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Morphism

In mathematics, particularly in category theory, a **morphism** is a structure-preserving map from one mathematical structure to another one of the same type. The notion of morphism recurs in much of contemporary mathematics. In set theory, morphisms are functions; in linear algebra, linear transformations; in group theory, group homomorphisms; in topology, continuous functions, and so on.

In category theory, *morphism* is a broadly similar idea: the mathematical objects involved need not be sets, and the relationships between them may be something other than maps, although the morphisms between the objects of a given category have to behave similarly to maps in that they have to admit an associative operation similar to function composition. A morphism in category theory is an abstraction of a homomorphism.[1]

The study of morphisms and of the structures (called "objects") over which they are defined is central to category theory. Much of the terminology of morphisms, as well as the intuition underlying them, comes from concrete categories, where the *objects* are simply *sets with some additional structure*, and *morphisms* are *structure-preserving functions*. In category theory, morphisms are sometimes also called **arrows**.

Contents

Definition

A category C consists of two classes, one of *objects* and the other of *morphisms*. There are two objects that are associated to every morphism, the *source* and the *target*. A morphism f with source X and target Y is written $f : X \to Y$, and is represented diagrammatically by an *arrow* from X to Y.

For many common categories, objects are sets (often with some additional structure) and morphisms are functions from an object to another object. Therefore, the source and the target of a morphism are often called *domain* and *codomain* respectively.

Morphisms are equipped with a partial binary operation, called *composition*. The composition of two morphisms f and g is defined precisely when the target of f is the source of g, and is denoted $g \circ f$ (or sometimes simply gf). The source of $g \circ f$ is the source of f, and the target of $g \circ f$ is the target of g. The composition satisfies two axioms:

Identity

For every object X, there exists a morphism $\mathrm{id}_X : X \to X$ called the **identity morphism** on X, such that for every morphism $f : A \to B$ we have $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$.

Associativity

$h \circ (g \circ f) = (h \circ g) \circ f$ whenever all the compositions are defined, i.e. when the target of f is the source of g, and the target of g is the source of h.

For a concrete category (a category in which the objects are sets, possibly with additional structure, and the morphisms are structure-preserving functions), the identity morphism is just the identity function, and composition is just ordinary composition of functions.

The composition of morphisms is often represented by a commutative diagram. For example,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g \circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

The collection of all morphisms from X to Y is denoted $\mathrm{Hom}_C(X,Y)$ or simply $\mathrm{Hom}(X, Y)$ and called the **hom-set** between X and Y. Some authors write $\mathrm{Mor}_C(X,Y)$, $\mathrm{Mor}(X, Y)$ or $\mathrm{C}(X, Y)$. Note that the term hom-set is something of a misnomer, as the collection of morphisms is not required to be a set. A category where $\mathrm{Hom}(X, Y)$ is a set for all objects X and Y is called locally small.

Note that the domain and codomain are in fact part of the information determining a morphism. For example, in the category of sets, where morphisms are functions, two functions may be identical as sets of ordered pairs (may have the same range), while having different codomains. The two functions are distinct from the viewpoint of category theory. Thus many authors require that the hom-classes $\mathrm{Hom}(X, Y)$ be disjoint. In practice, this is not a problem because if this disjointness does not hold, it can be assured by appending the domain and codomain to the morphisms (say, as the second and third components of an ordered triple).

Some special morphisms

Monomorphisms and epimorphisms

A morphism f: $X \to Y$ is called a monomorphism if $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$ for all morphisms g_1, g_2: $Z \to X$. A monomorphism can be called a *mono* for short, and we can use *monic* as an adjective.[2]

- A morphism f has a **left inverse** if there is a morphism g: $Y \to X$ such that $g \circ f = \mathrm{id}_X$. The left inverse g is also called a **retraction** of f.[2] Morphisms with left inverses are always monomorphisms, but the converse is not true in general; a monomorphism may fail to have a left inverse.
- A **split monomorphism** h: $X \to Y$ is a monomorphism having a left inverse g: $Y \to X$, so that $g \circ h = \mathrm{id}_X$. Thus $h \circ g$: $Y \to Y$ is idempotent; that is, $(h \circ g)^2 = h \circ (g \circ h) \circ g = h \circ g$.

- In concrete categories, a function that has a left inverse is injective. Thus in concrete categories, monomorphisms are often, but not always, injective. The condition of being an injection is stronger than that of being a monomorphism, but weaker than that of being a split monomorphism.

Dually to monomorphisms, a morphism $f: X \to Y$ is called an epimorphism if $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$ for all morphisms $g_1, g_2: Y \to Z$. An epimorphism can be called an *epi* for short, and we can use *epic* as an adjective.[2]

- A morphism f has a **right inverse** if there is a morphism $g: Y \to X$ such that $f \circ g = \mathrm{id}_Y$. The right inverse g is also called a **section** of f.[2] Morphisms having a right inverse are always epimorphisms, but the converse is not true in general, as an epimorphism may fail to have a right inverse.
- A **split epimorphism** is an epimorphism having a right inverse. If a monomorphism f splits with left inverse g, then g is a split epimorphism with right inverse f.
- In concrete categories, a function that has a right inverse is surjective. Thus in concrete categories, epimorphisms are often, but not always, surjective. The condition of being a surjection is stronger than that of being an epimorphism, but weaker than that of being a split epimorphism. In the category of sets, the statement that every surjection has a section is equivalent to the axiom of choice.

A morphism that is both an epimorphism and a monomorphism is called a **bimorphism**.

Isomorphisms

A morphism $f: X \to Y$ is called an isomorphism if there exists a morphism $g: Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. If a morphism has both left-inverse and right-inverse, then the two inverses are equal, so f is an isomorphism, and g is called simply the **inverse** of f. Inverse morphisms, if they exist, are unique. The inverse g is also an isomorphism, with inverse f. Two objects with an isomorphism between them are said to be isomorphic or equivalent.

While every isomorphism is a bimorphism, a bimorphism is not necessarily an isomorphism. For example, in the category of commutative rings the inclusion $\mathbf{Z} \to \mathbf{Q}$ is a bimorphism that is not an isomorphism. However, any morphism that is both an epimorphism and a *split* monomorphism, or both a monomorphism and a *split* epimorphism, must be an isomorphism. A category, such as **Set**, in which every bimorphism is an isomorphism is known as a **balanced category**.

Endomorphisms and automorphisms

A morphism $f: X \to X$ (that is, a morphism with identical source and target) is an endomorphism of X. A **split endomorphism** is an idempotent endomorphism f if f admits a decomposition $f = h \circ g$ with $g \circ h = \mathrm{id}$. In particular, the Karoubi envelope of a category splits every idempotent morphism.

An automorphism is a morphism that is both an endomorphism and an isomorphism. In every category, the automorphisms of an object always form a group, called the automorphism group of the object.

Examples

- In the concrete categories studied in universal algebra (groups, rings, modules, etc.), morphisms are usually homomorphisms. Likewise, the notions of automorphism, endomorphism, epimorphism, homeomorphism, isomorphism, and monomorphism all find use in universal algebra.

- In the category of topological spaces, morphisms are continuous functions and isomorphisms are called homeomorphisms.
- In the category of smooth manifolds, morphisms are smooth functions and isomorphisms are called diffeomorphisms.
- In the category of small categories, the morphisms are functors.
- In a functor category, the morphisms are natural transformations.

For more examples, see the entry category theory.

See also

- Normal morphism
- Zero morphism

Notes

1. "morphism" (https://ncatlab.org/nlab/show/morphism). nLab. Retrieved 2019-06-12.
2. Jacobson (2009), p. 15.

References

- Jacobson, Nathan (2009), *Basic algebra*, **2** (2nd ed.), Dover, ISBN 978-0-486-47187-7.
- Adámek, Jiří; Herrlich, Horst; Strecker, George E. (1990). *Abstract and Concrete Categories* (http://katmat.math.uni-bremen.de/acc/acc.pdf) (PDF). John Wiley & Sons. ISBN 0-471-60922-6. Now available as free on-line edition (4.2MB PDF).

External links

- Hazewinkel, Michiel, ed. (2001) [1994], "Morphism" (https://www.encyclopediaofmath.org/index.php?title=p/m064940), *Encyclopedia of Mathematics*, Springer Science+Business Media B.V. / Kluwer Academic Publishers, ISBN 978-1-55608-010-4
- "Category" (https://planetmath.org/?op=getobj&from=objects&id=965). *PlanetMath*.
- "TypesOfMorphisms" (https://planetmath.org/?op=getobj&from=objects&id=8114). *PlanetMath*.