

Isomorphism of categories

In category theory, two categories C and D are **isomorphic** if there exist functors $F : C \rightarrow D$ and $G : D \rightarrow C$ which are mutually inverse to each other, i.e. $FG = 1_D$ (the identity functor on D) and $GF = 1_C$.^[1] This means that both the objects and the morphisms of C and D stand in a one-to-one correspondence to each other. Two isomorphic categories share all properties that are defined solely in terms of category theory; for all practical purposes, they are identical and differ only in the notation of their objects and morphisms.

Isomorphism of categories is a very strong condition and rarely satisfied in practice. Much more important is the notion of equivalence of categories; roughly speaking, for an equivalence of categories we don't require that FG be *equal* to 1_D , but only *naturally isomorphic* to 1_D , and likewise that GF be naturally isomorphic to 1_C .

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Properties

As is true for any notion of isomorphism, we have the following general properties formally similar to an equivalence relation:

- any category C is isomorphic to itself
- if C is isomorphic to D , then D is isomorphic to C
- if C is isomorphic to D and D is isomorphic to E , then C is isomorphic to E .

A functor $F : C \rightarrow D$ yields an isomorphism of categories if and only if it is bijective on objects and on morphism sets.^[1] This criterion can be convenient as it avoids the need to construct the inverse functor G . (We use "bijection" informally here because, if a category is not concrete, we don't have such a notion.)

Examples

- Consider a finite group G , a field k and the group algebra kG . The category of k -linear group representations of G is isomorphic to the category of left modules over kG . The isomorphism can be described as follows: given a group representation $\rho : G \rightarrow \text{GL}(V)$, where V is a vector space over k , $\text{GL}(V)$ is the group of its k -linear automorphisms, and ρ is a group homomorphism, we turn V into a left kG module by defining

$$\left(\sum_{g \in G} a_g g \right) v = \sum_{g \in G} a_g \rho(g)(v)$$

for every v in V and every element $\sum a_g g$ in kG . Conversely, given a left kG module M , then M is a k vector space, and multiplication with an element g of G yields a k -linear automorphism of M (since g is invertible in kG), which describes a group homomorphism $G \rightarrow \text{GL}(M)$. (There are still several things to check: both these assignments are functors, i.e. they can be applied to maps between group representations resp. kG modules, and they are inverse to each other, both on objects and on morphisms). See also [Representation theory of finite groups#Representations, modules and the convolution algebra](#).

- Every [ring](#) can be viewed as a [preadditive category](#) with a single object. The [functor category](#) of all additive functors from this category to the [category of abelian groups](#) is isomorphic to the category of left modules over the ring.
- Another isomorphism of categories arises in the theory of Boolean algebras: the category of Boolean algebras is isomorphic to the category of [Boolean rings](#). Given a Boolean algebra B , we turn B into a Boolean ring by using the [symmetric difference](#) as addition and the meet operation \wedge as multiplication. Conversely, given a Boolean ring R , we define the join operation by $a \vee b = a + b + ab$, and the meet operation as multiplication. Again, both of these assignments can be extended to morphisms to yield functors, and these functors are inverse to each other.
- If C is a category with an initial object s , then the slice category $(s \downarrow C)$ is isomorphic to C . Dually, if t is a terminal object in C , the functor category $(C \downarrow t)$ is isomorphic to C . Similarly, if $\mathbf{1}$ is the category with one object and only its identity morphism (in fact, $\mathbf{1}$ is the [terminal category](#)), and C is any category, then the functor category $C^{\mathbf{1}}$, with objects functors $c: \mathbf{1} \rightarrow C$, selecting an object $c \in \text{Ob}(C)$, and arrows natural transformations $f: c \rightarrow d$ between these functors, selecting a morphism $f: c \rightarrow d$ in C , is again isomorphic to C .

See also

- [Equivalence of categories](#)

References

1. Mac Lane, Saunders (1998). *Categories for the Working Mathematician*. Graduate Texts in Mathematics. 5 (2nd ed.). Springer-Verlag. p. 14. ISBN 0-387-98403-8. MR 1712872 (<https://www.ams.org/mathscinet-getitem?mr=1712872>).

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This page was last edited on 28 January 2019, at 03:39 (UTC).

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