category theory

Contents

Context

Category theory

Mathematics

1. Idea

Category theory is a toolset for describing the general abstract structures in mathematics.

Paradigm. As opposed to set theory, category theory focuses not on elements $x, y, \cdots$ – called objects – but on the ***relations*** between these objects: the (homo)morphisms between them

$$x \xrightarrow{f} y \,.$$

Later this will lead naturally on to an infinite sequence of steps: first 2-category theory which focuses on relation between relations, morphisms between morphisms: 2-morphisms, then 3-category theory, etc. and to various variants, bicategories, Gray categories …. Eventually this leads to higher category theory, where one considers k-morphisms in all dimensions and to a wealth of interacting intuitions and concepts.

The concept that formalizes this is that of a category: a collection of arrows/morphism that can be composed if they are adjacent

$$\left(\begin{array}{c} x \\ \downarrow^f \\ y \\ \downarrow^g \\ z \end{array} \right) \mapsto \left(\begin{array}{c} x \\ \downarrow^{g \circ f} \\ z \end{array} \right) .$$

Examples. The archetypical example of a category is the category Set of sets and functions between sets.

The classical examples of categories are concrete categories whose objects are sets with extra structure and whose morphisms are structure preserving functions of sets, such as Top, Grp, Vect. These are the examples from which the term *category* derives: these categories literally *categorize* mathematical structures by packing structures of the same *type* (same category) and structure preserving mappings between them into a single whole structure, a category.

But it is far from the case that all categories are of this type. Categories are much more versatile than these classical examples suggest. After all, a category is just a quiver (a directed graph) with a notion of composition of its edges. As such it generalizes the concepts of monoid and poset. If the category is a groupoid, it generalizes the concept of group (in a sense called horizontal categorification) and also the concept of equivalence relation. Thinking of a category as a generalized poset is particularly useful when studying limits and adjunctions.

Archetypical examples of non-concrete categories are the fundamental groupoid of a topological space and the fundamental category of a directed space.

Terminology. Categories were named after the examples of concrete categories. As Saunders Mac Lane writes

> Now the discovery of ideas as general as these is chiefly the willingness to make a brash or speculative abstraction, in this case supported by the pleasure of purloining words from the philosophers: "Category" from Aristotle and Kant, "Functor" from Carnap (Logische Syntax der Sprache), and "natural transformation" from then current informal parlance.
>
> (Saunders Mac Lane, Categories for the Working Mathematician, 29–30).

However, the categories of category theory are way more general than these concrete categories, and the way Aristotle and Kant use the term in philosophy is not particularly related to what Eilenberg & Mac Lane did with it.

The basic trinity of concepts. Category theory reflects on itself. Categories are about collections of morphisms. And there are evident morphisms between categories: functors. And there are evident morphisms between functors: natural transformations.

This trinity of concepts

1. category
2. functor
3. natural transformation

is what category theory is built on.

In higher category theory this continues with

- k-transfors for all $k \in \mathbb{N}$.

Conceptual unification. A major driving force behind the development of category theory is its ability to abstract and unify concepts. General statements about categories apply to each specific concrete category of mathematical structures. The general notion of universal constructions in categories, such as representable functors, adjoint functors and limits, turns out to prevail throughout mathematics and manifest itself in myriads of special examples.

Abstract nonsense. This abstraction power of category theory has led Norman Steenrod to coin the term *abstract nonsense* or *general abstract nonsense* for it. It is being used as in "This property is not specific to this context, it already follows from general abstract nonsense". Peter Freyd expressed a similar feeling by his witticism:

> "Perhaps the purpose of categorical algebra is to show that which is trivial is trivially trivial."

But abstract nonsense still tends to meet with some resistance. In the preface of his 1965 book *Theory of Categories* Barry Mitchell writes:

> A number of sophisticated people tend to disparage category theory as consistently as others disparage certain kinds of classical music. When obliged to speak of a category they do so in an apologetic tone, similar to the way some say, "It was a gift – I've never even played it" when a record of Chopin Nocturnes is discovered in their possession. For this reason I add to the usual prerequisite that the reader have a fair amount of mathematical sophistication, the further prerequisite that he have no other kind.

The nPOV. The vast applicability and expressiveness of category theory leads to the observation that most structures in mathematics are best understood from a category theoretic or higher category theoretic viewpoint. This is the nPOV.

2. The central constructions

Presheaves

Much of the power of category theory rests in the fact that it reflects on itself. For instance that functors between two categories form themselves a category: the functor category.

This leads to the notion of presheaf categories and sheaf toposes. Much of category theory is topos theory.

Under Isbell duality this sets the stage for everything in mathematics related to space and algebra and their duality.

Universal constructions

Elementary as it is, the definition of a category supports a powerful set of constructions: universal constructions. These include

- limit and colimit;
- adjunction;
- Kan extension
- end and coend.

All these are special cases of each other and thus reflect different aspect of one single phenomenon. Applying category theory means applying these constructions in specific situations and using general abstract theorems for deducing statements about concrete contexts.

3. The central theorems

Category theory has a handful of central lemmas and theorems. Their proof is typically easy, sometimes almost tautological. Their power rests in the fact that they apply over and over again all over mathematics. Many concrete constructions get simplified by observing that they are but special realizations of these general abstract results in category theory. Among these central theorems are

- first and foremost: the Yoneda lemma;
- Isbell duality;
- the Grothendieck construction;
- the adjoint functor theorem;
- the monadicity theorem;
- Tannaka duality.

4. Applications

For a detailed list of applications see

- applications of (higher) category theory.

In pure mathematics

Apart from its general role in mathematics, category theory provides the high-level language for

- logic / type theory
- higher algebra
- higher geometry .

Outside of mathematics

Outside of pure mathematics, category theory finds major applications in

- fundamental physics – see higher category theory and physics
- theoretical computer science

and is increasingly finding applications in such diverse areas as chemistry, network theory and natural language processing. Further information can be found on the applied category theory page.

5. Contrast to theories of other objects

Category theory vs. set theory

Here set theory is assumed to be a theory of the usual concept of sets, that is *material* set theory.

No one of these is more fundamental than the other as a foundation of mathematics. Category theory is a holistic (structural) approach to mathematics that can (through such methods as Lawvere's ETCS) provide foundations of mathematics and (through algebraic set theory) reproduce all the different axiomatic set theories; elementary category theory does not need the concept of set to be formulated. Set theory is an analytic approach (element-wise) and can reproduce category theory by simply defining all the concepts in the usual way, as long as one include a technique to handle large categories (for instance by using classes instead of sets, or by including as an axiom that an uncountable inaccessible cardinal exists or even that Grothendieck universes exist).

Set theory		**Category theory**
membership relation		-
sets		categories
elements		objects
-		morphisms
functions		functors
equations between elements		isomorphisms between objects
equations between sets		equivalences between categories
equations between functions		natural transformations between functors

Lawvere pointed out that set theory is axiomatized by a binary membership relation while category theory is axiomatized by a ternary composition relation.

The process of going from sets to categories is a special case of categorification and the reverse process is a special case of decategorification.

For a philosophical consideration of foundations covering and comparing sets, structuralism (a la Bourbaki?) and categories, see the article

- Sets, categories and structuralism - Costas Drossos

Category theory vs. order theory

A category may be thought of as a categorification of a poset rather than of a set; much (but by no means all) of category theory also appears in order theory.

See category theory vs order theory for more discussion.

6. Theorems

- Yoneda lemma
- Isbell duality
- Grothendieck construction
- adjoint functor theorem
- monadicity theorem
- adjoint lifting theorem
- Tannaka duality
- Gabriel-Ulmer duality
- small object argument
- Freyd-Mitchell embedding theorem

7. Related concepts

- type theory
  - relation between category theory and type theory
- computer science
  - computational trinitarianism

higher category theory

- (0,1)-category theory
- 1-category theory, (∞,1)-category theory
- 2-category theory, (∞,2)-category theory
- (∞,n)-category theory

Some theorems in category theory are *folklore*.

8. References

History

The concepts of *category*, *functor* and *natural transformation* were introduced in

- Samuel Eilenberg, Saunders MacLane, *General Theory of Natural Equivalences*, Transactions of the American Mathematical Society Vol. 58, No. 2 (Sep., 1945), pp. 231-294 (JSTOR)

The reason for introducing categories was to introduce functors, and the reason for introducing functors was to introduce natural transformations (more specifically natural equivalences) in order to define what *natural* means in mathematics:

> If topology were publicly defined as the study of families of sets closed under finite intersection and infinite unions a serious disservice would be perpetrated on embryonic students of topology. The mathematical correctness of such a definition reveals nothing about topology except that its basic axioms can be made quite simple. And with category theory we are confronted with the same pedagogical problem. The basic axioms, which we will shortly be forced to give, are much too simple.
>
> A better (albeit not perfect) description of topology is that it is the study of continuous maps; and category theory is likewise better described as the theory of functors. Both descriptions are logically inadmissible as initial definitions, but they more accurately reflect both the present and the historical motivations of the subjects.
>
> It is not too misleading, at least historically, to say that categories are what one must define in order to define natural transformations.
>
> (from Freyd 64, page 1)

and

> Category theory is an embodiment of Klein's dictum that it is the maps that count in mathematics. If the dictum is true, then it is the functors between categories that are important, not the categories. And such is the case. Indeed, the notion of category is best excused as that which is necessary in order to have the notion of functor. But the progression does not stop here. There are maps between functors, and they are called natural transformations. And it was in order to define these that Eilenberg and MacLane first defined functors.
>
> (from Freyd 65, beginning of Part Two)

The paper Eilenberg-Maclane 45 was a clash of ideas from abstract algebra (Mac Lane) and topology/homotopy theory (Eilenberg). It was first rejected on the ground that it had no content but was later published. Since then category theory has flourished into almost all areas of mathematics, has found many applications outside mathematics and even attempts to build a foundations of mathematics.

This and much more history is recalled in

- Saunders MacLane, *Concepts and Categories in Perspective*, AMS (pdf)

Textbooks

Basic category theory

- Steve Awodey, *Category theory*.
- Colin McLarty, *Elementary categories, elementary toposes*.
- Peter Hilton (ed.) *Category Theory, Homology Theory and Their Applications III*, volume 99 of Lecture Notes in Mathematics (1969), Springer-Verlag Berlin-Heidelberg-New York.
- Masaki Kashiwara, Pierre Schapira, *Categories and Sheaves*
- Francis Borceux, *Handbook of Categorical Algebra*, vol 1–3.
- Saunders MacLane, *Lectures on category theory, Bowdoin Summer School 1969, Notes taken by Ellis Cooper*
- Saunders Mac Lane, *Categories for the working mathematician*, 2nd ed.
- Jiri Adamek, Horst Herrlich, George Strecker, *Abstract and concrete categories: the joy of cats* (pdf)
- Benjamin Pierce, *Basic category theory for computer scientists*.
- Michael Barr, Charles Wells, *Category theory for computing science*. free online, also in TAC Reprints.
- Bodo Pareigis, *Categories and functors*
- Peter Freyd, *Abelian Categories – An Introduction to the theory of functors*, originally published by Harper and Row, New York(1964), Reprints in Theory and Applications of Categories, No. 3, 2003 (TAC, pdf)
- Peter Freyd, *The theories of functors and models*, in Proceedings of *Symposium on the Theory of Models*, North Holland, 1965 (web)
- Peter Freyd and Andre Scedrov, *Categories, Allegories*, Mathematical Library Vol 39, North-Holland (1990). ISBN 978-0-444-70368-2.
- F. William Lawvere and Stephen Schanuel, *Conceptual Mathematics: A first introduction to categories*, 2nd Edition, Cambridge University Press 2009 (pdf)
- David Spivak, *Category theory for scientists* (arXiv:1302.6946)
- Tom Leinster, *Basic Category Theory*, 2014
- Emily Riehl, *Category theory in context*, 2016 (pdf)
- Brendan Fong, David Spivak, *An invitation to applied category theory*, 2018 (web, pdf)
- Robert Geroch, *Mathematical Physics*, Chicago 1985 (introduces categories by examples arising in mathematical physics)
- Marco Grandis, *Category Theory and Applications: A Textbook for Beginners*, 2018

Topos theory

Monographs on topos theory:

- Ieke Moerdijk, Saunders Mac Lane, *Sheaves in Geometry and Logic* (for which the *n*Lab currently provides detailed linked indexes)
- Peter Johnstone, *Topos theory*, 1977.
- Peter Johnstone, *Sketches of an Elephant*, vol 1–2 (3 forthcoming).
- Robert Goldblatt, *Topoi, the categorial analysis of logic*.
  free online
- Michael Barr and Charles Wells, *Toposes, triples and theories*. TAC Reprints
  (Here "triple" means" monad").

Higher category theory

- Carlos Simpson, *Homotopy Theory of Higher Categories* (pdf)
- Project description: higher categorical structures and their applications (pdf)
- Jacob Lurie, *Higher Topos Theory* (pdf)
- Tom Leinster, *Higher operads, higher categories*, math.CT/0305049
- Eugenia Cheng, Aaron Lauda, *Higher-dimensional categories: an illustrated guide book* free online

Towards homotopy theory:

- Birgit Richter, *From categories to homotopy theory* Cambridge studies in advanced mathematics, 2019 (webpage)

Foundations

The foundation of category theory in homotopy type theory (see at *internal category in homotopy type theory*) is discussed in

- Benedikt Ahrens, Chris Kapulkin, Michael Shulman, *Univalent categories and the Rezk completion* (arXiv:1303.0584)

Course notes

- Peter Johnstone, *Category Theory* , Lecture notes taken by David Mehrle, University of Cambridge 2015. (pdf)
- Tom Leinster, *Notes on basic category theory*
- Jaap van Oosten, *Basic category theory* (pdf)
- Bodo Pareigis, *Category theory*, dvi, ps, pdf
- Paolo Perrone, *Notes on Category Theory with examples from basic mathematics*. (arXiv)
- David Spivak, *Category theory for scientists* (arXiv:1302.6946)

Videos

- The Catsters, Videos on various topics in category theory. (YouTube link

Videos at an introductory level that cover basic concepts and constructions of category theory. The Catsters are Eugenia Cheng and Simon Willerton (anyone else?).

- Tom LaGatta, *Category theory* (video)

Enthusiastic, mostly nontechnical talk given by a probability theorist, made for an audience innocent of any exposure to category theory.

Relation to philosophy

Discussion of the relation to and motivation from the philosophy of mathematics includes

- Colin McLarty, *The Last Mathematician from Hilbert's Göttingen: Saunders Mac Lane as Philosopher of Mathematics*,Brit. J. Phil. Sci. 2007 (pdf)

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