higher category theory

Context

Higher category theory

Contents

1. Idea

Higher category theory is the generalization of category theory to a context where there are not only morphisms between objects, but generally k-morphisms between $(k-1)$-morphisms, for all $k \in \mathbb{N}$.

Higher category theory studies the generalization of ∞-groupoids – and hence, via the homotopy hypothesis, of topological spaces – to that of directed spaces and their *combinatorial or algebraic models* . It is to the theory of ∞-groupoids as category theory is to the theory of groupoids (and hence of groups).

These combinatorial or algebraic models are known as n-categories or, when $n \to \infty$, as ∞-categories or ω-categories, or, in more detail, as (n,r)-categories:

- the natural number n denotes the maximal dimension of *non-trivial* cells in the model,
- while the natural number r denotes the maximal dimension of the *directed* cells.

So an ordinary topological space or ∞-groupoid is an (∞,0)-category: it has cells of arbitrary dimension and all of them are reversible.

In contrast to that, a combinatorial or algebraic model for a directed space in which the 1-dimensional paths may not all be reversible is an (∞,1)-category: it still has cells of arbitrary dimension, but only those of dimension greater than 1 are guaranteed to be reversible.

Often it is convenient in practice to consider the case where the possible dimension n of non-trivial cells is finite. This is familiar from how a topological space that happens to have vanishing homotopy groups in dimension above some n – a homotopy n-type – is modeled by an n-groupoid. A fully directed version of this is an n-category, which is short for (n,n)-category: non-trivial cells up to and including dimension n, and all of them allowed to be non-reversible. Actually, it is possible to go on to an $(n, n+1)$-category, or $(n+1)$-poset; you can either consider than the n-cells are ordered, or else consider that there are irreversible $(n+1)$-cells which are indistinguishable. (Reversible indistinguishable $(n+1)$-cells are all identities and so might as well not exist.)

For low n very explicit algebraic models for n-categories are available but in their full generality become quickly rather untractable as n increases: the series starts with bicategory, tricategory and tetracategory. While bicategories have found plenty of applications, already the axioms of tricategories are rather involved and their theory mainly serves to produce the statement that there is a good semi-strictifications of tricategories called Gray-categories.

Indeed, there are many *strictified* models for higher categories: combinatorial or algebraic models that sacrifice full generality for a better concrete control. Notably there is a useful combinatorial/algebraic model for strict ∞-categories which, while falling short, already goes a long way towards describing general higher categorical structures. In fact, by Simpson's conjecture every ∞-category is equivalent to one that looks like a strict ∞-category except for possibly having weak unit laws.

The challenge of describing fully general ∞-categories is to achieve a combinatorial or algebraic control of all the higher composition rules of higher cells. One can distinguish roughly two orthogonal approaches to dealing with the problem:

in the algebraic definition of higher category an algebraic machinery is set up that allows to concretely handle the explicit *choices* of composites of cells. Such machinery usually involves operadic tools in one way or other. The most sophisticated definitions of this kind are the closely related Batanin ∞-category and Trimble ∞-category.

On the other hand, in the geometric definition of higher category a combinatorial machinery is set up that allows to guarantee *existence* of composites of cells. In the simplicial models for weak ∞-categories higher categories are characterized as simplicial sets with the extra property that certain composites exist. The issue here is to characterize these existence laws correctly.

The basic example for such “existence laws” is the *Kan-filler condition* that characterizes simplicial sets that are Kan complexes, the models for (∞,0)-categories. More general higher categories are obtained by relaxing the Kan condition in just the right way. For instance by simply restricting the Kan-condition to just a certain sub-set of all cells yields the definition of simplicial sets that are called quasi-categories. These model (∞,1)-categories.

The right further relaxation of the (weak) Kan filler condition is more involved. An approach to capture this has been given by Dominic Verity‘s definition of simplicial sets that are called complicial sets and weak complicial sets.

One expects that every algebraic definition of higher categories admits a construction called a nerve that maps it to a simplicial set that would constitute the corresponding geometric model.

Another approach to handle the geometric definition of higher categories is a recursive one that uses n-fold simplicial sets. This is based on the notion of complete Segal space, which is essentially a variation of the concept of quasi-category. Its advantage is that its definition may be applied recursively to yield the notion of n-fold complete Segal spaces. These model (∞,n)-categories for finite n.

Finally, a large supply of further models exists for (∞,1)-categories in terms of enriched category theory. Simplicially enriched model categories are a highly-developed toolkit for handling presentable (∞,1)-categories. Pretriangulated dg-enriched categories and A-∞ categories are a comparably highly developed toolkit for handling stable (∞,1)-categories.

2. Basic concepts

The basic concept on which higher category theory is built is the notion of ***k-morphism*** for all $k \in \mathbb{N}$, equipped with a notion of composition, such that ***coherence laws*** are satisfied.

This is what it’s all about.

3. Basic constructions

Higher presheaves

- higher topos theory

Higher universal constructions

- 2-limit
- (∞,1)-adjunction

- (∞,1)-Kan extension
  - (∞,1)-limit
- (∞,1)-Grothendieck construction

4. Basic theorems

- homotopy hypothesis-theorem
- delooping hypothesis-theorem
- periodic table
- stabilization hypothesis-theorem
- exactness hypothesis
- holographic principle

5. Applications

See

- applications of (higher) category theory.

Extended cobordisms

One major application of higher category theory and one of the driving forces in developing it has been extended functorial quantum field theory. This has recently led to what may become one of the central theorems of higher category theory, the proof of the cobordism hypothesis. This roughly characterizes the (∞,n)-category of cobordisms Bord_n as the free (∞,n)-category with duals on a single generator.

6. Models

There are many different *models* for bringing the abstract notion of higher category onto paper.

- (n × k)-category
- n-fold category
- (n,r)-category
  - Theta-space
  - ∞-category/∞-category
  - (∞,n)-category

    - n-fold complete Segal space
  - (∞,2)-category
  - (∞,1)-category
    - quasi-category
      - algebraic quasi-category
    - simplicially enriched category
    - complete Segal space
    - model category
    - internal category in homotopy type theory
  - (∞,0)-category/∞-groupoid
    - Kan complex
      - algebraic Kan complex
      - simplicial T-complex
  - n-category = (n,n)-category
    - 2-category, (2,1)-category
    - 1-category
    - 0-category
    - (-1)-category
    - (-2)-category
  - n-poset = (n−1,n)-category
    - 2-poset
  - n-groupoid = (n,0)-category
    - 2-groupoid, 3-groupoid
- categorification/decategorification
- geometric definition of higher category
  - Kan complex
  - quasi-category
  - simplicial model for weak ∞-categories
    - complicial set
    - weak complicial set
- algebraic definition of higher category
  - bicategory
  - bigroupoid

- tricategory
- tetracategory
- strict ∞-category
- Batanin ∞-category
- Trimble ∞-category
- Grothendieck-Maltsiniotis ∞-categories

- stable homotopy theory
  - symmetric monoidal category
  - symmetric monoidal (∞,1)-category
  - stable (∞,1)-category
    - dg-category
    - A-∞ category
    - triangulated category

1-categorical models

- homotopical category
- model category theory
- enriched category theory

7. Related concepts

- opetopic type theory

higher category theory

- (0,1)-category theory
- 1-category theory, (∞,1)-category theory
- 2-category theory, (∞,2)-category theory
- (∞,n)-category theory

8. References

For a very gentle introduction to higher category theory, try The Tale of *n*-Categories, which begins in “week73” of This Week’s Finds and goes on from there …; keep clicking the links.

For a slightly more formal but still pathetically easy introduction, try:

- John Baez, An Introduction to n-Categories, in

7th Conference on Category Theory and Computer Science, eds.

E. Moggi and G. Rosolini, Springer Lecture Notes in Computer Science vol. 1290, Springer, Berlin, 1997.

For a free introductory text on n-categories that's *full of pictures*, try this:

- Eugenia Cheng and Aaron Lauda, Higher-Dimensional Categories: An Illustrated Guidebook.

Tom Leinster has written about "comparative ∞-categoriology" (to borrow a term):

- Tom Leinster, *A Survey of Definitions of n-Category* (arXiv)
- Tom Leinster, *Higher Operads, Higher Categories* (arXiv)

A grand picture of the theory of higher categories is drawn in

- Carlos Simpson, *Homotopy Theory of Higher Categories* (pdf)

Another collection of discussions of definitions of higher categories is given at

- John Baez, Peter May Approaching Higher Category Theory

A brief useful survey of approaches to the definition of higher categories is provided by the set of slides

- Andre Joyal, Tim Porter, Peter May, *Weak categories* (pdf)

The theory of quasi-categories as $(\infty,1)$-categories has reached a point where it is well developed and being applied to a wealth of problems with

- Jacob Lurie, *Higher Topos Theory* (arXiv)

There's a lot more to add here, even if we restrict ourselves to very general texts. (More specialized stuff should go under more specialized subcategories!)

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