(\sigma) \equiv t(\theta, \phi) \not\equiv 0 \pmod{1}$, contrary to the assumption $t(\theta, \phi) \equiv 0$ for every ϕ. Therefore θ does lie in Hom $\{F \mid R, G\}$, and 15.4 is proved.

Finally, we assert that, under the pairing t,

$$\text{Annih Hom } \{R, G\} = 0. \tag{15.5}$$

For suppose instead that $t(\theta, \phi) \equiv 0 \pmod{1}$ for all θ and for some $\phi \neq 0$. Then for some $g_0 \in G$, $\phi(g_0) = u \neq 0$. The element u of F/R is the coset of some element w of F; as before, ϕ maps G into a finite subgroup of F/R, so that w

has a finite order m, modulo R. It is then possible to select in the free group F an independent basis with a first element z_0 such that $w = kz_0$ for some integer k. If z_0 has order n, modulo R, there is then a corresponding basis for R of elements y_α, with $y_0 = nz_0$. Now construct $\theta \in \mathrm{Hom}\ \{R, G\}$ by setting

$$\theta(y_0) = g_0, \qquad \theta(y_\alpha) = 0, \qquad y_\alpha \neq y_0.$$

This particular homomorphism carries R into the subgroup of G generated by g_0, so that the product $\sigma = \phi\theta$ carries R into the finite subgroup of F/R generated by $\phi(g_0) = u$. Since u is the coset of $w = kz_0$, this is contained in the subgroup of F/R generated by the coset of z_0. Furthermore $\sigma(nz_0) = u \neq 0$. Lemma 14.5 again applies to show that $t(\sigma) \equiv t(\theta, \phi) \not\equiv 0 \pmod 1$, counter to assumption.

Given the assertions (15.4) and (15.5) as to annihilators, it follows from Lemma 13.2 that the groups $\mathrm{Hom}\ \{R, G\}/\mathrm{Hom}\ \{F \mid R, G\}$ and $\mathrm{Hom}\ \{G, F/R\}$ are dually paired. Formula (15.2) is then a consequence of Theorem 13.5.

16. Two lemmas on homomorphisms

A generalized topological group G is said to have no arbitrarily small subgroups if there is in G an open set V containing 0 but containing no subgroups other than the group consisting of 0 alone.

LEMMA 16.1. *If the discrete group T has no elements of infinite order and the generalized topological group G has no arbitrarily small subgroups, while G_0 is the same group with the discrete topology, then* $\mathrm{Hom}\{T, G\}$ *and* $\mathrm{Hom}\{T, G_0\}$ *have the same topology.*

PROOF. $\mathrm{Hom}\ \{T, G\}$ and $\mathrm{Hom}\ \{T, G_0\}$ are algebraically identical. The hypotheses on T insure that every finite set of elements of T generates a finite subgroup of T. A complete set of neighborhoods U of 0 in $\mathrm{Hom}\ \{T, G\}$ may therefore be found thus: take a finite subgroup $T_0 \subset T$ and an open set V_0 in G containing 0, and let U consist of all homomorphisms θ with $\theta(T_0) \subset V_0$. In particular, if V_0 is contained in the special open set V of G which contains no proper subgroups, the subgroup $\theta(T_0)$ is zero, so that U consists of all θ with $\theta(T_0) = 0$. The special sets U so described also form a complete set of neighborhoods of 0 in $\mathrm{Hom}\ \{T, G_0\}$. Therefore the two topologies on the group are equivalent.

LEMMA 16.2. *Let $F \supset R$ be a free (discrete) group, $G' \supset G$ a discrete group, while* $\mathrm{Hom}\ \{F, G'; R, G\}$ *denotes the set of all homomorphisms $\phi \in$* $\mathrm{Hom}\ \{F, G'\}$ *with $\phi(R) \subset G$. Then*

(16.1) $$\mathrm{Hom}\ \{F, G'; R, G\}/\mathrm{Hom}\ \{F, G\} \cong \mathrm{Hom}\ \{F/R, G'/G\}.$$

PROOF. Any homomorphism of F/R into G'/G may be regarded as a homomorphism of F into G'/G which carries R into zero (Lemma 3.3), so that (16.1) becomes

(16.2) $$\mathrm{Hom}\ \{F, G'; R, G\}/\mathrm{Hom}\ \{F, G\} \cong \mathrm{Hom}\ \{F, G'/G; R, 0\}.$$

For each $\phi \in \text{Hom}\ \{F, G'\}$ let $\phi^{\#}$ be the corresponding homomorphism reduced modulo G, so that for $x \in F$, $\phi^{\#}(x)$ is the coset of $\phi(x)$, modulo G. The correspondence $\phi \to \phi^{\#}$ is a homomorphism mapping $\text{Hom}\ \{F, G'; R, G\}$ into $\text{Hom}\ \{F, G'/G; R, 0\}$. Furthermore $\phi^{\#} = 0$ if and only if $\phi(F) \subset G$, or $\phi \in \text{Hom}\ \{F, G\}$. Therefore $\phi \to \phi^{\#}$ provides an (algebraic) isomorphism of the left hand group in (16.2) to a subgroup of the right hand group.

Conversely, select a fixed basis z_α for the free group F, and for each coset $b \in G'/G$ pick a fixed representative element $\rho(b)$ in G'. For given $\sigma \in \text{Hom}\ \{F, G'/G; R, 0\}$, define a corresponding homomorphism $\phi = \phi(\sigma)$, for any $x = \sum k_\alpha z_\alpha \in F$, as

$$\phi(\sum_\alpha k_\alpha z_\alpha) = \sum_\alpha k_\alpha \rho[\sigma z_\alpha].$$

This is a homomorphism of F into G'. By construction, $\rho[\sigma z_\alpha]$ modulo G is just σz_α, hence $\phi(x)$, modulo G, is $\sigma(x)$, or $\phi^{\#} = \sigma$. This implies that $\phi(R) \subset G$, and so that each σ is the correspondent of some ϕ in the homomorphism $\phi \to \phi^{\#}$.

To show (16.2) bicontinuous, we first analyze the topology in the groups involved. By the definition of the topology in a factor group, we have to consider only open sets in $\text{Hom}\ \{F, G'; R, G\}$ which are unions of cosets of $\text{Hom}\ \{F, G\}$. If $z_1, \cdots, z_n$ is any finite selection from the fixed set of generators for F, the set $U(z_1, \cdots, z_n)$ consisting of all ϕ with $\phi(z_1) \equiv \cdots \equiv \phi(z_n) \equiv 0$ (mod G) is such an open set, and contains $\phi_0 = 0$. We assert that any open set V containing 0 which is a union of cosets contains one of these sets U. For, given V, there will be elements $x_1, \cdots, x_m$ in F such that V contains all ϕ with $\phi x_i = 0$. Select generators $z_1, \cdots, z_n$ such that each x_i can be expressed in terms of $z_1, \cdots, z_n$; then V contains all ϕ with $\phi z_i = 0$. Moreover, if $\phi z_i \equiv 0$ (mod G), there is a homomorphism ϕ_1 of F into G with $\phi z_i = \phi_1 z_i$; since $\phi - \phi_1 \in V$, since V is a union of cosets of $\text{Hom}\ \{F, G\}$, and since $\phi_1 \in \text{Hom}\ \{F, G\}$, we conclude that $\phi \in V$. Thus $V \supset U(z_1, \cdots, z_n)$.

A similar but simpler argument for $\text{Hom}\ \{F, G'/G; R, 0\}$ will show that every open set containing zero in this group contains all σ with $\sigma z_1 = \cdots = \sigma z_n = 0$, for a suitable set of the generators of F. The mapping $\sigma \to \phi$ carries open sets of this special type into the open sets $U(z_1, \cdots, z_n)$ described above, and conversely. This shows that the correspondence $\phi \to \phi^{\#}$ is continuous at 0, and hence everywhere.

17. Extensions of integers

Next we consider the case in which every element of H has finite order; we then write T instead of H for this group. The group of extensions of the integers by such a group T can be written as a group of characters. In case T is finite, the result is a generalization of Corollary 11.2, for in this case $\text{Char}\ T \cong T$.

THEOREM 17.1. *If T has only elements of finite order, and if I is the (additive) group of integers,*

$$\text{Ext}_f\ \{I, T\} = 0, \tag{17.1}$$

(17.2) $$\mathrm{Ext}\ \{I, T\} \cong \mathrm{Char}\ T.$$

The methods used to establish this result apply with equal force if I is replaced by any discrete group G which has no elements of finite order. The group Char T of homomorphisms of T into the group of reals modulo 1 must then be replaced by a group of homomorphisms of T into another group suitably constructed from G. In fact, any G with no elements of finite order can be embedded in an essentially unique discrete group G_∞ with the following properties:[19]

(i) G_∞ has no elements of finite order,

(ii) G_∞/G has only elements of finite order,

(iii) G_∞ is infinitely divisible.

For any $g \in G_\infty$ and any integer m there is then a unique $h = g/m$ in G_0 with $mh = g$. The (discrete) factor group G_∞/G is the analogue of the topological group P' of rationals modulo 1. Specifically, if $G = I$, $G_\infty = I_\infty$ is the group of rational numbers, and G_∞/G is the group P', but with a discrete topology. Since T has only elements of finite order, Char T is Hom $\{T, P'\}$. But P' clearly has no arbitrarily small subgroups, so that the latter group, by Lemma 16.1, is identical (algebraically and topologically) with Hom $\{T, I_\infty/I\}$. The exact generalization of Theorem 17.1 is thus

THEOREM 17.2. *If T has only elements of finite order, while G is discrete and has no elements of finite order, and G_∞ is defined as above,*

(17.3) $$\mathrm{Ext}_f\ \{G, T\} = 0,$$

(17.4) $$\mathrm{Ext}\ \{G, T\} \cong \mathrm{Hom}\ \{T, G_\infty/G\}.$$

The isomorphism is bicontinuous if G and G_∞/G are both discrete.

PROOF. If T is represented in the form $T = F/R$, for F free, the conclusions of this theorem can be reformulated, according to the fundamental theorem of Chapter II, as

(17.3a) $$\mathrm{Hom}_f\ \{R, G; F\} = \mathrm{Hom}\ \{F \mid R, G\},$$

(17.4a) $$\mathrm{Hom}\ \{R, G\}/\mathrm{Hom}\ \{F \mid R, G\} \cong \mathrm{Hom}\ \{F/R, G_\infty/G\}.$$

Observe first that any homomorphism $\theta \in \mathrm{Hom}\ \{R, G\}$ can be extended in a unique way to a homomorphism $\theta^* \in \mathrm{Hom}\ \{F, G_\infty\}$. For, since every element of $T = F/R$ has finite order, every $z \in F$ has a finite order modulo R. For each such z pick an integer m such that $mz \in R$, and define

(17.5) $$\theta^*(z) = (1/m)\theta(mz), \qquad z \in F,\ mz \in R.$$

This definition of θ^* is independent of the choice of m, and does yield a homomorphism of F into G_∞. Clearly it is the only such homomorphism extending the given θ.

Suppose now that $\theta \in \mathrm{Hom}_f\ \{R, G; F\}$. Each element $z \in F$ then generates a

[19] G_∞ could also be described as a tensor product; see §18.

finite subgroup of F/R, so θ can be extended to a homomorphism mapping z and R into G. This extension of θ must agree with the unique extension θ^*. This shows that $\theta^*(z) \in G$ for each z, so that θ^* is in fact a homomorphism of F into $G \subset G_\infty$, and $\theta \in \text{Hom}\,\{F \mid R, G\}$. This proves (17.3a).

As in §16, let Hom $\{F, G_\infty; R, G\}$ denote the group of all homomorphisms $\phi \in \text{Hom}\,\{F, G_\infty\}$ with $\phi(R) \subset G$. This is a topological group, under the usual specification (§1) that any open set in Hom $\{F, G_\infty; R, G\}$ is the intersection of this group with an open set in the topological group Hom $\{F, G_\infty\}$.

The correspondence $\phi \to \phi \mid R$ provides a bicontinuous isomorphism

$$\text{(17.6)} \qquad \text{Hom}\,\{F, G_\infty; R, G\} \cong \text{Hom}\,\{R, G\}.$$

For, by Lemma 3.4, $\phi \to \phi \mid R$ is a continuous homomorphism. It is an isomorphism because each $\theta \in \text{Hom}\,\{R, G\}$ has a unique extension $\theta^* = \phi \in \text{Hom}\,\{F, G_\infty; R, G\}$, by (17.5). This inverse correspondence is also continuous; for if U is the open set consisting of all ϕ with $\phi z_i = g_i$, for given $z_i \in F$ and $g_i \in G_\infty$, $i = 1, \cdots, n$, there is an open set U_m in Hom $\{R, G\}$ consisting of all θ with $\theta(mz_i) = mg_i$, where m is chosen so that each $mz_i \in R$ and each $mg_i \in G$. The correspondence $\theta \to \theta^*$ of (17.5) carries U_m into U. This proves (17.6).

The correspondence $\phi \to \phi \mid R$ maps the subgroup Hom $\{F, G\}$ of Hom $\{F, G_\infty; R, G\}$ onto Hom $\{F \mid R, G\}$. Hence (17.6) also yields an isomorphism

$$\text{Hom}\,\{F, G_\infty; R, G\}/\text{Hom}\,\{F, G\} \cong \text{Hom}\,\{R, G\}/\text{Hom}\,\{F \mid R, G\}.$$

On the other hand, Lemma 16.2 provides an isomorphism

$$\text{Hom}\,\{F, G_\infty; R, G\}/\text{Hom}\,\{F, G\} \cong \text{Hom}\,\{F/R, G_\infty/G\}.$$

These two combine to give the required isomorphism (17.4a).

It should be remarked that the results of this section can also be obtained by arguments directly on factor sets, without the interposition of the fundamental theorem of Chapter II. Specifically, to prove Theorem 17.2, one could consider an extension E of G by T, determined by a factor set $f(s, t)$ for $s, t \in T$. If $t \in T$ has order m, let $\phi_E(t) \equiv (1/m)\sum_i f(it, t) \pmod{G}$, where $i = 0, 1, \cdots, m - 1$. In this fashion E determines a homomorphism $\phi_E \in \text{Hom}\,\{T, G_\infty/G\}$. Conversely, given such a homomorphism ϕ, one may select for each $\phi(t) \in G_\infty/G$ a representative element $\phi'(t) \in G_\infty$ and construct the corresponding factor set as $f(s, t) = \phi'(s) + \phi'(t) - \phi'(s + t)$. These correspondences will establish (17.4). The device of constructing ϕ_E by summation over the terms of the factor set is an application of the so-called "Japanese homomorphism," as commonly used for (multiplicative) factor sets.

18. Tensor products

Some of our formulas can be expressed more easily by means of the tensor products introduced by Whitney [13]. If A and B are given discrete abelian groups the *tensor product* $A \circ B$ is a set whose elements are finite formal sums

$\sum a_i b_i$ of formal products $a_i b_i$, with each $a_i \epsilon A$, $b_i \epsilon B$. Two such elements are added simply by combining the two formal sums into a single sum. Two such elements are equal if and only if the second can be obtained from the first by a finite number of replacements of the forms $(a + a')b \leftrightarrow ab + a'b$, $a(b + b') \leftrightarrow ab + ab'$. The tensor product $A \circ B$ so defined is a discrete abelian group, and the multiplication $a \cdot b$ is a pairing of A and B to $A \circ B$.

In the special case when $B = G$ is a group containing no elements of finite order, and $A = R_0$ is the additive group of rational numbers, any sum $\sum a_i b_i$ can, by the distributive law, be rewritten as a single term $(r/s)b$, where s is a common denominator for the rational numbers a_i. This representation is essentially unique. Therefore $R_0 \circ G$ is simply the group G_∞ used in §17 above, and G_∞/G is $(R_0 \circ G)/G$ (for details, cf. Whitney [13], pp. 507–508).

The tensor product can equivalently be defined in terms of characters, in the following fashion:

THEOREM 18.1. *If A and B are (discrete) abelian groups,*

$$A \circ B \cong \text{Char Hom } \{B, \text{Char } A\}. \tag{18.1}$$

PROOF. This conclusion can also be written in the form

$$\text{Char } (A \circ B) \cong \text{Hom } \{B, \text{Char } A\}. \tag{18.2}$$

Since the group of characters is the group of homomorphisms into the group P of reals modulo 1, this conclusion is a special case (with $C = P$) of the following

LEMMA 18.2. *If A and B are discrete abelian groups, C any generalized (topological) abelian group, then there is a bicontinuous isomorphism*

$$\text{Hom } \{A \circ B, C\} \cong \text{Hom } \{B, \text{Hom } (A, C)\}. \tag{18.3}$$

PROOF. Let $\theta \epsilon \text{Hom } \{A \circ B, C\}$ be given. For each $b \epsilon B$, let $\phi_b(a) = \theta(ab)$. Then $\phi_b \epsilon \text{Hom } (A, C)$. Let $\omega_\theta(b) = \phi_b$. Then $\omega_\theta \epsilon \text{Hom } \{B, \text{Hom } (A, C)\}$, and the correspondence $\theta \to \omega_\theta$ is a homomorphism of Hom $\{A \circ B, C\}$ into Hom $\{B, \text{Hom } (A, C)\}$. One verifies readily that it is an (algebraic) isomorphism ($w_\theta = 0$ only if $\theta = 0$). Furthermore, it is an isomorphism onto the whole group Hom $\{B, \text{Hom } (A, C)\}$. For let any ω in the latter group be given, with $\omega(b) = \phi'_b \epsilon \text{Hom } (A, C)$ for each $b \epsilon B$. Then define

$$\theta_\omega(\sum a_i b_i) = \sum \phi'_{b_i}(a_i), \qquad a_i \epsilon A,\ b_i \epsilon B.$$

One verifies that θ_ω is uniquely defined, under the identifications $(a + a')b \to ab + a'b$, $a(b + b') \to ab + ab'$ used in the definition of $A \circ B$. Furthermore, $\theta_\omega \epsilon \text{Hom } \{A \circ B, C\}$, and $\theta_\omega \to \omega$ in the previously given correspondence. Therefore $\theta \to \omega_\theta$, $\omega \to \theta_\omega$ does yield the indicated isomorphism (18.3). The continuity of the isomorphism in both directions is readily established from these explicit formulas and the appropriate definitions of open sets in the given topologies of the groups concerned.

Chapter IV. Direct and Inverse Systems

The Čech homology groups for a space are defined as limits of certain "direct" and "inverse" systems of homology groups for finite coverings of the space (Chap. VI). In view of our representation of homology groups in terms of groups of homomorphisms and groups of group extensions we are led to consider limits of groups of this sort. We shall show that the limit of a group of homomorphisms is itself a group of homomorphisms (§21) and that the corresponding proposition holds in certain special cases for groups of group extensions (§22). In the general case, however, we must introduce a new group to represent the limit of a group of group extensions. This group can also be introduced as a limit of tensor products (§25).

19. Direct systems of groups

A directed set J is a partially ordered set of elements $\alpha, \beta, \gamma, \cdots$ such that for any two elements α and β there always exists an element γ with $\alpha < \gamma$, $\beta < \gamma$. For each index α in a directed set J let H_α be a (discrete) group, and for each pair $\alpha < \beta$, let $\phi_{\beta\alpha}$ be a homomorphism of H_α into H_β. If $\phi_{\gamma\alpha} = \phi_{\gamma\beta}\phi_{\beta\alpha}$ whenever $\alpha < \beta < \gamma$, the groups H_α are said to form a *direct system* with the *projections* $\phi_{\beta\alpha}$.[20]

Any direct system determines a unique (discrete) limit group $H = \underrightarrow{\operatorname{Lim}}\, H_\alpha$ as follows. Every element h_α of one of the groups H_α is regarded as an element h_α^* of the limit H, and two elements h_α^*, h_β^* are equal if and only if there is an index γ, $\alpha < \gamma$, $\beta < \gamma$, with $\phi_{\gamma\alpha}h_\alpha = \phi_{\gamma\beta}h_\beta$. Two elements h_α^* and h_β^* in H are added by finding some γ with $\alpha < \gamma$, $\beta < \gamma$; the sum is then the element $h_\gamma^* = (\phi_{\gamma\alpha}h_\alpha + \phi_{\gamma\beta}h_\beta)^*$. Under this addition and equality, the elements h_α^* form a group $H = \underrightarrow{\operatorname{Lim}}\, H_\alpha$. Each of the given groups H_α has a homomorphism $\phi_\alpha(h_\alpha) = h_\alpha^*$ into the limit group, and $\phi_\beta\phi_{\beta\alpha} = \phi_\alpha$, for $\alpha < \beta$.

In case each given projection $\phi_{\beta\alpha}$ is an isomorphism (of H_α into H_β), the limit group can be regarded as a "union" of the given groups: each group H_α has an isomorphic replica $\phi_\alpha H_\alpha$ within H, and H is simply the union of these subgroups.

A subset J' of the set J of indices α is said to be *cofinal* in J if for each index α there is in J' an α' with $\alpha < \alpha'$. The limit $\underrightarrow{\operatorname{Lim}}\, H_{\alpha'}$, taken over any such cofinal subset, is isomorphic to the original limit H.

20. Inverse systems of groups

For each index α in a directed set let A_α be a (generalized topological) group, and for each $\alpha < \beta$ let $\psi_{\alpha\beta}$ be a (continuous) homomorphism of A_β in A_α. If $\psi_{\alpha\beta}\psi_{\beta\gamma} = \psi_{\alpha\gamma}$ whenever $\alpha < \beta < \gamma$, the groups A_α are said to form an *inverse system* relative to the *projections* $\psi_{\alpha\beta}$. Each inverse system determines a limit group $A = \underleftarrow{\operatorname{Lim}}\, A_\alpha$. An element of this limit group is a set $\{a_\alpha\}$ of elements $a_\alpha \in A_\alpha$ which "match" in the sense that $\psi_{\alpha\beta}a_\beta = a_\alpha$ for each $\alpha < \beta$. The sum

[20] Direct (and inverse) systems were discussed in Steenrod [9], Lefschetz [7], Chap. I and II, and in Weil [12], Ch. I.

of two such sets is $\{a_\alpha\} + \{b_\alpha\} = \{a_\alpha + b_\alpha\}$; since the ψ's are homomorphisms, this sum is again an element of the group. This limit group A is a subgroup of the direct product of the groups A_α. The topology of the direct product $\prod A_\alpha$ thus induces (§1) a topology in $\varprojlim A_\alpha$; an open set in the latter group is the intersection with $\varprojlim A_\alpha$ of an open set of $\prod A_\alpha$. This makes $\varprojlim A_\alpha$ a generalized topological group. As before, a cofinal subset of the indices gives an isomorphic limit group.

Let each B_α be a subgroup of the corresponding group A_α of an inverse system, and assume, for $\alpha < \beta$, that $\psi_{\alpha\beta} B_\beta \subset B_\alpha$. Then the system B_α is an inverse system under the same projections $\psi_{\alpha\beta}$, and the limit $\varprojlim B_\alpha$ is, in natural fashion, a subgroup of $\varprojlim A_\alpha$. On the other hand, $\psi_{\alpha\beta}$ induces a homomorphism $\psi'_{\alpha\beta}$ of the (generalized topological) group A_β/B_β into A_α/B_α. Relative to these projections, the factor groups themselves form an inverse system A_α/B_α. The limit group of the latter system contains a homomorphic image of $\varprojlim A_\alpha$; if each a_α in A_α determines a coset a'_α in A_α/B_α, the map $\{a_\alpha\} \to \{a'_\alpha\}$ is a homomorphism of $\varprojlim A_\alpha$ into $\varprojlim (A_\alpha/B_\alpha)$ in which exactly the elements of $\varprojlim B_\alpha$ are mapped on zero. Thus we have

$$\varprojlim A_\alpha / \varprojlim B_\alpha \subset \varprojlim (A_\alpha/B_\alpha). \tag{20.1}$$

For compact topological subgroups this is an isomorphism:

LEMMA 20.1. *If the A_α form an inverse system relative to the $\psi_{\alpha\beta}$, and if each B_α is a compact topological subgroup of A_α with $\psi_{\alpha\beta}B_\beta \subset B_\alpha$, then*

$$\varprojlim A_\alpha / \varprojlim B_\alpha \cong \varprojlim (A_\alpha/B_\alpha). \tag{20.2}$$

PROOF. Consider any $c = \{c_\alpha\}$ in $\varprojlim (A_\alpha/B_\alpha)$, where $\psi'_{\alpha\beta} c_\beta = c_\alpha$ for each $\alpha < \beta$. Each $c_\alpha \in A_\alpha/B_\alpha$ is a coset of the compact topological subgroup B_α, hence itself is a compact Hausdorff subspace of the space A_α. Furthermore $\psi_{\alpha\beta}$ is a continuous mapping of the set c_β into c_α, for each $\alpha < \beta$. Since $\psi_{\alpha\gamma} = \psi_{\alpha\beta}\psi_{\beta\gamma}$, the sets c_α form an inverse system of compact non-empty Hausdorff spaces. Their limit space is therefore[21] non-vacuous. This means that there is a set of elements $a_\alpha \in c_\alpha$ with $\psi_{\alpha\beta} a_\beta = a_\alpha$ for $\alpha < \beta$. The element $\{a_\alpha\}$ in the group $\varprojlim A_\alpha$ is therefore an element which maps onto the given element $\{c_\alpha\}$ in the homomorphism $\{a_\alpha\} \to \{a'_\alpha\}$ used to establish (20.1). The continuity of (20.2), in both directions, follows readily.

There is also an "isomorphism" theorem for inverse systems.

LEMMA 20.2. *If the groups A_α form an inverse system relative to the projections $\psi_{\alpha\beta}$, while C_α form an inverse system (with the same set of indices) relative to projections $\phi_{\alpha\beta}$, and if σ_α are (bicontinuous) isomorphisms of A_α to C_α, for every α, such that the "naturality" condition $\sigma_\alpha\psi_{\alpha\beta} = \phi_{\alpha\beta}\sigma_\beta$ holds, then the groups $\varprojlim A_\alpha$ and $\varprojlim C_\alpha$ are bicontinuously isomorphic.*

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[21] See Lefschetz [7], Theorem 39.1 or Steenrod [9], p. 666. Observe, however, that the latter proof is incomplete, because of the gap in lines 10–11 on p. 666.

21. Inverse systems of homomorphisms

Consider the group of all homomorphisms of H into G. As in Chap. II, §12, each projection $\phi_{\beta\alpha}$ of a direct system of groups H_α will induce a "dual" homomorphism $\phi^*_{\alpha\beta}$ of Hom $\{H_\beta, G\}$ into Hom $\{H_\alpha, G\}$. Furthermore $\phi^*_{\alpha\beta}\phi^*_{\beta\gamma} = \phi^*_{\alpha\gamma}$ for all $\alpha < \beta < \gamma$, so that the groups Hom $\{H_\alpha, G\}$ form an inverse system relative to these dual projections.

THEOREM 21.1. *If the (discrete) groups H_α form a direct system, then*

$$\text{Hom}\ \{\underrightarrow{\text{Lim}}\ H_\alpha, G\} \cong \underleftarrow{\text{Lim}}\ \text{Hom}\ \{H_\alpha, G\}. \tag{21.1}$$

PROOF. Consider any element $\omega = \{\theta_\alpha\}$ in $\underleftarrow{\text{Lim}}$ Hom $\{H_\alpha, G\}$. To define a corresponding homomorphism θ_ω on $H = \underrightarrow{\text{Lim}}\ H_\alpha$, represent each element $h \in H$ as a projection $h = \phi_\alpha h_\alpha$ of some element $h_\alpha \in H_\alpha$, and set

$$\theta_\omega(h) = \theta_\omega(\phi_\alpha h_\alpha) = \theta_\alpha(h_\alpha), \qquad h = \phi_\alpha h_\alpha . \tag{21.2}$$

The "matching" requirement that $\theta_\alpha = \phi^*_{\alpha\beta}\theta_\beta$ for $\alpha < \beta$ readily shows that $\theta_\omega(h)$ has a unique value, independent of the representation $h = \phi_\alpha h_\alpha$ chosen. Furthermore, $\theta_\omega \in$ Hom $\{H, G\}$, and the correspondence $\omega \to \theta_\omega$ is an isomorphism.

Conversely, let any $\theta \in$ Hom $\{H, G\}$ be given, and define

$$\theta_\alpha(h_\alpha) = \theta(\phi_\alpha h_\alpha), \qquad h_\alpha \in H_\alpha . \tag{21.3}$$

If $\alpha < \beta$, $\phi^*_{\alpha\beta}\theta_\beta(h_\alpha) = \theta_\beta[\phi_{\beta\alpha}h_\alpha] = \theta[\phi_\beta\phi_{\beta\alpha}h_\alpha] = \theta(\phi_\alpha h_\alpha) = \theta_\alpha h_\alpha$; so $\phi^*_{\alpha\beta}\theta_\beta = \theta_\alpha$, and these θ's match. Therefore $\omega = \{\theta_\alpha\}$ is an element of the inverse limit group $\underleftarrow{\text{Lim}}$ Hom $\{H_\alpha, G\}$, and clearly θ_ω is the original homomorphism θ. The correspondence $\omega \to \theta_\omega$ therefore does establish the desired isomorphism (21.1). The continuity in both directions follows directly from the formulas (21.2) and (21.3) and the appropriate definition of neighborhoods of zero in the groups concerned.

22. Inverse systems of group extensions

Consider a direct system of discrete groups H_α. As in Chap. II, §12, each projection $\phi_{\beta\alpha}$ of H_α into H_β will induce a homomorphism $\phi^*_{\alpha\beta}$ of Ext $\{G, H_\beta\}$ into Ext $\{G, H_\alpha\}$. Furthermore $\phi^*_{\alpha\beta}\phi^*_{\beta\gamma} = \phi^*_{\alpha\gamma}$ for all $\alpha < \beta < \gamma$, so that the groups Ext $\{G, H_\alpha\}$ form an inverse system. Contrary to the situation in the previous section, the limit group $\underleftarrow{\text{Lim}}$ Ext $\{G, H_\alpha\}$ may not be isomorphic to Ext $\{G, \underrightarrow{\text{Lim}}\ H_\alpha\}$. An example to this effect will be given below. However, there are two important cases when "Lim" and "Ext" are interchangeable.

THEOREM 22.1. *If G is compact and topological, while the (discrete) groups H_α form a direct system, then*

$$\text{Ext}\ \{G, \underrightarrow{\text{Lim}}\ H_\alpha\} \cong \underleftarrow{\text{Lim}}\ \text{Ext}\ \{G, H_\alpha\}. \tag{22.1}$$

This is proved by repeated applications of Lemma 20.1 to the representation

(22.2) $$\text{Ext}\,\{G, H\} = \text{Fact}\,\{G, H\}/\text{Trans}\,\{G, H\},$$

where $H = \varinjlim H_\alpha$. Recall that any $f \in \text{Trans}\,\{G, H\}$ has the form

$$f(h, k) = g(h) + g(k) - g(h + k), \qquad h, k \in H.$$

Here $g \in G^H$ is any mapping of H into G. Clearly $f = 0$ if and only if $g \in \text{Hom}\,\{H, G\}$, so

(22.3) $$\text{Trans}\,\{G, H\} \cong G^H/\text{Hom}\,\{H, G\}.$$

The correspondence $g \to f$ is clearly continuous; since the isomorphism (22.3) is one-one and since the groups G^H and Hom $\{H, G\}$ are compact, by Lemma 3.1, the bicontinuity of (22.3) follows. Furthermore, this isomorphism is a "natural" one relative to homomorphisms, so that the isomorphism theorem for inverse systems (Lemma 20.2) gives

$$\varprojlim \text{Trans}\,\{G, H_\alpha\} \cong \varprojlim\,[G^{H_\alpha}/\text{Hom}\,\{H_\alpha, G\}].$$

In this representation the groups G^{H_α} and Hom $\{H_\alpha, G\}$ with the "dual" projections $\phi^*_{\alpha\beta}$ form inverse systems with the respective limits G^H and Hom $\{H, G\}$. Furthermore each group Hom $\{H_\alpha, G\}$ is compact and topological, so Lemma 20.1 gives

(22.4) $$\varprojlim \text{Trans}\,\{G, H_\alpha\} \cong \varprojlim G^{H_\alpha}/\varprojlim \text{Hom}\,\{H_\alpha, G\} = G^H/\text{Hom}\,\{H, G\} \cong \text{Trans}\,\{G, H\}.$$

On the other hand one may show exactly as in the proof of Theorem 21.1 on homomorphisms that there is a bicontinuous isomorphism

(22.5) $$\varprojlim \text{Fact}\,\{G, H_\alpha\} \cong \text{Fact}\,\{G, H\}.$$

Furthermore, each of the groups Trans $\{G, H_\alpha\}$ is compact and topological, so that Lemma 20.1 applies again to prove

$$\varprojlim\,[\text{Fact}/\text{Trans}] \cong \varprojlim \text{Fact}/\varprojlim \text{Trans}.$$

This, with (22.4) and (22.5), gives the desired conclusion.[22]

THEOREM 22.2. *If G is discrete and has no elements of finite order, while T_α is a direct systems of discrete groups with only elements of finite order, then*

(22.6) $$\text{Ext}\,\{G, \varinjlim T_\alpha\} \cong \varprojlim \text{Ext}\,\{G, T_\alpha\}.$$

The proof appeals directly to the result found in Theorem 17.2 of Chapter III, to the effect that

(22.7) $$\text{Ext}\,\{G, T_\alpha\} \cong \text{Hom}\,\{T_\alpha, G_\infty/G\}.$$

The groups Hom $\{T_\alpha, G_\infty/G\}$ will form an inverse system under the dual projections $\phi^*_{\alpha\beta}$; as in Theorem 21.1 we then have

$$\text{Hom}\,\{\varinjlim T_\alpha, G_\infty/G\} \cong \varprojlim \text{Hom}\,\{T_\alpha, G_\infty/G\}.$$

[22] Theorem 22.1 can also be proved by representing Ext by means of Char Hom $\{G, H\}$ as in Theorem 15.1. This argument, however, requires a tedious proof that the isomorphism established in the latter theorem is "natural," in the sense of §12.

But the group on the left is simply Ext $\{G, \varinjlim T_\alpha\}$, by another application of Theorem 17.2. The desired result should then follow by taking (inverse) limits on both sides in (22.7).

To carry out this argument, it is necessary to have the naturality condition which gives the isomorphism theorem (Lemma 20.2) for inverse systems. This naturality condition requires that the isomorphism (22.7) permute with the projections of the inverse systems. This is just a statement of the fact that the isomorphism (22.7) established in Theorem 17.2 is "natural" in the sense envisaged in §12. The proof of this naturality is straightforward, so details will be omitted.

COROLLARY 22.3. *If the discrete group G has only a finite number of generators, while T_α is a direct system of discrete groups with only elements of finite order, then*

$$\text{Ext}\ \{G, \varinjlim T_\alpha\} \cong \varprojlim \text{Ext}\ \{G, T_\alpha\}.$$

PROOF. Write G as $F \times L$ where F is free, L is finite (and thus compact). By (11.2) there is a "natural" isomorphism

$$\text{Ext}\ \{G, \varinjlim T_\alpha\} \cong \text{Ext}\ \{F, \varinjlim T_\alpha\} \times \text{Ext}\ \{L, \varinjlim T_\alpha\}.$$

The asserted result now follows by applying Theorem 22.2 to the first factor on the right, and Theorem 22.1 to the second, using Lemma 20.2.

We now show by an example that "Ext" and "Lim" do not necessarily commute. Let p be a fixed prime number, H the additive group of all rationals with denominator a power of p, and H_n the subgroup consisting of all multiples of $1/p^n$. Then $\varinjlim H_n = H$, since H is the union of the groups H_n. Furthermore H_n is a free group, so Ext $\{I, H_n\} = 0$, where I is the group of integers. On the other hand, Ext $\{I, \varinjlim H_n\} =$ Ext $\{I, H\}$ is a group computed in appendix B; it is decidedly not zero, in fact it is not even denumerable.

23. Contracted extensions

Before further consideration of the inverse limits of groups of extensions, we make a comparison of the group of extensions of a group G by a group H with the group of extensions by a subgroup H_0 of H. The identity mapping I of H_0 into H is a homomorphism, hence, as in §12, will give dual homorphisms

$$I^*: \text{Fact}\ \{G, H\} \to \text{Fact}\ \{G, H_0\}, \tag{23.1}$$

$$I^*: \text{Trans}\ \{G, H\} \to \text{Trans}\ \{G, H_0\}. \tag{23.2}$$

Specifically, I^* is the operation of "cutting off" a factor set $f \in \text{Fact}\ \{G, H\}$ to give a factor set $f_0 = I^*f \in \text{Fact}\ \{G, H_0\}$; $f_0(h, k)$ is defined only for $h, k \in H_0$, and always equals $f(h, k)$. Clearly I^* carries transformation sets into transformation sets, as in (23.2). Thus I^* also induces a dual homomorphism

$$I^*: \text{Ext}\ \{G, H\} \to \text{Ext}\ \{G, H_0\}. \tag{23.3}$$

This homomorphism may be visualized as follows: given E such that $G \subset E$

and $E/G = H$, there is an $E_0 \subset E$ such that $G \subset E_0$ and $E_0/G = H_0$. Then $I^*(E) = E_0$.

LEMMA 23.1. *If H_0 is a subgroup of the group H then for any group G the homomorphism I^* of* (23.3) *maps the group* Ext $\{G, H\}$ *onto* Ext $\{G, H_0\}$.

PROOF.[23] Represent H as F/R, where F is free. There is then a subgroup F_0 of F such that $R \subset F_0$ and $F_0/R = H_0$. By the fundamental theorem we have isomorphisms

$$\text{Ext}\,\{G, H\} \cong \text{Hom}\,\{R, G\}/\text{Hom}\,\{F \mid R, G\},$$

$$\text{Ext}\,\{G, H_0\} \cong \text{Hom}\,\{R, G\}/\text{Hom}\,\{F_0 \mid R, G\},$$

where Hom $\{F \mid R, G\} \subset$ Hom $\{F_0 \mid R, G\}$. According to the "naturality" theorem of §12 the homomorphism I^* between the groups on the left can be represented on the right as that correspondence which carries each coset of Hom $\{F \mid R, G\}$ into the coset of Hom $\{F_0 \mid R, G\}$ in which it is contained. This makes it obvious that the homomorphism is a mapping "onto."

LEMMA 23.2. *If $H_0 \subset H$, then the dual homomorphisms I^* of factor and transformation sets, as in* (23.1) *and* (23.2), *are mappings "onto."*

PROOF. Any element in Trans $\{G, H_0\}$ has the form

$$f(h, k) = g(h) + g(k) - g(h + k),$$

where g is an arbitrary function on H_0 to G. Let g^* be an arbitrary extension of g to H, and

$$f^*(h, k) = g^*(h) + g^*(k) - g^*(h + k).$$

Then f^* is a transformation set with $I^*f^* = f$. This proves that (23.2) is a mapping onto. Since (23.3) and (23.2) are mappings onto, the same holds for (23.1).

24. The group Ext*

Since limits do not always permute with groups of extensions, we now introduce a new group which is the limit of an inverse system of groups of group extensions.

Consider a discrete group T with only elements of finite order. The set $\{S_\alpha\}$ of all finite subgroups of T is a direct system, if $\alpha < \beta$ means that $S_\alpha \subset S_\beta$, and that the projection $I_{\beta\alpha}$ of S_α into S_β is simply the identity. The direct limit of $\{S_\alpha\}$ is the group T.

Let G be any generalized topological group. Since $\{S_\alpha\}$ is a direct system, it follows from a previous section that the groups Ext $\{G, S_\alpha\}$ form an inverse system with projections $I^*_{\alpha\beta}$. We define our new group as the limit of this system

[23] The lemma can also be proved directly in terms of the group extensions E, E_0, using a suitable transfinite induction.

$$\text{(24.1)} \qquad \mathrm{Ext}^* \{G, T\} = \varprojlim \mathrm{Ext} \{G, S_\alpha\}.$$

The two theorems of §22 as to cases in which "Ext" and "Lim" commute give at once

COROLLARY 24.1. *If G is compact and topological, or is discrete without elements of finite order, then*

$$\mathrm{Ext}^* \{G, T\} \cong \mathrm{Ext} \{G, T\}.$$

In the definition of Ext* we used the approximation of T by its finite subgroups S_α. However, any approximation by finite groups will give the same result:

THEOREM 24.2. *If T_α is any direct system of finite groups, the corresponding inverse system of groups* Ext $\{G, T_\alpha\}$ *has a limit*

$$\text{(24.2)} \qquad \varprojlim \mathrm{Ext} \{G, T_\alpha\} \cong \mathrm{Ext}^* \{G, \varinjlim T_\alpha\}.$$

PROOF. In case T_α is the system of all finite subgroups of the limit $T = \varinjlim T_\alpha$, this equation is simply the definition of Ext*. In general, $T = \varinjlim T_\alpha$ is a group in which every element has finite order. Each T_α has a homomorphic projection $T'_\alpha = \phi_\alpha T_\alpha$ into the limit T, and T is simply the union of these subgroups T'_α. The set of these subgroups T'_α is therefore cofinal in the set of all finite subgroups of T. The inverse system of the groups Ext $\{G, T'_\alpha\}$, relative to the "identity" projections $I^*_{\alpha\beta}$, is cofinal in the inverse system used to define Ext*, hence gives the same limit group,

$$\text{(24.3)} \qquad \mathrm{Ext}^* \{G, T\} \cong \varprojlim \mathrm{Ext} \{G, T'_\alpha\}.$$

An element f^* in this limit group can be represented (but not uniquely) as a set $\{f_\alpha\}$ of factor sets $f_\alpha \in$ Fact $\{G, T'_\alpha\}$ which "match" modulo transformation sets. This means that for each $\beta > \alpha$ there is a transformation set $t_{\alpha\beta} \in$ Trans $\{G, T'_\alpha\}$ such that

$$f_\alpha(h', k') = f_\beta(h', k') + t_{\alpha\beta}(h', k'), \qquad h', k' \in T'_\alpha .$$

Now each homomorphism ϕ_α of T_α into T'_α determines, as in §12, a dual homomorphism ϕ^*_α of Fact $\{G, T'_\alpha\}$ into Fact $\{G, T_\alpha\}$, defined so that $e_\alpha = \phi^*_\alpha f_\alpha$ is the factor set given by the equations

$$\text{(24.4)} \qquad e_\alpha(h, k) = f_\alpha(\phi_\alpha h, \phi_\alpha k), \qquad h, k \in T_\alpha .$$

If the f_α match, one readily proves that the corresponding e_α also match, modulo transformation sets. If the representation of f^* by $\{f_\alpha\}$ is changed by adding to each f_α a transformation set, the e_α's are changed accordingly by transformation sets. Therefore the correspondence

$$\text{(24.5)} \qquad f^* = \{f_\alpha\} \to e^* = \{\phi^*_\alpha f_\alpha\} = \omega f^*$$

carries each element f^* in $\varprojlim$ Ext $\{G, T'_\alpha\}$ into a well defined element e^* in $\varprojlim$ Ext $\{G, T_\alpha\}$. One verifies at once that this correspondence is a homomorphism.

Now we use the assumption that each T_α is finite. If $\phi_\alpha h_\alpha = 0$ for some $h_\alpha \in T_\alpha$, the definition of equality in a direct system shows that $\phi_{\beta\alpha} h_\alpha = 0$ for some $\beta > \alpha$. Since the whole group T_α is finite, we can select a single $\beta = \beta_0(\alpha) > \alpha$ which will do this for all h_α, so that

$$\phi_\alpha h_\alpha = 0 \quad \text{implies} \quad \phi_{\beta\alpha} h_\alpha = 0, \qquad \beta = \beta_0(\alpha).$$

Since $\phi_\beta \phi_{\beta\alpha} = \phi_\alpha$, ϕ_β is now an *isomorphism* of $\phi_{\beta\alpha} T_\alpha$ onto T'_α. Let ϕ_β^{-1} denote the inverse correspondence.

Next we show that ω, as defined by (24.5), is an isomorphism. For suppose $\omega f^* = 0$; every $\phi_\alpha^* f_\alpha$ is then a transformation set t_α. Using (24.4) and $\beta = \beta_0(\alpha)$, we then have, for any $h', k' \in T'_\alpha$,

$$f_\alpha(h', k') \equiv f_\beta(h', k') = e_\beta(\phi_\beta^{-1} h', \phi_\beta^{-1} k') = t_\beta(\phi_\beta^{-1} h', \phi_\beta^{-1} k').$$

This shows that f_α is a transformation set, hence that $f^* = \{f_\alpha\} = 0$ in $\text{Ext}^* \{G, T\}$.

To construct a correspondence inverse to ω, let $e^* = \{e_\alpha\}$ be a given element in $\varprojlim \text{Ext} \{G, T_\alpha\}$, where each $e_\alpha \in \text{Fact} \{G, T_\alpha\}$. Define

$$f_\alpha(h', k') = e_\beta(\phi_\beta^{-1} h', \phi_\beta^{-1} k'), \qquad \beta = \beta_0(\alpha) \tag{24.6}$$

for each $h', k' \in T'_\alpha$. Since the e_α's are known to match, we may verify that the replacement of β by any larger index γ in this definition will only alter f_α by a transformation set. To show that f_α and f_γ match properly for $\alpha < \gamma$, one then chooses $\beta > \beta_0(\alpha)$, $\beta > \beta_0(\gamma)$ in (24.6) and uses the given matching of the e_α's (modulo transformation sets). Finally, one verifies easily that the correspondence $\{e_\alpha\} \to \{f_\alpha\}$ of (24.6) is the inverse of the given correspondence ω of (24.5). This establishes the isomorphism (24.2) required in the theorem. The continuity, in both directions, follows from the formulae (24.5) and (24.6).

THEOREM 24.3. *If every element of T has finite order, the group* $\text{Ext}^* \{G, T\}$ *contains an everywhere dense subgroup isomorphic to* $\text{Ext} \{G, T\}/\text{Ext}_f \{G, T\}$.

This will be established by constructing a "natural" homomorphism of $\text{Ext} \{G, T\}$ into $\text{Ext}^* \{G, T\}$. To this end, let E be any extension of G by T determined by a factor set f. As in §23, f may be "cut off" to give a factor set f_α for any given finite subgroup $S_\alpha \subset T$. These factor sets match properly, so $\{f_\alpha\}$ determines a definite element in the inverse limit group $\text{Ext}^* \{G, T\}$. Alteration of f by a transformation set alters each f_α by the correspondingly "cut off" transformation set, hence does not alter the element $\{f_\alpha\} = f^*$ of Ext*. Therefore $f \to \{f_\alpha\}$ is a well defined homomorphism of Ext into Ext*. In case f lies in $\text{Ext}_f \{G, T\}$, each f_α is a transformation set, by the very definition of Ext_f, so that $\{f_\alpha\} = 0$. Conversely, if each f_α is a transformation set, $f \in \text{Ext}_f$. We thus have a (bicontinuous) isomorphism of Ext/Ext_f onto a subgroup of Ext*.

To show this subgroup everywhere dense in Ext* it will suffice, whatever the topology in G, to show the following: Given an element $f^* = \{f'_\alpha\}$ in $\text{Ext}^* \{G, T\}$ and a finite set J_0 of indices, there exists a factor set f in $\text{Fact} \{G, T\}$ such that

$f_\alpha - f'_\alpha$ is a transformation set for every index $\alpha \in J_0$. To prove this, choose a finite subgroup S_γ which contains all the groups S_α, for $\alpha \in J_0$. By Lemma 23.2, the given factor set f'_γ can be obtained by "cutting off" a suitable factor set f, so that $f_\gamma - f'_\gamma$ is the transformation set 0. The matching condition for the f'_α then shows that each difference $f_\alpha - f'_\alpha$ is also a transformation set, for $\alpha \in J_0$. This proves the property stated above, and with it, the theorem.

In many cases the subgroup considered in Theorem 24.3 is the whole group Ext*. It follows from previous considerations that this is the case when G is compact or when G is discrete and has no elements of finite order. Another important case is that when T is countable:

THEOREM 24.4. *If T is countable then*

$$\text{Ext}^* \{G, T\} \cong \text{Ext} \{G, T\}/\text{Ext}_f \{G, T\}. \tag{24.7}$$

PROOF. Since T is countable, the system of all finite subgroups of T used to define Ext* $\{G, T\}$ may be replaced by a cofinal sequence of finite subgroups T_n with $T_1 \subset T_2 \subset \cdots \subset T_n \subset \cdots \subset T$, with the identity projections $I_n : T_n \to T_{n+1}$. Therefore $\text{Ext}^* \{G, T\} = \varprojlim \text{Ext} \{G, T_n\}$. An element e^* of this group can then be represented as a sequence $\{f_n\}$ of factor sets $f_n \in \text{Fact} \{G, T_n\}$ which match, in the sense that, for some g_n,

$$f_{n+1}(h, k) = f_n(h, k) + [g_n(h) + g_n(k) - g_n(h + k)] \tag{24.8}$$

for all $h, k \in T_n$. The transformation set shown in brackets may be extended to all of T by extending g_n to a function g_n^* on T, as in Lemma 23.2. We introduce a new function $s_n(h) = g_1^*(h) + \cdots + g_{n-1}^*(h)$, for all $h \in T$, and a new family of factor sets

$$f'_n(h, k) = f_n(h, k) - [s_n(h) + s_n(k) - s_n(h + k)],$$

for $h, k \in T_n$.[24] Since f'_n differs from f_n by a transformation set, the given element e^* of Ext* has both representations $\{f_n\}$ and $\{f'_n\}$. But (24.8) also shows that f'_{n+1}, cut off at T_n, is exactly f'_n. Therefore these factor sets match exactly, and provide a composite factor set f of T in G. This factor set f is one which corresponds to the given element e^* of Ext* in the "natural" homomorphism of Ext into Ext* as constructed in Theorem 24.3, so this homomorphism maps Ext on all of Ext*, as asserted in (24.7).

25. Relation to tensor products

The group Ext* introduced in this chapter is closely related to the tensor product. Since an early form ([5]) of our results was formulated in terms of tensor products, we shall briefly state the connection. Let G be any group, A a compact zero-dimensional group, $\{A_\alpha\}$ the family of all open and closed subgroups of A. Then the groups A/A_α and $G \circ (A/A_\alpha)$ both form inverse

[24] This construction is an exact group theoretic analog of a similar matching process for chains, as devised by Steenrod ([9], p. 692).

systems. The modified tensor product $G\bullet A$ is defined as the limit of the groups $G\circ(A/A_\alpha)$.

Now let the group T with all elements of finite order be represented in terms of a free group F as $T = F/R$. Each finite subgroup S_α then has a representation F_α/R, and the fundamental theorem of Chapter II asserts that

$$\text{(25.1)} \qquad \text{Ext}\ \{G, S_\alpha\} \cong \text{Hom}\ \{R, G\}/\text{Hom}\ \{F_\alpha \mid R, G\}.$$

The groups on both sides here form inverse systems, relative to the identity as projections. Furthermore, the isomorphism of (25.1) permutes with these projections, so that the limits of the two direct systems in (25.1) are also isomorphic. In view of the definition of Ext*, this gives

$$\text{(25.2)} \qquad \text{Ext}^*\ \{G, T\} \cong \varprojlim\ [\text{Hom}\ \{R, G\}/\text{Hom}\ \{F_\alpha \mid R, G\}].$$

Now if I is the group of integers, any element $\sigma = \sum g_i\phi_i$ in the tensor product $G\circ\text{Hom}\ \{R, I\}$ determines in natural fashion the homomorphism $\theta \in \text{Hom}\ \{R, G\}$ with $\theta(r) = \sum \phi_i(r)g_i$. By a somewhat lengthy argument, this correspondence $\sigma \rightarrow \theta$ can be used to "factor out" the G in (25.2) to give

$$\text{(25.3)} \qquad \text{Ext}^*\ \{G, T\} \cong \varprojlim\ G\circ[\text{Hom}\ \{R, I\}/\text{Hom}\ \{F_\alpha \mid R, I\}].$$

The group in brackets here is Ext $\{I, F_\alpha/R\}$, by the fundamental theorem on group extensions. According to Theorem 17.1 it can be expressed as Char S_α. Therefore (25.3) is[25]

$$\text{(25.4)} \qquad \text{Ext}^*\ \{G, T\} \cong \varprojlim\ (G\circ\text{Char}\ S_\alpha).$$

But the group Char S_α can, by the theory of characters (Lemma 13.2, Theorem 13.5), be rewritten as a factor group Char T/Annih S_α, where the subgroups of the form Annih S_α in Char T are exactly the open and closed subgroups in the zero-dimensional group Char T. Thus (25.4) may be restated in terms of the modified tensor product, as

$$\text{(25.5)} \qquad \text{Ext}^*\ \{G, T\} \cong G\bullet\text{Char}\ T.$$

The use of the "modified" tensor product is therefore equivalent to the use of the group Ext*.

CHAPTER V. ABSTRACT COMPLEXES

Turning now to the topological applications, we will establish the fundamental theorem on the decomposition of the homology groups of an infinite complex in terms of the integral cohomology groups of the complex. This theorem will be obtained in several closely related forms (Theorems 32.1, 32.2 and 34.2) for three different types of homology groups. The largest (or "longest") homology group is that consisting of infinite cycles, with coefficients in G, reduced modulo

[25] This argument requires an application of the isomorphism theorem for inverse systems, and hence rests on the fact that the isomorphism of Theorem 17.1 is "natural" in the sense of §12.

the subgroup of actual boundaries. Since the latter subgroup is not in general closed, this homology group will be only a generalized topological group. This suggests the introduction of the shorter "weak" homology group, which consists of cycles modulo "weak" boundaries; i.e. those cycles which can be regarded as boundaries in any finite portion of the complex. The fundamental theorem for this type of homology group uses the group Ext_f which has been already analyzed. Finally, the group of cycles modulo the closure of the group of boundaries gives (following Lefschetz) a homology group which is always topological; for this we derive a corresponding form of the fundamental theorem. Furthermore, the standard duality between homology and cohomology groups enables us to deduce a corresponding theorem (Theorem 33.1) for the cohomology groups with coefficients in an arbitrary discrete group G.

The fundamental theorem expresses a homology group by means of a group of homomorphisms and a group of group extensions; the latter can also be represented by groups of homomorphisms, as in the basic theorem of Chapter II. The requisite connection between cycles of the homology group and homomorphisms is provided by the Kronecker index (§29).

26. Complexes

The complexes considered here will be abstract cell complexes[26] satisfying a star finiteness condition. More precisely, we consider a collection K of abstract elements σ^q called *cells*. With each cell there is associated an integer q called the dimension of σ^q. (There is no restriction requiring the dimension to be non-negative.) To any two cells σ_i^{q+1}, σ_j^q there corresponds an integer $[\sigma_i^{q+1}:\sigma_j^q]$, called the *incidence number*. K will be called a *star finite complex* provided the incidence numbers satisfy the following two conditions:

(26.1) *Given* σ_j^q, $[\sigma_i^{q+1}:\sigma_j^q] \neq 0$ *only for a finite number of indices* i;

(26.2) *Given* σ_j^{q+1} *and* σ_k^{q-1}, $\sum_i [\sigma_j^{q+1}:\sigma_i^q][\sigma_i^q:\sigma_k^{q-1}] = 0$.

Condition (26.1) is the star finiteness condition. It insures that the summation in (26.2) is finite.

If we consider the "incidence" matrices of integers

$$A^q = \| [\sigma_j^{q+1}:\sigma_i^q] \|$$

we can rewrite the two conditions as follows:

(26.1′) $\qquad A^q$ *is column finite*;

(26.2′) $\qquad A^qA^{q-1} = 0.$

Actually we could have defined a complex as a collection of matrices $\{A^q\}$, $q = 0, \pm 1, \pm 2, \cdots$, such that (26.1′) and (26.2′) hold; we must assume then that the columns of A^q have the same set of labels as do the rows of A^{q-1}, in

[26] Essentially like those introduced by A. W. Tucker, for the case of finite complexes. Homology and cohomology are treated as in Whitney [14].

order to form the product $A^q A^{q-1}$. A q-cell will be then either a column of A^q or the corresponding row of A^{q-1}.

A subset L of the cells of K is called an *open subcomplex* if L contains with each q-cell all incident $(q + 1)$-cells; that is, if σ_i^q in L and $[\sigma_j^{q+1} : \sigma_i^q] \neq 0$ imply $\sigma_j^{q+1} \in L$. The incidence matrix A_L^q of L is then the submatrix obtained from A^q by deleting all rows and all columns belonging to cells not in L. Conditions (26.1) and (26.2) automatically hold in L, the latter because of the requirement that L be "open."

A subset $L \subset K$ *is a closed subcomplex* if L contains with each q-cell all incident $(q - 1)$-cells; that is, if $\sigma_j^q \in L$ and $[\sigma_j^q : \sigma_k^{q-1}] \neq 0$ imply $\sigma_k^{q-1} \in L$. The incidence matrix of L is obtained as before, and the conditions (26.1) and (26.2) again hold in L. Whenever L is a closed subcomplex, its complement $K - L$ is an open one, and vice versa.

A subset L of K will be called *q-finite* if L contains only a finite number of q-cells. Because K is star-finite, every $(q - 1)$-cell is contained in a q-finite open subcomplex of K.

27. Homology and cohomology groups

Let G be an abelian group. A q-dimensional *chain* c^q in K with coefficients in G is a function which associates to every q-cell σ_i^q in K an element g_i of G. We write c^q as a formal infinite sum

$$c^q = \sum_i g_i \sigma_i^q .$$

The sum of two chains $\sum g_i \sigma_i^q$ and $\sum h_i \sigma_i^q$ is the chain $\sum (g_i + h_i)\sigma_i^q$, and the chains form a group denoted by $C^q(K, G)$. If $g_i \neq 0$ for only a finite number of indices i then the chain c^q is *finite*. The finite chains form a subgroup $\mathcal{C}_q(K, G)$ of C^q.

The *coboundary* δc^q of a finite chain $c^q = \sum g_j \sigma_j^q$ is defined as

$$\delta c^q = \sum_i \Big(\sum_j [\sigma_i^{q+1} : \sigma_j^q] g_j\Big) \sigma_i^{q+1}.$$

Because of (26.1) δc^q is a finite $(q + 1)$-chain, while, because of (26.2), $\delta\delta c^q = 0$. The operation δ is a homomorphic mapping of $\mathcal{C}_q$ into $\mathcal{C}_{q+1}$. The kernel of this homomorphism is a subgroup $\mathcal{Z}_q(K, G)$ of $\mathcal{C}_q$. The chains of $\mathcal{Z}_q$ are called (finite) *cocycles*:

$$\mathcal{Z}_q(K, G) = [\text{all finite } q\text{-chains } c^q \text{ with } \delta c^q \doteq 0].$$

A *coboundary* is a q-chain of the form δd^{q-1} for some $d^{q-1} \in \mathcal{C}_{q-1}$; these coboundaries form a subgroup

$$\mathcal{B}_q(K, G) = [\text{all finite chains } \delta d^{q-1}].$$

From the relation $\delta\delta = 0$ it follows that $\mathcal{B}_q \subset \mathcal{Z}_q$. The corresponding factor group

$$\mathcal{H}_q(K, G) = \mathcal{Z}_q(K, G)/\mathcal{B}_q(K, G)$$

is called the q^{th} *cohomology* group of finite cocycles of K with coefficients in G. We also define the *co-torsion* group $\mathcal{T}_q(K, G)$ as the subgroup of all elements of finite order in $\mathcal{H}_q(K, G)$.

For a chain $c^q = \sum g_i\sigma_i^q$ of $C^q(K, G)$ we also define the *boundary*

$$\partial c^q = \sum_j \left(\sum_i [\sigma_i^q : \sigma_j^{q-1}] g_i\right)\sigma_j^{q-1}.$$

It again follows from (26.1) that ∂c^q is a well defined chain of $C^{q-1}(K, G)$ and from (26.2) that $\partial\partial c^q = 0$. The operation ∂ is a homomorphic mapping of C^q into C^{q-1}. The kernel of this homomorphism is a subgroup $Z^q(K, G)$ of C^q. The chains of Z^q are called *cycles*:

$$Z^q(K, G) = [\text{all chains } c^q \text{ with } \partial c^q = 0].$$

The chains of the form ∂d^{q+1} where $d^{q+1} \in C^{q+1}$ are the *boundaries*. They form a subgroup

$$B^q(K, G) = [\text{all chains } c^q = \partial d^{q+1}].$$

Because $\partial\partial = 0$ it follows that $B^q \subset Z^q$. The group

$$H^q(K, G) = Z^q(K, G)/B^q(K, G)$$

is called the q^{th} *homology* group of K with coefficients in G.

Let L be a (closed or open) subcomplex of K. Each chain c^q in K, considered as a function on the q-cells, defines a corresponding chain c_L^q in L. If $c^q = \sum g_i\sigma_i^q$, $c_L^q = \sum' g_i\sigma_i^q$ is the sum found by deleting all terms $g_i\sigma_i^q$ for which σ_i^q is not in L. If L is open, then $\partial_L(c_L^q) = (\partial c^q)_L$, so that one can establish the following facts.

LEMMA 27.1. *$c^q \in Z^q(K, G)$ if and only if $c_L^q \in Z^q(L, G)$ for every q-finite open subcomplex L of K.*

LEMMA 27.2. *If $c^q \in B^q(K, G)$ then $c_L^q \in B^q(L, G)$, provided L is an open subcomplex of K.*

A statement analogous to Lemma 27.1 concerning B^q is not generally true. In this connection we define the group $B_w^q(K, G)$ of the *weak boundaries* as follows: $c^q \in B_w^q(K, G)$ provided $c_L^q \in B^q(L, G)$ for every q-finite open subcomplex L of K. For each such open subcomplex L we can construct a subcomplex L' consisting of all q-cells of L, all those $(q + 1)$-cells of L which lie on coboundaries of q-cells of L, and all $(q + i)$-cells of K, for $i > 1$. This subcomplex L' is open, is both q and $(q + 1)$-finite, and has $B^q(L, G) = B^q(L', G)$. Hence we conclude that $c^q \in B_w^q(K, G)$ if and only if $c_L^q \in B^q(L, G)$ for every open subcomplex L of K which is both q- and $(q + 1)$-finite. Clearly $B^q = B_w^q$ when K itself is q-finite.

It follows from Lemmas 27.1 and 27.2 that

$$B^q(K, G) \subset B_w^q(K, G) \subset Z^q(K, G).$$

The factor group

$$H^q_w(K, G) = Z^q(K, G)/B^q_w(K, G)$$

will be called the *weak* q^{th} *homology group* of K with coefficients in G. Clearly $H^q = H^q_w$ when K is q-finite

LEMMA 27.3. *$c^q \in B^q_w(K, G)$ if and only if for each finite subset M of K there is a chain c^q_1 in $K - M$ such that $c^q - c^q_1 \in B^q(K, G)$.*

PROOF. Suppose that $c^q \in B^q_w$. Given the finite set M there is a q-finite open subcomplex L containing M. Since $c^q_L \in B^q(L, G)$ there is a d^{q+1} in L such that $(\partial d^{q+1})_L = c^q_L$. Set $c^q_1 = c^q - \partial d^{q+1}$. Clearly $c^q - c^q_1 \in B^q$ and $(c^q_1)_L = c^q_L - (\partial d^{q+1})_L = 0$, hence $c^q_1 \subset K - L \subset K - M$.

Suppose now that c^q satisfies the condition of Lemma 27.3. Given a q-finite open subcomplex L of K there is a c^q_1 in $K - L$ such that $c^q - c^q_1 \in B^q(K, G)$. There is then a d^{q+1} such that $\partial d^{q+1} = c^q - c^q_1$. Since L is open we have

$$\partial_L(d^{q+1}_L) = (\partial d^{q+1})_L = c^q_L - (c^q_1)_L = c^q_L ;$$

therefore $c^q_L \in B^q(L, G)$.

28. Topology in the homology groups

The group of q-chains $C^q(K, G)$ is isomorphic with $\prod_i G_i$, where $G = G_i$ and the set of indices i is in a 1-1 correspondence with the set of q-cells σ^q_i. Hence, if G is a generalized topological group, we can consider $C^q(K, G)$ as a generalized topological group, under the direct product topology, as defined in §1. If G is topological or compact, then $C^q(K, G)$ is also topological or compact, as the case may be.

The boundary operator ∂, regarded as a homomorphism of C^q into C^{q-1}, is continuous. Since Z^q is the group mapped into 0 by ∂, we obtain

LEMMA 28.1. *If G is topological then $Z^q(K, G)$ is a closed subgroup of $C^q(K, G)$.*

From Lemma 27.3 we deduce

LEMMA 28.2. $B^q(K, G) \subset B^q_w(K, G) \subset \bar{B}^q(K, G)$.

The homology groups $H^q = Z^q/B^q$ and $H^q_w = Z^q/B^q_w$ as factor groups of generalized topological groups are generalized topological groups; this is the way they will be considered in the rest of this paper. Even in the case when G and consequently Z^q is topological the groups H^q and H^q_w may be only generalized topological groups, for B^q and B^q_w need not be closed subgroups of Z^q.

If G is compact and topological, then $Z^q(K, G)$ and $C^{q+1}(K, G)$ are compact; since $B^q(K, G)$ is a continuous image of C^{q+1} (under the operation ∂), $B^q(K, G)$ is compact and therefore closed (see Lemma 1.1). Consequently we obtain

LEMMA 28.3. *If G is compact and topological, then $B^q(K, G) = B^q_w(K, G) = \bar{B}^q(K, G)$, $H^q(K, G) = H^q_w(K, G)$ and the groups are all compact and topological.*

Despite the fact that $\mathcal{C}_q$ is a subgroup of the generalized topological group C^q we consider $\mathcal{C}_q$ discrete and consequently the cohomology groups $H_q(K, G)$ are taken discrete.

29. The Kronecker index

Let G be a generalized topological group, H a discrete group and assume that a product $\phi(g, h) \epsilon J$ is given pairing G and H to a group J (see §13).

Given two chains

$$c^q \epsilon C^q(K, G), \qquad d^q \epsilon \mathcal{C}_q(K, H),$$

we define the Kronecker index as

$$c^q \cdot d^q = \sum_i \phi(g_i, h_i) \epsilon J;$$

the summation is finite since d^q is a finite chain. We verify at once that in this way the groups $C^q(K, G)$ and $\mathcal{C}_q(K, H)$ are paired to J.

Given $c^{q+1} \epsilon C^{q+1}(K, G)$ and $d^q \epsilon \mathcal{C}_q(K, H)$ we have

$$(29.1) \qquad (\partial c^{q+1}) \cdot d^q = c^{q+1} \cdot (\delta d^q).$$

This is a restatement of the associative law for matrix multiplication, since the operator ∂ is essentially a postmultiplication by the incidence matrix, while the coboundary operator δ is a premultiplication by the same matrix.

We now examine the annihilators relative to the Kronecker index.

$$(29.2) \qquad \mathcal{Z}_q(K, H) \subset \text{Annih } B^q_w(K, G) \subset \text{Annih } B^q(K, G).$$

$$(29.3) \qquad Z^q(K, G) \subset \text{Annih } \mathcal{B}_q(K, H).$$

Proof. Let $z^q \epsilon B^q_w$ and $w^q \epsilon \mathcal{Z}_q$. Since w^q is finite there is a finite subset M of K such that $w^q \subset M$. In view of Lemma 27.3 there is a cycle $z^q_1 \subset K - M$ and a chain $c^{q+1} \epsilon C^{q+1}(K, G)$ such that $\partial c^{q+1} = z^q - z^q_1$. Consequently

$$z^q \cdot w^q = (z^q - z^q_1) \cdot w^q = (\partial c^{q+1}) \cdot w^q = c^{q+1} \cdot \delta w^q = c^{q+1} \cdot 0 = 0.$$

Therefore $\mathcal{Z}_q \subset \text{Annih } B^q_w$. The proof of (29.3) is analogous.

It follows from (29.2) and (29.3) that

(29.4) *$H^q(K, G)$ and $\mathcal{H}_q(K, H)$ are paired to J,*

(29.5) *$H^q_w(K, G)$ and $\mathcal{H}_q(K, H)$ are paired to J.*

Lemma 29.1. *If G and H are dually paired to J then, relative to the Kronecker index,*

(29.6) *$C^q(K, G)$ and $\mathcal{C}_q(K, H)$ are dually paired to J,*

$$(29.7) \qquad \mathcal{Z}_q(K, H) = \text{Annih } B^q_w(K, G) = \text{Annih } B^q(K, G),$$

$$(29.8) \qquad Z^q(K, G) = \text{Annih } \mathcal{B}_q(K, H).$$

Proof. Given $c^q = \sum g_i \sigma^q_i \neq 0$ in C^q, we have $g_{i_0} \neq 0$ for some i_0. Select $h \epsilon H$ so that $\phi(g_{i_0}, h) \neq 0$. Consider the chain $d^q = h\sigma^q_{i_0}$. Then $c^q \cdot d^q = \phi(g_{i_0}, h) \neq 0$. This proves that Annih $\mathcal{C}_q(K, H) = 0$. Similarly we prove that Annih $C^q(K, G) = 0$. This establishes (29.6).

Let $d^q \in \text{Annih } B^q(K, G)$. Hence $c^{q+1} \cdot (\delta d^q) = (\partial c^{q+1}) \cdot d^q = 0$ for every c^{q+1}, and therefore $\delta d^q = 0$, in view of (29.6). This shows that $\text{Annih } B^q \subset \mathcal{Z}_q$, which, together with (29.2), gives (29.7).

The proof of (29.8) is analogous to the previous one.

We remark that even when the pairing of the coefficient groups G and H is dual, the pairing (29.4) or (29.5) of the homology and cohomology groups need not be dual, as observed by Whitney ([14], p. 42).

We shall be especially interested in the pairing of G with the group I of integers to G by means of the product $\phi(g, m) = mg$. This pairing has the property that $\text{Annih } I = 0$. This is half of the definition of a dual pairing; the other half ($\text{Annih } G = 0$) may fail in case the order of every element in G divides a fixed integer m. Nevertheless the argument for Lemma 29.1 shows in this case that

$$\text{(29.6')} \qquad \text{Annih } \mathcal{C}_q(K, I) = 0,$$

$$\text{(29.8')} \qquad Z^q(K, G) = \text{Annih } \mathcal{B}_q(K, I).$$

We now introduce a subgroup of the group of cycles by the following definition:

$$\text{(29.9)} \qquad A^q(K, G) = \text{Annih } \mathcal{Z}_q(K, I);$$

in other words, $c^q \in A^q(K, G)$ if and only if $c^q \cdot w^q = 0$ for every finite integral cocycle w^q. The position of this group A^q may be described as follows:

$$\text{(29.10)} \qquad B^q_w(K, G) \subset A^q(K, G) \subset Z^q(K, G).$$

By (29.2) we have $\mathcal{Z}_q \subset \text{Annih } B^q_w$; consequently $B^q_w \subset \text{Annih } \mathcal{Z}_q = A^q$. Since $\mathcal{B}_q \subset \mathcal{Z}_q$, we have $A^q = \text{Annih } \mathcal{Z}_q \subset \text{Annih } \mathcal{B}_q = Z^q$ by (29.8').

LEMMA 29.2. *If G is a topological group, $A^q(K, G)$ is closed.*

This follows immediately from the continuity of the Kronecker index.

In case G is topological, the various subgroups of cycles of $C^q(K, G)$ are therefore related as follows:

$$B^q \subset B^q_w \subset \bar{B}^q \subset A^q = \bar{A}^q \subset Z^q = \bar{Z}^q \subset C^q.$$

30. Construction of homomorphisms

The essential device of this chapter is that of using the Kronecker index to generate homomorphisms. For a given chain $c^q \in C^q(K, G)$ define θ_{c^q} by

$$\text{(30.1)} \qquad \theta_{c^q}(d^q) = c^q \cdot d^q, \qquad d^q \in \mathcal{C}_q(K, I).$$

LEMMA 30.1. *The correspondence $c^q \to \theta_{c^q}$ establishes an isomorphism*

$$C^q(K, G) \cong \text{Hom } \{\mathcal{C}_q(K, I), G\}.$$

PROOF. It is clear that $\theta_{c^q} \in \text{Hom } \{\mathcal{C}_q, G\}$, and that the correspondence $c^q \to \theta_{d^q}$ preserves sums. Also, if $\theta_{c^q} = 0$ then $c^q \cdot d^q = 0$ for all $d^q \in \mathcal{C}_q$ and consequently $c^q = 0$. Conversely, given $\theta \in \text{Hom } \{\mathcal{C}_q, G\}$, define

$$\text{(30.2)} \qquad c^q = \sum_i \theta(\sigma^q_i)\sigma^q_i.$$

Clearly $c^q \in C^q(K, G)$, while, for any given $d^q = \sum h_i \sigma_i^q \in \mathcal{C}_q$, we have

$$\theta_{c^q}(d^q) = c^q \cdot d^q = \sum h_i \theta(\sigma_i^q) = \theta(\sum h_i \sigma_i^q) = \theta(d^q).$$

This establishes the algebraic part of the Lemma.

We now recall that

$$C^q(K, G) \cong \prod_i G_i$$

where $G_i = G$ and the subscripts i are in a 1-1 correspondence with the q-cells σ_i^q. On the other hand, since the $\{\sigma_i^q\}$ constitute a set of generators for $\mathcal{C}_q(K, I)$, we have

$$\text{Hom } \{\mathcal{C}_q(K, I), G\} \cong \prod_i G_i.$$

Both these isomorphisms are bicontinuous, hence the combined isomorphism, which is precisely the isomorphism $c^q \leftrightarrow \theta_{c^q}$, is also bicontinuous.

LEMMA 30.2. *$\mathcal{Z}_q(K, I)$ is a direct factor of $\mathcal{C}_q(K, I)$.*

PROOF. The coboundary operator δ maps $\mathcal{C}_q$ onto $\mathcal{B}_{q+1}$ and the kernel is $\mathcal{Z}_q$. Hence $\mathcal{C}_q$ is a group extension of $\mathcal{Z}_q$ by $\mathcal{B}_{q+1}$. As a subgroup of the free group $\mathcal{C}_{q+1}$ the group $\mathcal{B}_{q+1}$ is free (Lemma 4.1) and therefore the group extension is trivial (Theorem 7.2). Hence $\mathcal{C}_q$ is the direct product of $\mathcal{Z}_q$ and a subgroup isomorphic with $\mathcal{B}_{q+1}$.

THEOREM 30.3. *$A^q(K, G)$ is a direct factor of $Z^q(K, G)$ and of $C^q(K, G)$.*

PROOF. Since $A^q \subset Z^q$ it will be sufficient to show that A^q is a direct factor of C^q. In the group Hom $\{\mathcal{C}_q(K, I), G\}$ consider the subgroup A of those homomorphisms that annihilate Z_q. Since $\mathcal{Z}_q$ is a direct factor of $\mathcal{C}_q$, A is a direct factor of Hom $\{\mathcal{C}_q, G\}$. However, under the isomorphism $\theta_{c^q} \to c^q$ of Lemma 30.1 the group A is mapped onto $A^q(K, G) = \text{Annih } \mathcal{Z}_q$, hence the conclusion. This proof also shows (Lemma 3.3) that

$$A^q(K, G) \cong \text{Hom } \{\mathcal{C}_q(K, I)/\mathcal{Z}_q(K, I), G\}. \tag{30.3}$$

Theorem 30.3 leads to the following direct product decompositions of the homology groups:

$$H^q(K, G) \cong (Z^q/A^q) \times (A^q/B^q), \tag{30.4}$$

$$H_w^q(K, G) \cong (Z^q/A^q) \times (A^q/B_w^q). \tag{30.5}$$

We proceed with the study of the first factor, Z^q/A^q.

THEOREM 30.4. *The correspondence $c^q \to \theta_{c^q}$ establishes an isomorphism*

$$Z^q(K, G)/A^q(K, G) \cong \text{Hom } \{\mathcal{H}_q(K, I), G\}.$$

PROOF. Since $Z^q = \text{Annih } \mathcal{B}_q$, by (29.8′), it follows that under the isomorphism $c^q \to \theta_{c^q}$ the group Z^q is mapped onto the subgroup of Hom $\{\mathcal{C}_q, G\}$ consisting of those homomorphisms annihilating $\mathcal{B}_q$. By Lemma 3.3 the latter subgroup can be identified with Hom $\{\mathcal{C}_q/\mathcal{B}_q, G\}$, so $Z^q \cong \text{Hom } \{\mathcal{C}_q/\mathcal{B}_q, G\}$. On the other hand, $\mathcal{Z}_q/\mathcal{B}_q$ is a direct factor of $\mathcal{C}_q/\mathcal{B}_q$, so that Lemma 3.4 shows that Hom $\{\mathfrak{Z}_q/\mathfrak{B}_q, G\}$ is a factor group of Hom $\{\mathfrak{C}_q/\mathfrak{B}_q, G\}$, corresponding to the subgroup consisting of homomorphisms annihilating $\mathfrak{Z}_q/\mathfrak{B}_q$. This subgroup in turn corresponds to the subgroup A^q of Z^q, hence

$$Z^q/A^q \cong \text{Hom } \{\mathfrak{Z}_q/\mathfrak{B}_q, G\}.$$

This is the desired conclusion.

31. Study of A^q

The correspondence $c^q \to \theta_{c^q}$ of Lemma 30.1 maps the group A^q of annihilators of cocycles onto the group of those homomorphisms of $\mathfrak{C}_q$ into G which carry $\mathfrak{Z}_q$ into zero. As observed in Lemma 3.3, the latter group is isomorphic to Hom $\{\mathfrak{C}_q/\mathfrak{Z}_q, G\}$. Since $\mathfrak{C}_q/\mathfrak{Z}_q \cong \mathfrak{B}_{q+1}$, this gives the isomorphism

$$A^q(K, G) \cong \text{Hom } \{\mathfrak{B}_{q+1}(K, I), G\}. \tag{31.1}$$

An examination of this construction shows that the homomorphism corresponding to a given $z^q \in A^q$ is determined as follows. For each $d^{q+1} \in \mathfrak{B}_{q+1}$ choose a $d^q \in \mathfrak{C}_q(K, I)$ for which $\delta d^q = d^{q+1}$, and define[27]

$$\phi_{z^q}(d^{q+1}) = z^q \cdot d^q.$$

Because z^q is in A^q, this result is independent of the choice of d^q for given d^{q+1}. Furthermore ϕ_{z^q} is a homomorphism of $\mathfrak{B}_{q+1}$ into G, and it is obtained from θ_{z^q} by the process indicated above, for one has

$$\phi_{z^q}(\delta d^q) = \theta_{z^q}(d^q).$$

We therefore have the following result.

LEMMA 31.1. *The correspondence $z^q \to \phi_{z^q}$ establishes the (bicontinuous) isomorphism* (31.1).

The properties of this isomorphism can be collected in the following

THEOREM 31.2. *The isomorphism $z^q \to \phi_{z^q}$ induces the isomorphisms*

$$A^q(K, G)/B^q(K, G) \cong \text{Hom } \{\mathfrak{B}_{q+1}, G\}/\text{Hom } \{\mathfrak{Z}_{q+1} \mid \mathfrak{B}_{q+1}, G\},$$

$$B^q_w(K, G)/B^q(K, G) \cong \text{Hom}_f \{\mathfrak{B}_{q+1}, G; \mathfrak{Z}_{q+1}\}/\text{Hom } \{\mathfrak{Z}_{q+1} \mid \mathfrak{B}_{q+1}, G\},$$

$$A^q(K, G)/B^q_w(K, G) \cong \text{Hom } \{\mathfrak{B}_{q+1}, G\}/\text{Hom}_f \{\mathfrak{B}_{q+1}, G; \mathfrak{Z}_{q+1}\},$$

where $\mathfrak{B}_{q+1} = \mathfrak{B}_{q+1}(K, I)$ and $\mathfrak{Z}_{q+1} = \mathfrak{Z}_{q+1}(K, I)$.

PROOF. We shall show that the groups $B^q(K, G)$ and $B^q_w(K, G)$ are mapped onto Hom $\{\mathfrak{Z}_{q+1} \mid \mathfrak{B}_{q+1}, G\}$ and Hom$_f$ $\{\mathfrak{B}_{q+1}, G; \mathfrak{Z}_{q+1}\}$, respectively.

Assume that $z^q \in B^q(K, G)$; then $\partial z^{q+1} = z^q$ for some $z^{q+1} \in C^{q+1}(K, G)$. Define

$$\phi^*(d^{q+1}) = z^{q+1} \cdot d^{q+1}; \qquad d^{q+1} \in \mathfrak{C}_{q+1}.$$

[27] Notice the analogy with the definition of the so-called "linking coefficient" (cf. Lefschetz [7], Ch. III).

Clearly $\phi^* \in \operatorname{Hom}\{\mathcal{C}_{q+1}, G\}$. If $d^{q+1} = \delta d^q$ then

$$\phi^*(d^{q+1}) = z^{q+1}\cdot d^{q+1} = z^{q+1}\cdot\delta d^q = \partial z^{q+1}\cdot d^q = z^q\cdot d^q = \phi_{z^q}(d^{q+1}).$$

Hence ϕ^* is an extension of ϕ_{z^q} to $\mathcal{C}_{q+1}$ and in particular also to $\mathcal{Z}_{q+1}$.

Suppose conversely that ϕ_{z^q} can be extended to $\mathcal{Z}_{q+1}$. Since $\mathcal{Z}_{q+1}$ is a direct factor of $\mathcal{C}_{q+1}$ (Lemma 30.2) we may then find an extension ϕ^* of ϕ_{z^q} to $\mathcal{C}_{q+1}$. Define

$$z^{q+1} = \sum_i \phi^*(\sigma_i^{q+1})\sigma_i^{q+1}.$$

Clearly $z^{q+1} \in C^{q+1}(K, G)$ and $z^{q+1}\cdot\sigma_j^{q+1} = \phi^*(\sigma_j^{q+1})$ and hence $z^{q+1}\cdot d^{q+1} = \phi^*(d^{q+1})$ for all $d^{q+1} \in \mathcal{C}_{q+1}$. Consequently

$$\partial z^{q+1}\cdot\sigma_j^q = z^{q+1}\cdot\delta\sigma_j^q = \phi^*(\delta\sigma_j^q) = \phi_{z^q}(\delta\sigma_j^q) = z^q\cdot\sigma_j^q.$$

Since this holds for every σ_j^q we have $\partial z^{q+1} = z^q \in B^q(K, G)$.

Suppose $z^q \in B_w^q(K, G)$. In view of Lemma 5.1 it is sufficient to prove that if the cocycle $md^{q+1} \in \mathcal{B}_{q+1}(K)$ then $\phi_{z^q}(md^{q+1})$ is divisible by m. Let $\delta d^q = md^{q+1}$ and let M be a finite subset of K such that $d^q \subset M$. In view of Lemma 27.3 there is a chain $z_1^q \subset K - M$ such that $z^q - z_1^q = z_2^q \in B^q(K, G)$. It follows that $z_2^q \in A^q$ and so that $z_1^q \in A^q$, hence $\phi_{z_1^q}$ and $\phi_{z_2^q}$ are defined and $\phi_{z^q} = \phi_{z_1^q} + \phi_{z_2^q}$. Since $z_2^q \in B^q(K, G)$, then, as we just proved, $\phi_{z_2^q}$ can be extended to Z_{q+1} and therefore $\phi_{z_2^q}(md^{q+1})$ must be divisible by m. Since $d^q \subset M$ and $z_1^q \subset K - M$ we have $\phi_{z_1^q}(md^{q+1}) = z_1^q\cdot d^q = 0$. Hence $\phi_{z^q}(md^{q+1})$ is divisible by m.

Suppose conversely that ϕ_{z^q} can be extended to every subgroup of $\mathcal{Z}_{q+1}(K, I)$ of finite order over $\mathcal{B}_{q+1}(K, I)$. Then, as in Lemma 5.2, ϕ_{z^q} can also be extended to every subgroup $\mathfrak{D}$ of $\mathcal{Z}_{q+1}(K, I)$ such that $\mathfrak{D}/\mathcal{B}_{q+1}$ has a finite number of generators. Now let L be any open subcomplex of K which is both q and $(q + 1)$-finite; there is then an extension of ϕ_{z^q} to the group $\mathfrak{D}_L$ generated by $\mathcal{B}_{q+1}(K, I)$ and $\mathcal{Z}_{q+1}(L, I)$. But in the complex L the homomorphism ϕ_{y^q} induced by $y^q = z_L^q$ agrees on $\mathcal{B}_{q+1}(L, I)$ with the homomorphism ϕ_{z^q}. Therefore $\phi_{y^q} \in \operatorname{Hom}\{\mathcal{B}_{q+1}(L, I), G\}$ has an extension to $\mathcal{Z}_{q+1}(L, I)$. In view of what we proved before, we therefore have $y^q = z_L^q \in B^q(L, G)$. Since this holds for each L considered, $z^q \in B_w^q(K, G)$. This concludes the proof of Theorem 31.2.

In this theorem the factor homomorphism groups on the right can be reinterpreted as groups of group extensions, in accord with the results of Chapter II.

THEOREM 31.3. *The isomorphism $z^q \leftrightarrow \phi_{z^q}$ combined with the isomorphisms establishing relations between group extensions and homomorphisms lead to the following isomorphisms*:

(31.2) $$A^q(K, G)/B^q(K, G) \cong \operatorname{Ext}\{G, \mathcal{H}_{q+1}\},$$

(31.3) $$B_w^q(K, G)/B^q(K, G) \cong \operatorname{Ext}_f\{G, \mathcal{H}_{q+1}\},$$

(31.4) $$A^q(K, G)/B_w^q(K, G) \cong \operatorname{Ext}\{G, \mathcal{T}_{q+1}\}/\operatorname{Ext}_f\{G, \mathcal{T}_{q+1}\}$$

where $\mathcal{H}_{q+1} = \mathcal{H}_{q+1}(K, I)$ and $\mathcal{T}_{q+1} = \mathcal{T}_{q+1}(K, I)$ is the corresponding co-torsion group.

The isomorphisms established so far have all been bicontinuous.

32. Computation of the homology groups

As we have shown in §29, the Kronecker index establishes a pairing of the group $H^q(K, G)$ or $H^q_w(K, G)$ with the group $\mathcal{H}_q(K, I)$, the values of the products being in the group G. Accordingly we define the following subhomology groups:

$$Q^q(K, G) = \text{Annih } \mathcal{H}^q(K, I) \text{ in } H^q(K, G), \tag{32.1}$$

$$Q^q_w(K, G) = \text{Annih } \mathcal{H}_q(K, I) \text{ in } H^q_w(K, G). \tag{32.2}$$

We verify at once that $Q^q = A^q/B^q$ and $Q^q_w = A^q/B^q_w$. Consequently the results of the last two sections furnish the following two basic theorems:

THEOREM 32.1. *For a star finite complex K the homology group $H^q(K, G)$ of infinite cycles with coefficients in a generalized topological group G can be expressed in terms of the integral cohomology groups $\mathcal{H}_q = \mathcal{H}_q(K, I)$ and $\mathcal{H}_{q+1} = \mathcal{H}_{q+1}(K, I)$ of finite cocycles. The explicit relation is*

$$H^q(K, G) \cong \text{Hom } \{\mathcal{H}_q, G\} \times \text{Ext } \{G, \mathcal{H}_{q+1}\}. \tag{32.3}$$

More explicitly, H^q has a subgroup Q^q, defined by (32.1), *where*

$$Q^q(K, G) \textit{ is a direct factor of } H^q(K, G), \tag{32.4}$$

$$Q^q(K, G) \cong \text{Ext } \{G, \mathcal{H}_{q+1}\}, \tag{32.5}$$

$$H^q(K, G)/Q^q(K, G) \cong \text{Hom } \{\mathcal{H}_q, G\}. \tag{32.6}$$

THEOREM 32.2. *For a star finite complex K the weak homology group $H^q_w(K, G)$ of infinite cycles with coefficients in a generalized topological group G can be expressed in terms of the integral cohomology group $\mathcal{H}_q = \mathcal{H}_q(K, I)$ and the integral co-torsion group $\mathcal{T}_{q+1} = \mathcal{T}_{q+1}(K, I)$ of finite cocycles. The explicit relation is*

$$H^q_w(K, G) \cong \text{Hom } \{\mathcal{H}_q, G\} \times (\text{Ext } \{G, \mathcal{T}_{q+1}\}/\text{Ext}_f \{G, \mathcal{T}_{q+1}\}). \tag{32.7}$$

More explicitly, H^q_w has a subgroup Q^q_w, defined by (32.2), *where*

$$Q^q_w(K, G) \textit{ is a direct factor of } H^q_w(K, G), \tag{32.8}$$

$$Q^q_w(K, G) \cong \text{Ext } \{G, \mathcal{T}_{q+1}\}/\text{Ext}_f \{G, \mathcal{T}_{q+1}\}, \tag{32.9}$$

$$H^q_w(K, G)/Q^q_w(K, G) \cong \text{Hom } \{\mathcal{H}_q, G\}. \tag{32.10}$$

Both factors in (32.3) and (32.7) are generalized topological groups and the isomorphisms are bicontinuous.

If G is topological then by Corollary 3.2 the group Hom $\{\mathcal{H}_q, G\}$ is topological. If we also assume that mG is a closed subgroup of G for $m = 2, 3, \cdots$ then Corollary 11.6 shows that $\text{Ext}_f \{G, \mathcal{T}_{q+1}\}$ is a closed subgroup of Ext $\{G, \mathcal{T}_{q+1}\}$. Consequently we obtain

THEOREM 32.3. (Steenrod [9]). *If G is a topological group and mG is a closed subgroup of G for $m = 2, 3, \cdots$ then $H^q_w(K, G)$ is topological.*

The expressions for Q^q and Q^q_w can be simplified if additional information concerning the group G is available. If G is infinitely divisible then, by Corollary 11.4, Ext $\{G, H\} = 0$ for all H and therefore

COROLLARY 32.4. *If G is infinitely divisible then $Q^q(K, G) = Q^q_w(K, G) = 0$ and*

$$H^q(K, G) = H^q_w(K, G) \cong \text{Hom}\ \{\mathcal{H}_q, G\}.$$

From Theorem 17.2 we deduce

COROLLARY 32.5. *If G has no elements of finite order then*

$$Q^q_w(K, G) \cong \text{Ext}\ \{G, \mathcal{T}_{q+1}\}.$$

If, in addition, G is discrete then

$$Q^q_w(K, G) \cong \text{Hom}\ \{\mathcal{T}_{q+1}, G_\infty/G\}.$$

In particular, if $G = I$ then, by Theorem 17.1, $Q^q_w(K, I) \cong \text{Char}\ \mathcal{T}_{q+1}$ and therefore

$$H^q_w(K, I) \cong \text{Hom}\ \{\mathcal{H}_q, I\} \times \text{Char}\ \mathcal{T}_{q+1}. \tag{32.11}$$

THEOREM 32.6. *If G is compact and topological then $H^q(K, G) = H^q_w(K, G)$ is compact and topological and*

$$Q^q(K, G) = Q^q_w(K, G) \cong \text{Ext}\ \{G, \mathcal{T}_{q+1}\} \cong \text{Char Hom}\ \{G, \mathcal{T}_{q+1}\}. \tag{32.12}$$

This is a consequence of Corollary 11.7 and Theorem 15.1. Since G is compact, $\mathcal{T}_{q+1}$ discrete, and only continuous homomorphisms are taken in Hom $\{G, \mathcal{T}_{q+1}\}$, it follows that in the formula (32.12) for $Q^q(K, G)$ we may replace G by G/G_0 where G_0 is the component of 0 in G.

COROLLARY 32.7. *If $\mathcal{H}_{q+1}(K, I)$ has a finite number of generators then $B^q(K, G) = B^q_w(K, G)$ and*

$$H^q(K, G) = H^q_w(K, G) \cong \text{Hom}\ \{\mathcal{H}_q, G\} \times \text{Ext}\ \{G, \mathcal{T}_{q+1}\}. \tag{32.13}$$

In fact, since $\text{Ext}_f\ \{G, \mathcal{H}_{q+1}\} = 0$ (Corollary 11.3) it follows from (31.3) that $B^q = B^q_w$. Since also $\text{Ext}_f\ \{G, \mathcal{T}_{q+1}\} = 0$, formula (32.13) follows from Theorem 32.2.

In particular, Corollary 32.7 applies if K is a finite complex (cf. Alexandroff-Hopf [1], Ch. V and Steenrod [9], p. 675).

33. Computation of the cohomology groups

We start out with a brief review of the duality between homology and cohomology. Let G be a discrete group and $\hat{G} = \text{Char}\ G$ compact and topological. Since $\hat{G}$ and G are dually paired to the group P of reals mod 1 (see §13) the Kronecker index $c^q \cdot d^q \in P$ is defined as in §29 for $c^q \in C^q(K, \hat{G})$ and $d^q \in \mathcal{C}_q(K, G)$. Since the pairing of $\hat{G}$ and G is dual (Theorem 13.5) we have by Lemma 29.1

(33.1) *$C^q(K, \hat{G})$ and $\mathcal{C}_q(K, G)$ are dually paired to P,*

$$\mathcal{Z}_q(K, G) = \text{Annih}\ B^q(K, \hat{G}); \qquad Z^q(K, \hat{G}) = \text{Annih}\ \mathcal{B}_q(K, G). \tag{33.2}$$

These formulas, Theorem 13.7, Lemma 13.2 and Theorem 13.5 imply that the Kronecker index defines a dual pairing of $\mathcal{H}_q(K, G)$ and $H^q(K, \hat{G})$ to P and that

$$\mathcal{H}_q(K, G) \cong \text{Char}\ H^q(K, \text{Char}\ G). \tag{33.3}$$

Using this result and the formulas established in the previous section for $H^q(K, \text{Char } G)$ we could write down a formula expressing $\mathcal{H}_q(K, G)$. For convenience we first define a subcohomology group

$$\mathcal{P}_q(K, G) = \text{Annih } Q^q(K, \text{Char } G) \text{ in } \mathcal{H}_q(K, G), \tag{33.4}$$

in order to get a more detailed form for our result.

THEOREM 33.1. *For a star finite complex K the cohomology group $\mathcal{H}_q(K, G)$ of finite cocycles with coefficients in a discrete group G can be expressed in terms of the cohomology group $\mathcal{H}_q = \mathcal{H}_q(K, I)$ and the integral co-torsion group $\mathcal{T}_{q+1} = \mathcal{T}_{q+1}(K, I)$. The explicit relation is*

$$\mathcal{H}_q(K, G) \cong (G \circ \mathcal{H}_q) \times \text{Hom } \{\text{Char } G, \mathcal{T}_{q+1}\}. \tag{33.5}$$

More explicitly, $\mathcal{H}_q(K, G)$ has a subgroup $\mathcal{P}_q(K, G)$, defined by (33.4), *where*

$$\mathcal{P}_q(K, G) \text{ is a direct factor of } \mathcal{H}_q(K, G), \tag{33.6}$$

$$\mathcal{P}_q(K, G) \cong G \circ \mathcal{H}_q \tag{33.7}$$

$$\mathcal{H}_q(K, G)/\mathcal{P}_q(K, G) \cong \text{Hom } \{\text{Char } G, \mathcal{T}_{q+1}\}. \tag{33.8}$$

PROOF. Since Q^q is a direct factor of H^q it follows from the character theory that $\mathcal{P}_q = \text{Annih } Q^q$ is a direct factor of $\mathcal{H}_q(K, G) = \text{Char } H^q$. It also follows that

$$\mathcal{P}_q \cong \text{Char } (H^q/Q^q), \qquad \mathcal{H}_q(K, G)/\mathcal{P}_q(K, G) \cong \text{Char } Q^q.$$

The first formula and (32.6) imply

$$\mathcal{P}_q(K, G) \cong \text{Char Hom } \{\mathcal{H}_q, \text{Char } G\},$$

which in view of Theorem 18.1 gives (33.7). The second formula combined with (32.12) proves (33.8).

If G has no elements of finite order, then Char G is connected and therefore Hom $\{\text{Char } G, \mathcal{T}_{q+1}\} = 0$. From (33.7) and (33.8) we therefore obtain

COROLLARY 33.2. *If G has no elements of finite order then*

$$\mathcal{H}_q(K, G) = \mathcal{P}_q(K, G) \cong G \circ \mathcal{H}_q(K, I).$$

We now proceed to give an intrinsic characterization of the subgroup $\mathcal{P}_q$ of $\mathcal{H}_q(K, G)$. A cocycle $w^q \in \mathcal{Z}_q(K, G)$ will be called *pure* if it is a linear combination of integral cocycles, as

$$w_q = \sum_{i=1}^{k} g_i w_i^q, \quad g_i \in G, \qquad w_i^q \in \mathcal{Z}_q(K, I).$$

LEMMA 33.3. *The group $\mathcal{P}_q(K, G)$ is the subgroup of $\mathcal{H}_q(K, G)$ determined by the pure cocycles.*

PROOF. Let $\mathcal{S}$ be the subgroup of $\mathcal{Z}_q(K, G)$ consisting of all the pure cocycles. It may be shown that $\mathcal{B}_q(K, G) \subset \mathcal{S}$. In order to prove that $\mathcal{S}/\mathcal{B}_q(K, G) = \mathcal{P}_q(K, G)$ we must prove that $\mathcal{S}/\mathcal{B}_q(K, G) = \text{Annih } Q^q(K, \hat{G})$ where $\hat{G} = \text{Char } G$.

This is equivalent to proving that $Q^q(K, \hat{G}) = \text{Annih } (\mathcal{S}/\mathcal{B}_q(K, G))$, which reduces to the formula

$$A^q(K, \hat{G}) = \text{Annih } \mathcal{S},$$

that we now propose to establish.

Let $z^q \in A^q(K, \hat{G})$ and let $w^q \in \mathcal{S}$. Since $w^q = \sum g_i w_i^q$, where $w_i^q \in \mathcal{Z}_q(K, I)$ and since $z^q \cdot w_i^q = 0$ by the definition of A^q, it follows that $z^q \cdot w^q = 0$.

Suppose now that c^q lies in $C^q(K, \hat{G})$ but not in $A^q(K, \hat{G})$. There is then a $w_1^q \in \mathcal{Z}_q(K, I)$ such that $c^q \cdot w_1^q = \hat{g} \neq 0$ where $\hat{g} \in \hat{G}$. Pick $g \in G$ so that $\hat{g}(g) \neq 0$ and define $w^q = gw_1^q$. Clearly $w^q \in \mathcal{S}$ is a pure cocycle and $c^q \cdot w^q = \hat{g}(g) \neq 0$, hence c^q is not in Annih $\mathcal{S}$. This concludes the proof of the Lemma.

Using the description of $\mathcal{P}_q(K, G)$ given in the Lemma we could easily establish the isomorphism $\mathcal{P}_q \cong G \circ \mathcal{H}_q(K, I)$ directly, using the definition of the tensor product. This was the procedure adopted by Čech [3] who essentially has proved all the results of this section. Our main improvement is that our isomorphisms are given explicitly and invariantly, while Čech used generators and relations throughout.

34. The groups H_t^q

The fact that the groups H^q and H_w^q may not be topological groups even though the coefficient group G is chosen to be topological induced Lefschetz and others to introduce the following group, for a topological coefficient group G,

$$H_t^q(K, G) = Z^q(K, G)/\bar{B}^q(K, G)$$

as a standard homology group for K.

The relation of this group to the groups previously considered is immediate:

$$H_t^q \cong H^q/\bar{0} \cong H_w^q/\bar{0}. \tag{34.1}$$

Theorem 32.3 can now be reformulated as follows.

THEOREM 34.1 (Steenrod [9]) *If G is topological and mG is closed for $m = 2, 3, \cdots$ then $H_w^q(K, G) = H_t^q(K, G)$.*

Since G is a topological group, $A^q(K, G)$ is a closed subgroup of $Z^q(K, G)$ (Lemma 29.2) and consequently $\bar{B}^q \subset A^q$. It follows that the Kronecker index can be defined for elements of $H_t^q(K, G)$ and $\mathcal{H}_q(K, I)$. We define a subhomology group

$$Q_t^q(K, G) = \text{Annih } \mathcal{H}_q(K, I) \text{ in } H_t^q(K, G). \tag{34.2}$$

THEOREM 34.2. *For a star finite complex K the topological homology group $H_t^q(K, G)$ of infinite cycles with coefficients in a topological group G can be expressed in terms of the integral cohomology group $\mathcal{H}_q = \mathcal{H}_q(K, I)$ and the integral co-torsion group $\mathcal{T}_{q+1} = \mathcal{T}_{q+1}(K, I)$ of finite cocycles. The explicit relation is*

$$H_t^q(K, G) \cong \text{Hom } \{\mathcal{H}_q, G\} \times (\text{Ext } \{G, \mathcal{T}_{q+1}\}/\bar{0}). \tag{34.3}$$

More explicitly, H_t^q *has a subgroup* Q_t^q, *defined by* (34.2), *where*

(34.3) $\quad Q_t^q(K, G)$ *is a direct factor of* $H_t^q(K, G)$,

(34.5) $\quad Q_t^q(K, G) \cong \text{Ext}\,\{G, \mathcal{T}_{q+1}\}/\bar{0}$,

(34.6) $\quad H_t^q(K, G)/Q_t^q(K, G) \cong \text{Hom}\,\{\mathcal{H}_q, G\}$.

PROOF. From the direct product decomposition (30.5) we obtain

$$H_t^q \cong (Z^q/A^q) \times [(A^q/B_w^q)/\bar{0}].$$

Consequently $Q_t^q = Q_w^q/\bar{0}$ is a direct factor. Since $Q_w^q \cong \text{Ext}\,\{G, \mathcal{T}_{q+1}\}/\text{Ext}_f\,\{G, \mathcal{T}_{q+1}\}$ and since, by Corollary 11.6, $\overline{\text{Ext}_f}\,\{G, \mathcal{T}_{q+1}\} = \bar{0}$, we obtain (34.5). Formula (34.6) follows from Theorem 30.4.

It might be interesting to notice that, while the groups $H^q(K, G)$ and $H_w^q(K, G)$ were algebraically independent of the choice of the topology in G, the group $H_t^q(K, G)$ depends both algebraically and topologically upon the topology chosen in G.

35. Universal coefficients

The results of the previous three sections can be summarized in the following fashion.

UNIVERSAL COEFFICIENT THEOREM. *In a star finite complex K the integral cohomology groups of finite cocycles determine all the homology and cohomology groups that were defined for a star finite complex, specifically*:

The groups G, $\mathcal{H}_q(K, I)$ and $\mathcal{H}_{q+1}(K, I)$ determine the generalized topological homology group $H^q(K, G)$ of infinite cycles with coefficients in a generalized topological group G.

The groups G, $\mathcal{H}_q(K, I)$ and $\mathcal{T}_{q+1}(K, I)$ determine:

(a) *the generalized topological weak homology group $H_w^q(K, G)$ of infinite cycles with coefficients in a generalized topological group G*;

(b) *the topological homology group $H_t^q(K, G)$ of infinite cycles with coefficients in a topological group G*;

(c) *the discrete cohomology group $\mathcal{H}_q(K, G)$ of finite cocycles with coefficients in a discrete group G.*

This shows that the group I of integers is a universal coefficient group for the homology theory of the complex K. Since the group P of reals mod 1 is the group of characters of I we have in view of (33.3) the fact that $\mathcal{H}_q(K, I) \cong \text{Char}\, H^q(K, P)$; therefore all the groups can be expressed in terms of $H^q(K, P)$ and $H^{q+1}(K, P)$, so that P is also universal.

Given a closed subcomplex L of K one often has to consider the relative groups of K mod L. However, the complexes used here are so general that $K - L$ is also a complex and the usual groups of K mod L coincide with the groups of $K - L$ as we have defined them. Consequently all our formulas remain valid in the relative theory.

36. Closure finite complexes

Closure finite complexes are obtained by replacing condition (26.1) in the definition of a complex by the following

(36.1) *Given* σ_i^q, $[\sigma_i^q : \sigma_k^{q-1}] \neq 0$ *for only a finite number of indices* k.

Simplicial complexes are all closure finite.

In a closure finite complex we consider finite cycles and infinite cocycles and obtain the discrete homology groups $\mathcal{H}^q(K, G)$ and the topologized cohomology groups $H_q(K, G)$, $H_q^w(K, G)$ and $H_q^t(K, G)$. All our development can be repeated with the modification of interchanging homology and cohomology groups and replacing $q + 1$ by $q - 1$. For instance formula (32.3) will take the form:

$$H_q(K, G) \cong \text{Hom}\,\{\mathcal{H}^q(K, I), G\} \times \text{Ext}\,\{G, \mathcal{H}^{q-1}(K, I)\}.$$

Instead of repeating the arguments for closure finite complexes we can use the previous results for star finite complexes and apply them to closure finite complexes by means of the concept of the dual complex. If the complex K is described by the incidence matrices A^q, the dual complex K^* will be defined by the transposed matrices

$$B^q = (A^{-q})'$$

The dual of a star finite complex is closure finite and vice versa. Also $(K^*)^* = K$. Moreover by passing from a complex to its dual, the boundary operation becomes the coboundary, and vice versa. Hence the homology and cohomology group are interchanged, and our formulas apply.

A locally finite (i.e. both closure and star finite) complex carries therefore two homology theories, namely, the theory of a star finite complex and the theory of a closure finite one. In the case of a manifold the Poincaré duality establishes a relation between the two theories. In general the theories are unrelated and in any specific problem we only use one at a time. We will quote two examples to this effect.

A) In the following chapter we define for every compact metric space a complex called the fundamental complex. This complex is locally finite, but its closure finite theory is trivial, while its star finite theory is extremely useful for the study of the underlying space.

B) Let us consider two infinite polyhedra represented as two locally finite complexes K and K'. Given a continuous mapping f of K into K' it is well known that f induces homomorphisms: 1°) of the groups of finite cycles of K into the corresponding groups of K', 2°) of the groups of infinite cocycles of K' into the corresponding groups of K. This explains why in problems connected with continuous mappings (like Hopf's mapping theorem and its generalizations; see [4]) we use only finite cycles and infinite cocycles, or in other words we use only the closure finite theory of K and K'.

CHAPTER VI. TOPOLOGICAL SPACES

Here we formulate our results for the homology groups of a space. In the case of a compact metric space, Steenrod has shown that the homology groups can all be expressed as corresponding homology groups of the fundamental complex of the space, so that the results of Chapter V apply directly (§44). For a general space, the Čech homology groups are obtained as (direct or inverse) limits, so that the decomposition of the homology group is obtained as a limit of the known decompositions for the homology groups of finite complexes, and here the techniques developed in Chapter IV apply. The results obtained for a general space are not as complete as those for complexes, partly because the limit of a set of direct sums apparently need not be a direct sum, and partly because "Lim" and "Ext" do not permute, so that the group Ext* discussed in Chap. IV is requisite. We also discuss (§45) Steenrod's homology groups of "regular" cycles.

37. Chain transformations

Let $K = \{\sigma_i^q\}$ and $K' = \{\tau_j^q\}$ be two star finite complexes. Suppose also that for every integer q there is given a matrix of integers,

$$B^q = \| b_{ij}^q \|$$

with rows indexed by the q-cells of K, columns by the q-cells of K', and with only a finite number of non-zero entries in each column.

Given a q-chain $c^q = \sum g_i \sigma_i^q \in C^q(K, G)$ in K, define

$$Tc^q = \sum_j \left(\sum_i g_i b_{ij}^q\right) \tau_j^q.$$

The column finiteness condition implies that the summation $\sum_i g_i b_{ij}^q$ is finite and therefore that Tc^q is a well defined element of $C^q(K', G)$. We thus obtain homomorphisms (one for each q and G)

$$T: \quad C^q(K, G) \to C^q(K', G).$$

Given a finite q-chain $d^q = \sum g_j \tau_j^q \in \mathcal{C}_q(K', G)$ in K', define

$$T^* d^q = \sum_i \left(\sum_j g_j b_{ij}^q\right) \sigma_i^q$$

This time the column finiteness of B^q implies that T^*d^q is finite; hence we obtain homomorphisms

$$T^*: \quad \mathcal{C}_q(K', G) \to \mathcal{C}_q(K, G).$$

T^* is called the *dual* of T.

It can be verified at once that if c^q is a chain in K and d^q is a finite chain in K' then

$$(37.1) \qquad (Tc^q) \cdot d^q = c^q \cdot (T^* d^q),$$

whenever the coefficients are such that the Kronecker index has a meaning (§29).

T is called a *chain transformation* of K into K' if $\partial Tc^q = T(\partial c^q)$ for every q chain; that is, if

$$\partial T = T\partial. \tag{37.2}$$

It can be shown that this condition is equivalent to the requirement that

$$\delta T^* = T^*\delta. \tag{37.3}$$

It follows that a chain transformation T maps the groups Z^q, A^q, B^q_w and B^q of K homomorphically into the corresponding groups of K'. Similarly T^* maps the groups of K' into the corresponding groups of K. In particular a chain transformation induces homomorphisms of the homology groups

$$T\colon\ H^q(K, G) \to H^q(K', G), \tag{37.4}$$

$$T^*\colon\ \mathcal{H}_q(K', G) \to \mathcal{H}_q(K, G), \tag{37.5}$$

and of the corresponding subgroups defined by (32.1) and (33.4)

$$T\colon\ Q^q(K, G) \to Q^q(K', G), \tag{37.6}$$

$$T^*\colon\ \mathcal{P}_q(K', G) \to \mathcal{P}_q(K, G). \tag{37.7}$$

38. Naturality

We are now in a position to give a precise meaning to the fact that the isomorphisms established in Chapter V are all "natural."

THEOREM 38.1. *If T is a chain transformation of a complex K into K', then T permutes with the isomorphisms established in Theorems* 30.4 *and* 31.2, *provided the application of T in any group is taken to mean the application of the appropriate transformation induced by T on that group.*

PROOF. If the homomorphism established in Theorem 30.4 be denoted by μ (or by μ', for K'), then we have the homomorphisms

$$\begin{array}{ccc} Z^q(K) & \xrightarrow{\mu} & \mathrm{Hom}\,\{\mathcal{H}_q, G\} \\ \downarrow T & & \downarrow T_h^{**} \\ Z^q(K') & \xrightarrow{\mu'} & \mathrm{Hom}\,\{\mathcal{H}'_q, G\}, \end{array}$$

where T_h^{**} is the homomorphism of $\mathrm{Hom}\,\{\mathcal{H}_q(K, I), G\}$ into $\mathrm{Hom}\,\{\mathcal{H}_q(K', I), G\}$, induced as in §12 by the dual chain transformation T^*. The theorem then asserts that

$$\mu' T = T_h^{**}\mu.$$

To show this, take $c^q \in Z^q(K, G)$. The corresponding homomorphism $\theta = \mu c^q$ is then defined, for each cocycle d^q in $\mathcal{Z}_q(K)$, by $\theta(d^q) = c^q \cdot d^q$ (cf. §30). Then $\theta' = T_h^{**}\theta$ is, according to the definition of T_h, simply $\theta'(d'^q) = \theta(T^*d'^q)$. Hence, for any cocycle d'^q,

$$\theta'(d'^q) = \theta(T^*d'^q) = c^q\cdot(T^*d'^q) = (Tc^q)\cdot d'^q.$$

In the other direction, Tc^q maps under μ' into the homomorphism ϕ', defined for $d'^q \in Z_q(K')$ by

$$\phi'(d'^q) = (Tc^q)\cdot d'^q.$$

The formulas show that $\phi' = \mu' Tc^q$ and $\theta' = T_h^{**}\mu c^q$ are in fact identical, as required by Theorem 38.1.

To treat Theorem 31.2, let τ (or τ') denote the homomorphism of $A^q(K, G)$ onto Hom $\{\mathcal{B}_{q+1}(K, I), G\}$ given in that theorem, while η is the map of the latter group onto Ext $\{G, \mathcal{H}_{q+1}\}$. The figure is

$$\begin{array}{ccccc} A^q & \xrightarrow[\tau]{} & \text{Hom } \{\mathcal{B}_{q+1}, G\} & \xrightarrow[\eta]{} & \text{Ext } \{G, \mathcal{H}_{q+1}\} \\ \downarrow T & & \downarrow T_h^{**} & & \downarrow T_e^{**} \\ A'^q & \xrightarrow[\tau']{} & \text{Hom } \{\mathcal{B}'_{q+1}, G\} & \xrightarrow[\eta']{} & \text{Ext } \{G, \mathcal{H}'_{q+1}\} \end{array}$$

where T_h^{**}, T_e^{**} are again the induced homomorphisms. If $z^q \in A^q(K, G)$ is given, $\phi = \tau z^q$ is defined on each coboundary δd^q as $\phi(\delta d^q) = z^q \cdot d^q$, while $\phi' = T_h^{**}\phi$ is defined in turn as

$$\phi'(\delta d'^q) = \phi(T^*\delta d'^q) = \phi(\delta T^* d'^q) = z^q\cdot(T^* d'^q).$$

On the other hand, $\chi = \tau'(Tz^q)$ is defined on a coboundary $\delta d'^q$ of K' as

$$\chi(\delta d'^q) = (Tz^q)\cdot d'^q = z^q\cdot(T^*d'^q).$$

The results are identical, so $T_h^{**}\tau = \tau' T$. Now the "naturality" theorem for group extensions showed that T permutes with η, as in $T_e^{**}\eta = \eta' T_h^{**}$. Combination of these results gives

$$(\eta'\tau')T = T_e^{**}(\eta\tau).$$

This is the required commutativity condition, for $\eta\tau$ is the isomorphism envisaged in Theorem 31.3.

39. Čech's homology groups

We now briefly outline Čech's method of defining the homology and cohomology groups for a space X. Let U_α be a finite open covering of X and N_α the nerve of U_α. If U_β is a refinement of U_α we write $\alpha < \beta$. For $\alpha < \beta$ we have a chain transformation $T_{\alpha\beta}$: $N_\beta \to N_\alpha$ defined as follows: for each open set of the covering U_β select a set of U_α containing it; this maps the vertices of N_β into the vertices of N_α and leads to a simplicial mapping $T_{\alpha\beta}$. This chain transformation is not defined uniquely, but the induced homomorphisms

$$T_{\alpha\beta}: \quad H^q(N_\beta, G) \to H^q(N_\alpha, G),$$

$$T_{\beta\alpha}^*: \quad \mathcal{H}_q(N_\alpha, G) \to \mathcal{H}_q(N_\beta, G)$$

are unique. Using the directed system of all the finite open coverings of X we define[28]

$$\mathcal{H}^q(X, G) = \varprojlim H^q(N_\alpha, G) \tag{39.1}$$

$$\mathcal{H}_q(X, G) = \varinjlim \mathcal{H}_q(N_\alpha, G). \tag{39.2}$$

In (39.2) the groups are all discrete. In (39.1) G can be any generalized topological group and $\mathcal{H}^q(X, G)$, as an inverse limit of generalized topological groups, also is a generalized topological group. If G has the property that each of its subgroups mG $(m = 2, 3 \cdots)$ is closed in G, the finiteness of each N_α implies that $H^q(N_\alpha, G)$ and hence $\mathcal{H}^q(X, G)$ is topological. If G does not have this property, it would still be possible to consider the group

$$\varprojlim H^q_t(N_\alpha, G) = \varprojlim [H^q(N_\alpha, G)/\bar{0}].$$

This group is always topological but its relation to the other groups is rather obscure.

In view of (37.6) the subgroups $Q^q(N_\alpha, G)$ of $H^q(N_\alpha, G)$ form an inverse system. We define

$$\mathcal{Q}^q(X, G) = \varprojlim Q^q(N_\alpha, G). \tag{39.3}$$

Clearly $\mathcal{Q}^q$ is a subgroup of $\mathcal{H}^q(X, G)$.

Similarly, in view of (37.7), the subgroups $\mathcal{P}_q(N_\alpha, G)$ of $\mathcal{H}_q(N_\alpha, G)$ form a direct system so we define

$$\mathcal{P}_q(X, G) = \varinjlim \mathcal{P}_q(N_\alpha, G). \tag{39.4}$$

$\mathcal{P}_q$ is a subgroup of $\mathcal{H}_q(X, G)$.

LEMMA 39.1. *The Kronecker index establishes a pairing of* $\mathcal{H}^q(X, G)$ *and* $\mathcal{H}_q(X, I)$ *with values in* G; *under this pairing*

$$\mathcal{Q}^q(X, G) = \text{Annih } \mathcal{H}_q(X, I).$$

LEMMA 39.2. *Let G be discrete and* $\hat{G} = \text{Char } G$. *The Kronecker index establishes a dual pairing of* $\mathcal{H}^q(X, \hat{G})$ *and* $\mathcal{H}_q(X, G)$ *with values in the group P of reals* mod 1; *under this pairing*

$$\mathcal{H}_q(X, G) \cong \text{Char } \mathcal{H}^q(X, \hat{G})$$

$$\mathcal{P}_q(X, G) = \text{Annih } \mathcal{Q}^q(X, \hat{G}).$$

Both lemmas have been established for each of the complexes N_α. The passage to the limit is possible in view of formula (37.1.)

In $\mathcal{H}_q(X, G)$ we also consider the subgroup $\mathcal{T}_q(X, G)$ of all elements of finite

[28] For more detail see Lefschetz [7]. Although the definition of the homology and cohomology groups given here is valid for any space X, it is well known that its interest is restricted to compact spaces only. This is due to the fact that only in compact spaces is the family of finite open coverings cofinal with the family of all open coverings.

order. Since each approximating group $\mathcal{H}_q(N_\alpha, G)$ has a finite set of generators, one can show, by arguments resembling those of §24, that

$$\mathcal{T}_q(X, G) = \underrightarrow{\mathrm{Lim}}\, \mathcal{T}_q(N_\alpha, G).$$

40. Formulas for a general space

Using the formulas for complexes and applying a straightforward passage to the limit we obtain here some relations for $\mathcal{H}^q(X, G)$ and $\mathcal{H}_q(X, G)$ in terms of the groups $\mathcal{H}_q(X, I)$ and $\mathcal{T}_{q+1}(X, I)$. The results are not as complete as in the case of a complex.

THEOREM 40.1. *For a space X and a generalized topological coefficient group G the subgroup $\mathfrak{Q}^q$ of the Čech homology group is expressible, in terms of a co-torsion group, as*

$$\mathfrak{Q}^q(X, G) \cong \mathrm{Ext}^* \{G, \mathcal{T}_{q+1}(X, I)\}, \tag{40.1}$$

while the corresponding factor group $\mathcal{H}^q(X, G)/\mathfrak{Q}^q(X, G)$ is isomorphic to a subgroup of Hom $\{\mathcal{H}_q(X, I), G\}$.

If G/mG is compact and topological for $m = 2, 3, \cdots$ then

$$\mathcal{H}^q(X, G)/\mathfrak{Q}^q(X, G) \cong \mathrm{Hom}\, \{\mathcal{H}_q(X, I), G\}. \tag{40.2}$$

PROOF. For each nerve N_α we have (Theorem 32.1)

$$Q^q(N_\alpha, G) \cong \mathrm{Ext}\, \{G, \mathcal{T}_{q+1}(N_\alpha, I)\}$$

The groups on either side form inverse systems and it follows from Theorem 38.1 and Lemma 20.2 that the limits of these systems are isomorphic,

$$\mathfrak{Q}^q(X, G) \cong \underleftarrow{\mathrm{Lim}}\, \mathrm{Ext}\, \{G, \mathcal{T}_{q+1}(N_\alpha, I)\}.$$

However since $\mathcal{T}_{q+1}(X, I) = \underrightarrow{\mathrm{Lim}}\, \mathcal{T}_{q+1}(N_\alpha, I)$ and the groups $\mathcal{T}_{q+1}(N_\alpha, I)$ are finite it follows from Theorem 24.2 that the limit on the right is Ext* $\{G, \mathcal{T}_{q+1}\}$. This proves formula (40.1).

From Theorem 32.1 we also have

$$H^q(N_\alpha, G)/Q^q(N_\alpha, G) \cong \mathrm{Hom}\, \{\mathcal{H}_q(N_\alpha, I), G\},$$

and again the limits of the two inverse systems are isomorphic in view of Theorem 38.1. Consequently from Theorem 21.1 we get

$$\underleftarrow{\mathrm{Lim}}\, [H^q(N_\alpha, G)/Q^q(N_\alpha, G)] \cong \mathrm{Hom}\, \{\mathcal{H}_q(X, I), G\}.$$

Now it follows from (20.1) (Chap. IV) that the group

$$\mathcal{H}^q(X, G)/\mathfrak{Q}^q(X, C) = \underleftarrow{\mathrm{Lim}}\, H^q_\alpha/\underleftarrow{\mathrm{Lim}}\, Q^q_\alpha$$

is isomorphic with a subgroup of the group $\underleftarrow{\mathrm{Lim}}\, (H^q_\alpha/Q^q_\alpha)$. This proves the second assertion of the theorem. The subgroup will turn out to be the whole group whenever we are able to prove that $Q^q(N_\alpha, G)$ are compact topological groups.

Suppose now that G/mG is compact and topological for $m = 2, 3, \cdots$.

Given a cyclic group T of order $m \geqq 2$ we have Ext $\{G, T\} \cong G/mG$ (Corollary 11.2) and consequently Ext $\{G, T\}$ is compact and topological. It follows that Ext $\{G, T\}$ is compact and topological for every finite group T. In particular the groups

$$Q^q(N_\alpha, G) \cong \text{Ext } \{G, \mathcal{T}_{q+1}(N_\alpha, I)\}$$

are all compact and topological.

This completes the proof of the theorem. Notice that if G/mG is compact and topological for $m = 2, 3, \cdots$ then the group $\mathfrak{Q}^q(X, G)$, as a limit of compact topological groups, is compact and topological.

If G is discrete and has no elements of finite order, or if $\mathcal{T}_{q+1}$ is countable, then by Theorem 24.4 and Corollary 24.1, the group Ext* in (40.1) may be replaced by Ext/Ext_f. In particular if $G = I$ then by Theorems 17.1 and 40.1,

$$\mathfrak{Q}^q(X, I) \cong \text{Char } \mathcal{T}_{q+1}(X, I), \tag{40.3}$$

$$\mathcal{H}^q(X, I)/\mathfrak{Q}^q(X, I) \cong \text{Hom } \{\mathcal{H}_q(X, I), I\}. \tag{40.4}$$

THEOREM 40.2. *The Čech homology group $\mathcal{H}^q(X, G)$ of a space X over a compact topological group G has a subgroup $\mathfrak{Q}^q$, with factor group $\mathcal{H}^q/\mathfrak{Q}^q$, both expressible in terms of integral cohomology groups of X as*

$$\mathfrak{Q}^q(X, G) \cong \text{Char Hom } \{G, \mathcal{T}_{q+1}(X, I)\}, \tag{40.5}$$

$$\mathcal{H}^q(X, G)/\mathfrak{Q}^q(X, G) \cong \text{Hom } \{\mathcal{H}_q(X, I), G\}. \tag{40.6}$$

PROOF. From Theorem 40.1 we have $\mathfrak{Q}^q \cong \text{Ext}^* \{G, \mathcal{T}_{q+1}\}$. However since G is compact topological we have $\text{Ext}^* \{G, \mathcal{T}_{q+1}\} \cong \text{Ext } \{G, \mathcal{T}_{q+1}\}$ (Corollary 24.1) and $\text{Ext } \{G, \mathcal{T}_{q+1}\} \cong \text{Char Hom } \{G, \mathcal{T}_{q+1}\}$ (Theorem 15.1). This proves formula (40.5). We recall here that only continuous homomorphisms are considered. Formula (40.6) is a consequence of (40.2).

THEOREM 40.3. *The Čech cohomology groups $\mathcal{H}_q \supset \mathcal{P}_q$ of a space X over a discrete coefficient group G can be expressed, in part, in terms of the integral cohomology groups as*

$$\mathcal{P}_q(X, G) \cong G \circ \mathcal{H}_q(X, I), \tag{40.7}$$

$$\mathcal{H}_q(X, G)/\mathcal{P}_q(X, G) \cong \text{Hom } \{\text{Char } G, \mathcal{T}_{q+1}(X, I)\}. \tag{40.8}$$

PROOF. Let $\hat{G} = \text{Char } G$. Since $\mathcal{H}_q(X, G) \cong \text{Char } \mathcal{H}^q(X, \hat{G})$ and $\mathcal{P}_q = \text{Annih } \mathfrak{Q}^q(X, \hat{G})$ we have

$$\mathcal{P}_q(X, G) \cong \text{Char } [\mathcal{H}^q(X, \hat{G})/\mathfrak{Q}^q(X, \hat{G})],$$

and using Theorems 40.2 and 18.1 we get

$$\mathcal{P}_q(X, G) \cong \text{Char Hom } \{\mathcal{H}_q(X, I), \text{Char } G\} \cong G \circ \mathcal{H}_q(X, I).$$

This formula could have been proved directly, passing to the limit with $\mathcal{P}_q(N_\alpha, G) \cong G \circ \mathcal{H}_q(N_\alpha, I)$. Since also $\mathcal{H}_q/\mathcal{P}_q \cong \text{Char } \mathfrak{Q}^q(X, \text{Char } G)$, formula (40.8) is a consequence of Theorem 40.2.

The theorems and proofs carry over without change to the homology theory of X modulo a closed subset. Another generalization can be obtained by replacing the space X by a net of complexes, as defined by Lefschetz ([7] Ch. VI).

We are unable to answer the question whether $\mathfrak{Q}^q(X, G)$ and $\mathcal{P}_q(X, G)$ are direct factors of $\mathcal{H}^q(X, G)$ and $\mathcal{H}_q(X, G)$. This is why we do not obtain expressions for $\mathcal{H}^q(X, G)$ and $\mathcal{H}_q(X, G)$ in terms of $\mathcal{H}_q(X, I)$ and $\mathcal{T}_{q+1}(X, I)$. The best we achieve in the case of a general space X is a description of the subgroups $\mathfrak{Q}^q$ and $\mathcal{P}_q$ and of the corresponding factor groups, leaving the direct product proposition undecided.[29]

In the following sections of this chapter we shall discuss the case when X is a compact metric space, using the method of the fundamental complex. In this case we are able to obtain complete results, including the direct product decomposition.

41. The case $q = 0$

Before we proceed with the treatment of compact metric spaces we will discuss some details connected with the definition of the homology and cohomology groups for the dimension zero.

Let K be a finite simplicial complex. If we assume that there are no cells of dimension less than zero then every 0-chain will be a 0-cycle and the groups $H^0(K, G)$ and $\mathcal{H}_0(K, G)$ will be isomorphic to the product of G by itself n times, n being the number of components of K.

An alternate procedure is to consider K "augmented" by a single (-1)-cell σ^{-1} such that $[\sigma_i^0 : \sigma^{-1}] = 1$ for all σ_i^0. In this case, given a 0-chain $c^0 = \sum g_i \sigma_i^0$, we have $\partial c^0 = (\sum g_i)\sigma^{-1}$ and consequently c^0 is a cycle if and only if $\sum g_i = 0$. The cohomology group gets affected also because the cocycle $\sum \sigma_i^0$ that was not a coboundary in the first approach is a coboundary in the augmented complex, since $\delta\sigma^{-1} = \sum \sigma_i^0$. It turns out that $H^0(K, G)$ and $\mathcal{H}_0(K, G)$ are isomorphic to the product of G by itself $n - 1$ times.

In defining the groups $\mathcal{H}^0(X, G)$ and $\mathcal{H}_0(X, G)$ for a space we again have two alternatives according as the nerves N_α are augmented or not.

Both the augmented and unaugmented complexes are abstract complexes in the sense of Ch. V and therefore all our previous results hold for either definition of $\mathcal{H}^0$ and $\mathcal{H}_0$. However in the discussion of compact metric spaces that follows there is an advantage in considering the nerves as augmented complexes, so as to have $\mathcal{H}^0(X, G) = \mathcal{H}_0(X, G) = 0$ if X is a connected space.

42. Fundamental complexes

Let X be a compact metric space. There is then a sequence U_n $(n = 0, 1, \cdots)$ of finite open coverings of X such that U_n is a refinement of U_{n-1} and every finite

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[29] Steenrod [9] §10 brings an argument, which if correct would settle the question positively. Unfortunately an error occurs on p. 681, line 5. The error was noticed by C. Chevalley, who has also constructed an example showing that the argument could not be corrected in the general case. If X is metric compact, Steenrod's argument can be corrected to give the desired direct product decomposition (see §44 below).

open covering of X has some U_n as a refinement. This last property asserts that in the directed family of all the finite open coverings of X the sequence $\{U_n\}$ constitutes a cofinal subfamily and therefore the Čech homology and cohomology group can be equivalently defined using only the sequence of coverings U_n. We shall assume that U_0 is a covering consisting of only one set, namely X itself, so that the nerve N_0 of U_0 is a vertex. For each n we select a projection $T_n:N_n \to N_{n-1}$ of the nerve of U_n into the nerve of U_{n-1}. The projections $N_n \to N_{n-k}$ we define by transitivity.

We now define the fundamental complex K of X as follows. The complexes N_n for $n = 0, 1, \cdots$ shall be disjoint subcomplexes of K. For each $n = 1, 2, \cdots$ and each simplex σ^q of N_n we introduce a new $(q + 1)$-cell $\mathfrak{D}\sigma^q$ whose boundary is $T_n\sigma^q - \sigma^q - \mathfrak{D}\partial\sigma^q$. This formula gives a recursive definition of the incidence numbers.

In order to give a more intuitive picture of K we may consider each of the nerves N_n as a geometric simplicial complex, the projection T_n can then be regarded as a continuous simplicial transformation; that is, as linear on every simplex σ^q of N_n, while $\mathfrak{D}\sigma^q$ can be visualized as a deformation prism consisting of intervals joining each point of σ^q with its image under T_n. With this interpretation K becomes a geometric complex and the cells $\mathfrak{D}\sigma^q$ can be subdivided so as to furnish a simplicial subdivision of K. It is clear from this picture that K can be contracted to a point, namely by moving every point up its projection lines towards the vertex N_0.

The complex K is countable and is locally finite; i.e., both closure and star finite. Viewing K as a closure finite complex, we can define finite cycles and infinite cocycles. However, since K is contractible all the homology group with finite cycles will vanish. Using the results of Ch. V we conclude that the cohomology groups with infinite cocycles also will vanish. Consequently, regarded as a closure finite complex, the structure of K is trivial. If we approach K as a star finite complex we obtain cohomology groups with finite cocycles and homology groups with infinite cycles. Regarded this way the complex K furnishes a true picture of the combinatorial structure of the space X.

43. Relations between a space and its fundamental complex

THEOREM 43.1. *The compact metric space X and its fundamental complex K are linked by isomorphisms*

$$\mathcal{H}^q(X, G) \cong H_w^{q+1}(K, G), \tag{43.1}$$

$$\mathcal{H}_q(X, G) \cong \mathcal{H}_{q+1}(K, G). \tag{43.2}$$

We shall restrict ourselves here to indicate the definitions of the isomorphisms without going into the complete proof, which involves lengthy but straightforward calculations.[30]

Let $\mathbf{z}^q$ be an element of $\mathcal{H}^q(X, G)$. Then $\mathbf{z}^q$ can be represented by a sequence

[30] This proof is closely related to one given by Steenrod; see [10], §4.

of cycles $z_n^q \in Z^q(N_n, G)$ such that $z_{n-1}^q - T_n z_n^q \in B^q(N_{n-1}, G)$. For each $n = 1, 2, \cdots$ select a chain c_{n-1}^{q+1} in N_{n-1} such that

$$\partial c_{n-1}^{q+1} = z_{n-1}^q - T_n z_n^q,$$

and consider the chain

$$z^{q+1} = \sum_{n=1}^{\infty} c_{n-1}^{q+1} + \sum_{n=1}^{\infty} \mathfrak{D} z_n^q.$$

We verify that

$$\begin{aligned}\partial z^{q+1} &= \sum_{n=1}^{\infty} (z_{n-1}^q - T_n z_n^q) + \sum_{n=1}^{\infty} (T_n z_n^q - z_n^q - \mathfrak{D}\partial z_n^q)\\ &= z_0^q - \sum_{n=1}^{\infty} \mathfrak{D}\partial z_n^q = 0,\end{aligned}$$

since $\partial z_n^q = 0$, while $z_0^q = 0$ for $q \geqq 0$, $z_0^0 = 0$ by §41. Consequently z^{q+1} is a cycle of K. If instead of $\{c_n^{q+1}\}$ we use a sequence $\{\bar{c}_n^{q+1}\}$ to define a cycle $\bar{z}^{q+1}$, then

$$z^{q+1} - \bar{z}^{q+1} = \sum_{n=1}^{\infty} (c_{n-1}^{q+1} - \bar{c}_{n-1}^{q+1})$$

Each term $c_{n-1}^{q+1} - \bar{c}_{n-1}^{q+1}$ is a finite cycle and therefore bounds in K, therefore $z^{q+1} - \bar{z}^{q+1}$ is a weakly bounding cycle and z^{q+1} determines uniquely an element $\mathbf{z}^{q+1} \in H_w^{q+1}(K, G)$. We define

$$\phi(\mathbf{z}^q) = \mathbf{z}^{q+1}.$$

Now let $\mathbf{w}^q \in \mathcal{H}_q(X, G)$. The element $\mathbf{w}^q$ can be represented for suitable n by a single cocycle $w^q \in \mathcal{Z}_q(N_n, G)$. We verify that $\mathfrak{D}w^q$ is then a $(q + 1)$-cocycle of K. Using the formula

$$\delta w^q = \mathfrak{D}T_n^* w^q - \mathfrak{D}w^q \text{ in } K,$$

and the fact that $\mathfrak{D}$ and δ commute we show that $\mathfrak{D}w^q$ determines uniquely an element $\mathbf{w}^{q+1}$ of $\mathcal{H}_q(K, G)$. We define

$$\psi(\mathbf{w}^q) = \mathbf{w}^{q+1}.$$

We also notice that the pair of isomorphisms ϕ, ψ preserves the Kronecker index

$$\phi(\mathbf{z}^q)\cdot\psi(\mathbf{w}^q) = \mathbf{z}^q\cdot\mathbf{w}^q. \tag{43.3}$$

If X_0 is a closed subset of X then every covering U_n of X determines a covering of X_0 whose nerve L_n is a subcomplex of the nerve N_n of U_n. The subcomplex

$$L = \sum_{n=1}^{\infty} L_n + \sum_{n=1}^{\infty} \mathfrak{D}L_n$$

of K is then a fundamental complex of X_0. The isomorphisms (43.1) and (43.2) of Theorem 43.1 can be generalized as follows

$$\mathcal{H}^q(X \bmod X_0, G) \cong H_w^{q+1}(K \bmod L, G) \tag{43.1'}$$

$$\mathcal{H}_q(X - X_0, G) \cong \mathcal{H}_{q+1}(K - L, G). \tag{43.2'}$$

44. Formulas for a compact metric space

Using the fundamental complex and the results of Ch. V we shall now establish theorems for a compact metric space quite analogous to the ones proved for a complex in Ch. V.

THEOREM 44.1. *The Čech homology groups of a compact metric space X over a generalized topological coefficient group G can be expressed in terms of the integral cohomology groups $\mathcal{H}_q = \mathcal{H}_q(X, I)$, $\mathcal{T}_{q+1} = \mathcal{T}_{q+1}(X, I)$ as*

$$\mathcal{H}^q(X, G) \cong \text{Hom}\,\{\mathcal{H}_q, G\} \times (\text{Ext}\,\{G, \mathcal{T}_{q+1}\}/\text{Ext}_f\,\{G, \mathcal{T}_{q+1}\}).$$

More precisely, in terms of the subhomology group $\mathfrak{Q}^q$ of (39.3) *we have*

$$\mathfrak{Q}^q(X, G) \text{ is a direct factor of } \mathcal{H}^q(X, G), \tag{44.1}$$

$$\mathfrak{Q}^q(X, G) \cong \text{Ext}\,\{G, \mathcal{T}_{q+1}\}/\text{Ext}_f\,\{G, \mathcal{T}_{q+1}\}, \tag{44.2}$$

$$\mathcal{H}^q(X, G)/\mathfrak{Q}^q(X, G) \cong \text{Hom}\,\{\mathcal{H}_q, G\}. \tag{44.3}$$

To prove the theorem we use the fact that the Kronecker intersection is preserved under the pair of isomorphisms ϕ, ψ of the previous section. Consequently, since

$$\mathfrak{Q}^q(X, G) = \text{Annih}\, \mathcal{H}_q(X, I) \text{ in } \mathcal{H}^q(X, G),$$
$$Q_w^{q+1}(K, G) = \text{Annih}\, \mathcal{H}_{q+1}(K, I) \text{ in } H_w^{q+1}(K, G),$$

we have

$$\phi[\mathfrak{Q}^q(X, G)] = Q_w^{q+1}(K, G),$$

and the theorem becomes a consequence of Theorems 43.1 and 32.2.

THEOREM 44.2. *The Čech cohomology groups of a compact metric space X with coefficients in a discrete group G can be expressed in terms of the integral cohomology groups $\mathcal{H}_q = \mathcal{H}_q(X, I)$, $\mathcal{T}_{q+1} = \mathcal{T}_{q+1}(X, I)$ as*

$$\mathcal{H}_q(X, G) \cong (G \circ \mathcal{H}_q) \times \text{Hom}\,\{\text{Char}\, G, \mathcal{T}_{q+1}\}.$$

More precisely, in terms of the subgroup $\mathcal{P}_q$ of (39.4), *we have*

$$\mathcal{P}_q(X, G) \text{ is a direct factor of } \mathcal{H}_q(X, G), \tag{44.4}$$

$$\mathcal{P}_q(X, G) \cong G \circ \mathcal{H}_q, \tag{44.5}$$

$$\mathcal{H}_q(X, G)/\mathcal{P}_q(X, G) \cong \text{Hom}\,\{\text{Char}\, G, \mathcal{T}_{q+1}\}. \tag{44.6}$$

To prove the theorem we notice that

$$\mathcal{P}_q(X, G) = \text{Annih } \mathfrak{Q}^q(X, \text{Char } G) \text{ in } \mathcal{H}_q(X, G),$$

$$\mathcal{P}_{q+1}(K, G) = \text{Annih } Q_w^{q+1}(K, \text{Char } G) \text{ in } \mathcal{H}_{q+1}(K, G),$$

and therefore

$$\psi[\mathcal{P}_q(X, G)] = \mathcal{P}_{q+1}(K, G)$$

and the theorem becomes a consequence of Theorems 43.1 and 33.1.

All these results remain valid for the homologies of X modulo a closed subset.

We now proceed to compare the results obtained here for the metric compact case with the results of §40 concerning general spaces.

Statements (44.1) and (44.4) contain a positive solution for the direct product problem which is still unsolved for the general space. Formula (44.3) was proved in (40.2) for general spaces only under the additional condition that G/mG be compact and topological for $m = 2, 3, \cdots$. Formula (44.2) was proved for general spaces under the form

$$\mathfrak{Q}^q(X, G) \cong \text{Ext}^* \{G, \mathcal{T}_{q+1}(X, I)\}$$

which is equivalent to (44.2) because

$$\text{Ext}^* \{G, T\} \cong \text{Ext } \{G, T\}/\text{Ext}_f \{G, T\}$$

for countable groups T with only elements of finite order (Theorem 24.4) and the group $\mathcal{T}_{q+1}(X, I) \cong \mathcal{T}_{q+2}(K, I)$ is countable for a compact metric X, since K is countable.

Formulas (44.5) and (44.6) coincide with the ones proved in Theorem 40.3 for a general space.

45. Regular cycles

Using the concept of a "regular cycle" Steenrod ([10]) has defined a new homology group $H^q(X, G)$ of "regular" cycles, for a compact metric space X. This group is useful especially in the case when X is a subset of the n-sphere S^n, because it provides information about the structure of the open set $S^n - X$.

Steenrod ([10], Theorem 7) has proved that if K denotes a fundamental complex of X then

$$H^q(X, G) \cong H^q(K, G). \tag{45.1}$$

From this, using Theorems 43.1 and 32.1 we derive the formula

$$H^q(X, G) \cong \text{Hom } \{\mathcal{H}_{q-1}(X, I), G\} \times \text{Ext } \{G, \mathcal{H}_q(X, I)\}, \tag{45.2}$$

for $q > 0$. This formula expresses $H^q(X, G)$ in terms of $\mathcal{H}_{q-1}(X, I)$ and $\mathcal{H}_q(X, I)$ and hence shows that, essentially, $H^q(X, G)$ is no *new* invariant.

Let us specialize formula (45.2), assuming that $q = 1$, and that X is connected. We have then $\mathcal{H}_0(X, I) = 0$ and therefore

$$H^1(X, G) \cong \text{Ext } \{G, \mathcal{H}_1(X, I)\}. \tag{45.3}$$

Let us further assume $G = I$ and that X is one of the solenoids Σ. Since Σ is a connected, compact abelian group we have $H^1(\Sigma, P) \cong \Sigma$ (Steenrod [9], Theorem 15) where P (Steenrod's $\mathfrak{X}$) is the group of reals mod 1. Further, since Char $I \cong P$ we have $H_1(\Sigma, I) \cong \text{Char } H^1(\Sigma, P) \cong \text{Char } \Sigma$. Hence finally

$$H^1(\Sigma, I) \cong \text{Ext}\,\{I, \text{ Char } \Sigma\}. \tag{45.4}$$

This group will be explicitly computed in Appendix B; it was the starting point of this investigation (see introduction).

Steenrod has defined a subgroup $\tilde{H}^q(X, G)$ of $H^q(X, G)$ by considering regular cycles that are sums of finite cycles. He has also proved that under the isomorphism (45.1) this group is mapped onto the subgroup $B^q_w(K, G)/B^q(K, G)$ of $H^q(K, G)$.

We shall now show that, for $q > 1$,

$$\tilde{H}^q(X, G) \cong \text{Ext}_f\,\{G, \mathcal{H}_q(X, I)\}. \tag{45.5}$$

$$H^q(X, G)/\tilde{H}^q(X, G) \cong \mathcal{H}^{q-1}(X, G). \tag{45.6}$$

In fact, from Theorems 31.3 and 43.1 we deduce that $B^q_w(K, G)/B^q(K, G) \cong \text{Ext}_f\,\{G, \mathcal{H}_{q+1}(K, I)\} \cong \text{Ext}_f\,\{G, \mathcal{H}_q(X, I)\}$. This proves (45.5). In order to prove (45.6) notice that $H^q(X, G)/\tilde{H}^q(X, G) \cong H^q(K, G)/[B^q_w(K, G)/B^q(K, G)] \cong H^q_w(K, G) \cong \mathcal{H}^{q-1}(X, G)$.

Formulas (45.5) and (45.6) provide a splitting of $H^q(X, G)$ different from the one used in (45.2). The isomorphism (45.6) was established by Steenrod [10], who has also shown that $\tilde{H}^q$ can be computed using G and $\mathcal{H}_q(X, I)$, without however getting the explicit formula (45.5).

From (45.5) we immediately deduce the theorem of Steenrod that $\tilde{H}^q(X, G) = 0$ and $H^q(X, G) \cong \mathcal{H}^{q-1}(X, G)$ whenever $\mathcal{H}_q(X, I)$ has a finite number of generators.

Appendix A. Coefficient Groups with Operators

In many topological investigations it is convenient to construct homology groups $H^q(K, G)$ in cases when G is not just a group, but a ring or even a field. More generally, G can be allowed to be a group with operators. We show here that our results extend unchanged to such cases, and in particular, that the resulting homology groups are still completely determined by the integral cohomology groups.

G is called a *group with operators* Ω if G is a generalized topological group, Ω a space, and if to each element $\omega \in \Omega$ and each $g \in G$ there is assigned an element $\omega g \in G$ (the result of operating on g with ω), in such wise that

(i) ωg is a continuous function of the pair (ω, g),

(ii) $\omega(g_1 + g_2) = \omega g_1 + \omega g_2 \qquad (g_1, g_2 \in G)$.

It then follows that each element ω determines a (continuous) homomorphism $g \to \omega g$ of G into G; however, distinct elements of Ω need not determine distinct homomorphisms. The set Ω may have a discrete topology, or may even consist of just one operator ω.

If both G_1 and G_2 have operators Ω, a homomorphism (or isomorphism) ϕ of G_1 into G_2 is said to be *Ω-allowable* if $\phi[\omega g_1] = \omega[\phi g_1]$ for all $g_1 \epsilon G_1$, $\omega \epsilon \Omega$.

If G has operators Ω, a subgroup $S \subset G$ is said to be *allowable* if $\omega(S) \subset S$ for all $\omega \epsilon \Omega$. The operators Ω may then be applied in natural fashion to the factor group G/S, by setting $\omega(g + S) = \omega g + S$. Then G/S is a group with operators Ω, and the natural homomorphism of G on G/S is allowable.[31]

If G is a group with operators Ω, the various groups introduced as functions of G in Chapters I–IV are also groups with operators. Specifically, let H be a discrete group, and for each $\theta \epsilon \text{Hom}\,\{H, G\}$ define $\omega\theta$ as $[\omega\theta](h) = \omega[\theta(h)]$. Then $\omega\theta \epsilon \text{Hom}\,\{H, G\}$, and

(A.1) $$\text{Hom}\,\{H, G\} \text{ has operators } \Omega.$$

Furthermore, if $H = F/R$, where F is free, the groups $\text{Hom}\,\{F \mid R, G\}$ and $\text{Hom}_f\,\{R, G; F\}$ are allowable subgroups of $\text{Hom}\,\{R, G\}$, so

(A.2) $$\text{Hom}\,\{R, G\}/\text{Hom}\,\{F \mid R, G\} \text{ has operators } \Omega.$$

Again, let f be a factor set of H in G, and define another factor set ωf by taking $[\omega f](h, k)$ as $\omega[f(h, k)]$. Then Ω becomes a space of operators for the group $\text{Fact}\,\{G, H\}$. Furthermore $\text{Trans}\,\{G, H\}$ is an allowable subgroup; therefore

(A.3) $$\text{Ext}\,\{G, H\} \text{ has operators } \Omega.$$

In similar fashion one concludes that $\text{Ext}_f\,\{G, H\}$ and Ext/Ext_f have operators Ω.

As another case, take $\phi \epsilon \text{Hom}\,\{G, H\}$ and define a homomorphism $\omega\phi \epsilon \text{Hom}\,\{G, H\}$ by setting $[\omega\phi](g) = \phi[\omega(g)]$ for each $g \epsilon G$. If G is compact or discrete, one may show that $\omega\phi$ is a continuous function of ω and ϕ. In this case, and for any generalized topological group H,

(A.4) $$\text{Hom}\,\{G, H\} \text{ has operators } \Omega.$$

In particular, if G is discrete or compact,

(A.5) $$\text{Char}\, G \text{ has operators } \Omega.$$

Given these interpretations of all our basic groups as groups with operators, we next demonstrate that the various isomorphisms between these groups, as established in Chapters II–IV, are allowable. In particular, an inspection of the construction used to establish the fundamental Theorem 10.1 of Chapter II proves

(A.6) *The isomorphism*

$$\text{Ext}\,\{G, H\} \cong \text{Hom}\,\{R, G\}/\text{Hom}\,\{F \mid R, G\},$$

where $H = F/R$, F free, is allowable.

[31] Practically all the elementary formal facts about groups and homomorphisms apply to operator groups and allowable homomorphisms.

The same conclusion holds for the other isomorphisms stated in that theorem. Also, the isomorphism Ext $\{G, H\} \cong$ Char Hom $\{G, H\}$ established in Theorem 15.1 for compact topological G and discrete H is allowable. The proof of this fact depends essentially on showing that the "trace" used in that theorem has the commutation property,

$$t(\omega\theta, \phi) = t(\theta, \omega\phi), \text{ for any } \theta \in \text{Hom } \{R, G\}, \text{ and } \phi \in \text{Hom } \{G, H\}.$$

The allowability of the other isomorphisms in Chapters II–IV is similarly established. The proofs are closely analogous to the "naturality" proofs of §12, except that here the operators apply to G, while in §12 the operator T applied to H.

Now turn to the homology groups. Let c^q be a chain in the star finite complex K, with coefficients chosen in the group G with operators Ω. For each $\omega \in \Omega$, define

$$\omega(c^q) = \omega(\sum_i g_i \sigma_i^q) = \sum_i (\omega g_i)\sigma_i^q;$$

since the result is a chain, and since the requisite continuity holds, the group $C^q(K, G)$ of q-chains has operators Ω. Moreover, $\omega\partial = \partial\omega$, so that both $Z^q(K, G)$ and $B^q(K, G)$ are allowable subgroups of C^q. Therefore

(A.7) $H^q(K, G)$ has operators Ω.

The essential tool in establishing the isomorphisms of Chap. V is the Kronecker index $c^q \cdot d^q$ for $d^q \in \mathcal{C}_q(K, I)$, $c^q \in C^q(K, G)$. We verify at once that

$$\omega(c^q \cdot d^q) = (\omega c^q) \cdot d^q \qquad (\text{all } \omega \in \Omega). \tag{A.8}$$

Since the subgroup A^q of Z^q was defined as a certain annihilator under this Kronecker index (see (29.9)), it follows at once that A^q is an allowable subgroup of Z^q. Furthermore, the proof that A^q is a direct factor of C^q depended on a decomposition of $\mathcal{C}_q$ as a direct product $\mathcal{C}_q = \mathcal{Z}_q \times \mathfrak{D}_q$, for a suitably chosen group $\mathfrak{D}_q$. In the notation of Lemma 16.2, we then had, by means of the Kronecker index (see the proof of Theorem 30.3)

$$C^q \cong \text{Hom } \{\mathcal{C}_q, G\} \cong \text{Hom } \{\mathcal{C}_q, G; \mathfrak{D}_q, 0\} \times \text{Hom } \{\mathcal{C}_q, G; \mathcal{Z}_q, 0\}.$$

On the right both factors are allowable subgroups, and the isomorphism to the direct product is allowable;[32] furthermore, the second factor is the one which corresponds to the subgroup A^q of C^q. Therefore C^q has a representation of the form $C^q = A^q \times D^q$, where D^q is an allowable subgroup, complementary to A^q. A similar decomposition holds for Z^q and thus for its factor group $H^q = Z^q/B^q$. In terms of the homology subgroup $Q^q = A^q/B^q$ determined by A^q, this proves

(A.9) *The isomorphism* $H^q \cong (H^q/Q^q) \times Q^q$ *is allowable.*

The further analysis of these two factors, as carried out in Chapter V, all depended on the Kronecker index. In view of the property (A.8) of this index,

[32] If A and B are two groups with operators Ω the direct product $A \times B$ has operators Ω defined by $\omega(a, b) = (\omega(a), \omega(b))$ for $\omega \in \Omega$.

and the property (A.6) of the basic group-extension theorem, we have
(A.10) *The isomorphisms*

$$H^q(K, G)/Q^q(K, G) \cong \text{Hom}\ \{\mathcal{H}_q, G\},$$
$$Q^q(K, G) \cong \text{Ext}\ \{G, \mathcal{H}_{q+1}\}$$

are allowable, as is the isomorphism $H^q \cong \text{Hom} \times \text{Ext}$, *obtained by combining* (A.9) *and* (A.10).

Similar remarks apply to the representation of the "weak" homology group H^q_w (Theorem 32.2), which is a factor group of H^q by an allowable subgroup. The same holds for the topologized homology group H^q_t (i.e., the isomorphisms of Theorem 34.2 are allowable), for in any topological group G with operators Ω, the continuity of the operators insures that the subgroup $\bar{0} \subset G$ is allowable (recall that $H^q_t = H^q/\bar{0}$).

Turn next to the analysis of the cohomology groups. The groups $\mathcal{C}_q(K, G)$, with G discrete, again have operators in Ω, under the natural definition. As in the case of the homology groups, we have

(A.11) $\mathcal{H}_q(K, G)$ has operators Ω.

The representation of these groups depended on duality; i.e., on the Kronecker index $c^q \cdot d^q$, for $c^q \in Z^q(K, \text{Char}\ G)$, $d^q \in Z^q(K, G)$. Given the various definitions of the effect of an operator ω, one shows easily that

$$(\omega c^q) \cdot d^q = c^q \cdot (\omega d^q) \qquad (\text{all } \omega \in \Omega).$$

From this formula one may deduce that the well known isomorphism $\mathcal{H}_q(K, G) \cong \text{Char}\ H^q(K, \text{Char}\ G)$ is allowable. Thence it follows that the isomorphisms of Theorem 33.1 representing $\mathcal{H}_q$ are allowable.

These considerations yield the following

ADDENDUM TO THE UNIVERSAL COEFFICIENT THEOREM. *If K is any star finite complex, G a group with operators Ω, then the homology groups of K (and, if G is discrete, the finite cohomology groups) with coefficients in G all have operators Ω. All these groups with their operators are determined by the group G (with its operators) and the cohomology groups of the finite integral cocycles of K.*

A similar discussion applies to the results of Chap. VI.

In many important cases the operators form a ring (or even a field). Let us assume then that Ω is a generalized topological ring; that is, a ring which is a generalized topological group under addition and in which the multiplication is continuous. Then G is called an *Ω-modulus* if G is a generalized topological group with operators Ω (i.e., conditions (i) and (ii) above hold) such that

(iii) $$(\omega_1\omega_2)g = \omega_1(\omega_2 g). \qquad (\text{for } \omega_i \in \Omega,\ g \in G),$$

(iv) $$(\omega_1 + \omega_2)g = \omega_1 g + \omega_2 g \qquad (\text{for } \omega_i \in \Omega,\ g \in G).$$

In other words, addition and multiplication of operators are determined in the natural fashion from G.

If the standard coefficient group G is now assumed to be an Ω-modulus, simple arguments will show that all the groups with operators Ω as described above are in fact Ω-moduli. Since the basic isomorphisms are still Ω-allowable, we conclude that the addendum to the universal coefficient group theorem still holds in these circumstances.

It is sometimes convenient to use a set Ω of operators in which only the addition or only the multiplication of operators is defined. More generally, we may consider a space Ω in which only certain sums $\omega_1 + \omega_2$ and products $\omega_1\omega_2$ are defined (and continuous); we then require that conditions (iii) and (iv) above hold only when the terms $\omega_1\omega_2$ or $\omega_1 + \omega_2$ are defined. The derived groups satisfy similar assumptions, and the universal coefficient theorem still holds.

If the coefficient group G is locally compact, one can always take the operators to form a ring, for any such group G has its endomorphism ring Ω_G as a natural ring of operators. Specifically, Ω_G is the additive group Hom $\{G, G\}$ of endomorphisms of G, with its usual topology (§3), and the multiplication $\omega_1\omega_2$ of two endomorphisms is defined by (iii) above. The requisite continuity properties of $\omega_1\omega_2$ and ωg are readily established, in virtue of the local compactness of G. Furthermore, if Ω is any other space of operators on G, each $\omega \in \Omega$ determines uniquely an endomorphism $\bar{\omega} \in \Omega_G$ with $\bar{\omega}g = \omega g$ for each g. The correspondence $\omega \longrightarrow \bar{\omega}$ is a continuous mapping of Ω into Ω_G which preserves whatever sums and products may be present in Ω (assumed to satisfy (iii) and (iv)). Thus, any group, derived from G, which is an Ω_G-modulus is also a group with operators Ω, and any isomorphism between groups which is Ω_G-allowable is Ω-allowable. This indicates, that, for locally compact groups, one may restrict attention to operators of the ring Ω_G.

The most useful case is that in which the coefficient group is a field F, which is its own ring of operators. In this case all the homology groups, groups of homomorphisms, etc., become F-moduli; that is, vector spaces over F.

All these remarks suggest the following rather negative conclusion: *although in many applications it is convenient to consider a homology theory over coefficients which form more than merely a group, no new topological invariants can be so obtained.*

Appendix B. Solenoids

Here we compute the one-dimensional homology group $H^1(\Sigma, I)$ of regular cycles for the solenoid[33] Σ, or the isomorphic group Ext $\{I, \text{Char } \Sigma\}$ (see (45.4)).

A solenoid is uniquely determined by a Steinitz G-number; that is, by a formal (infinite) product $G = \prod p_i^{e_i}$ of distinct primes with exponents e_i which are non-negative integers or ∞. Any such number G can be represented (in many ways) as a formal product $G = a_1a_2 \cdots a_n \cdots$ of ordinary integers a_i; if G is not an ordinary integer, we can take each $a_i \geqq 2$. Given such a representation of G, take replicas P_n of the additive group P of real numbers modulo 1, and let ϕ_n be the homomorphism which wraps P_n a_{n-1} times around P_{n-1}. The

[33] Solenoids were studied by L. Vietoris, Math. Annalen 97 (1927), p. 459, and more in detail by D. van Dantzig, Fundam. Math. 15 (1930), pp. 102–135. See also L. Pontrjagin [8], p. 171.

P_n then form an inverse system of groups, relative to the homomorphisms $\psi_{n+m,n} = \phi_{n+1} \cdots \phi_{n+m}$, and the solenoid Σ_G is defined as the limit $\Sigma_G = \varprojlim P_n$. Therefore Char $\Sigma_G = \varinjlim$ Char P_n, where the groups form a direct system under the dual correspondences ϕ_n^*. Here Char P_n is an isomorphic replica I_n of the additive group of integers, and ϕ_n^* maps I_n into I_{n+1} by multiplying each $x \in I_n$ by a_n. Therefore Char $\Sigma_G = \varinjlim I_n$ is a subgroup N_G of the additive group of rational numbers, consisting of all rationals of the form a/d_n, with a an integer and $d_n = a_1 \cdots a_{n-1}$. Alternatively, N_G consists of all rationals r/s with s a "divisor" of G; hence N_G and Σ_G are uniquely determined by G, and are independent of the representation $G = a_1a_2 \cdots a_n \cdots$.

A Steinitz G-number which is not an ordinary integer also determines a certain topological ring. Set $G = a_1a_2 \cdots a_n \cdots$, $d_n = a_1 \cdots a_{n-1}$. In the ring I of integers, introduce as neighborhoods of zero the sets (d_n) of all multiples of d_n. Since the intersection of all these (d_n) is the zero element of I, these neighborhoods make I a topological ring. It can be embedded in a unique fashion in a minimal complete topological ring $I_G \supset I$, so that every element of I_G is a limit of a sequence of integers, under the given topology. This is one of the b_ν-adic rings introduced by D. van Dantzig.[34] The additive group of I_G can be alternatively described as a limit of an inverse sequence; specifically, the factor group $I/(d_{n+1})$ has a natural homomorphism into $I/(d_n)$, and the limit group is $I_G \cong \varprojlim I/(d_n)$. In the special case when $G = p^\infty$ is an infinite power of a prime p, I_G is the ordinary ring of p-adic integers.

THEOREM. *If G is any Steinitz G-number which is not an ordinary integer, Σ_G the corresponding solenoid, and I_G the corresponding complete ring containing the ring I of integers, then*

$$\text{(B.1)} \qquad \operatorname{Ext}\{I, \operatorname{Char} \Sigma_G\} \cong I_G/I.$$

PROOF. As above, Char Σ_G is a group N_G of rationals, generated by the numbers $r_n = 1/d_n$ with relations $a_nr_{n+1} = r_n$. Therefore N_G can be represented as F/R, where F is a free group with generators $z_1, z_2, \cdots$, and R the subgroup with generators $y_n = a_nz_{n+1} - z_n$, $n = 1, 2, \cdots$. By the fundamental theorem on group extensions

$$\text{(B.2)} \qquad \operatorname{Ext}\{I, \operatorname{Char} \Sigma_G\} \cong \operatorname{Hom}\{R, I\}/\operatorname{Hom}\{F \mid R, I\}.$$

Let $\theta \in \operatorname{Hom}\{R, I\}$ and set

$$x(\theta) = \operatorname{Lim}_{n \to \infty} [\theta y_1 + d_2\theta y_2 + \cdots + d_n\theta y_n].$$

Then $x(\theta)$ is a well-defined element of I_G, and $\theta \to x(\theta)$ is a homomorphic mapping of Hom $\{R, I\}$ into I_G and thus, derivatively, into I_G/I. We assert that the kernel of the latter mapping is Hom $\{F \mid R, I\}$.

Assume first that $\theta \in \operatorname{Hom}\{F \mid R, I\}$, and let θ^* be an extension of θ to F. Then

$$\theta(y_1 + d_2y_2 + \cdots + d_ny_n) = -\theta^*z_1 + d_{n+1}\theta^*z_{n+1},$$

[34] Math. Annalen 107 (1932), pp. 587–626; Compositio Math. 2 (1935), pp. 201–223.

so that the limit $x(\theta)$ is $-\theta^* z_1$, which is an integer in I. Conversely, suppose that $x(\theta) \epsilon I$, and set $x(\theta) = -c_1$. We then have

$$\theta y_1 + d_2 \theta y_2 + \cdots + d_n \theta y_n \equiv -c_1 \pmod{d_{n+1}}.$$

By successive applications of this condition we find integers c_n with $\theta y_n = a_n c_{n+1} - c_n$. The homomorphism $\theta^* z_n = c_n$ then provides an extension of θ to F, so that $\theta \epsilon \operatorname{Hom} \{F \mid R, I\}$.

Every element in I_G is a limit of integers, hence has the form Lim $[b_1 + d_2 b_2 + \cdots + d_n b_n]$; therefore $\theta \rightarrow x(\theta)$ is a mapping onto I_G. We thus have

(B.3) $$\operatorname{Hom} \{R, I\}/\operatorname{Hom} \{F \mid R, I\} \cong I_G/I.$$

The correspondence is topological, as one may readily verify that both (generalized topological) groups carry the trivial topology in which the only open sets are zero and the whole space. Thus (B.2) and (B.3) prove the isomorphism (B.1).

By cardinal number considerations, one shows that the group I_G/I is uncountable, hence not void. The formula (B.1) gives at once all the special properties of the homology group of the solenoid, as found by Steenrod [10] in his partial determination of this group.

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