WIKIPEDIA

Subfunctor

In category theory, a branch of mathematics, a **subfunctor** is a special type of functor that is an analogue of a subset.

Definition

Let **C** be a category, and let F be a contravariant functor from **C** to the category of sets **Set**. A contravariant functor G from **C** to **Set** is a **subfunctor** of F if

1. For all objects c of **C**, $G(c) \subseteq F(c)$, and
2. For all arrows $f: c' \to c$ of **C**, $G(f)$ is the restriction of $F(f)$ to $G(c)$.

This relation is often written as $G \subseteq F$.

For example, let **1** be the category with a single object and a single arrow. A functor F: **1** → **Set** maps the unique object of **1** to some set S and the unique identity arrow of **1** to the identity function 1_S on S. A subfunctor G of F maps the unique object of **1** to a subset T of S and maps the unique identity arrow to the identity function 1_T on T. Notice that 1_T is the restriction of 1_S to T. Consequently, subfunctors of F correspond to subsets of S.

Remarks

Subfunctors in general are like global versions of subsets. For example, if one imagines the objects of some category **C** to be analogous to the open sets of a topological space, then a contravariant functor from **C** to the category of sets gives a set-valued presheaf on **C**, that is, it associates sets to the objects of **C** in a way that is compatible with the arrows of **C**. A subfunctor then associates a subset to each set, again in a compatible way.

The most important examples of subfunctors are subfunctors of the Hom functor. Let c be an object of the category **C**, and consider the functor Hom(−, c). This functor takes an object c' of **C** and gives back all of the morphisms $c' \to c$. A subfunctor of Hom(−, c) gives back only some of the morphisms. Such a subfunctor is called a sieve, and it is usually used when defining Grothendieck topologies.

Open subfunctors

Subfunctors are also used in the construction of representable functors on the category of ringed spaces. Let F be a contravariant functor from the category of ringed spaces to the category of sets, and let $G \subseteq F$. Suppose that this inclusion morphism $G \to F$ is representable by open immersions, i.e., for any representable functor Hom(−, X) and any morphism Hom(−, X) → F, the fibered product $G \times_F \mathrm{Hom}(-, X)$ is a representable functor Hom(−, Y) and the morphism $Y \to X$ defined by the Yoneda lemma is an open immersion. Then G is called an **open subfunctor** of F. If F is covered by representable open subfunctors, then, under certain conditions, it can be shown that F is representable. This is a useful technique for the construction of ringed spaces. It was discovered and exploited heavily by Alexandre Grothendieck, who applied it especially to the case of schemes. For a formal statement and proof, see Grothendieck, *Éléments de géométrie algébrique*, vol. 1, 2nd ed., chapter 0, section 4.5.