AUGMENTED VIRTUAL DOUBLE CATEGORIES

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ABSTRACT. In this article the notion of virtual double category (also known as fc-multicategory) is extended as follows. While cells in a virtual double category classically have a horizontal multi-source and single horizontal target, the notion of augmented virtual double category introduced here extends the latter notion by including cells with empty horizontal target as well.

Any augmented virtual double category comes with a *built-in* notion of "locally small object" and we describe advantages of using augmented virtual double categories as a setting for formal category rather than 2-categories, which are classically *equipped* with a notion of "admissible object" by means of a yoneda structure in the sense of Street and Walters.

An object is locally small precisely if it admits a horizontal unit, and we show that the notions of augmented virtual double category and virtual double category coincide in the presence of all horizontal units. Without assuming the existence of horizontal units we show that most of the basic theory for virtual double categories, such as that of restriction and composition of horizontal morphisms, extends to augmented virtual double categories. We introduce and study in augmented virtual double categories the notion of "pointwise" composition of horizontal morphisms, which formalises the classical composition of profunctors given by the coend formula.

Contents

Introduction

Analogous to the generalisation of monoidal category to multicategory, Burroni in [Bur71] generalised the notion of double category to that of virtual double category [CS10] (Burroni used the term 'multicatégorie'). A virtual double category consists of objects A, B, $C, \dots$, two types of morphism $f \colon A \to C$ and $J \colon A \nrightarrow B$ (which we will draw vertically and horizontally respectively) and cells ϕ of the form as on the left below, each with a single morphism $K \colon C \nrightarrow D$ as horizontal target and a (potentially empty) path $\bar{J} = (A_0 \xrightarrow{J_1} A_1 \cdots A_{n-1} \xrightarrow{J_n} A_n)$ of morphisms as horizontal source.

$$
\begin{array}{ccc}
A_0 \xrightarrow{J_1} A_1 \cdots A_{n-1} \xrightarrow{J_n} A_n \\
f \downarrow \qquad \Downarrow \phi \qquad \downarrow g \\
C \xrightarrow[K]{} D
\end{array}
\qquad\qquad
\begin{array}{c}
A_0 \xrightarrow{J_1} A_1 \cdots A_{n-1} \xrightarrow{J_n} A_n \\
f \searrow \quad \Downarrow \psi \quad \swarrow g \\
C
\end{array}
$$

The present article introduces the notion of 'augmented virtual double category', which extends that of virtual double category by including cells ψ as on the right above, with empty horizontal targets. The prototypical augmented virtual double category Prof has as morphisms functors $f \colon A \to C$ and profunctors $J \colon A^{\mathrm{op}} \times B \to \mathsf{Set}$ between categories A, B, $C, \dots$ that need not be locally small, i.e. need not have all hom-sets isomorphic to objects in Set. As does any augmented virtual double category, Prof contains a virtual double category $U(\mathsf{Prof})$ consisting of cells of the form ϕ above only (see Example 1.4 below).

In contrast to the vertical morphisms, horizontal morphisms are not equipped with composition in either notion of virtual double category. In both Prof and $U(\mathsf{Prof})$ for example the composite of two profunctors along a properly large category does not exist in general. A fortuitous path $\underline{J} = (J_1, \dots, J_n)$ of horizontal morphisms however may still admit a composite $(J_1 \odot \cdots \odot J_n)$, defined as such by a universal cell $(J_1, \dots, J_n) \Rightarrow (J_1 \odot \cdots \odot J_n)$. Likewise an object A may admit a horizontal unit morphism $I_A \colon A \nrightarrow A$ defined by a universal cell $(A) \Rightarrow I_A$, whose horizontal source is the empty path at A. E.g. $A \in \mathsf{Prof}$ admits a horizontal unit I_A if and only if A is locally small, in which case I_A consists of its hom-sets.

A fundamental advantage of working with an augmented virtual double category $\mathcal{K}$ is that its collection of vertical morphisms form a 2-category $V(\mathcal{K})$, whose cells are those of the form ψ above with empty horizontal source $\bar{J} = (A_0)$. In contrast vertical morphisms in a virtual double category only form a category $a\ priori$. E.g. while Prof contains all

natural transformations $\psi\colon f \Rightarrow g$ between functors f and $g\colon A_0 \to C$, only those with C locally small can be canonically identified with cells in the virtual double category $U(\mathsf{Prof})$ contained in Prof, namely the cells ϕ above with $\underline{J} = (A_0)$ and $K = I_C$; for details see Example 2.6 below.

One of the main results of this paper (Theorem 10.1 below) asserts that the notions of virtual double category and augmented virtual double category are equivalent whenever all horizontal units exist; such (augmented) virtual double categories we will call unital virtual double categories. In any unital virtual double category cells $\psi\colon f \Rightarrow g$ as above, with $\underline{J} = (A_0)$, correspond precisely to cells of the form

$$
\begin{array}{ccc}
A_0 & \xrightarrow{I_{A_0}} & A_0 \\
f \downarrow & \Downarrow & \downarrow g \\
C & \xrightarrow[I_C]{} & C
\end{array}
$$

(see Corollary 5.10 below).

A further advantage of using augmented virtual double categories is that they allow for suppressing all 'unit coherence cells', such as λ in the composite on the left-hand side below, which are often used in compositions of cells in unital virtual double categories. Indeed using the language of augmented virtual double categories the cell ψ in the left-hand side, with the horizontal unit I_C as horizontal target, corresponds to the cell ψ' in the right-hand side, such that the two composites below coincide. Moreover the right-hand side allows us to consider this composite even when the horizontal unit I_C does not exist. Thus proofs of results for unital virtual double categories can both be significantly shortened as well as be generalised to proofs that apply to (not necessarily unital) augmented virtual double categories. Parts of Lemma 5.4, Corollary 5.7 and Lemma 8.1 below are obtained in this way from analogous results in [CS10].

$$
\begin{array}{ccccc}
A_0 & \xrightarrow{J} & A_n & \xrightarrow{H} & B_m \\
f \downarrow & \Downarrow \psi & g \downarrow \Downarrow \phi & & \downarrow h \\
C & \xrightarrow[I_C]{} & C & \xrightarrow[K]{} & D \\
\mathrm{id}_C \downarrow & & \Downarrow \lambda & & \downarrow \mathrm{id}_D \\
C & & \xrightarrow[K]{} & & D
\end{array}
\qquad = \qquad
\begin{array}{ccccc}
A_0 & \xrightarrow{J} & A_n & \xrightarrow{H} & B_m \\
f \diagdown & \Downarrow \psi' & g \Downarrow \phi & \diagup h & \\
& C & \xrightarrow[K]{} & D &
\end{array}
$$

Our main purpose for augmented virtual double categories is to use them as a convenient "double dimensional" setting for the internalisation of the notion of yoneda embedding, thus giving an alternative to the classical 2-categorical approach of Street and Walters' yoneda structures [SW78]. While such work will have to appear as a sequel to the present paper (for a draft see Sections 4 and 5 of [Kou15]) we will, after having given the outline of this paper below, close this introduction by broadly describing its ideas and some of its benefits.

This article is largely based on Sections 1, 2 and 3 of the draft [Kou15]. Since the material presented here is significantly more streamlined as well as expanded in several ways, the author encourages readers to consult the present article rather than the latter sections. The first version of [Kou15] used the term "hypervirtual double category" for what is the main notion of this article, where presently we use "augmented virtual double category" instead, as suggested to the author by Robert Paré.

OUTLINE. We start by introducing the notion of augmented virtual double category in Section 1. Examples are given in Section 2, including the augmented virtual double category $\mathcal{V}$-Prof of $\mathcal{V}$-enriched profunctors (Example 2.4); $(\mathcal{V}, \mathcal{V}')$-Prof of $\mathcal{V}$-enriched profunctors between $\mathcal{V}'$-categories, where $\mathcal{V}' \supset \mathcal{V}$ is a universe enlargement of $\mathcal{V}$ in the sense of Section 3.11 of [Kel82] (Example 2.7); $\mathcal{V}$-sProf of small $\mathcal{V}$-enriched profunctors in the sense of [DL07] (Example 2.8); Prof($\mathcal{E}$) of profunctors internal to a category $\mathcal{E}$ with pullbacks (Example 2.9); Rel($\mathcal{E}$) of relations in a category $\mathcal{E}$ with pullbacks (Example 2.10) and spFib($\mathcal{K}$) of split two-sided fibrations in a finitely complete 2-category $\mathcal{K}$ (Example 2.11). In Section 3 the 2-category of augmented virtual double categories, the functors between them and their transformations is introduced, and its equivalences are characterised as functors that are full, faithful and essentially surjective in the appropriate sense.

In Section 4 the notion of restriction $K(f, g)\colon A \nrightarrow B$ of a horizontal morphism $K\colon C \nrightarrow D$ along vertical morphisms $f\colon A \to C$ and $g\colon B \to D$, that was introduced in Section 7 of [CS10] for virtual double categories, is translated to augmented virtual double categories as well as expanded to include that of 'nullary restriction' $C(f, g)\colon A \nrightarrow B$ of an object C along morphisms $f\colon A \to C$ and $g\colon B \to C$. Both types of restriction are defined by cells with a certain universal property; such cells are called 'cartesian', while 'weakly cocartesian' cells satisfy a vertical dual property. Full and faithful morphisms are defined in terms of cartesian cells, and the horizontal unit $I_A\colon A \nrightarrow A$ of an object A is defined to be the nullary restriction $I_A := A(\mathrm{id}_A, \mathrm{id}_A)$.

In Section 5 the 'companion' $f_*\colon A \nrightarrow C$ and 'conjoint' $f^*\colon C \nrightarrow A$ of a vertical morphism $f\colon A \to C$ are introduced as the nullary restrictions $f_* := C(f, \mathrm{id}_C)$ and $f^* := C(\mathrm{id}_C, f)$; they can be thought of as the horizontal morphisms that are respectively "isomorphic" and "adjoint" to f. Unlike similar definitions for unital virtual double categories given in [CS10] we need not require that the horizontal unit I_C exists. Analogous to observations for double categories in Section 4 of [Shu08] we prove that the companion f_* can be equivalently defined in three ways: by a cartesian cell $(f_*) \Rightarrow (A)$, by a weakly cocartesian cell $(A) \Rightarrow (f_*)$, or by a pair of cells $(f_*) \Rightarrow (A)$ and $(A) \Rightarrow (f_*)$ satisfying certain "companion identities"; a horizontal dual result holds for the conjoint f^*. These identities and their horizontal duals imply that companions, conjoints and horizontal units are preserved by any functor of augmented virtual double categories. Horizontal units I_A can both be regarded as the companion and conjoint of the identity id_A; we prove that their defining cells $(I_A) \Rightarrow (A)$ and $(A) \Rightarrow (I_A)$ are both cartesian as well as weakly cocartesian. We prove lemmas that relate the notions of nullary restriction, horizontal unit and full and faithful morphism. We describe adjunctions and absolute left liftings in the 2-category $V(\mathcal{K})$ in terms of companions and conjoints in $\mathcal{K}$.

In Section 6 we consider horizontal morphisms $J\colon A \nrightarrow B$ that are representable by a vertical morphism $f\colon A \to B$, i.e. $J \cong f_*$. Given an augmented virtual double category $\mathcal{K}$ in Theorem 6.5 we describe its locally full sub-augmented virtual double category $\mathsf{Rep}(\mathcal{K})$ of representable horizontal morphisms in terms of its vertical 2-category $V(\mathcal{K})$.

In Section 7 we study composites $(J_1 \odot \cdots \odot J_n)$ of paths $\underline{J} = (J_1, \ldots, J_n)$ of horizontal morphisms. As described above, these are defined by universal cells $\underline{J} \Rightarrow (J_1 \odot \cdots \odot J_n)$. The main lemma of Section 8 proves that, for morphisms $A \xrightarrow{f} C \overset{K}{\nrightarrow} D \xleftarrow{g} B$, the restriction $K(f, g)$ and the composite $f^* \odot K \odot g_*$ coincide. This translates and extends Theorem 7.16 of [CS10] from unital virtual double categories to augmented virtual double categories; here too we need not require the existence of any horizontal units. Internalising the composition of profunctors given by the "coend formula", in Section 9 we introduce and study 'pointwise' horizontal composites. Informally, a horizontal composite is pointwise whenever any of its restrictions are again horizontal composites. Finally in Section 10 we prove the equivalence of the notions of virtual double category and augmented virtual double category in the presence of all horizontal units.

MOTIVATION: INTERNALISING YONEDA EMBEDDINGS. Following Wood [Woo82] and Grandis and Paré [GP08], who used 'proarrow equipments' and double categories respectively to formalise parts of classical category theory, recently certain unital virtual double categories have been used to study formal category theory in less well behaved settings, as follows. Cruttwell and Shulman in [CS10] internalise the notion of fully faithful morphism in the unital virtual double category $\mathbb{M}\mathsf{od}(\mathbb{X})$ of 'modules' in a virtual double category $\mathbb{X}$, while Riehl and Verity in [RV17] internalise the notions of fully faithful morphism, 'exact square' and (pointwise) Kan extension in the unital virtual double category $\underline{\mathrm{Mod}}_{\mathcal{K}}$ of modules between ∞-categories in the homotopy 2-category of a '∞-cosmos' $\mathcal{K}$.

In line with the previous our goal for augmented virtual double categories is to use them as a setting for internalising the notion of yoneda embedding, as we will now sketch roughly. We start by recalling the classical internalisation of yoneda embeddings, in the form of a yoneda structure on a 2-category [SW78]. First let us recall some details of the classical theory of yoneda embeddings. Given a properly large, locally small category A, recall from [FS95] that the category $PA := \mathsf{Set}^{A^{\mathrm{op}}}$ of small set valued presheaves $A^{\mathrm{op}} \to \mathsf{Set}$ is necessarily locally properly large. Thus the natural 2-dimensional environment for classical yoneda embeddings $yA\colon A \to PA$ for such A, that map each $x \in A$ to the representable presheaf $A(-, x)$, is the 2-category Cat of locally large categories, functors and natural transformations. Using that any functor $f\colon A \to B$, with small hom-sets $B(fx, y)$ for all $x \in A$ and $y \in B$, induces a functor $B(f, 1)\colon B \to PA$ given by $B(f, 1)(y) = B(f-, y)$, Yoneda's lemma can be rephrased internally to Cat as follows: the canonical natural transformation

$$A \xrightarrow{f} B$$
$$yA \searrow \overset{\chi^f}{\Rightarrow} \swarrow B(f,1)$$
$$PA$$

exhibits f as the 'absolute left lifting' (see e.g. [SW78]) of yA along $B(f,1)$.

A yoneda structure on a 2-category $\mathcal{C}$ formalises the previous as follows. Firstly it postulates a right ideal[1] of 'admissible morphisms' $f\colon A \to C$ in $\mathcal{C}$, which internalises the smallness condition on the functors f above; an object A is then called admissible whenever its identity morphism id_A is so. Secondly it provides a morphism $yA\colon A \to PA$ for each admissible object A, internalising the yoneda embedding, together with a cell χ^f as above for each admissible $f\colon A \to B$. The cells χ^f are required to satisfy three axioms [SW78]:

(1) χ^f exhibits $B(f,1)$ as the left Kan extension of f along yA (together with (3) below this formalises yA being dense);

(2) χ^f exhibits f as the absolute left lifting of yA along $B(f,1)$ (as above);

(3) roughly, the assignment $f \mapsto \chi_f$ is pseudofunctorial.

The stronger notion of 'good yoneda structure' on a finitely complete 2-category $\mathcal{C}$, introduced by Weber [Web07], is defined as above except for replacing axioms (1) and (3) with the following stronger axiom:

(4) if any cell ϕ in $\mathcal{C}$, of the form as below and with f admissible, exhibits f as the absolute left lifting of yA along g then it exhibits g as the pointwise left Kan extension of yA along f (in the sense of [Str74]).

$$A \xrightarrow{\ f\ } B$$
$$yA \searrow \ \overset{\phi}{\Rightarrow}\ \swarrow g$$
$$PA$$

While yoneda embeddings for 2-categories, that is categories enriched in Cat, combine to form a yoneda structure, satisfying axioms (1)–(3) above, they do not satisfy axiom (4); this is explained in Remark 9 of [Wal18].

The main idea of internalising the notion of yoneda embedding in an augmented virtual double category $\mathcal{K}$ is the following. Instead of postulating a notion of admissible morphism in $\mathcal{K}$ we internalise that notion, simply by regarding all horizontal morphisms of $\mathcal{K}$ to be admissible; consequently a vertical morphism $f\colon A \to C$ (respectively an object A) is considered admissible whenever its companion $f_*\colon A \nrightarrow C$ (respectively its horizontal unit $I_A\colon A \nrightarrow A$) exists. Compare the prototypal example $\mathcal{K} = \mathsf{Prof}$, in which all horizontal morphisms are small-set-valued profunctors. This allows for a simpler notion of yoneda embedding: we do not have to specify an ideal of admissible morphisms and, instead of having to provide a full coherent family of yoneda embeddings as in a yoneda structure, we may simply consider a single yoneda embedding $y\colon A \to \widehat{A}$ in $\mathcal{K}$, as follows. To exhibit

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[1] A right ideal of a category is a class $\mathcal{I}$ of morphisms closed under precomposition: if f and g are composable then $g \in \mathcal{I}$ implies $g \circ f \in \mathcal{I}$.

y as a yoneda embedding amounts to providing, for each horizontal morphism $J\colon A \nrightarrow B$ in $\mathcal{K}$, a cell χ as below on the left, satisfying the following 'yoneda' and 'density' axioms, analogous to (2) and (4) above:

(y) χ is a 'cartesian cell' (see Section 4 below), thus exhibiting J as the restriction of the object $\widehat{A}$ along the morphisms y and J^λ;

(d) any cartesian cell ϕ in $\mathcal{K}$, as on the right, defines g as a pointwise left Kan extension of y along J (in the sense of Section 4.2 of [Kou15]).

$$
\begin{array}{ccc}
A \xrightarrow{\ J\ } B & \qquad\qquad & A \xrightarrow{\ J\ } B \\
{\scriptstyle y}\searrow \ \Downarrow\!{\scriptstyle\chi}\ \nearrow{\scriptstyle J^\lambda} & & {\scriptstyle y}\searrow \ \Downarrow\!{\scriptstyle\phi}\ \nearrow{\scriptstyle g} \\
\widehat{A} & & \widehat{A}
\end{array}
$$

In $\mathcal{K} = \mathsf{Prof}$ the yoneda embedding $y := yA$ for a locally small category A is defined as before, with $\widehat{A} = \mathsf{Set}^{A^{\mathrm{op}}}$, while the functors J^λ are defined by $J^\lambda y = J(-, y)$ for $y \in B$. The components $\chi_{x,y}\colon J(x,y) \to \widehat{A}(yx, J^\lambda y)$ of χ, which axiom (y) requires to be isomorphisms, are supplied by Yoneda's lemma.

To conclude this motivation we list some benefits of using augmented virtual double categories $\mathcal{K}$ to internalise the notion of yoneda embedding $y\colon A \to \widehat{A}$.

- If all nullary restrictions $\widehat{A}(y, g)$ exist in $\mathcal{K}$, as they do in all well known examples (e.g. $\mathcal{K} = \mathsf{Prof}$), then the assignment $J \mapsto J^\lambda$ induces an equivalence between morphisms of the forms $A \nrightarrow B$ and $B \to \widehat{A}$. In contrast the assignment $f \mapsto B(f, 1)$ induced by a yoneda structure is in general not essentially surjective onto morphisms $B \to PA$ (e.g. take $A = 1 = B$ the terminal category in $\mathcal{C} = \mathsf{Cat}$).

- Several types of yoneda embedding satisfy the axioms (1)–(3) of a yoneda structure but their appropriate notion of admissible morphism does not form a right ideal, so that the theory of yoneda structures does not apply fully. For a well-known example consider a closed symmetric monoidal, small complete category $\mathcal{V}$. The appropriate notion of admissible $\mathcal{V}$-functor for the $\mathcal{V}$-enriched yoneda embeddings $y\colon A \to \widehat{A}^{\mathrm{s}}$, where $\widehat{A}^{\mathrm{s}}$ denotes the $\mathcal{V}$-category of 'small $\mathcal{V}$-presheaves on A' in the sense of [DL07], does not form a right ideal. In contrast, it is not hard to prove that these y do form yoneda embeddings in the augmented virtual double category $\mathcal{V}$-sProf (Example 2.8 below), so that the theory of [Kou15] applies fully. Likewise yoneda embeddings induced by a 'KZ doctrine', as studied by Walker in [Wal18], do not satisfy the right ideal property; they too are likely to form yoneda embeddings in some appropriately chosen augmented virtual double category.

- As noted previously $\mathcal{V}$-enriched yoneda embeddings, in the classical sense of e.g. Section 2.4 of [Kel82], form yoneda structures that do not in general satisfy axiom (4). On the other hand they do satisfy a stronger, i.e. $\mathcal{V}$-enriched, version of axiom (1). Thus neither notion of yoneda structure captures the notion of $\mathcal{V}$-enriched yoneda

embedding exactly. The augmented virtual double category $(\mathcal{V}, \mathcal{V}')$-Prof, as described in Example 2.7 below, is the right setting in this case: therein axioms (y) and (d) capture correctly the $\mathcal{V}$-enriched notion of yoneda embedding. Roughly this is because the pointwise notion of Kan extension in axiom (d) above, when considered in $(\mathcal{V}, \mathcal{V}')$-Prof, coincides with the classical notion of $\mathcal{V}$-enriched Kan extension (see Section 4.4 of [Kou15]).

- Regarding all horizontal morphisms of augmented virtual double categories as admissible allows us to prove results that assert admissibility of morphisms or objects. For instance consider any full and faithful morphism $h\colon C \to E$ in an augmented virtual double category that has all restrictions of the form $K(f, g)$. Lemma 5.14 below proves that h 'reflects admissibility', that is C is admissible (i.e. the horizontal unit I_C exists) whenever E is. Even though inside a unital virtual double category, i.e. with all horizontal units, our notion of full and faithful coincides with that of [CS10], notice that in unital virtual double categories this result is meaningless. For another example remember that (good) yoneda structures provide a yoneda embedding for each admissible object. Inside augmented virtual double categories a weak converse holds: given a yoneda embedding $y\colon A \to \widehat{A}$ the horizontal unit of A exists whenever all nullary restrictions of the form $\widehat{A}(y, g)$ exist; see Section 5.1 of [Kou15].

- For an example of a formalisation of a more involved result, similar to those of the previous item, let $f\colon A \to C$ be a $\mathcal{V}$-functor and $f^\sharp\colon \widehat{A}^{\mathrm{s}} \to \widehat{C}^{\mathrm{s}}$ be given by left Kan extending small $\mathcal{V}$-presheaves on A along f. In our terms Proposition 3.3 of [DL07] can be rephrased as follows: $f^\sharp$ has a right adjoint if and only if f is admissible (in other words its companion $f_*\colon A \nrightarrow C$ exists) in the augmented virtual double category $\mathcal{V}$-sProf (Example 2.8 below). This result partially formalises to any $f\colon A \to C$ in a general augmented virtual double category $\mathcal{K}$, assuming that the yoneda embeddings $y_A\colon A \to \widehat{A}$ and $y_C\colon C \to \widehat{C}$ exist: the morphism $f^\sharp$ can then be internalised as being the left Kan extension of $y_C \circ f$ along y_A and the implications

$$f^\sharp \text{ has a right adjoint} \quad \Leftrightarrow \quad (y_C \circ f)_* \text{ exists} \quad \Rightarrow \quad f_* \text{ exists}$$

hold under mild conditions on $\mathcal{K}$; see Section 5.2 of [Kou15].

- Axiom (y) above allows us to capture a monoidal variant of Yoneda's lemma as follows. Recall that a monoidal structure $\otimes$ on a category A induces a monoidal structure $\widehat{\otimes}$ on its category of presheaves $\widehat{A} := \mathsf{Set}^{A^{\mathrm{op}}}$ that is given by Day convolution [Day70]

$$(p \mathbin{\widehat{\otimes}} q)(x) := \int^{u,v \in A} A(x, u \otimes v) \times pu \times qv, \qquad \text{where } p, q \in \widehat{A}. \qquad (1)$$

With respect to $\widehat{\otimes}$ the yoneda embedding $y\colon A \to \widehat{A}$ forms a monoidal functor, i.e. it admits a coherent family of isomorphisms $\bar{y}\colon yx \mathbin{\widehat{\otimes}} yy \cong y(x \otimes y)$ where $x, y \in A$. Thus

a monoidal functor $(y, \bar{y}) \colon (A, \otimes) \to (\widehat{A}, \widehat{\otimes})$ satisfies the following monoidal variant of the yoneda axiom (y) for profunctors: any lax monoidal profunctor $J \colon A \nrightarrow B$ (i.e. equipped with coherent maps $\bar{J} \colon J(x_1, y_1) \times J(x_2, y_2) \to J(x_1 \otimes x_2, y_1 \otimes y_2))$ induces a lax monoidal functor $J^\lambda \colon B \to \widehat{A}$ such that $\widehat{A}(y-, J^\lambda-) \cong J$ as lax monoidal profunctors. In detail, we can take J^λ to be as defined before: $J^\lambda y := J(-, y)$, and take the coherence morphisms $\bar{J}^\lambda \colon J^\lambda y_1 \widehat{\otimes} J^\lambda y_2 \Rightarrow J^\lambda(y_1 \otimes y_2)$ to be induced by the composites

$$A(x, u \otimes v) \times J(u, y_1) \times J(v, y_2)$$
$$\xrightarrow{\mathrm{id} \times \bar{J}} A(x, u \otimes v) \times J(u \otimes v, y_1 \otimes y_2) \to J(x, y_1 \otimes y_2),$$

where the unlabelled morphism is induced by the functoriality of J in A.

In fact, we may consider the augmented virtual double category MonProf of lax monoidal functors and lax monoidal profunctors between (possibly large) monoidal categories, and show that the monoidal yoneda embedding $(y, \bar{y})$ satisfies both axioms (y) and (d) therein. As described in the first item above, together these axioms imply an equivalence between lax monoidal profunctors $A \nrightarrow B$ and lax monoidal functors $B \to \widehat{A}$.[1] We remark that the monoidal yoneda embeddings $(y, \bar{y})$ do not combine to form a yoneda structure on the two 2-categories consisting of either lax or colax monoidal functors between (possibly large) monoidal categories: this is because colax monoidal structures on a functor $f \colon A \to B$ correspond to *lax* monoidal structures on the corresponding functor $B(f, 1) \colon B \to \widehat{A}$ and similarly lax monoidal structures on f do in general not induce monoidal structures on $B(f, 1)$.

- One of the main results of [Kou15] formalises Day convolution for monoidal structures to algebraic structures defined by monads T on augmented virtual double categories $\mathcal{K}$. More precisely, given a yoneda embedding $y \colon A \to \widehat{A}$ in $\mathcal{K}$ it gives conditions under which a T-algebra structure $a \colon TA \to A$ on A induces such a structure $\widehat{a}$ on $\widehat{A}$, in a way that makes y into a yoneda embedding in the augmented virtual double category T-Alg of T-algebras. Formalising a result of Im and Kelly [IK86] one can then show, for instance, that y, as a T-morphism, defines $(\widehat{A}, \widehat{a})$ as the free cocompletion of (A, a) in T-Alg; see Section 5.4 of [Kou15].

1. Augmented virtual double categories

The definition of augmented virtual double category below uses the notion of *directed graph*, by which we mean a parallel pair of functions

$$A = \left(A_1 \underset{t}{\overset{s}{\rightrightarrows}} A_0 \right)$$

---

[1]This observation is not new: it follows from Pisani's study of exponentiable multicategories in Section 2 of [Pis14].

from a class A_1 of *edges* to a class A_0 of *vertices*. An edge e with $(s,t)(e) = (x,y)$ is denoted $x \xrightarrow{e} y$; the vertices x and y are called its *source* and *target*. Any category $\mathcal{C}$ has an underlying graph $\mathcal{C}_1 \rightrightarrows \mathcal{C}_0$ with $\mathcal{C}_1$ and $\mathcal{C}_0$ the classes of morphisms and objects of $\mathcal{C}$ respectively. Conversely, remember that any graph A generates a *free category* $\mathsf{fc}\, A$, with as objects the vertices of A and as morphisms $x \to y$ (possibly empty) paths $\underline{e} = (x = x_0 \xrightarrow{e_1} x_1 \xrightarrow{e_2} \cdots \xrightarrow{e_n} x_n = y)$ of edges in A; we write $|\underline{e}| := n$ for their lengths. Composition in $\mathsf{fc}\, A$ is given by concatenation

$$(\underline{e}, \underline{f}) \mapsto \underline{e} \frown \underline{f} := (x_0 \xrightarrow{e_1} \cdots \xrightarrow{e_n} x_n = y_0 \xrightarrow{f_1} \cdots \xrightarrow{f_m} y_m)$$

of paths, while the empty path (x) forms the identity at $x \in A_0$.

1.1. NOTATION. For any integer $n \geq 1$ we write $n' := n - 1$.

1.2. DEFINITION. *An* augmented virtual double category $\mathcal{K}$ *consists of*

- *a class $\mathcal{K}_0$ of* objects A, B, ...

- *a category $\mathcal{K}_\mathrm{v}$ with $\mathcal{K}_{\mathrm{v}0} = \mathcal{K}_0$, whose morphisms $f \colon A \to C$, $g \colon B \to D$, ... are called* vertical morphisms;

- *a directed graph $\mathcal{K}_\mathrm{h}$ with $\mathcal{K}_{\mathrm{h}0} = \mathcal{K}_0$, whose edges are called* horizontal morphisms *and denoted by slashed arrows $J \colon A \nrightarrow B$, $K \colon C \nrightarrow D$, ...;*

- *a class of* cells ϕ, ψ, ... *that are of the form*

$$
\begin{array}{ccc}
A_0 & \xrightarrow{J} & A_n \\
f \downarrow & \Downarrow \phi & \downarrow g \\
C & \xrightarrow[K]{} & D
\end{array}
\tag{2}
$$

where $\underline{J}$ and $\underline{K}$ are (possibly empty) paths in $\mathcal{K}_\mathrm{h}$ with $|\underline{K}| \leq 1$;

- *for any path of cells*

$$
\begin{array}{ccccccccc}
A_{10} & \xrightarrow{J_1} & A_{1m_1} & \xrightarrow{J_2} & A_{2m_2} & & A_{n'm_{n'}} & \xrightarrow{J_n} & A_{nm_n} \\
f_0 \downarrow & \Downarrow \phi_1 & \downarrow f_1 & \Downarrow \phi_2 & \downarrow f_2 & \cdots & f_{n'} \downarrow & \Downarrow \phi_n & \downarrow f_n \\
C_0 & \xrightarrow[K_1]{} & C_1 & \xrightarrow[K_2]{} & C_2 & & C_{n'} & \xrightarrow[K_n]{} & C_n
\end{array}
\tag{3}
$$

of length $n \geq 1$ and a cell ψ as on the left below, a vertical composite *as on the right;*

$$
\begin{array}{ccc}
C_0 & \xrightarrow{K_1 \frown K_2 \frown \cdots \frown K_n} & C_n \\
h \downarrow & \Downarrow \psi & \downarrow k \\
E & \xrightarrow[L]{} & F
\end{array}
\qquad
\begin{array}{ccc}
A_{10} & \xrightarrow{J_1 \frown J_2 \frown \cdots \frown J_n} & A_{nm_n} \\
h \circ f_0 \downarrow & \Downarrow \psi \circ (\phi_1, \ldots, \phi_n) & \downarrow k \circ f_n \\
E & \xrightarrow[L]{} & F
\end{array}
\tag{4}
$$

- horizontal identity cells *as on the left below, one for each* $J\colon A \nrightarrow B$;

$$
\begin{array}{ccc}
A \xrightarrow{(J)} B & \qquad & A \xrightarrow{(A)} A \\
\mathrm{id}_A \Big\downarrow \quad \Big\Downarrow \mathrm{id}_J \ \Big\downarrow \mathrm{id}_B & & f \Big\downarrow \quad \Big\Downarrow \mathrm{id}_f \ \Big\downarrow f \\
A \xrightarrow[(J)]{} B & & C \xrightarrow[(C)]{} C
\end{array}
$$

- vertical identity cells *as on the right above, one for each* $f\colon A \to C$, *with empty horizontal source* (A) *and target* (C), *that are preserved by vertical composition:* $\mathrm{id}_h \circ (\mathrm{id}_f) = \mathrm{id}_{h \circ f}$; *we write* $\mathrm{id}_A := \mathrm{id}_{\mathrm{id}_A}$.

The vertical composition above is required to satisfy the associativity axiom

$$
\chi \circ \big(\psi_1 \circ (\phi_{11}, \dots, \phi_{1m_1}), \dots, \psi_n \circ (\phi_{n1}, \dots, \phi_{nm_n})\big)
$$
$$
= \big(\chi \circ (\psi_1, \dots, \psi_n)\big) \circ (\phi_{11}, \dots, \phi_{nm_n}), \quad (5)
$$

whenever the left-hand side makes sense, as well as the unit axioms

$$
\mathrm{id}_C \circ (\phi) = \phi, \quad \mathrm{id}_K \circ (\phi) = \phi, \quad \phi \circ (\mathrm{id}_A) = \phi, \quad \phi \circ (\mathrm{id}_{J_1}, \dots, \mathrm{id}_{J_n}) = \phi
$$

and
$$
\psi \circ (\phi_1, \dots, \phi_i, \mathrm{id}_{f_i}, \phi_{i+1}, \dots, \phi_n) = \psi \circ (\phi_1, \dots, \phi_i, \phi_{i+1}, \dots, \phi_n)
$$

whenever these make sense and where, in the last axiom, $0 \le i \le n$ *(in the cases* $i = 0$ *and* $i = n$ *the identity cells* id_{f_0} *and* id_{f_n} *are respectively the first and last cell in the path that is composed with* ψ*).*

For a cell ϕ as in (2) above we call the vertical morphisms f and g its *vertical source* and *target* respectively, the path of horizontal morphisms $\underline{J} = (J_1, \dots, J_n)$ its *horizontal source* and $\underline{K}$ its *horizontal target*. We write $|\phi| := (|\underline{J}|, |\underline{K}|)$ for the *arity* of ϕ. An $(n, 1)$-ary cell will be called *unary*, $(n, 0)$-ary cells *nullary* and $(0, 0)$-ary cells *vertical*.

When writing down paths $(J_1, \dots, J_n)$ of length $n \le 1$ we will often leave out parentheses and simply write $J_1 := (A_0 \xrightarrow{J_1} A_1)$ or $A_0 := (A_0)$. Likewise in the composition of cells: $\psi \circ \phi_1 := \psi \circ (\phi_1)$. We will often denote unary cells simply by $\phi\colon (J_0, \dots, J_n) \Rightarrow K$ and nullary cells by $\psi\colon (J_0, \dots, J_n) \Rightarrow C$, leaving out their vertical source and target. When drawing compositions of cells it is often helpful to depict them in full detail and, in the case of nullary cells, draw their empty horizontal target as a single object, as shown below.

$$
\begin{array}{ccccc}
A_0 & A_0 \xrightarrow{J_1} A_1 \cdots A_{n'} \xrightarrow{J_n} A_n & A_0 & A_0 \xrightarrow{J_1} A_1 \cdots A_{n'} \xrightarrow{J_n} A_n \\
f \Big(\Downarrow \psi \Big) g & \searrow \ \Downarrow\psi \ \swarrow & f \diagup \ \Downarrow \phi \ \diagdown g & f \Big\downarrow \qquad \Downarrow\phi \qquad \Big\downarrow g \\
C & f \quad C \quad g & C \xrightarrow{K} D & C \xrightarrow[K]{} D
\end{array}
$$

A cell with identities as vertical source and target is called *horizontal*. A horizontal cell $\phi\colon J \Rightarrow K$ with unary horizontal source is called *invertible* if there exists a horizontal

cell $\psi \colon K \Rightarrow J$ such that $\phi \circ \psi = \mathrm{id}_K$ and $\psi \circ \phi = \mathrm{id}_J$; in that case we write $\phi^{-1} := \psi$. When drawing diagrams we shall often depict identity morphisms by equal signs (=), while in identity cells we will leave out the arrows ($\Downarrow \mathrm{id}$), leaving them empty instead. Because composition of cells is associative we will leave out bracketings when writing down composites.

For convenience we use the 'whisker' notation from 2-category theory and define

$$h \circ (\phi_1, \ldots, \phi_n) := \mathrm{id}_h \circ (\phi_1, \ldots, \phi_n) \qquad \text{and} \qquad \psi \circ f := \psi \circ \mathrm{id}_f,$$

whenever the right-hand side makes sense. Moreover, for any path

$$
\begin{array}{ccccccccc}
A_0 & \xrightarrow{J_1} & A_1 & \cdots & A_{n'} & \xrightarrow{J_n} & A_n & \xrightarrow{H_1} & B_1 & \cdots & B_{m'} & \xrightarrow{H_m} & B_m \\
f \downarrow & & & \Downarrow \phi & & & \downarrow g & & \Downarrow \psi & & & & \downarrow h \\
C & & \xrightarrow[\underline{K}]{} & & & D & & \xrightarrow[\underline{L}]{} & & & G
\end{array}
$$

with $|\underline{K}| + |\underline{L}| \leq 1$ we define the *horizontal composite* $\phi \odot \psi \colon \underline{J} \frown \underline{H} \Rightarrow \underline{K} \frown \underline{L}$ by

$$\phi \odot \psi := \mathrm{id}_{\underline{K} \frown \underline{L}} \circ (\phi, \psi),$$

where $\mathrm{id}_{\underline{K} \frown \underline{L}}$ is to be interpreted as the identity $\mathrm{id}_C \colon C \to C$ in the case that $\underline{K} \frown \underline{L} = (C)$. The following lemma follows easily from the associativity and unit axioms for vertical composition.

1.3. LEMMA. *Horizontal composition* $(\phi, \psi) \mapsto \phi \odot \psi$, *as defined above, satisfies the* associativity *and* unit axioms

$$(\phi \odot \psi) \odot \chi = \phi \odot (\psi \odot \chi), \qquad (\mathrm{id}_f \odot \phi) = \phi \qquad and \qquad (\phi \odot \mathrm{id}_g) = \phi$$

whenever these make sense. Moreover, horizontal and vertical composition satisfy the interchange axioms

$$\big(\psi \circ (\phi_1, \ldots, \phi_n)\big) \odot \big(\chi \circ (\xi_1, \ldots, \xi_m)\big) = (\psi \odot \chi) \circ (\phi_1, \ldots, \phi_n, \xi_1, \ldots, \xi_m)$$

and $\qquad \psi \circ \big(\phi_1, \ldots, (\phi_{i'} \odot \phi_i), \ldots, \phi_n\big) = \psi \circ (\phi_1, \ldots, \phi_{i'}, \phi_i, \ldots, \phi_n)$

whenever they make sense.

The following examples relate augmented virtual double categories to some classical 2-dimensional categorical notions. Further examples are given in the next section.

1.4. EXAMPLE. By restricting to augmented virtual double categories in which all nullary cells are vertical identities, that is id_f for some vertical morphism $f \colon A \to C$, we recover the classical notion of *virtual double category*, in the sense of [CS10] or Section 5.1 of [Lei04] (where it is called fc-multicategory). Virtual double categories were originally introduced by Burroni [Bur71] who called them 'multicatégories'. It follows that every augmented virtual double category $\mathcal{K}$ contains a virtual double category $U(\mathcal{K})$ consisting of its objects, vertical and horizontal morphisms, and unary cells.

1.5. EXAMPLE. Augmented virtual double categories with no horizontal morphisms, so that all cells are vertical, correspond precisely to 2-*categories*, with the compositions $\circ$ and $\odot$ corresponding to the vertical and horizontal composition in 2-categories respectively. Thus every augmented virtual double category $\mathcal{K}$ contains a *vertical* 2-*category* $V(\mathcal{K})$, consisting of its objects, vertical morphisms and vertical cells. As remarked in the Introduction virtual double categories do not canonically contain 2-categories of vertical morphisms unless they have all horizontal units (see Proposition 6.1 of [CS10]). In Theorem 10.1 below we will see that the notions of augmented virtual double category and virtual double category coincide in the presence of horizontal units (see Definition 7.1).

1.6. EXAMPLE. Restricting to augmented virtual double categories in which all vertical morphisms are identities and all nullary cells are identity cells

$$
\begin{array}{c}
A \\
\left(\!\!\left\Vert \Downarrow \mathrm{id}_A \right\Vert\!\!\right) \\
A,
\end{array}
$$

recovers the notion of *multicategory* (see e.g. Section 2.1 of [Lei04]). Similarly augmented virtual double categories in which all vertical morphisms are identities and all *vertical* cells are identity cells can be regarded as multicategories $\mathcal{K}$ equipped with a bimodule $\mathcal{K} \nrightarrow 1$, in the sense of Definition 2.3.6 of [Lei04], where 1 denotes the terminal multicategory.

1.7. EXAMPLE. Let us recall the notion of a *horizontal unit* $I_A\colon A \nrightarrow A$ for an object A in a virtual double category $\mathcal{K}$ from e.g. Section 8 of [Her00] or Section 5 of [CS10]; see also Section 7 below. It is defined by an *cocartesian cell* η_A as on the left below, satisfying the following universal property: any cell ϕ in $\mathcal{K}$, with A an object in its horizontal source as in the middle below, factors uniquely through η_A as a cell ϕ' as shown, where the empty cells denote paths of identity cells.

$$
\begin{array}{ccccc}
A & & X_0 \xrightarrow{J} A \xrightarrow{H} Y_m & & \begin{array}{c} X_0 \xrightarrow{J} A \xrightarrow{H} Y_m \\ \| \quad \|\Downarrow \eta_A\| \quad \| \\ X_0 \xrightarrow{J} A \xrightarrow{I_A} A \xrightarrow{H} Y_m \\ f\searrow \quad \Downarrow \phi' \quad \swarrow g \\ C \xrightarrow{K} D \end{array} \\
\|\Downarrow \eta_A\| & & f\searrow \quad \Downarrow\phi \quad \swarrow g & = & \\
A \xrightarrow{I_A} A & & C \xrightarrow{K} D & &
\end{array} \qquad (6)
$$

We call a virtual double category *unital* if each of its objects admits a horizontal unit. Choosing a cocartesian cell η_A for each object A in a unital virtual double category $\mathcal{K}$ allows $\mathcal{K}$ to be regarded as an augmented virtual double category $N(\mathcal{K})$, as we shall now explain. The augmented virtual double category $N(\mathcal{K})$ has as objects, morphisms and unary cells the objects, morphisms and cells of $\mathcal{K}$ while the nullary cells ξ of $N(\mathcal{K})$, of the

shape as on the left below, are the cells ξ of $\mathcal{K}$ that are of the shape as on the right.[1]

$$A_0 \xrightarrow{J_1} A_1 \;\cdots\; A_{n'} \xrightarrow{J_n} A_n$$

$$A_0 \xrightarrow{J_1} A_1 \;\cdots\; A_{n'} \xrightarrow{J_n} A_n$$

Composition $\psi \circ (\phi_1, \ldots, \phi_n)$ in $N(\mathcal{K})$, with $\phi_i \colon \underline{J}_i \Rightarrow \underline{K}_i$ as in (3) and $\psi \colon \underline{K}_1 \frown \cdots \frown \underline{K}_n \Rightarrow \underline{L}$ as in (4), is defined as the composite in $\mathcal{K}$

$$\psi \circ (\phi_1, \ldots, \phi_n) := \psi' \circ (\phi_1, \ldots, \phi_n)$$

where ψ' is defined as follows. Writing $\underline{\eta_\phi}$ for the path of cocartesian cells $(\eta_{\phi_1}, \ldots, \eta_{\phi_n})$, where $\eta_{\phi_i} := \eta_{C_{i'}}$ if ϕ_i is nullary with horizontal target $C_{i'}$ and $\eta_{\phi_i} := \mathrm{id}_{K_i}$ if ϕ_i is unary with horizontal target $K_i \colon C_{i'} \nrightarrow C_i$, the cell ψ' is the unique factorisation in $\psi = \psi' \circ \underline{\eta_\phi}$. This factorisation ψ' exists by the universal property of the $\eta_{C_{i'}}$, and it contains a unit $\overline{I}_{C_{i'}}$ in its horizontal source for each nullary cell ϕ_i in $\underline{\phi}$. Finally the horizontal identity cells id_J in $N(\mathcal{K})$ are simply those of $\mathcal{K}$, while the vertical identity cells id_f in $N(\mathcal{K})$, one for each $f \colon A \to C$, are the composites $\mathrm{id}_f := \eta_C \circ f$ in $\mathcal{K}$. That the composition for $N(\mathcal{K})$ as defined above satisfies the associativity and unit axioms is a straightforward consequence of those axioms in $\mathcal{K}$, combined with the uniqueness of the factorisations ψ'.

In Theorem 10.1 below we will see that the assignment $\mathcal{K} \mapsto N(\mathcal{K})$ is part of an equivalence between unital virtual double categories and augmented virtual double categories that have all horizontal units.

Every augmented virtual double category has a horizontal dual as follows.

1.8. DEFINITION. *Let $\mathcal{K}$ be an augmented virtual double category. The* horizontal dual *of $\mathcal{K}$ is the augmented virtual double category $\mathcal{K}^{\mathrm{co}}$ that has the same objects and vertical morphisms, that has a horizontal morphism $J^{\mathrm{co}} \colon A \nrightarrow B$ for each $J \colon B \nrightarrow A$ in $\mathcal{K}$, and a cell ϕ^{co} as on the left below for each cell ϕ in $\mathcal{K}$ as on the right.*

$$A_0 \xrightarrow{J_1^{\mathrm{co}}} A_1 \;\cdots\; A_{n'} \xrightarrow{J_n^{\mathrm{co}}} A_n$$

$$A_n \xrightarrow{J_n} A_{n'} \;\cdots\; A_1 \xrightarrow{J_1} A_0$$

Identities and compositions in $\mathcal{K}^{\mathrm{co}}$ are induced by those of $\mathcal{K}$:

$$\mathrm{id}_{J^{\mathrm{co}}} := (\mathrm{id}_J)^{\mathrm{co}}, \quad \mathrm{id}_f := (\mathrm{id}_f)^{\mathrm{co}} \quad and \quad \psi^{\mathrm{co}} \circ (\phi_1^{\mathrm{co}}, \ldots, \phi_n^{\mathrm{co}}) := \big(\psi \circ (\phi_n, \ldots, \phi_1)\big)^{\mathrm{co}}.$$

We end this section with a remark on the associativity of composition of cells in augmented virtual double categories.

---

[1] Notice that a cell $\xi \colon (J_1, \ldots, J_n) \Rightarrow I_C$ in $\mathcal{K}$ as above appears as a cell in $N(\mathcal{K})$ in two ways: once as a unary cell $\xi \colon (J_1, \ldots, J_n) \Rightarrow I_C$ and once as a nullary cell $\xi \colon (J_1, \ldots, J_n) \Rightarrow C$.

1.9. REMARK. Consider a configuration of composable cells as in the scheme below, where ϕ_2 is a nullary cell and the other cells are unary. Notice that there are two ways of vertically composing these cells if we compose the top rows first: in that case the cell ϕ_2 can be composed either with ψ_1 or with ψ_2. In contrast, if we start by first composing the bottom two rows then there is only one way to form the composite.

This example shows why the associativity axiom (5) for vertical composition has to be "read from left to right": when read in the other direction, in general, there might be multiple ways in which the cells $(\phi_1, \ldots, \phi_m)$ of top row can be "distributed" over the cells $(\psi_1, \ldots, \psi_n)$ in the middle row.
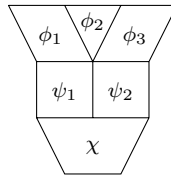
Formally the above observation is a manifestation of the fact that, when regarded as monoids, augmented virtual double categories $\mathcal{K}$ (with a fixed vertical category $\mathcal{K}_v = \mathcal{C}$) are monoids in a *skew-monoidal category*, in the sense of Szlachányi [Szl12], instead of monoids in an ordinary monoidal category. This is made precise in [Kou19].

2. Examples

Our main source of augmented virtual double categories will be virtual double categories, as will be explained in this section. Briefly, given a virtual double category $\mathcal{K}$ we will consider 'monoids' and 'bimodules' in $\mathcal{K}$, as recalled from Section 5.3 of [Lei04] (or Section 2 of [CS10]) in the definition below, and these arrange into a virtual double category $\mathsf{Mod}(\mathcal{K})$. The latter admits all horizontal units so that we can apply Example 1.7, thus obtaining an augmented virtual double category $(N \circ \mathsf{Mod})(\mathcal{K})$. Often we will then consider a sub-augmented virtual double category of $(N \circ \mathsf{Mod})(\mathcal{K})$ by "restricting the size of bimodules". For instance, while the canonical notion of bimodule between large categories (i.e. categories internal to a category Set' of 'large sets') is a profunctor $J \colon A \nrightarrow B$ with images $J(a, b)$ that are possibly large, we take the viewpoint (see Example 2.6 below) that it is preferable to consider profunctors with all images $J(a, b)$ small.

2.1. DEFINITION. [Leinster] *Let $\mathcal{K}$ be a virtual double category.*

- *A monoid A in $\mathcal{K}$ is a quadruple $A = (A, \alpha, \bar{\alpha}, \tilde{\alpha})$ consisting of a horizontal morphism $\alpha \colon A \nrightarrow A$ in $\mathcal{K}$ equipped with* multiplication *and* unit *cells*
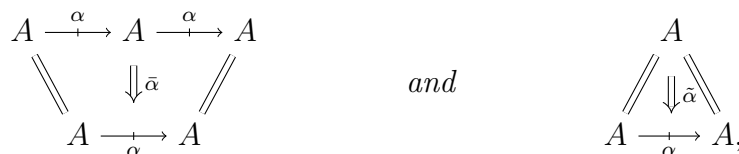
that satisfy the associativity axiom $\bar{\alpha} \circ (\bar{\alpha}, \mathrm{id}_\alpha) = \bar{\alpha} \circ (\mathrm{id}_\alpha, \bar{\alpha})$ *and the unit axioms* $\bar{\alpha} \circ (\tilde{\alpha}, \mathrm{id}_\alpha) = \mathrm{id}_\alpha = \bar{\alpha} \circ (\mathrm{id}_\alpha, \tilde{\alpha})$.

- *A morphism* $A \to C$ *of monoids is a vertical morphism* $f \colon A \to C$ *in* $\mathcal{K}$ *that is equipped with a cell*

$$
\begin{array}{ccc}
A & \xrightarrow{\alpha} & A \\
f \downarrow & \Downarrow \bar{f} & \downarrow f \\
C & \xrightarrow{\gamma} & C
\end{array}
$$

satisfying the associativity and unit axioms $\bar{\gamma} \circ (\bar{f}, \bar{f}) = \bar{f} \circ \bar{\alpha}$ *and* $\tilde{\gamma} \circ f = \bar{f} \circ \tilde{\alpha}$.

- *A* bimodule $A \nrightarrow B$ *between monoids is a horizontal morphism* $J \colon A \nrightarrow B$ *in* $\mathcal{K}$ *that is equipped with left and right* action *cells*

$$
\begin{array}{ccccc}
A & \xrightarrow{\alpha} & A & \xrightarrow{J} & B \\
& \Big\| & \Downarrow \lambda & \Big/\Big/ & \\
& A & \xrightarrow[J]{} & B &
\end{array}
\qquad and \qquad
\begin{array}{ccccc}
A & \xrightarrow{J} & B & \xrightarrow{\beta} & B \\
& \Big\| & \Downarrow \rho & \Big/\Big/ & \\
& A & \xrightarrow[J]{} & B, &
\end{array}
$$

satisfying the usual associativity, unit and compatibility axioms for bimodules:

$$
\begin{aligned}
\lambda \circ (\bar{\alpha}, \mathrm{id}_J) &= \lambda \circ (\mathrm{id}_\alpha, \lambda); & \rho \circ (\mathrm{id}_J, \bar{\beta}) &= \rho \circ (\rho, \mathrm{id}_\beta); \\
\lambda \circ (\tilde{\alpha}, \mathrm{id}_J) &= \mathrm{id}_J = \rho \circ (\mathrm{id}_J, \tilde{\beta}); & \rho \circ (\lambda, \mathrm{id}_\beta) &= \lambda \circ (\mathrm{id}_\alpha, \rho).
\end{aligned}
$$

- *A cell*

$$
\begin{array}{ccccccc}
A_0 & \xrightarrow{J_1} & A_1 & \cdots & A_{n'} & \xrightarrow{J_n} & A_n \\
f \downarrow & & & \Downarrow \phi & & & \downarrow g \\
C & & & \xrightarrow[K]{} & & & D
\end{array}
$$

of bimodules, where $n \geq 1$, *is a cell* ϕ *in* $\mathcal{K}$ *between the underlying morphisms satisfying the* external equivariance *axioms*

$$
\begin{aligned}
\phi \circ (\lambda_{J_1}, \mathrm{id}_{J_2}, \ldots, \mathrm{id}_{J_n}) &= \lambda_K \circ (\bar{f}, \phi) \\
\phi \circ (\mathrm{id}_{J_1}, \ldots, \mathrm{id}_{J_{n'}}, \rho_{J_n}) &= \rho_K \circ (\phi, \bar{g})
\end{aligned}
$$

and the internal equivariance *axioms*

$$
\begin{aligned}
\phi \circ (\mathrm{id}_{J_1}, \ldots, \mathrm{id}_{J_{i''}}, \rho_{J_{i'}}, \mathrm{id}_{J_i}, \mathrm{id}_{J_{i+1}}, \ldots, \mathrm{id}_{J_n}) \\
= \phi \circ (\mathrm{id}_{J_1}, \ldots, \mathrm{id}_{J_{i''}}, \mathrm{id}_{J_{i'}}, \lambda_{J_i}, \mathrm{id}_{J_{i+1}}, \ldots, \mathrm{id}_{J_n})
\end{aligned}
$$

for $2 \leq i \leq n$.

- *A cell*

$$
\begin{array}{c}
A \\
{}^{f}\diagup \quad \Downarrow\phi \quad \diagdown^{g} \\
C \xrightarrow[K]{} D
\end{array}
$$

of bimodules is a cell ϕ in $\mathcal{K}$ between the underlying morphisms satisfying the external equivariance *axiom* $\lambda \circ (\bar{f}, \phi) = \rho \circ (\phi, \bar{g})$.

Monoids in $\mathcal{K}$, their morphisms and bimodules, as well as the cells between them, form a virtual double category $\mathsf{Mod}(\mathcal{K})$, *whose composition and identities are simply those of $\mathcal{K}$. In fact the assignment $\mathcal{K} \mapsto \mathsf{Mod}(\mathcal{K})$ extends to an endo-2-functor on the 2-category* $\mathsf{VirtDblCat}$ *of virtual double categories; see Proposition 3.9 of [CS10].*

As is shown in Proposition 5.5 of [CS10] the virtual double category $\mathsf{Mod}(\mathcal{K})$ has all horizontal units, in the sense of Example 1.7. Indeed the unit I_A for a monoid $A = (A, \alpha, \bar{\alpha}, \tilde{\alpha})$ in $\mathcal{K}$ is the bimodule $I_A := \alpha \colon A \nrightarrow A$, whose actions λ and ρ are both given by multiplication $\bar{\alpha} \colon (\alpha, \alpha) \Rightarrow \alpha$. The cocartesian cell $\eta_A \colon A \Rightarrow \alpha$ is the unit cell $\eta_A := \tilde{\alpha}$: the factorisation of a cell ϕ through η_A, that is of the form as in (6), is obtained by composing ϕ with the right or left action of A on either bimodule $J_n \colon X_{n'} \nrightarrow A$ or $H_1 \colon A \nrightarrow Y_1$ in its horizontal source.

2.2. EXAMPLE. Let $\mathcal{K}$ be a virtual double category. Applying Example 1.7 to the unital virtual double category $\mathsf{Mod}(\mathcal{K})$ of bimodules in $\mathcal{K}$ we obtain the augmented virtual double category $(N \circ \mathsf{Mod})(\mathcal{K})$ whose objects, morphisms and unary cells are the same as those of $\mathsf{Mod}(\mathcal{K})$, as in Definition 2.1, while the nullary cells ξ of $(N \circ \mathsf{Mod})(\mathcal{K})$, of the shape as on the left below, are cells of bimodules ξ of the shape as on the right, where $\gamma \colon C \nrightarrow C$ is the horizontal unit bimodule for the monoid $C = (C, \gamma, \bar{\gamma}, \tilde{\gamma})$ as described above.

$$
\begin{array}{ccc}
A_0 \xrightarrow{J_1} A_1 \cdots A_{n'} \xrightarrow{J_n} A_n & \qquad & A_0 \xrightarrow{J_1} A_1 \cdots A_{n'} \xrightarrow{J_n} A_n \\
{}_{f}\searcol \quad \Downarrow\xi \quad \swarrow_{g} & & {}_{f}\big\downarrow \qquad \Downarrow\xi \qquad \big\downarrow^{g} \\
C & & C \xrightarrow[\gamma]{} C
\end{array}
$$

The remainder of this section consists of examples of (augmented) virtual double categories. They can be split into two kinds: Examples 2.4—2.8 are examples of enriched structures, while Examples 2.9—2.11 are examples of internal structures.

2.3. NOTATION. Throughout this article we assume given a category Set' of *large sets*, as well as a full subcategory $\mathsf{Set} \subsetneq \mathsf{Set}'$ of *small sets*, such that the collection of morphisms of Set forms an object in Set'. A large set $A \in \mathsf{Set}'$ will be called *properly large* if it is not isomorphic to any small set.

2.4. EXAMPLE. Let $\mathcal{V} = (\mathcal{V}, \otimes, I)$ be a monoidal category. The virtual double category $\mathcal{V}$-Mat of $\mathcal{V}$-*matrices* has large sets and functions as objects and vertical morphisms, while a horizontal morphism $J\colon A \nrightarrow B$ is a $\mathcal{V}$-matrix, given by a family $J(x, y)$ of $\mathcal{V}$-objects indexed by pairs $(x, y) \in A \times B$. A cell

$$
\begin{array}{ccc}
A_0 \xrightarrow{J_1} A_1 \;\cdots\; A_{n'} \xrightarrow{J_n} A_n \\
f\downarrow \qquad\quad \Downarrow\phi \qquad\quad \downarrow g \\
C \xrightarrow{\qquad K \qquad} D
\end{array}
$$

in $\mathcal{V}$-Mat is a family of $\mathcal{V}$-maps

$$
\phi_{(x_0,\ldots,x_n)}\colon J_1(x_0, x_1) \otimes \cdots \otimes J_n(x_{n'}, x_n) \to K(fx_0, gx_n)
$$

indexed by $(n+1)$-tuples $(x_0, \ldots, x_n) \in A_0 \times \cdots \times A_n$, where the tensor product is taken to be the monoidal unit I in the case that $n = 0$.

The augmented virtual double category $\mathcal{V}$-Prof $:= (N \circ \mathsf{Mod})(\mathcal{V}\text{-Mat})$ of monoids and bimodules in $\mathcal{V}$-Mat is that of large $\mathcal{V}$-*enriched* categories, $\mathcal{V}$-functors, $\mathcal{V}$-profunctors and $\mathcal{V}$-natural transformations. In some more detail: a $\mathcal{V}$-profunctor $J\colon A \nrightarrow B$, between $\mathcal{V}$-categories A and B, consists of a family of $\mathcal{V}$-objects $J(x, y)$, indexed by pairs of objects $x \in A$ and $y \in B$, that is equipped with associative and unital actions

$$
\lambda\colon A(x_1, x_2) \otimes J(x_2, y) \to J(x_1, y) \qquad \text{and} \qquad \rho\colon J(x, y_1) \otimes B(y_1, y_2) \to J(x, y_2)
$$

satisfying the usual compatibility axiom for bimodules; see e.g. Section 3 of [Law73]. If $\mathcal{V}$ is closed symmetric monoidal, so that it can be considered as enriched over itself, then $\mathcal{V}$-profunctors $J\colon A \nrightarrow B$ can be identified with $\mathcal{V}$-functors of the form $J\colon A^{\mathrm{op}} \otimes B \to \mathcal{V}$, where A^{op} denotes the dual of A (see e.g. Section 1.4 of [Kel82]). In Example 4.2 we will see that $\mathcal{V}$-Prof has all horizontal units so that by Theorem 10.1 it can equivalently be regarded as a virtual double category.

A vertical cell $\phi\colon f \Rightarrow g$ in $\mathcal{V}$-Prof, between $\mathcal{V}$-functors f and $g\colon A \to C$, is a $\mathcal{V}$-natural transformation $f \Rightarrow g$ in the usual sense; see for instance Section 1.2 of [Kel82]. We conclude that the vertical 2-category $V(\mathcal{V}\text{-Prof})$ contained in $\mathcal{V}$-Prof (Example 1.5) equals the 2-category $\mathcal{V}$-Cat of $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations.

Taking $\mathcal{V} = \mathsf{Set}$ in the above we obtain the augmented virtual double category Set-Prof of locally small (i.e. Set-enriched) categories, functors, Set-*profunctors* $J\colon A^{\mathrm{op}} \times B \to \mathsf{Set}$ and transformations. Likewise Set'-Prof is the augmented virtual double category of categories (possibly with large hom-sets), functors, Set'-*profunctors* $J\colon A^{\mathrm{op}} \times B \to \mathsf{Set}'$ and transformations. We will call a Set'-category A *locally properly large* if $A(x_1, x_2)$ is properly large for some $x_1, x_2 \in A$. Likewise a Set'-profunctor $J\colon A \nrightarrow B$ is *properly large* if $J(x, y)$ is properly large for some $x \in A$ and $y \in B$.

2.5. EXAMPLE. A *quantale* $\mathcal{V}$ (see e.g. Section II.1.10 of [HST14]) is a complete lattice equipped with a monoid structure $\otimes$ that preserves suprema on both sides. Equivalently,

a quantale can be thought of as a thin category $\mathcal{V}$ that is complete (hence cocomplete) and equipped with a closed monoidal structure.

The extended positive real line $\mathcal{V} = [0, \infty]$ for example, equipped with the reversed order $\geq$, forms a quantale whose monoid structure is given by addition $(+, 0)$ while its closed structure is truncated subtraction $[x, y] := \max(y - x, 0)$. Categories enriched in $[0, \infty]$ form Lawvere's paradigmatic example of enriched category theory [Law73]: they can be regarded as *generalised metric spaces*, that is sets A equipped with a (not necessarily symmetric) distance function $A \times A \to [0, \infty]$ (which we again denote by A). Both vertical morphisms $f \colon A \to C$ and horizontal morphisms $J \colon A \nrightarrow B$ in $[0, \infty]$-Prof are required to be *non-expanding*, that is $A(x_1, x_2) \geq C(fx_1, fx_2)$ and

$$A(x_1, x_2) + J(x_2, y) \geq J(x_1, y) \qquad \text{and} \qquad J(x, y_1) + B(y_1, y_2) \geq J(x, y_2)$$

respectively, for all $x_1, x_2, x \in A$ and $y, y_1, y_2 \in B$.

Notice that, because quantales $\mathcal{V}$ are thin categories, their induced augmented virtual double categories $\mathcal{V}$-Prof are *locally thin*: any cell in $\mathcal{V}$-Prof is uniquely determined by its (horizontal and vertical) sources and targets. Locally thin augmented virtual double categories of the form $\mathcal{V}$-Prof, where $\mathcal{V}$ is a quantale, form a natural setting for the study of 'monoidal topology' [HST14], see for instance [Kou18].

The following example motivates our choice of augmented virtual double categories as the optimal 'double dimensional' environment for classical category theory.

2.6. EXAMPLE. Taking $\mathcal{V} = \mathsf{Set}'$ in Example 2.4, we write $(\mathsf{Set}, \mathsf{Set}')$-Prof for the locally full sub-augmented virtual double category of Set'-Prof that is generated by Set-profunctors. In detail: $(\mathsf{Set}, \mathsf{Set}')$-Prof consists of all Set'-categories and functors, only those profunctors $J \colon A \nrightarrow B$ with $J(x, y) \in \mathsf{Set}$ for all $(x, y) \in A \times B$, and all cells between such Set-profunctors (including the nullary and vertical cells).

Thus we have a chain of sub-augmented virtual double categories

$$\mathsf{Set}\text{-Prof} \subsetneqq (\mathsf{Set}, \mathsf{Set}')\text{-Prof} \subsetneqq \mathsf{Set}'\text{-Prof},$$

and we take the view that the classical theory of locally small categories is best considered in $(\mathsf{Set}, \mathsf{Set}')$-Prof, motivated as follows. Recall from [FS95] that, for a locally small category A, the category $\mathsf{Set}^{A^{\mathrm{op}}}$ of presheaves on A is locally small if and only if A is essentially small. Thus, on one hand, presheaves on a locally small category A in general do not form an object in Set-Prof, while they do form one in $(\mathsf{Set}, \mathsf{Set}')$-Prof. On the other hand, writing $\mathrm{y} \colon A \to \mathsf{Set}^{A^{\mathrm{op}}}$ for the yoneda embedding, Yoneda's lemma supplies, for each horizontal morphism $J \colon A \nrightarrow B$ in $(\mathsf{Set}, \mathsf{Set}')$-Prof, a functor $J^{\lambda} \colon B \to \mathsf{Set}^{A^{\mathrm{op}}}$ equipped with a natural isomorphism of Set-profunctors $J \cong \mathsf{Set}^{A^{\mathrm{op}}}(\mathrm{y}, J^{\lambda})$[1]; of course such a J^{λ} does not exist for the properly large profunctors J contained in Set'-Prof. Thus the objects in $(\mathsf{Set}, \mathsf{Set}')$-Prof are "large enough" for it to contain all presheaf categories $\mathsf{Set}^{A^{\mathrm{op}}}$ with A locally small while its horizontal morphisms are "small enough" to allow

---

[1]Indeed, take $J^{\lambda}(y) := J(-, y)$ for $y \in B$.

for a simple universal property of the yoneda embeddings $y\colon A \to \mathsf{Set}^{A^{\mathrm{op}}}$, that is given in terms of *all* horizontal morphisms of $(\mathsf{Set}, \mathsf{Set}')$-$\mathsf{Prof}$ instead of a certain subclass of "admissible" ones, the latter such as in the definition of yoneda structure [SW78]; in particular this universal property is straightforward to formalise.

For an example of an advantage of working in the augmented virtual double category $(\mathsf{Set}, \mathsf{Set}')$-$\mathsf{Prof}$ rather than in the virtual double category $U\left((\mathsf{Set}, \mathsf{Set}')\text{-}\mathsf{Prof}\right)$ that it contains (Example 1.4) notice that, for any two functors f and $g\colon A \to C$ into a locally properly large category C, the natural transformations $\phi\colon f \Rightarrow g$ are contained in the former but cannot be considered the latter. Indeed in $(\mathsf{Set}, \mathsf{Set}')$-$\mathsf{Prof}$ they exist as vertical cells $\phi\colon f \Rightarrow g$, but these are removed when passing to $U\left((\mathsf{Set}, \mathsf{Set}')\text{-}\mathsf{Prof}\right)$. And while such natural transformations correspond to cells in Set'-Prof of the form below, where I_C is the 'unit profunctor' given by the hom-sets $I_C(x, y) = C(x, y)$, the properly large profunctor I_C is not contained in $(\mathsf{Set}, \mathsf{Set}')$-$\mathsf{Prof}$ (see Example 4.6) and thus neither in $U\left((\mathsf{Set}, \mathsf{Set}')\text{-}\mathsf{Prof}\right)$.

$$
\begin{array}{c}
A \\
f \swarrow \;\; \Downarrow\phi \;\; \searrow g \\
C \xrightarrow[I_C]{} C
\end{array}
$$

Considering Set'-categories is one way of dealing with the size of the categories $\mathsf{Set}^{A^{\mathrm{op}}}$ of presheaves on locally small categories A. Another way is to restrict to 'small' presheaves on A instead, as recalled in Example 2.8 below. The next example generalises the construction of $(\mathsf{Set}, \mathsf{Set}')$-$\mathsf{Prof}$ above to the enriched setting.

2.7. EXAMPLE. Analogous to the previous example we can consider sub-augmented virtual double categories $(\mathsf{Ab}, \mathsf{Ab}')$-$\mathsf{Prof} \subset \mathsf{Ab}'$-$\mathsf{Prof}$, $(\mathsf{Cat}, \mathsf{Cat}')$-$\mathsf{Prof} \subset \mathsf{Cat}'$-$\mathsf{Prof}$, etc., where $\mathsf{Ab} \subset \mathsf{Ab}'$, $\mathsf{Cat} \subset \mathsf{Cat}'$, etc., are embeddings obtained by considering abelian groups, categories, etc., in both categories of sets Set and Set' respectively. Again we prefer to work in e.g. $(\mathsf{Ab}, \mathsf{Ab}')$-$\mathsf{Prof}$ instead of Ab-Prof or Ab'-Prof, for reasons similar to the ones given in the previous example.

More generally we will follow Kelly's approach in Section 3.11 of [Kel82], which is based on [Day70], and enrich both in a monoidal category $\mathcal{V}$ as well as in a 'universe enlargement' of $\mathcal{V}$, as follows. A *universe enlargement* of a large (not necessarily closed) monoidal category $\mathcal{V}$ is a monoidal full embedding $\mathcal{V} \subset \mathcal{V}'$ of $\mathcal{V}$ into a closed[1] monoidal category $\mathcal{V}'$ that satisfies the following axioms:

(a) $\mathcal{V}'$ is locally large, that is $\mathcal{V}'(x', y') \in \mathsf{Set}'$ for all $x', y' \in \mathcal{V}'$;

(b) $\mathcal{V}'$ is large complete and large cocomplete;

(c) $\mathcal{V} \subset \mathcal{V}'$ preserves all limits.

---

[1] $\mathcal{V}' = (\mathcal{V}', \otimes', I')$ is closed if, for every object $x' \in \mathcal{V}'$, the endofunctor $x' \otimes -$ has a right adjoint $[x', -]$; see e.g. Section 1.5 of [Kel82].

One can show that the embeddings $\mathsf{Set} \subset \mathsf{Set}'$, $\mathsf{Ab} \subset \mathsf{Ab}'$ and $\mathsf{Cat} \subset \mathsf{Cat}'$ are universe enlargements in the above sense, as long as Set has infinite sets. More generally Kelly shows that the yoneda embedding $\mathrm{y}\colon \mathcal{V} \to \mathsf{Set}'^{\mathcal{V}^{\mathrm{op}}}$ defines the category $\mathsf{Set}'^{\mathcal{V}^{\mathrm{op}}}$ of Set'-presheaves on $\mathcal{V}$ as a universe enlargement of $\mathcal{V}$, with the monoidal structure $\otimes'$ on the category $\mathsf{Set}'^{\mathcal{V}^{\mathrm{op}}}$ given by 'Day convolution' [Day70] (or see (1) above). If $\mathcal{V}$ is closed monoidal then, besides (a)—(c) above, the yoneda embedding $\mathrm{y}\colon \mathcal{V} \to \mathsf{Set}'^{\mathcal{V}^{\mathrm{op}}}$ also is a closed monoidal embedding, that is $\mathrm{y}([x,y]) \cong [\mathrm{y}x, \mathrm{y}y]'$ coherently for all x, $y \in \mathcal{V}$. In that case, as is shown in Section 3.12 of [Kel82], the factorisation of y through the full subcategory $\mathcal{V}' \subset \mathsf{Set}'^{\mathcal{V}^{\mathrm{op}}}$ of Set'-presheaves that preserve all large limits in $\mathcal{V}^{\mathrm{op}}$ is a universe enlargement $\mathcal{V} \subset \mathcal{V}'$ that, besides preserving all limits, preserves large colimits as well.

Returning to a universe enlargement $\mathcal{V} \subset \mathcal{V}'$ with $\mathcal{V}$ not necessarily closed, consider a $\mathcal{V}'$-profunctor $J\colon A \nrightarrow B$ in $\mathcal{V}'$-Prof (see Example 2.4). We will call J a $\mathcal{V}$-profunctor whenever $J(x,y)$ is a $\mathcal{V}$-object for all pairs $x \in A$ and $y \in B$. Analogous to the definition of $(\mathsf{Set}, \mathsf{Set}')$-$\mathsf{Prof}$ in the previous example we denote by $(\mathcal{V}, \mathcal{V}')$-$\mathsf{Prof}$ the sub-augmented virtual double category of $\mathcal{V}'$-Prof that consists of all $\mathcal{V}'$-categories and $\mathcal{V}'$-functors, as well as $\mathcal{V}$-profunctors and their transformations.

In the next example we recall the notion of 'small $\mathcal{V}$-profunctor' and show that such profunctors form a sub-augmented virtual double category of $\mathcal{V}$-Prof (Example 2.4). In doing so we use the classical coend formula that defines compositions of $\mathcal{V}$-profunctors, which we first recall briefly. Let $\underline{J} = (A_0 \overset{J_1}{\nrightarrow} A_1, \ldots, A_{n'} \overset{J_n}{\nrightarrow} A_n)$ be a non-empty path of $\mathcal{V}$-profunctors and let $x \in A_0$ and $y \in A_n$ be objects. Inspired by Mac Lane's construction of ends as limits in Section IX.5 of [ML98] we consider the following diagram functor $\underline{J}^{\S}(x,y)\colon \underline{J}^{\mathrm{S}} \to \mathcal{V}$. The objects of J^{S} are of two kinds: they are either n'-tuples of pairs $\big((v_1, w_1), (v_2, w_2), \ldots, (v_{n'}, w_{n'})\big)$, with each pair (v_i, w_i) objects in A_i, or they are n'-tuples $(u_1, u_2, \ldots, u_n)$ of objects $u_i \in A_i$. The non-identity morphisms of J^{S} are the legs of spans of the form

$$(v_1, v_2, \ldots, v_{n'}) \leftarrow \big((v_1, w_1), (v_2, w_2), \ldots, (v_{n'}, w_{n'})\big) \to (w_1, w_2, \ldots, w_{n'});$$

consequently in any composable pair of morphisms in J^{S} either morphism necessarily is an identity. Having defined J^{S} we next denote by $\underline{J}^{\S}(x,y)$ the diagram $J^{\mathrm{S}} \to \mathcal{V}$ that maps each span above to the following span in $\mathcal{V}$.

$$J_1(x, v_1) \otimes A_1(v_1, w_1) \otimes J_2(w_1, v_2) \otimes A_2(v_2, w_2) \otimes \cdots \otimes A_n(v_{n'}, w_{n'}) \otimes J_n(w_{n'}, y)$$

$$\mathrm{id} \otimes \lambda \otimes \cdots \otimes \lambda \qquad\qquad \rho \otimes \cdots \otimes \rho \otimes \mathrm{id}$$

$$J_1(x, v_1) \otimes J_2(v_1, v_2) \otimes \cdots \otimes J_n(v_{n'}, y) \qquad\qquad J_1(x, w_1) \otimes J_2(w_1, w_2) \otimes \cdots \otimes J_n(w_{n'}, y)$$

In the case $\underline{J} = (J_1, J_2)$ the colimits of $(J_1, J_2)^{\S}(x,y)$, if they exist for all $x \in A_0$ and $y \in A_2$, combine to form the composite $\mathcal{V}$-profunctor '$J_2 \circ J_1\colon A_0 \nrightarrow A_2$' as defined in Section 3 of [Law73]. For general $\underline{J}$, if $\mathcal{V}$ is closed symmetric monoidal, so that each $J_i\colon A_{i'} \nrightarrow A_i$ can be identified with a $\mathcal{V}$-functor $J_i\colon A_{i'}^{\mathrm{op}} \otimes A_i \to \mathcal{V}$, then the colimit of

$\underline{J}^{\S}(x,y)$ is easily checked to coincide with the iterated *coend* on the left-hand side below; for the definition of the dual notion 'end' see e.g. Section 2.1 of [Kel82]. We will use the coend notation

$$\int\limits^{u_1 \in A_1} \cdots \int\limits^{u_{n'} \in A_{n'}} J_1(x, u_1) \otimes \cdots \otimes J_n(u_{n'}, y) := \operatorname{colim} \underline{J}^{\S}(x, y)$$

for the colimit of $\underline{J}^{\S}(x,y)$ regardless of whether the monoidal category $\mathcal{V}$ is closed symmetric.

2.8. EXAMPLE. A $\mathcal{V}$-profunctor $J \colon A \nrightarrow B$ in $\mathcal{V}$-Prof (Example 2.4) is called *small* if for each object $y \in B$ there exists a small sub-$\mathcal{V}$-category $A_y \subseteq A$ such that, for each $x \in A$, the coend below exists and

$$J(x, y) \cong \int\limits^{x' \in A_y} A(x, x') \otimes J(x', y).$$

For example if $\mathcal{V} = \mathsf{Set}$ and A is any large set seen as a discrete category, then a Set-profunctor $J \colon A \nrightarrow B$, with B any category, is small precisely if for each $y \in B$ the set

$$\{x \in A \mid J(x, y) \neq \emptyset\}$$

is small, which in that case we can take as A_y. In general notice that any $\mathcal{V}$-profunctor $J \colon A \nrightarrow B$ is small whenever A is a small $\mathcal{V}$-category.

   We denote by $\mathcal{V}$-sProf $\subseteq \mathcal{V}$-Prof the sub-augmented virtual double category consisting of all $\mathcal{V}$-categories, all $\mathcal{V}$-functors, only small $\mathcal{V}$-profunctors, and all cells between them (including the nullary and vertical ones). We will see in Example 4.7 that $\mathcal{V}$-sProf has horizontal units and, in Example 9.3, that, unlike $\mathcal{V}$-Prof and $(\mathcal{V}, \mathcal{V}')$-Prof, it has all horizontal composites (see Section 7) whenever $\mathcal{V}$ is small cocomplete such that its monoidal product $\otimes$ preserves large colimits on both sides. Thus in that case $\mathcal{V}$-sProf is a pseudo double category in the sense of [GP99] (or see Section 7 below).

   To see that, when $\mathcal{V}$ is closed symmetric monoidal, the above notion agrees with the usual notion of smallness for $\mathcal{V}$-profunctors notice that, by equation (4.25) of [Kel82], for each $y \in B$ the isomorphisms above exhibit the $\mathcal{V}$-presheaf $J(-, y) \colon A^{\mathrm{op}} \to \mathcal{V}$ as the left Kan extension of $J(-, y)$ along the inclusion $A_y \subseteq A$. Hence each $J(-, y)$ is an 'accessible' $\mathcal{V}$-presheaf in the sense of Proposition 4.83 of [Kel82]; more recently (e.g. [DL07]) such $\mathcal{V}$-presheaves have been termed *small*. Assuming that the $\mathcal{V}$-category $[B, \mathcal{V}]$ of $\mathcal{V}$-functors $B \to \mathcal{V}$ exists (see Section 2 of [Kel82]), it follows that $J \colon A \nrightarrow B$ is small in the above sense precisely if the corresponding $\mathcal{V}$-functor $A^{\mathrm{op}} \to [B, \mathcal{V}]$ is 'pointwise small' in the sense of [DL07].

2.9. EXAMPLE. Let $\mathcal{E}$ be a category with pullbacks. The augmented virtual double category $\mathsf{Span}(\mathcal{E})$ of *spans* in $\mathcal{E}$ has as objects and vertical morphisms the objects and morphisms of $\mathcal{E}$, while its horizontal morphisms $J \colon A \nrightarrow B$ are spans $A \leftarrow J \to B$ in $\mathcal{E}$. A

unary cell ϕ in $\mathsf{Span}(\mathcal{E})$, as on the left below, is a morphism $\phi\colon J_1 \times_{A_1} \cdots \times_{A_{n'}} J_n \to K$ in $\mathcal{E}$ lying over f and g, where the wide pullback is taken to be A_0 if the horizontal source of ϕ is empty. Nullary cells in $\mathsf{Span}(\mathcal{E})$ on the other hand are uniquely determined by their boundary: a cell ψ as in the middle exists precisely if the square on the right commutes.

$$
\begin{array}{ccc}
A_0 \xrightarrow{\;J\;} A_n & \qquad A_0 \xrightarrow{\;J\;} A_n \\
f \downarrow \quad \Downarrow\phi \quad \downarrow g & \qquad f \searrow \;\Downarrow\psi\; \swarrow g \\
C \xrightarrow[K]{} D & \qquad C
\end{array}
$$

$$
\begin{array}{c}
J_1 \times_{A_1} \cdots \times_{A_{n'}} J_n \\
\swarrow \qquad \searrow \\
A_0 \qquad\qquad A_n \\
f \searrow \qquad \swarrow g \\
C
\end{array}
$$
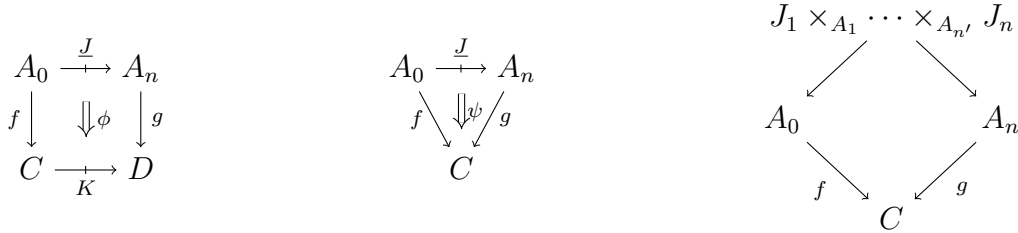
The virtual double category $U(\mathsf{Span}(\mathcal{E}))$ (Example 1.4) contained in $\mathsf{Span}(\mathcal{E})$ is the same as that considered in Example 2.7 of [CS10]. The augmented virtual double category $\mathsf{Prof}(\mathcal{E}) := (N \circ \mathsf{Mod})(U(\mathsf{Span}(\mathcal{E})))$ of monoids and bimodules in $U(\mathsf{Span}(\mathcal{E}))$ is that of *internal* categories, functors, profunctors and transformations in $\mathcal{E}$. The vertical 2-category $V(\mathsf{Prof}(\mathcal{E}))$ contained inside $\mathsf{Prof}(\mathcal{E})$ (Example 1.5) is the 2-category $\mathsf{Cat}(\mathcal{E})$ of internal categories, functors and transformations in $\mathcal{E}$; the latter in the classical sense of [Str74].

We will see that $\mathsf{Span}(\mathcal{E})$ has all horizontal units (Example 4.3) and composites (Example 7.3), so that it can be equivalently regarded as a pseudo double category (see Section 7).

2.10. EXAMPLE. As in the previous example let $\mathcal{E}$ be a category with pullbacks. A span $A \xleftarrow{j_0} J \xrightarrow{j_1} B$ in $\mathcal{E}$ is called a *relation* (see e.g. [CKS84]) if any two horizontal cells $\phi, \psi\colon H \Rightarrow J$ in $\mathsf{Span}(\mathcal{E})$ are equal, that is j_0 and j_1 are jointly monic. We denote by $\mathsf{Rel}(\mathcal{E}) \subseteq \mathsf{Span}(\mathcal{E})$ the sub-augmented virtual double category generated by the relations in $\mathcal{E}$. Like $\mathsf{Span}(\mathcal{E})$, $\mathsf{Rel}(\mathcal{E})$ has all horizontal units (see Example 4.3), so that it can be equivalently regarded as a virtual double category by Theorem 10.1. Notice that $\mathsf{Rel}(\mathcal{E})$ is a locally thin augmented virtual double category in the sense of Example 2.5.

We remark that in order to be able to arrange relations in $\mathcal{E}$ into a bicategory or a pseudo double category (see [GP99] or Section 7 below) one needs $\mathcal{E}$ to be *regular* (see e.g. [CKS84]); in contrast, to form $\mathsf{Rel}(\mathcal{E})$ as an (augmented) virtual double category it suffices that $\mathcal{E}$ has pullbacks.

2.11. EXAMPLE. Let $\mathcal{K}$ be a finitely complete 2-category, that is $\mathcal{K}$ has all finite conical limits as well as cotensors with the "walking arrow" category $\mathbf{2} := (0 \to 1)$. 'Split bifibrations' in $\mathcal{K}$, introduced in [Str74] and recently called *split two-sided fibrations*, can be regarded as profunctors internal to $\mathcal{K}_0$, the category underlying $\mathcal{K}$, as follows. For $A \in \mathcal{K}$ the cotensor $\Phi A := [\mathbf{2}, A]$ is defined by a cell

$$
\begin{array}{c}
\Phi A \\
d_0 \left(\;\Downarrow\; \right) d_1 \\
A
\end{array}
$$

whose universal property induces on the span $A \xleftarrow{d_0} \Phi A \xrightarrow{d_1} A$ the structure of a category internal to $\mathcal{K}_0$. In fact, Proposition 2 of [Str74] shows that choosing a cotensor ΦA for each $A \in \mathcal{K}$ induces a functor $\Phi \colon \mathcal{K}_0 \to \mathsf{Cat}(\mathcal{K}_0) = V(\mathsf{Prof}(\mathcal{K}_0))$ (Example 2.9). Given objects $A, B \in \mathcal{K}$, an internal profunctor $J \colon \Phi A \nrightarrow \Phi B$ in $\mathsf{Prof}(K_0)$ is precisely a 'split bifibration' $A \leftarrow J \to B$ in $\mathcal{K}$ in the sense of [Str74], which follows easily from Proposition 12 therein. Likewise horizontal cells $\phi \colon J \Rightarrow K$ in $\mathsf{Prof}(K_0)$, with $K \colon \Phi A \nrightarrow \Phi B$, are morphisms of bifibrations in the sense of [Str74].

In light of the above we denote by $\mathsf{spFib}(\mathcal{K})$ the augmented virtual double category whose objects and vertical morphisms are those of $\mathcal{K}_0$, and whose horizontal morphisms $J \colon A \nrightarrow B$ are profunctors $J \colon \Phi A \nrightarrow \Phi B$ internal to $\mathcal{K}_0$. Cells in $\mathsf{spFib}(\mathcal{K})$, with vertical source $f \colon A_0 \to C$ and target $g \colon A_n \to D$, are cells in $\mathsf{Prof}(\mathcal{K}_0)$ with vertical source $\Phi f \colon \Phi A_0 \to \Phi C$ and target $\Phi g \colon \Phi A_n \to \Phi D$; their compositions are defined as in $\mathsf{Prof}(\mathcal{K}_0)$.

3. The 2-category of augmented virtual double categories

Having introduced the notion of augmented virtual double categories next we consider the functors between them, as well as their transformations.

3.1. DEFINITION. *A functor $F \colon \mathcal{K} \to \mathcal{L}$ between augmented virtual double categories consists of a functor $F \colon \mathcal{K}_v \to \mathcal{L}_v$ as well as assignments mapping the horizontal morphisms and cells of $\mathcal{K}$ to those of $\mathcal{L}$, as shown below, in a way that preserves vertical composition and identity cells strictly.*

$$J \colon A \nrightarrow B \qquad \mapsto \qquad FJ \colon FA \nrightarrow FB$$

$$
\begin{array}{ccc}
A_0 \xrightarrow{J_1} A_1 \cdots A_{n'} \xrightarrow{J_n} A_n & & FA_0 \xrightarrow{FJ_1} FA_1 \cdots FA_{n'} \xrightarrow{FJ_n} FA_n \\
f\downarrow \quad \Downarrow\phi \quad \downarrow g & \mapsto & Ff\downarrow \quad \Downarrow F\phi \quad \downarrow Fg \\
C \xrightarrow{\quad K \quad} D & & FC \xrightarrow{\quad FK \quad} FD
\end{array}
$$

$$
\begin{array}{ccc}
A_0 \xrightarrow{J_1} A_1 \cdots A_{n'} \xrightarrow{J_n} A_n & & FA_0 \xrightarrow{FJ_1} FA_1 \cdots FA_{n'} \xrightarrow{FJ_n} FA_n \\
f \searrow \Downarrow\phi \swarrow g & \mapsto & Ff \searrow \Downarrow F\phi \swarrow Fg \\
C & & FC
\end{array}
$$

Notice that $F \colon \mathcal{K} \to \mathcal{L}$ preserving vertical composition $\circ$ implies that F preserves horizontal composition $\odot$ (see Lemma 1.3).

3.2. DEFINITION. *A transformation $\xi \colon F \Rightarrow G$ of functors $F, G \colon \mathcal{K} \to \mathcal{L}$ of augmented virtual double categories consists of a natural transformation $\xi \colon F_v \Rightarrow G_v$ as well as a*

family of $(1,1)$-ary cells

$$
\begin{array}{ccc}
FA & \xrightarrow{FJ} & FB \\
\xi_A \downarrow & \Downarrow \xi_J \ \ \downarrow \xi_B & \\
GA & \xrightarrow[GJ]{} & GB
\end{array}
$$

in $\mathcal{L}$, one for each $J\colon A \nrightarrow B \in \mathcal{K}$, that satisfies the naturality axiom

$$G\phi \circ \xi_{\underline{J}} = \xi_{\underline{K}} \circ F\phi$$

for all cells $\phi\colon \underline{J} \Rightarrow \underline{K}$ in $\mathcal{K}$, where $\xi_{\underline{J}} \coloneqq (\xi_{J_1}, \ldots, \xi_{J_n})$ if $\underline{J} = (J_1, \ldots, J_n)$ and $\xi_{\underline{J}} \coloneqq \xi_A$ if $\underline{J} = (A)$.

In Example 1.4 we saw that restricting to augmented virtual double categories with only vertical identity cells as nullary cells recovers the notion of virtual double category. Likewise, under this restriction the definitions above reduce to that of functor and transformation for virtual double categories as given in Section 3 of [CS10]. The latter combine into a 2-category of virtual double categories which we denote VirtDblCat. Remember that every augmented virtual double category $\mathcal{K}$ contains a 2-category $V(\mathcal{K})$ (Example 1.5) and a virtual double category $U(\mathcal{K})$ (Example 1.4). In the following proposition, which is easily checked, 2-Cat denotes the 2-category of 2-categories, strict 2-functors and 2-natural transformations.

3.3. PROPOSITION. *Augmented virtual double categories, the functors between them and their transformations form a 2-category* AugVirtDblCat. *Both the assignments $\mathcal{K} \mapsto V(\mathcal{K})$ and $\mathcal{K} \mapsto U(\mathcal{K})$ extend to strict 2-functors*

$$V\colon \mathsf{AugVirtDblCat} \to \mathsf{2\text{-}Cat} \qquad and \qquad U\colon \mathsf{AugVirtDblCat} \to \mathsf{VirtDblCat}.$$

3.4. EXAMPLE. Every lax monoidal functor $F\colon \mathcal{V} \to \mathcal{W}$ between monoidal categories induces a functor $F\text{-Mat}\colon \mathcal{V}\text{-Mat} \to \mathcal{W}\text{-Mat}$ between the virtual double categories of matrices in $\mathcal{V}$ and $\mathcal{W}$ (Example 2.4) in the evident way. Likewise the components of any monoidal transformation $\xi\colon F \Rightarrow G$ form the cell-components of an induced transformation $\xi\text{-Mat}\colon F\text{-Mat} \Rightarrow G\text{-Mat}$. The assignments $F \mapsto F\text{-Mat}$ and $\xi \mapsto \xi\text{-Mat}$ combine to form a strict 2-functor $(-)\text{-Mat}\colon \mathsf{MonCat}_l \to \mathsf{VirtDblCat}$, where MonCat_l denotes the 2-category of monoidal categories, lax monoidal functors and monoidal transformations.

3.5. EXAMPLE. Similarly any pullback-preserving functor $F\colon \mathcal{D} \to \mathcal{E}$, between categories with pullbacks, induces a functor $\mathsf{Span}(F)\colon \mathsf{Span}(\mathcal{D}) \to \mathsf{Span}(\mathcal{E})$ between the augmented virtual double categories of spans in $\mathcal{D}$ and $\mathcal{E}$ (see Example 2.9). This too extends to a strict 2-functor $\mathsf{Span}(-)\colon \mathsf{Cat}_{\mathrm{pb}} \to \mathsf{AugVirtDblCat}$, where $\mathsf{Cat}_{\mathrm{pb}}$ denotes the 2-category of categories with pullbacks, pullback-preserving functors and all natural transformations between them.

By an equivalence of augmented virtual double categories we, of course, mean an internal equivalence in the 2-category AugVirtDblCat. The goal of the remainder of this

section is to prove that, like in classical category theory (see e.g. Section IV.4 of [ML98]), giving an equivalence $\mathcal{K} \simeq \mathcal{L}$ of augmented virtual double categories is the same as giving a functor $F\colon \mathcal{K} \to \mathcal{L}$ that is 'full, faithful and essentially surjective'. The following definitions generalise analogous definitions for functors between double categories given in Section 7 of [Shu08].

We start with the notions full and faithful. Let $F\colon \mathcal{K} \to \mathcal{L}$ be a functor between augmented virtual double categories. Its restriction $J \mapsto FJ$ to horizontal morphisms preserves sources and targets, so that it extends to an assignment

$$\underline{J} = (J_1, \ldots, J_n) \mapsto F\underline{J} := (FJ_1, \ldots, FJ_n)$$

on paths. Likewise, for any pair of morphisms $f\colon A_0 \to C$ and $g\colon A_n \to D$ in $\mathcal{K}$, together with paths $\underline{J}\colon A_0 \nrightarrow A_n$ and $\underline{K}\colon C \nrightarrow D$ where $|K| \leq 1$, the functor F restricts to an assignment below, between classes of cells with sources and targets as shown.

$$
\left\{
\begin{array}{ccc}
A_0 & \xrightarrow{\underline{J}} & A_n \\
f\downarrow & \Downarrow\phi & \downarrow g \\
C & \xrightarrow{\underline{K}} & D
\end{array}
\right\}
\quad \xrightarrow{\ F\ } \quad
\left\{
\begin{array}{ccc}
FA_0 & \xrightarrow{F\underline{J}} & FA_n \\
Ff\downarrow & \Downarrow\psi & \downarrow Fg \\
FC & \xrightarrow{F\underline{K}} & FD
\end{array}
\right\}
$$

3.6. DEFINITION. *A functor $F\colon \mathcal{K} \to \mathcal{L}$ between augmented virtual double categories is called* locally faithful *(resp.* locally full*) if, for any $f\colon A_0 \to C$, $g\colon A_n \to D$, $\underline{J}\colon A_0 \nrightarrow A_n$ and $\underline{K}\colon C \nrightarrow D$ with $|K| \leq 1$ in $\mathcal{K}$, the assignment above is injective (resp. surjective). If moreover the restriction $F\colon \mathcal{K}_\mathrm{v} \to \mathcal{L}_\mathrm{v}$, to the vertical categories, is faithful (resp. full), then F is called* faithful *(resp.* full*).*

The following is Definition 7.6 of [Shu08] applied verbatim to the setting of augmented virtual double categories.

3.7. DEFINITION. *A functor $F\colon \mathcal{K} \to \mathcal{L}$ of augmented virtual double categories is called* essentially surjective *if we can simultaneously make the following choices:*

- *for each object $A \in \mathcal{L}$, an object $A' \in \mathcal{K}$ and an isomorphism $\sigma_A\colon FA' \cong A$;*

- *for each horizontal morphism $J\colon A \nrightarrow B$ in $\mathcal{L}$, a morphism $J'\colon A' \nrightarrow B'$ in $\mathcal{K}$ and an invertible cell*

$$
\begin{array}{ccc}
FA' & \xrightarrow{FJ'} & FB' \\
\sigma_A\downarrow & \Downarrow\sigma_J & \downarrow\sigma_B \\
A & \xrightarrow{\ J\ } & B.
\end{array}
$$

3.8. PROPOSITION. *A functor $F\colon \mathcal{K} \to \mathcal{L}$ between augmented virtual double categories is part of an equivalence $\mathcal{K} \simeq \mathcal{L}$ if and only if it is full, faithful and essentially surjective.*

PROOF (SKETCH). The 'only if'-part is straightforward; we will sketch the 'if'-part. First, because F is essentially surjective, we can choose objects $A' \in \mathcal{K}$, for each $A \in \mathcal{L}$, and horizontal morphisms $J' \colon A' \nrightarrow B' \in \mathcal{K}$, for each $J \colon A \nrightarrow B \in \mathcal{L}$, as in the definition above, together with isomorphisms $\sigma_A \colon FA' \cong A$ and $\sigma_J \colon FJ' \cong J$. Using the full and faithfulness of F these choices can be extended to a functor $(-)' \colon \mathcal{L} \to \mathcal{K}$ as follows: for each vertical morphism $f \colon A \to C$ in $\mathcal{L}$ we define $f' \colon A' \to C'$ to be the unique map in $\mathcal{K}$ such that $Ff = \sigma_C^{-1} \circ f \circ \sigma_A$, and for each cell $\phi \colon \underline{J} \Rightarrow \underline{K}$ in $\mathcal{L}$ we define $\phi' \colon \underline{J}' \Rightarrow \underline{K}'$ to be the unique cell in $\mathcal{K}$ such that $F\phi' = \sigma_{\underline{K}}^{-1} \circ \phi \circ \sigma_{\underline{J}}$, where the notation $\sigma_{\underline{J}}$ is as in Definition 3.2. Using that F is faithful it is easily checked that these assignments preserve the composition and identities of $\mathcal{L}$.

Finally the isomorphisms $(\sigma_A)_{A \in \mathcal{L}}$ and $(\sigma_J)_{J \in \mathcal{L}}$ combine to form a transformation $\sigma \colon F \circ (-)' \cong \mathrm{id}_\mathcal{L}$. Conversely, a transformation $\eta \colon \mathrm{id}_\mathcal{K} \cong (-)' \circ F$ is obtained by defining η_A, where $A \in \mathcal{K}$, to be unique with $F\eta_A = \sigma_{FA}^{-1}$ and defining η_J, where $J \colon A \nrightarrow B$ in $\mathcal{K}$, such that $F\eta_J = \sigma_{FJ}^{-1}$.   ∎

4. Restriction of horizontal morphisms

In this section we consider the restriction of horizontal morphisms along vertical morphisms, a construction that is often used in the study of formal category theory internal to (generalised) double categories. Restrictions of horizontal morphisms are defined by 'cartesian cells' as in the following definition, which generalises the notions of $(1,1)$-ary cartesian cell considered in Section 7 of [CS10], for virtual double categories, and in Section 4 of [Shu08], for double categories, to (n,m)-ary cartesian cells where $n, m \leq 1$.

4.1. DEFINITION. *A cell $\psi \colon \underline{J} \Rightarrow \underline{K}$ with $|\underline{J}| \leq 1$, as in the right-hand side below, is called* cartesian *if any cell χ, as on the left-hand side, factors uniquely through ψ as a cell ϕ as shown.*

$$
\begin{array}{ccc}
X_0 \xrightarrow{H_1} X_1 \cdots X_{n'} \xrightarrow{H_n} X_n & & X_0 \xrightarrow{H_1} X_1 \cdots X_{n'} \xrightarrow{H_n} X_n \\
h \downarrow \qquad\qquad\qquad\quad \downarrow k & & h \downarrow \qquad \Downarrow \phi \qquad \downarrow k \\
A \qquad \Downarrow \chi \qquad B & = & A \xrightarrow[\;\;J\;\;]{} B \\
f \downarrow \qquad\qquad\qquad\quad \downarrow g & & f \downarrow \qquad \Downarrow \psi \qquad \downarrow g \\
C \xrightarrow[\;\;\underline{K}\;\;]{} D & & C \xrightarrow[\;\;\underline{K}\;\;]{} D
\end{array}
$$

Vertically dual, provided that $|\underline{J}| = 1$ the cell ϕ is called weakly cocartesian[1] *if any cell χ factors uniquely through ϕ as a cell ψ as shown.*

If a $(1,n)$-ary cartesian cell ψ of the form above exists then its horizontal source $J \colon A \nrightarrow B$ is called the *restriction* of $\underline{K} \colon C \nrightarrow D$ along f and g, and denoted $\underline{K}(f,g) \coloneqq J$. If $\underline{K} = (C \xrightarrow{K} D)$ then we call $K(f,g)$ *unary*; in the case that $\underline{K} = (C)$ we call $C(f,g)$

---

[1] The stronger notion of *cocartesian* cell will be defined in Definition 7.1.

nullary. Restrictions of the form $\underline{K}(f, \mathrm{id})$ and $\underline{K}(\mathrm{id}, g)$ are called restrictions *on the left* and *right.* We will call the nullary restriction $C(\mathrm{id}, \mathrm{id})\colon C \nrightarrow C$ the *(horizontal) unit* of the object C and denote it $I_C := C(\mathrm{id}, \mathrm{id})$; if I_C exists then we call C *unital.* In Section 7 below we will see that the horizontal morphisms I_C form the units for composition of horizontal morphisms. In Theorem 10.1 we will see that the notions of an augmented virtual double category with all horizontal units is equivalent to that of a virtual double category with all horizontal units; the latter in the sense of Section 5 of [CS10]. Consequently we call an (augmented) virtual double category a *unital virtual double category* whenever it has all horizontal units. Recall that, as motivated in the Introduction, an advantage of taking augmented virtual double categories as a setting for the formalisation of yoneda embeddings, rather than 2-categories, is that the "built-in" notion of unital object can be taken to replace the notion of 'admissible' object as defined by a yoneda structure [SW78].

By their universal property any two cartesian cells defining the same restriction factor through each other as invertible horizontal cells. We will often not name cartesian cells, but simply depict them as on the left below.

$$
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
{\scriptstyle f}\downarrow & \text{cart} & \downarrow{\scriptstyle g} \\
C & \xrightarrow[\underline{K}]{} & D
\end{array}
\qquad\qquad
\begin{array}{ccc}
X_0 & \xrightarrow{H} & X_n \\
{\scriptstyle h}\downarrow & \text{cocart} & \downarrow{\scriptstyle k} \\
A & \xrightarrow[J]{} & B
\end{array}
$$

If the weakly cocartesian cell on the right above exists then we call its horizontal target J the *extension* of $\underline{H}$ along h and k. Like restrictions, extensions are unique up to isomorphism. When considered in a virtual double category, by restricting the factorisations of Definition 4.1 to unary cells χ, our notion of cocartesian cell coincides with that of weakly cocartesian cell considered in Remark 5.8 of [CS10]. We shall see in Corollary 8.5 that the extension of $\underline{H}$ along h and k above coincides with the 'horizontal composite' $(A(\mathrm{id}, h) \odot H_1 \odot \cdots \odot H_n \odot B(k, \mathrm{id}))$ whenever it exists, where $A(\mathrm{id}, h)\colon A \nrightarrow X_0$ and $B(k, \mathrm{id})\colon X_n \nrightarrow B$ are nullary restrictions as defined above. Analogously, in Lemma 8.1 we will see that the restriction of $\underline{K}$ along f and g, defined by the cartesian cell on the left above, coincides with the composite $(C(f, \mathrm{id}) \odot \underline{K} \odot D(\mathrm{id}, g))$.

The following examples describe restrictions and horizontal units in various augmented double categories. At the end of this section weakly cocartesian cells of a certain shape are characterised in $\mathcal{V}$-Prof (Example 2.4), $\mathsf{Span}(\mathcal{E})$ (Example 2.9) and $\mathsf{Rel}(\mathcal{E})$ (Example 2.10).

4.2. EXAMPLE. In the augmented virtual double category $\mathcal{V}$-Prof of $\mathcal{V}$-profunctors (Example 2.4) unary restrictions $K(f, g)$ are indeed obtained by restricting the profunctor K: they consist of the family of $\mathcal{V}$-objects $K(fx, gy)$, for all $x \in A$, $y \in B$. Likewise the nullary restriction $C(f, g)$ of two $\mathcal{V}$-functors f and g, with common target C, is given by the hom-objects $C(fx, gy)$; in particular $I_C(x, y) = C(x, y)$ defines the unit profunctor I_C. From this it easily follows that a cell $\psi\colon \underline{J} \Rightarrow \underline{K}$ in $\mathcal{V}$-Prof, as in Definition 4.1 above, is cartesian if and only if all its components $\psi_{x,y}\colon \underline{J}(x, y) \to \underline{K}(fx, gy)$ are invertible.

4.3. EXAMPLE. Let $\mathcal{E}$ be a category with pullbacks. One easily checks that in $\mathsf{Span}(\mathcal{E})$ (Example 2.9) the restriction of a span $K\colon C \nrightarrow D$ along morphisms $f\colon A \to C$ and $g\colon B \to D$ is the "wide pullback" of the diagram $A \xrightarrow{f} C \leftarrow K \to D \xleftarrow{g} B$. Similarly the nullary restriction $C(f, g)$ is obtained by pulling back the cospan $A \xrightarrow{f} C \xleftarrow{g} B$, while horizontal units are unit spans $I_A = (A \xleftarrow{\mathrm{id}} A \xrightarrow{\mathrm{id}} A)$. It is clear that the latter spans form relations in $\mathcal{E}$ so that, by Lemma 4.5 below, they also form nullary restrictions and horizontal units in $\mathsf{Rel}(\mathcal{E}) \subseteq \mathsf{Span}(\mathcal{E})$, the sub-augmented virtual double category of relations in $\mathcal{E}$ (Example 2.10).

4.4. EXAMPLE. For any isomorphism $f\colon A \to C$ in an augmented virtual double category the vertical identity cell id_f is cartesian. Similarly invertible vertical cells of the form $s \cong \mathrm{id}_A$ or $\mathrm{id}_A \cong s$, where $s\colon A \to A$, are cartesian: factorisations through $s \cong \mathrm{id}_A$ for instance are obtained by composing on the left with its inverse $\mathrm{id}_A \cong s$.

The following straightforward lemma is useful for constructing restrictions in locally full sub-augmented virtual double categories. A vertically dual result holds for weakly cocartesian cells, see Lemma 9.4 below.

4.5. LEMMA. *Any locally full and faithful functor $F\colon \mathcal{K} \to \mathcal{L}$ (Definition 3.6) reflects cartesian cells, that is a cell $\phi \in \mathcal{K}$ is cartesian whenever its image $F\phi$ is cartesian in $\mathcal{L}$.*

4.6. EXAMPLE. For every universe enlargement $\mathcal{V} \subset \mathcal{V}'$ (Example 2.7) the locally full embedding $(\mathcal{V}, \mathcal{V}')\text{-}\mathsf{Prof} \hookrightarrow \mathcal{V}'\text{-}\mathsf{Prof}$ preserves cartesian cells. Since $K(f, g)$ is a $\mathcal{V}$-profunctor whenever K is, it follows that $(\mathcal{V}, \mathcal{V}')\text{-}\mathsf{Prof}$ has all unary restrictions. Similarly the nullary restriction $C(f, g)\colon A \nrightarrow B$ exists in $(\mathcal{V}, \mathcal{V}')\text{-}\mathsf{Prof}$ whenever the hom-objects $C(fx, gy)$ are isomorphic to $\mathcal{V}$-objects for all $x \in A$ and $y \in B$, and a $\mathcal{V}'$-category C is unital in $(\mathcal{V}, \mathcal{V}')\text{-}\mathsf{Prof}$ whenever all its hom-objects are isomorphic to $\mathcal{V}$-objects. For example, in $(\mathsf{Set}, \mathsf{Set}')\text{-}\mathsf{Prof}$ (Example 2.6) all nullary restrictions $C(f, g)$ exist as soon as the category C is locally small. We will see in Example 5.6 below that for the aforementioned sufficient condition for 'one-sided' nullary restrictions $C(f, \mathrm{id})$ and $C(\mathrm{id}, g)$ in $(\mathcal{V}, \mathcal{V}')\text{-}\mathsf{Prof}$, as well as that for unitality of $\mathcal{V}'$-categories, are necessary conditions as well.

In the next example we denote by I the unit $\mathcal{V}$-category, consisting of a single object $*$ and hom-object $I(*, *) = I$, the unit of $\mathcal{V}$. We can identify $\mathcal{V}$-functors $I \to A$ with objects in A and $\mathcal{V}$-profunctors $I \nrightarrow I$ with $\mathcal{V}$-objects; cells between such profunctors can be identified with $\mathcal{V}$-maps.

4.7. EXAMPLE. The locally full embedding $\mathcal{V}\text{-}\mathsf{sProf} \hookrightarrow \mathcal{V}\text{-}\mathsf{Prof}$ (Example 2.8) reflects cartesian cells by the previous lemma; we claim that it preserves cartesian cells as well. To see this consider any cartesian cell $\psi\colon J \to \underline{K}$ in $\mathcal{V}\text{-}\mathsf{sProf}$, which defines a small $\mathcal{V}$-profunctor $J\colon A \nrightarrow B$ as the restriction $\underline{K}(f, g)$ say. By Example 4.2 it suffices to show that the components $\psi_{(x,y)}\colon J(x, y) \to K(fx, gy)$ are invertible for all $x \in A$ and $y \in B$. Notice that the cartesian cell $\phi\colon J(x, y) \Rightarrow J$, which restricts J along $x\colon I \to A$ and $y\colon I \to B$, is reflected by the embedding. It follows from Lemma 4.15 below that the composite $\psi \circ \phi\colon J(x, y) \Rightarrow K$, which consists of the single component $\psi_{(x,y)}$, is a cartesian

cell in $\mathcal{V}$-sProf that defines $J(x, y)\colon I \nrightarrow I$ as the restriction of $\underline{K}$ along $fx\colon I \to C$ and $gy\colon I \to D$. But the latter restriction is reflected by the embedding too so that, by Example 4.2 and the fact that restrictions are unique up to isomorphism, we may conclude that $\psi_{(x,y)}$ is invertible.

It follows from the above that the restriction $\underline{K}(f, g)$ exists in $\mathcal{V}$-sProf if and only if $\underline{K}(f, g)$, when constructed in $\mathcal{V}$-Prof as $\underline{K}(f, g)(x, y) = \underline{K}(fx, gy)$, is a small $\mathcal{V}$-profunctor; in that case the two restrictions coincide. Clearly this is so for all unary restrictions $K(\mathrm{id}, g)$ on the right. It is easy to show that all unit $\mathcal{V}$-profunctors I_C are small too so that, using Corollary 4.16 below, we conclude that $\mathcal{V}$-sProf has all nullary restrictions $C(\mathrm{id}, g)$ on the right as well.

To see that $\mathcal{V}$-sProf does not have restrictions $\underline{K}(f, \mathrm{id})$ on the left in general take $\mathcal{V} = \mathsf{Set}$ and consider the terminal endoprofunctor $1\colon 1 \nrightarrow 1$ on the terminal category $1 = \{*\}$, i.e. $1(*, *)$ is the singleton set. It follows from the characterisation of small Set-profunctors given in Example 2.8 that the restriction of 1 along a terminal functor $!\colon A \to 1$, where A is any properly large set regarded as a discrete category, is not small.

4.8. EXAMPLE. Unary restrictions in the augmented virtual double category $(N \circ \mathsf{Mod})(\mathcal{K})$ of bimodules in a virtual double category $\mathcal{K}$ (Example 2.2) can be created in $\mathcal{K}$. For a bimodule $(K, \lambda, \rho)\colon C \nrightarrow D$ and monoid morphisms $(f, \bar{f})\colon A \to C$ and $(g, \bar{g})\colon B \to D$ this means that the restriction $K(f, g)$ in $\mathcal{K}$, if it exists, admits a bimodule structure that is unique in making its defining cartesian cell into a cartesian cell in $(N \circ \mathsf{Mod})(\mathcal{K})$. Proving this is straightforward; see Proposition 11.10 of [Shu08] for the analogous result in the case of pseudo double categories.

Similarly the nullary restriction $C(f, g)$ in $(N \circ \mathsf{Mod})(\mathcal{K})$, of a monoid $C = (C, \gamma, \bar{\gamma}, \tilde{\gamma})$ and along monoid morphisms $(f, \bar{f})\colon A \to C$ and $(g, \bar{g})\colon B \to C$, can be created in $\mathcal{K}$ from the restriction $\gamma(f, g)$, if it exists. In particular every monoid $A = (A, \alpha, \bar{\alpha}, \tilde{\alpha})$ has a horizontal unit given by $I_A = (\alpha, \bar{\alpha}, \bar{\alpha})$ in $(N \circ \mathsf{Mod})(\mathcal{K})$; in other words $(N \circ \mathsf{Mod})(\mathcal{K})$ is a unital virtual double category.

4.9. EXAMPLE. By the previous example unary restrictions of internal profunctors in $\mathsf{Prof}(\mathcal{E})$ (Example 2.9) can be created as in $\mathsf{Span}(\mathcal{E})$, that is as wide pullbacks (Example 4.3). Since the embedding $\mathsf{spFib}(\mathcal{K}) \hookrightarrow \mathsf{Prof}(\mathcal{K}_0)$ (Lemma 2.11) is locally full and faithful as well as surjective on horizontal morphisms, by Lemma 4.5 above the restrictions of split two-sided fibrations in $\mathcal{K}$ are given by wide pullbacks as well; this partially recovers Corollary 13 of [Str74].

It is clear from e.g. [CS10], [Kou14], [Kou18] and [Shu08] that the notion of a (generalised) double category that has all restrictions is a useful one. In [CS10] virtual double categories are called 'virtual equipments' if they have all restrictions and all horizontal units—a term derived from Wood's 'bicategories equipped with proarrows' [Woo82]. As we have seen in the examples above some important augmented virtual double categories do not have all nullary restrictions (e.g. $(\mathcal{V}, \mathcal{V}')$-Prof) or only have restrictions on the right (e.g. $\mathcal{V}$-sProf). This is why we consider the following generalisations of the notion of 'equipment' as appropriate for augmented virtual double categories.

Notion	Example(s)
virtual double category w/ restrictions	$\mathcal{V}$-Mat
augmented virtual equipment	$(\mathcal{V}, \mathcal{V}')$-Prof
unital virtual double category w/ restrictions on the right	$\mathcal{V}$-sProf
unital virtual equipment	Rel$(\mathcal{E})$ Prof$(\mathcal{E})$ spFib$(\mathcal{K})$ $\mathcal{V}$-Prof
pseudo double category w/ restrictions on the right	$\mathcal{V}$-sProf ($\mathcal{V}$ is small cocomplete and closed)
equipment	Span$(\mathcal{E})$ Rel$(\mathcal{E})$ ($\mathcal{E}$ is regular) Prof$(\mathcal{E})$ ($\mathcal{E}$ has coequalisers pres. by pullback) spFib$(\mathcal{K})$ ($\mathcal{K}$ has coequalisers pres. by pullback) $\mathcal{V}'$-Prof ($\mathcal{V}'$ large cocomplete and closed)

Table 4.1: Most examples of Section 2 grouped according to whether they have all unary restrictions $K(f, g)$ ('equipment') and/or all horizontal units I_A ('unital'). In the bottom two rows a 'pseudo double category'/'equipment' is a unital virtual double category/unital virtual equipment that has all 'horizontal composites', see Section 7; in these examples horizontal composites are in fact 'pointwise' in the sense of Section 9.

4.10. DEFINITION. *An augmented virtual double category* $\mathcal{K}$ *is said to have* restrictions on the left (resp. right) *if it has all unary restrictions of the form* $K(f, \mathrm{id})$ *(resp.* $K(\mathrm{id}, g)$). *An* augmented virtual equipment *is an augmented virtual double category that has all unary restrictions* $K(f, g)$. *A* unital virtual equipment *is a unital virtual double category that has all restrictions* $\underline{K}(f, g)$.

For a unital virtual double category $\mathcal{K}$ to be a unital virtual equipment it suffices that $\mathcal{K}$ has all unary restrictions, by Corollary 4.16 below. Under the equivalence of Theorem 10.1 our notion of unital virtual equipment coincides with that of 'virtual equipment' studied in [CS10]. Table 4.1 lists most of the (generalised) equipments that are considered in this paper.

The following example demonstrates the relation between cartesian vertical identity cells and full and faithful vertical morphisms.

4.11. EXAMPLE. In the augmented virtual double category $(\mathcal{V}, \mathcal{V}')$-Prof (Example 2.7) the identity cell id_f of a $\mathcal{V}'$-functor $f\colon A \to C$ is cartesian if f is full and faithful. Indeed

if the actions $\bar{f}\colon A(x,y) \to C(fx,fy)$ of f on hom-objects are invertible then the unique factorisation of a cell $\chi\colon (H_1,\dots,H_n) \Rightarrow C$ through id_f, as in Definition 4.1, is obtained by composing the components of χ with the inverses of $\bar{f}$. The converse holds as soon as the nullary restriction $C(f,f)\colon A \nrightarrow A$ exists in $(\mathcal{V},\mathcal{V}')$-$\mathsf{Prof}$: the inverses of $\bar{f}$ can then be recovered as the components of the factorisation of the cartesian cell that defines $C(f,f)$ through id_f.

In view of the previous we make the following definition.

4.12. DEFINITION. *A vertical morphism $f\colon A \to C$ is called* full and faithful *if its identity cell* id_f *is cartesian.*

In Section 8 of [CS10] a notion of full and faithfulness is introduced for morphisms of monoids, in terms of the unital virtual double category $\mathsf{Mod}(\mathcal{K})$ of bimodules in a virtual double category $\mathcal{K}$ (Example 2.2). Under the equivalence of Theorem 10.1 this notion coincides with ours, as follows from the discussion following Corollary 5.10.

4.13. EXAMPLE. Isomorphisms are full and faithful by Example 4.4.

The converse to the following lemma holds as soon as $\mathcal{K}$ has 'all weakly cocartesian paths of $(0,1)$-ary cells', see Proposition 7.12 below.

4.14. LEMMA. *If $f\colon A \to C$ is full and faithful in the augmented virtual double category $\mathcal{K}$ then it is full and faithful in the 2-category $V(\mathcal{K})$ (Example 1.5): for any $X \in \mathcal{K}$ the functor $V(\mathcal{K})(X,f)\colon V(\mathcal{K})(X,A) \to V(\mathcal{K})(X,C)$, given by postcomposition with f, is full and faithful (see e.g. [Str74]).*

Cartesian cells satisfy the following pasting lemma. As a consequence, taking restrictions is 'pseudofunctorial' in the sense that $\underline{K}(f,g)(h,k) \cong \underline{K}(f \circ h, g \circ k)$ and $K(\mathrm{id},\mathrm{id}) \cong K$.

4.15. LEMMA. [Pasting lemma] *If the cell ϕ in the composite below is cartesian then the composite $\phi \circ \psi$ is cartesian if and only if ψ is.*

$$
\begin{array}{ccc}
X & \xrightarrow{H} & Y \\
h\downarrow & \Downarrow\psi & \downarrow k \\
A & \xrightarrow{\ \ \ } & B \\
 & J & \\
f\downarrow & \Downarrow\phi & \downarrow g \\
C & \xrightarrow{\ \ \ } & D \\
 & K &
\end{array}
$$

Restricting to the case where the cartesian cell ϕ above defines a horizontal unit I_C we find that nullary restrictions can be obtained as unary restrictions of horizontal units, as in the following corollary. Consequently in an augmented virtual equipment all nullary restrictions $C(f,g)$ exist whenever the object C is unital.

4.16. COROLLARY. *Let $f\colon A \to C$ and $g\colon B \to C$ be morphisms into a unital object C. The nullary restriction $C(f,g)$ exists if and only if the unary restriction $I_C(f,g)$ does, and in that case they are isomorphic.*

$$
\begin{array}{ccc}
A \xrightarrow{\;J\;} B & & A \xrightarrow{\;J\;} B \\
f \searrow \Downarrow\phi \swarrow g & = & f\downarrow \quad \Downarrow\phi' \quad \downarrow g \\
C & & C \xrightarrow[I_C]{} C \\
& & \Vert \quad \mathrm{cart} \quad \Vert \\
& & C
\end{array}
$$

In detail a nullary cell ϕ, as on the left-hand side above, is cartesian if and only if its factorisation ϕ' through I_C is cartesian.

Horizontal composition with the (co)unit of an adjunction preserves nullary cartesian cells as follows.

4.17. LEMMA. *In an augmented virtual double category $\mathcal{K}$ consider the composite below. If η is the unit of an adjunction $f \dashv g$ (in $V(\mathcal{K})$) then ϕ is cartesian precisely if the composite is so. A horizontally dual result holds for composition on the right with a counit.*

$$
\begin{array}{c}
A \xrightarrow{\;J\;} B \\
h \searrow \quad \Downarrow\phi \\
C \quad \diagdown k \\
\downarrow f \\
\Downarrow\eta \; E \\
\diagdown g \\
C
\end{array}
$$

PROOF. Consider the commutative diagram of assignments below, between collections of cells in $\mathcal{K}$ that are of the shape as shown.

$$
\left\{
\begin{array}{c}
X \xrightarrow{\;H\;} Y \\
p \searrow \Downarrow\chi \diagup q \\
A \\
h \searrow \quad B \\
C \diagdown k \\
f \searrow \downarrow \\
E
\end{array}
\right\}
\xleftarrow{\;\phi\,\circ\,-\;}
\left\{
\begin{array}{c}
X \xrightarrow{\;H\;} Y \\
p\downarrow \; \Downarrow\psi \; \downarrow q \\
A \xrightarrow[\;J\;]{} B
\end{array}
\right\}
\xrightarrow{\;((\eta\circ h)\odot(g\circ\phi))\,\circ\,-\;}
\left\{
\begin{array}{c}
X \xrightarrow{\;H\;} Y \\
p \searrow \Downarrow\xi \diagup q \\
B \\
A \diagdown k \\
h \searrow \downarrow \\
E \diagdown g \\
C
\end{array}
\right\}
$$

$$(\eta\circ h\circ p)\odot(g\circ-)$$

Notice that the ϕ is cartesian precisely if the top left assignment is a bijection, and that the composite $(\eta \circ h) \odot (g \circ \phi)$ is cartesian precisely if the top right assignment is a bijection. The proof follows from the fact that the bottom assignment is a bijection: its inverse is given by composing the cells ξ on the right with the counit of $f \dashv g$. ∎

Closing this section we characterise weakly cocartesian cells (of a certain shape) in the virtual double categories $\mathcal{V}$-Prof, $\mathsf{Span}(\mathcal{E})$ and $\mathsf{Rel}(\mathcal{E})$.

4.18. EXAMPLE. Consider cells ϕ in $\mathcal{V}$-Prof (Example 2.4) of the form below, where I denotes the unit $\mathcal{V}$-category as recalled before Example 4.7. Notice that such cells correspond precisely to cocones $\underline{H}^{\S}(*, *) \Rightarrow J$, where $J := J(*, *)$ and $\underline{H}^{\S}(*, *)$ is the diagram defining the iterated coend $\int^{u_1 \in X_1} \cdots \int^{u_{n'} \in X_{n'}} H_1(*, u_1) \otimes \cdots \otimes H_n(u_{n'}, *)$; see the definition preceding Example 2.8. Indeed, the internal equivariance axioms satisfied by such ϕ (Definition 2.1) correspond precisely to the naturality of such cocones. Weakly cocartesian cells thus correspond to colimiting cocones, that is the cell ϕ below is weakly cocartesian in $\mathcal{V}$-Prof precisely if it defines the $\mathcal{V}$-object J as the afore-mentioned coend.

$$
\begin{array}{ccc}
I & \xrightarrow{H_1} X_1 \ \cdots \ X_{n'} \xrightarrow{H_n} & I \\
\| & \Downarrow \phi & \| \\
I & \xrightarrow[J]{} & I
\end{array}
$$

4.19. EXAMPLE. In $\mathsf{Span}(\mathcal{E})$ (Example 2.9) a horizontal cell ϕ, of the form as below, is weakly cocartesian if and only if its underlying morphism $\phi \colon H_1 \times_{X_1} \cdots \times_{X_{n'}} H_n \to J$ is an isomorphism.

$$
\begin{array}{ccc}
X_0 & \xrightarrow{H_1} X_1 \ \cdots \ X_{n'} \xrightarrow{H_n} & X_n \\
\| & \Downarrow \phi & \| \\
X_0 & \xrightarrow[J]{} & X_n
\end{array}
$$

4.20. EXAMPLE. Consider a horizontal cell ϕ in the virtual double category $\mathsf{Rel}(\mathcal{E})$ (Example 2.10) of the form as above. Recall that it is given by a morphism of spans $\phi \colon H_1 \times_{X_1} \cdots \times_{X_{n'}} H_n \to J$ as in the bottom left commuting triangle in the diagram below, where the relation $X_0 \xleftarrow{j_0} J \xrightarrow{j_1} X_n$ is drawn as the (jointly monic) pair of morphisms $(j_0, j_1) \colon J \rightarrowtail (X_0, X_n)$; composition in this diagram is defined in the obvious way.

$$
\begin{array}{ccc}
H_1 \times_{X_1} \cdots \times_{X_{n'}} H_n & \xrightarrow{\chi} & K \\
\phi \downarrow \quad \searrow^{\psi} & & \downarrow (k_0, k_1) \\
J \xrightarrow[(j_0, j_1)]{} (X_0, X_n) & \xrightarrow[(f, g)]{} & (C, D)
\end{array}
$$

The cell ϕ is weakly cocartesian (Definition 4.1) if for any relation $C \xleftarrow{k_0} K \xrightarrow{k_1} D$ and any morphism of spans $\chi \colon H_1 \times_{X_1} \cdots \times_{X_{n'}} H_n \to K$, as in the commuting trapezium on the

right in the diagram above, there exists a (necessarily unique) lift ψ making the diagram commute. Such lifts exist in particular when ϕ is a *strong epimorphism* in the sense of Section 3 of [CKS84]: in that case lifts $\psi\colon J \to K$ exist in any commuting square of the form $(k_0, k_1) \circ \chi = (p, q) \circ \phi$ where $C \overset{p}{\leftarrow} J \overset{q}{\to} D$ is any span.

5. Companions and conjoints

Here we study nullary restrictions of the forms $C(f, \mathrm{id})$ and $C(\mathrm{id}, f)$ where $f\colon A \to C$ is any vertical morphism. These have been called respectively 'horizontal companions' and 'horizontal adjoints' in the setting of double categories [GP04]; we follow Section 7 of [CS10] and call them 'companions' and 'conjoints'. We remark that the latter only defines companions and conjoints in virtual double categories that have all horizontal units; in contrast the definition for augmented virtual double categories below does not require horizontal units. As foreshadowed in the discussion following Definition 4.1, companions and conjoints can be regarded as building blocks for restrictions and extensions as will be explained in Section 8.

5.1. DEFINITION. *Let* $f\colon A \to C$ *be a vertical morphism in an augmented virtual double category. The nullary restriction* $C(f, \mathrm{id})\colon A \nrightarrow C$ *is called the* companion *of* f *and denoted* f_*. *Likewise* $C(\mathrm{id}, f)\colon C \nrightarrow A$ *is called the* conjoint *of* f *and denoted* f^*.

Notice that the notions of companion and conjoint are interchanged when moving from $\mathcal{K}$ to its horizontal dual $\mathcal{K}^{\mathrm{co}}$ (Definition 1.8).

5.2. EXAMPLE. It follows from Example 4.2 that the companion f_* and conjoint f^* of a $\mathcal{V}$-functor $f\colon A \to C$ in $\mathcal{V}$-Prof are the representable $\mathcal{V}$-profunctors given by $f_*(x, z) = C(fx, z)$ and $f^*(z, x) = C(z, fx)$. Companions and conjoints in $(\mathcal{V}, \mathcal{V}')$-Prof (Example 2.7) are characterised in Example 5.6 below.

5.3. EXAMPLE. From Example 4.3 it follows that in the augmented virtual double categories $\mathsf{Span}(\mathcal{E})$ (Example 2.9) and $\mathsf{Rel}(\mathcal{E})$ (Example 2.10) the companion and conjoint of a morphism $f\colon A \to C$ are the relations $f_* = (A \overset{\mathrm{id}}{\leftarrow} A \overset{f}{\to} C)$ and $f^* = (C \overset{f}{\leftarrow} A \overset{\mathrm{id}}{\to} A)$ in $\mathcal{E}$.

While the companion and conjoint of a morphism $f\colon A \to C$ have been defined as nullary restrictions along f, the following lemma and its horizontal dual show that they can equivalently be defined as extensions. More precisely it gives, for a horizontal morphism $J\colon A \nrightarrow C$, a bijective correspondence between cartesian cells ψ defining J as the companion of f and weakly cocartesian cells ϕ defining J as the extension of (A) along id_A and f, in such a way that each corresponding pair (ψ, ϕ) satisfies the identities below; these are called the *companion identities*. Horizontally dual identities are satisfied by pairs of corresponding cartesian and weakly cocartesian cells defining a conjoint; these are called the *conjoint identities*. In Corollary 8.3 below we will see that any weakly cocartesian cell defining a companion or conjoint satisfies the stronger notion of 'cocartesian cell' in the sense of Section 7.

5.4. LEMMA. *Consider the factorisation of a vertical identity cell on the left below. The following conditions are equivalent: ψ is cartesian; the identity on the right below holds; ϕ is weakly cocartesian.*

PROOF (SKETCH). Notice that both sides of the identity on the right coincide after composing them with ψ below or with ϕ above. Hence the identity itself follows when ψ is cartesian or ϕ is weakly cocartesian, by the uniqueness of factorisations through (weakly co)cartesian cells.

Conversely if both identities hold then the unique factorisation of any nullary cell χ through ψ, as in Definition 4.1 but with $g = \mathrm{id}_C$, is obtained by composing χ on the left with ϕ; this shows that ψ is cartesian. Unique factorisations through ϕ are likewise obtained by composing with ψ on the right, showing that ϕ is weakly cocartesian. ∎

As an immediate consequence we find that, unlike functors between virtual double categories, functors of augmented virtual double categories preserve companions, conjoints and horizontal units.

5.5. COROLLARY. *Any functor of augmented virtual double categories preserves the cartesian and weakly cocartesian cells that define companions, conjoints or horizontal units.*

PROOF. This follows immediately from the fact that functors preserve vertical composition strictly, so that the companion and conjoint identities of (the horizontal dual of) the previous lemma are preserved. ∎

5.6. EXAMPLE. In Example 4.6 we saw that a $\mathcal{V}'$-functor $f \colon A \to C$ has a companion f_* in $(\mathcal{V}, \mathcal{V}')$-Prof as soon as all hom-objects $C(fx, z)$ are isomorphic to $\mathcal{V}$-objects. Using the previous lemma we can prove the converse, as follows. If the companion f_* exists in $(\mathcal{V}, \mathcal{V}')$-Prof, as a $\mathcal{V}$-profunctor $f_* \colon A \nrightarrow C$, then consider cells $\psi \colon f_* \Rightarrow C$ and $\phi \colon A \Rightarrow f_*$ as in the lemma. It is straightforward to check that the companion identities for ϕ and ψ imply that the composites below are inverses for the components $f_*(x, z) \to C(fx, z)$ of ψ, thus showing that $C(fx, z) \cong f_*(x, z)$, the latter of which are $\mathcal{V}$-objects for all $x \in A$ and $z \in C$.

$$C(fx, y) \cong I \otimes' C(fx, y) \xrightarrow{\phi_x \otimes' \mathrm{id}} J(x, fx) \otimes' C(fx, y) \xrightarrow{\rho} J(x, y)$$

Horizontally dual, the conjoint $f^* \colon C \nrightarrow A$ exists in $(\mathcal{V}, \mathcal{V}')$-Prof if and only if the hom-objects $C(z, fx)$ are isomorphic to $\mathcal{V}$-objects.

The companion identities of the lemma above, together with the conjoint identities, directly imply the following. The analogous result for unital virtual equipments was proved as Theorem 7.20 of [CS10].

5.7. COROLLARY. *Let $f\colon A_0 \to C$ and $g\colon A_n \to D$ be morphisms such that the conjoint f^* and the companion g_* exist. Horizontally composing with the cartesian cells defining f^* and g_* gives a bijection between cells of the form*

$$
\begin{array}{ccc}
A_0 & \xrightarrow{\ J\ } & A_n \\
f \downarrow & \Downarrow\phi & \downarrow g \\
C & \xrightarrow[\ K\ ]{} & D
\end{array}
\qquad and \qquad
\begin{array}{ccc}
C & \xrightarrow{\ f^* \frown J \frown g_*\ } & D \\
\| & \Downarrow\psi & \| \\
C & \xrightarrow[\ K\ ]{} & D.
\end{array}
$$

5.8. EXAMPLE. As was recalled in Example 2.10, a relation J internal to a category $\mathcal{E}$ is a span $J\colon A \nrightarrow B$ in $\mathcal{E}$ such that any two horizontal cells $\phi, \psi\colon H \Rightarrow J$ in $\mathsf{Span}(\mathcal{E})$ coincide. Since $\mathsf{Span}(\mathcal{E})$ has all companions and conjoints (Example 5.3), by the corollary the latter is equivalent to asking that any two cells $\phi, \psi\colon H \Rightarrow J$, of the same shape but not necessarily horizontal, coincide in $\mathsf{Span}(\mathcal{E})$.

Now consider a unary restriction $K(f, g)$ of a relation K in $\mathsf{Span}(\mathcal{E})$ (Example 4.3). By the universal property of $K(f, g)$ and the preceding it follows that $K(f, g)$ is again a relation and thus, using Lemma 4.5, forms the restriction of K in $\mathsf{Rel}(\mathcal{E})$. Since $\mathsf{Rel}(\mathcal{E})$ has all nullary restrictions as well (Example 4.3), we conclude that $\mathsf{Rel}(\mathcal{E})$ is a unital virtual equipment.

Since horizontal units I_C are a special kind of companions, i.e. $I_C := C(\mathrm{id}, \mathrm{id}) = (\mathrm{id}_C)_*$ (see the discussion following Definition 4.1), they too are defined by pairs (ψ, ϕ) of cells satisfying two 'horizontal unit identities', as the lemma below explains. It also shows that the cells ψ and ϕ are both cartesian as well as weakly cocartesian; in Lemma 7.6 below we will see that they are 'cocartesian' as well, in the sense of Section 7.

5.9. LEMMA. *Consider cells ψ and ϕ as in the identities below, and assume that either identity holds. The following conditions are equivalent:* (a) *ψ is cartesian;* (b) *ψ is weakly cocartesian;* (c) *both identities hold;* (d) *ϕ is cartesian;* (e) *ϕ is weakly cocartesian.*

$$
\left(
\begin{array}{c}
A \\
\Downarrow\mathrm{id}_A \\
A
\end{array}
\right)
\overset{(A)}{=\!=}
\begin{array}{c}
A \\
\diagup \Downarrow\phi \diagdown \\
A \xrightarrow[\ J\ ]{} A \\
\diagdown \Downarrow\psi \diagup \\
A
\end{array}
\qquad
\begin{array}{c}
A \xrightarrow{\ J\ } A \\
\| \ \Downarrow\mathrm{id}_J \ \| \\
A \xrightarrow[\ J\ ]{} A
\end{array}
\overset{(J)}{=\!=}
\begin{array}{c}
A \xrightarrow{\ J\ } A \\
\diagdown \Downarrow\psi \diagup \\
A \\
\diagup \Downarrow\phi \diagdown \\
A \xrightarrow[\ J\ ]{} A
\end{array}
$$

Consequently any cell of the form as ψ or ϕ above is cartesian if and only if it is weakly cocartesian.

PROOF. We will show that under the assumption of the identity (A) the implications
(a) $\Leftrightarrow$ (c) $\Leftrightarrow$ (e) and (a) $\Rightarrow$ (d) $\Rightarrow$ (c) hold while, under the assumption of (J), either (a)
or (d) implies (A). Vertically dual, one similarly shows that (e) $\Rightarrow$ (b) $\Rightarrow$ (c) under the
assumption of (A), while (e) or (b) implies (J) $\Rightarrow$ (A). Together these complete the proof
of the main assertion.

Assuming (A) first notice that (a), (c) and (e) are equivalent by Lemma 5.4, using the
fact that $\phi \circ \psi = \phi \odot \psi$ by the interchange axioms (Lemma 1.3). Applying the pasting
lemma to (A) shows that (a) $\Rightarrow$ (d).

Next we show that under the assumption of (d) the identities (A) and (J) are equivalent
so that, in particular, (d) $\Rightarrow$ (c) follows from (A). If (d) holds, that is ϕ is cartesian, then
there exists a unique cell ψ' such that $\mathrm{id}_J = \phi \circ \psi'$. Because $\phi \circ \psi' \circ \phi = \phi$ and ϕ is
cartesian, $\psi' \circ \phi = \mathrm{id}_A$ follows. If (A) holds then $\psi = \psi \circ \phi \circ \psi' = \psi'$ follows, so that
$\mathrm{id}_J = \phi \circ \psi' = \phi \circ \psi$, which is (J). On the other hand if (J) then $\psi = \psi' \circ \phi \circ \psi = \psi'$, so
that $\mathrm{id}_A = \psi' \circ \phi = \psi \circ \phi$, which is (A).

It remains to prove that (a) implies (J) $\Rightarrow$ (A). If (a) holds then id_A factors as
$\mathrm{id}_A = \psi \circ \phi'$; assuming (J) we then have $\phi = \phi \circ \psi \circ \phi' = \phi'$ so that (A) follows. For the
final assertion notice that any (weakly co)cartesian cell of the form ϕ or ψ can be used
to obtain a factorisation of either form (A) or (J), so that the equivalence follows from
applying the main assertion.   ∎

The following is similar to Corollary 5.7.

5.10. COROLLARY. *Let A and C be unital objects. Vertically composing with the cartesian
cells $I_A \Rightarrow A$ and $C \Rightarrow I_C$ that define the horizontal units I_A and I_C gives a bijection
between cells of the form*

$$
\begin{array}{ccc}
A & & A \xrightarrow{I_A} A \\
f \left(\Downarrow \phi \right) g & \qquad and \qquad & f \downarrow \ \Downarrow \psi \ \downarrow g \\
C & & C \xrightarrow{I_C} C
\end{array}
$$

which preserves cartesian cells.

Restricting to vertical identity cells $\phi = \mathrm{id}_f$ in the above we find that choosing a
horizontal unit I_A for each unital object A in an augmented virtual double category
extends uniquely to a functorial assignment

$$
(f \colon A \to C) \qquad \mapsto \qquad
\begin{array}{c}
A \xrightarrow{I_A} A \\
f \downarrow \ \Downarrow I_f \ \downarrow f \\
C \xrightarrow{I_C} C,
\end{array}
$$

where f is full and faithful (Definition 4.12) if and only if I_f is cartesian.

The remainder of this section records some useful properties of companions, conjoints
and horizontal units. The first of these is an immediate consequence of the pasting lemma
for cartesian cells (Lemma 4.15).

5.11. LEMMA. *Let $f\colon A \to C$ and $h\colon C \to E$ be morphisms and assume that the companion $h_*\colon C \nrightarrow E$ exists. The companion $(h \circ f)_*$ exists if and only if the restriction $h_*(f, \mathrm{id})$ does, and in that case they are isomorphic.*

In detail, the cell χ above is cartesian if and only if the cell ψ is.

5.12. LEMMA. *Let $f\colon A \to C$, $g\colon B \to C$ and $h\colon C \to E$ be morphisms and assume that h is full and faithful. The nullary restriction $C(f, g)$ exists if and only if $E(h \circ f, h \circ g)\colon A \nrightarrow C$ does, and in that case they are isomorphic.*

In detail, the cell χ above is cartesian if and only if the cell ψ is.

PROOF. Because h is full and faithful its identity cell is cartesian by Definition 4.12. The proof follows immediately from applying the pasting lemma (Lemma 4.15) to the factorisation above. ∎

Recall that any isomorphism $h\colon C \to E$ is full and faithful (Example 4.13), so that taking $g = h^{-1}$ in previous lemma gives the following.

5.13. COROLLARY. *Let $f\colon A \to C$ and $h\colon C \to E$ be morphisms and assume that h is an isomorphism. The companion $(h \circ f)_*$ exists if and only if the nullary restriction $C(f, h^{-1})$ does, and in that case they are isomorphic.*

Together with Lemma 5.9, Lemma 5.12 implies the following.

5.14. LEMMA. *Consider the factorisation on the left below. Any two of the following properties imply the third:*

(a) *the cell χ is cartesian (defining J as the nullary restriction $E(h, h)$);*

(b) *the cell ψ is cartesian (defining J as the horizontal unit of C);*

(c) *the morphism h is full and faithful.*

Moreover if (a) *holds then* (b) *is equivalent to*

(b') *the factorisation* id', *as on the right below, is cartesian.*

Consequently any full and faithful morphism $h\colon C \to E$ in an augmented virtual equipment (Definition 4.10) 'reflects unitality': if the target E is unital then so is the source C.

PROOF. We first prove the second assertion: if χ is cartesian then (b) $\Leftrightarrow$ (b'). Combining both identities above gives $\chi \circ \mathrm{id}' \circ \psi = \chi$, so that $\mathrm{id}' \circ \psi = \mathrm{id}_J$ by uniqueness of factorisations through χ. But this is identity (J) in Lemma 5.9, which asserts that ψ is cartesian precisely if id' is, that is (b) $\Leftrightarrow$ (b').

The main assertion now follows easily. Taking $f = \mathrm{id}_C = g$ in Lemma 5.12 we find that (c) implies (a) $\Leftrightarrow$ (b). Conversely assume (a) and (b): by the previous (b') follows so that both χ and id' in the identity on the right above are cartesian. Applying the pasting lemma (Lemma 4.15) we find that id_h is cartesian, showing that h is full and faithful.

For the final assertion notice that E being unital in an augmented virtual equipment implies that the nullary restriction $E(h, h)$ exists, by Corollary 4.16. Applying the main assertion we find that a cartesian cell defining the horizontal unit of C can be obtained by factorising the cartesian cell that defines $E(h, h)$ through id_h. ∎

5.15. COROLLARY. *Let $h\colon C \to E$ be a full and faithful morphism in an augmented virtual double category $\mathcal{K}$ and assume that the nullary restriction $E(h, h)$ exists. A functor $F\colon \mathcal{K} \to \mathcal{L}$ preserves the full and faithful morphism h if and only if it preserves the cartesian cell defining $E(h, h)$.*

In Corollary 8.2 we will see that F preserves $E(h, h)$ whenever the companion h_* and the conjoint h^* exist in $\mathcal{K}$.

PROOF. Factorising the cartesian cell χ that defines $E(h, h)$ through the vertical identity cell of h we obtain $\chi = \mathrm{id}_h \circ \psi$ as in the previous lemma, where ψ is the cartesian cell defining $E(h, h)$ as the horizontal unit of C. Applying F to both factorisations considered in the previous lemma we obtain

$$F\chi = \mathrm{id}_{Fh} \circ F\psi \qquad \text{and} \qquad \mathrm{id}_{Fh} = F\chi \circ F\mathrm{id}'_h,$$

where F preserves the cartesian cells ψ and id'_h by Corollary 5.5. Applying the pasting lemma for cartesian cells (Lemma 4.15) to these identities we conclude that $F\chi$ is cartesian precisely if id_{Fh} is. ∎

Recall from Example 1.5 that the objects, vertical morphisms and vertical cells of any augmented virtual double category $\mathcal{K}$ form a 2-category $V(\mathcal{K})$. The next lemmas reformulate the notions of adjunction and *absolute left lifting* (see Section 1 of [SW78] or Section 2.4 of [Web07]) in $V(\mathcal{K})$ in terms of companions in $\mathcal{K}$.

5.16. LEMMA. *In an augmented virtual double category $\mathcal{K}$ let $f\colon A \to C$ be a vertical morphism whose companion f_* exists. Consider vertical cells η and ε below as well as their factorisations through f_*, as shown.*

The following are equivalent:

(a) (η, ε) *defines an adjunction $f \dashv g$ in $V(\mathcal{K})$;*

(b) (η', ε') *satisfies the conjoint identities (horizontally dual to the companion identities of Lemma 5.4), thus defining f_* as the conjoint of g in $\mathcal{K}$.*

PROOF. We claim that the triangle identities for η and ε in $V(\mathcal{K})$ are equivalent to the conjoint identities $\eta' \circ \varepsilon' = \mathrm{id}_g$ and $\eta' \odot \varepsilon' = \mathrm{id}_{f_*}$ in $\mathcal{K}$. Indeed we have

$$(f \circ \eta) \odot (\varepsilon \circ f) = \mathrm{id}_f \;\Leftrightarrow\; (f \circ \eta' \circ \mathrm{cocart}) \odot (\mathrm{cart} \circ \varepsilon' \circ f) = \mathrm{id}_f$$
$$\Leftrightarrow\; \mathrm{cart} \circ (\eta' \odot \varepsilon') \circ \mathrm{cocart} = \mathrm{id}_f \;\Leftrightarrow\; \eta' \odot \varepsilon' = \mathrm{id}_{f_*},$$

where the second equivalence follows from the interchange axioms (Lemma 1.3), and the third from the vertical companion identity $\mathrm{cart} \circ \mathrm{id}_{f_*} \circ \mathrm{cocart} = \mathrm{id}_f$ together with the uniqueness of factorisations through (co)cartesian cells. Likewise

$$(\eta \circ g) \odot (g \circ \varepsilon) = \mathrm{id}_g \;\Leftrightarrow\; (\eta' \circ \mathrm{cocart} \circ g) \odot (g \circ \mathrm{cart} \circ \varepsilon') = \mathrm{id}_g$$
$$\Leftrightarrow\; \eta' \circ (\mathrm{cocart} \odot \mathrm{cart}) \circ \varepsilon' = \mathrm{id}_g \;\Leftrightarrow\; \eta' \circ \varepsilon' = \mathrm{id}_g,$$

where we used the horizontal companion identity $\mathrm{cocart} \odot \mathrm{cart} = \mathrm{id}_{f_*}$. ∎

The converse of the following holds whenever $\mathcal{K}$ has 'all weakly cocartesian paths of $(0, 1)$-ary cells', see Proposition 7.12 below.

5.17. LEMMA. *In an augmented virtual double category $\mathcal{K}$ consider the factorisation be-low. The vertical cell ψ defines j as the absolute left lifting of f along g in $V(\mathcal{K})$ whenever its factorisation ψ' is cartesian in $\mathcal{K}$.*

PROOF. Consider the diagram of assignments between collections of cells in $\mathcal{K}$ below, where cart denotes the cartesian cell that defines j_*. That it commutes follows from the identity above and the horizontal companion identity (Lemma 5.4).

By definition the vertical cell ψ defines j as the absolute left lifting of f along g in $V(\mathcal{K})$ when the bottom assignment is a bijection, so that the proof follows from the fact that both top assignments are bijections whenever ψ' is cartesian.   ∎

6. Representable horizontal morphisms

In this section we study horizontal morphisms $J\colon A \nrightarrow B$ that are 'represented' by vertical morphisms $f\colon A \to B$ in the sense that $J \cong f_*$; see the definition below. Given an augmented virtual double category $\mathcal{K}$, the main result (Theorem 6.5) of this section characterises the sub-augmented virtual double category of $\mathcal{K}$ generated by representable horizontal morphisms, in terms of the strict double category $(Q \circ V)(\mathcal{K})$ of 'quintets' in the vertical 2-category $V(\mathcal{K})$ (Example 1.5); see Example 6.3 below.

Generalising the fact that lax monoidal profunctors (as described in the Introduction) that are representable can be identified with colax monoidal functors, Theorem 6.5 can be used to obtain a correspondence between representable 'horizontal T-morphisms' and 'colax T-morphisms', where T is any 'monad' on an augmented virtual double category; this is done in Section 6.4 of [Kou15].

6.1. DEFINITION. *A vertical morphism $j\colon A \to B$ is said to* represent *the horizontal morphism $J\colon A \nrightarrow B$ if there exists a cartesian cell as on the left below, that is J forms the companion of j; in this case we say that J is* representable. *Horizontally dual, J is called* oprepresentable *whenever there exists a cartesian cell as on the right.*
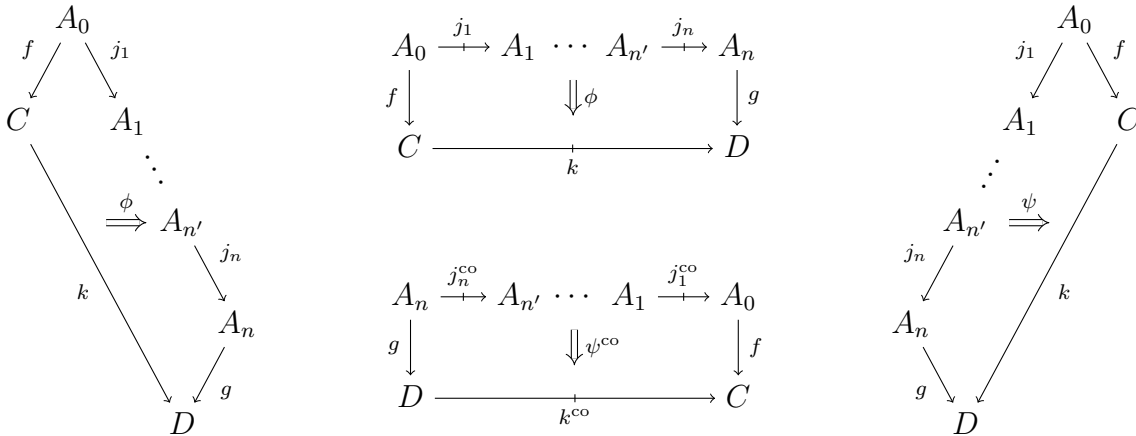
$$
\begin{array}{ccc}
A \xrightarrow{\;J\;} B & \qquad\qquad & A \xrightarrow{\;J\;} B \\
j \searrow \;\text{cart}\; \nearrow & & \searrow \;\text{cart}\; \swarrow g \\
B & & A
\end{array}
$$

For an augmented virtual double category $\mathcal{K}$ we write $\mathsf{Rep}(\mathcal{K}) \subseteq \mathcal{K}$ for the sub-augmented virtual double category that consists of all objects, all vertical morphisms, only those horizontal morphisms that are representable, and all cells between them. The subcategory $\mathsf{opRep}(\mathcal{K})$ generated by the oprepresentable horizontal morphisms is defined analogously; notice that $\mathsf{opRep}(\mathcal{K}) = (\mathsf{Rep}(\mathcal{K}^{\mathrm{co}}))^{\mathrm{co}}$ where $\mathcal{K}^{\mathrm{co}}$ denotes the horizontal dual of $\mathcal{K}$ (Definition 1.8). Because functors of augmented virtual double categories preserve companions and conjoints (Corollary 5.5), they preserve (op)representable horizontal morphisms as well; whence the following.

6.2. PROPOSITION. *The assignments $\mathcal{K} \mapsto \mathsf{Rep}(\mathcal{K})$ and $\mathcal{K} \mapsto \mathsf{opRep}(\mathcal{K})$ extend to strict 2-endofunctors Rep and opRep on $\mathsf{AugVirtDblCat}$.*

In [Ehr63] Ehresmann defined for any 2-category $\mathcal{C}$ a strict double category $Q(\mathcal{C})$ of 'quintets' in $\mathcal{C}$. The following example describes $Q(\mathcal{C})$ as an augmented virtual double category.

6.3. EXAMPLE. Let $\mathcal{C}$ be a 2-category. The augmented virtual double category $Q(\mathcal{C})$ of *quintets in $\mathcal{C}$* has as objects those of $\mathcal{C}$ while both its vertical and horizontal morphisms are morphisms in $\mathcal{C}$. A unary cell ϕ in $Q(\mathcal{C})$, as in the middle below, is a cell ϕ in $\mathcal{C}$ as on the left, while the nullary cells of $Q(\mathcal{C})$ are cells in $\mathcal{C}$ as on the left but with $k = \mathrm{id}_C$. Composition in $Q(\mathcal{C})$ is induced by that of $\mathcal{C}$ in the evident way.

$$
\begin{array}{ccc}
A_0 \xrightarrow{j_1} A_1 \cdots A_{n'} \xrightarrow{j_n} A_n \\
f \downarrow \qquad \Downarrow \phi \qquad \downarrow g \\
C \xrightarrow{\;\;k\;\;} D
\end{array}
$$

$$
\begin{array}{ccc}
A_n \xrightarrow{j_n^{\mathrm{co}}} A_{n'} \cdots A_1 \xrightarrow{j_1^{\mathrm{co}}} A_0 \\
g \downarrow \qquad \Downarrow \psi^{\mathrm{co}} \qquad \downarrow f \\
D \xrightarrow{\;k^{\mathrm{co}}\;} C
\end{array}
$$

We abbreviate $Q^{\mathrm{co}}(\mathcal{C}) \coloneqq (Q(\mathcal{C}^{\mathrm{co}}))^{\mathrm{co}}$, where $\mathcal{C}^{\mathrm{co}}$ denotes the 2-category obtained by reversing the direction of the cells in $\mathcal{C}$. Thus, to each morphism $j\colon A \to B$ in $\mathcal{C}$ there is a horizontal morphism $j^{\mathrm{co}}\colon B \nrightarrow A$ in $Q^{\mathrm{co}}(\mathcal{C})$, and to each cell ϕ as on the left below there is a unary cell ϕ^{co} in $Q^{\mathrm{co}}(\mathcal{C})$ as on the right.

6.4. PROPOSITION. *The assignments* $\mathcal{C} \mapsto Q(\mathcal{C})$ *and* $\mathcal{C} \mapsto Q^{\mathrm{co}}(\mathcal{C})$ *above extend to strict 2-functors* $Q\colon 2\text{-}\mathsf{Cat} \to \mathsf{AugVirtDblCat}$ *and* $Q^{\mathrm{co}}\colon 2\text{-}\mathsf{Cat} \to \mathsf{AugVirtDblCat}$.

PROOF. The image $QF\colon Q\mathcal{C} \to Q\mathcal{D}$ of a strict 2-functor $F\colon \mathcal{C} \to \mathcal{D}$ is simply given by letting F act on objects, morphisms and cells. The image $Q\xi\colon QF \Rightarrow QG$ of a 2-natural transformation $\xi\colon F \Rightarrow G$ is given by $(Q\xi)_A \coloneqq \xi_A$ on objects, while the naturality cell $(Q\xi)_j\colon Fj \Rightarrow Gj$ in $Q(\mathcal{D})$, for $j\colon A \nrightarrow B$ in $Q(\mathcal{C})$, is the quintet given by the naturality square $Gj \circ \xi_A = \xi_B \circ Fj$. Finally $\mathcal{C} \mapsto Q^{\mathrm{co}}(\mathcal{C})$ is extended by the composite of strict 2-functors $Q^{\mathrm{co}} \coloneqq (-)^{\mathrm{co}} \circ Q \circ (-)^{\mathrm{co}}$. ∎

Remember that any augmented virtual double category $\mathcal{K}$ contains a 2-category $V(\mathcal{K})$ of vertical morphisms and cells; see Example 1.5. We denote by $(Q \circ V)_*(\mathcal{K}) \subseteq (Q \circ V)(\mathcal{K})$ the sub-augmented virtual double category generated by all vertical morphisms, those horizontal morphisms $j\colon A \nrightarrow B$ that correspond to morphisms $j\colon A \to B$ that admit companions in $\mathcal{K}$, and all quintets between them. Because functors between augmented virtual double categories preserve cartesian cells that define companions (Corollary 5.5), this gives a sub-2-endofunctor $(Q \circ V)_* \subseteq Q \circ V$ on $\mathsf{AugVirtDblCat}$. The sub-2-endofunctor $(Q^{\mathrm{co}} \circ V)^*$ is defined likewise, by mapping each $\mathcal{K}$ to the sub-augmented virtual double category $(Q^{\mathrm{co}} \circ V)^*(\mathcal{K}) \subseteq (Q^{\mathrm{co}} \circ V)(\mathcal{K})$ that is generated by horizontal morphisms $j^{\mathrm{co}}\colon B \nrightarrow A$ corresponding to vertical morphism $j\colon A \to B$ that admit conjoints in $\mathcal{K}$.

6.5. THEOREM. *Let* $\mathcal{K}$ *be an augmented virtual double category. Choosing for each* $j\colon A \nrightarrow B$ *in* $(Q \circ V)_*(\mathcal{K})$, *corresponding to* $j\colon A \to B$ *in* $\mathcal{K}$, *a cartesian cell*

$$
\begin{array}{ccc}
A & \xrightarrow{\;j_*\;} & B \\
{\scriptstyle j}\searrow & \Downarrow{\scriptstyle \varepsilon_j} & \!\!\!\!\!\nearrow \\
& B &
\end{array}
$$

in $\mathcal{K}$ *that defines the companion of* j *induces and equivalence* $(-)_*\colon (Q \circ V)_*(\mathcal{K}) \xrightarrow{\simeq} \mathsf{Rep}(\mathcal{K})$ *of augmented virtual double categories as follows. Restricting to the identity on objects and vertical morphisms,* $(-)_*$ *maps each horizontal morphism* $j\colon A \nrightarrow B$ *in* $(Q \circ V)_*(K)$ *to its chosen companion* j_*, *while a cell* ϕ *of* $(Q \circ V)_*(K)$, *as in the left-hand side below, is mapped to the unique factorisation* ϕ_* *as shown; here* $\underline{\varepsilon_k} \coloneqq \varepsilon_k$ *if* ϕ *is unary and* $\underline{\varepsilon_k} \coloneqq \mathrm{id}_C$ *otherwise.*

Letting $\mathcal{K}$ *vary in the above, the functors* $(-)_*$ *combine to form a pseudonatural transformation* $(-)_*\colon (Q \circ V)_* \Rightarrow \mathsf{Rep}$ *of strict 2-endofunctors on* $\mathsf{AugVirtDblCat}$.

Analogously, choosing cartesian cells in $\mathcal{K}$ *that define conjoints induces an equivalence* $(Q^{\mathrm{co}} \circ V)^*(\mathcal{K}) \simeq \mathsf{opRep}(\mathcal{K})$. *Their underlying functors too combine to form a pseudonatural*

transformation $(-)^* \colon (Q^{\mathrm{co}} \circ V)^* \Rightarrow \mathsf{opRep}$.

$$(7)$$

PROOF. We will construct the functors $(-)_* \colon (Q \circ V)_*(\mathcal{K}) \to \mathsf{Rep}(\mathcal{K})$; show that they are full, faithful and essentially surjective, so that they are part of equivalences by Proposition 3.8; and prove that they are pseudonatural in $\mathcal{K}$. Horizontally dual, the functors $(-)^* \colon (Q^{\mathrm{co}} \circ V)^*(\mathcal{K}) \to \mathsf{opRep}(\mathcal{K})$ can then be defined as the composites $(-)^* \coloneqq (-)^{\mathrm{co}} \circ (-)_* \circ (-)^{\mathrm{co}}$, where we use that companions in $\mathcal{K}^{\mathrm{co}}$ correspond to conjoints in $\mathcal{K}$, so that $((Q \circ V)_*(\mathcal{K}^{\mathrm{co}}))^{\mathrm{co}} = (Q^{\mathrm{co}} \circ V)^*(\mathcal{K})$ and $(\mathsf{Rep}(\mathcal{K}^{\mathrm{co}}))^{\mathrm{co}} = \mathsf{opRep}(\mathcal{K})$.

It is clear that $\phi \mapsto \phi_*$ preserves identities. To see that it preserves composites $\psi \circ (\phi_1, \ldots, \phi_n)$ too consider the following equation where, as in (7) above, each cell denoted $\underline{\varepsilon}$ is either an identity or one of the chosen cartesian cells $\underline{\varepsilon} = \varepsilon_k$. The identities follow from (7) above and the definition of composition in $(Q \circ V)(\mathcal{K})$. We conclude that $\psi_* \circ (\phi_{1*}, \ldots, \phi_{n*})$ and $\big(\psi \circ (\phi_1, \ldots, \phi_n)\big)_*$ coincide after composition with the cell $\underline{\varepsilon}$ used in the definition of ψ_*. By uniqueness of factorisations through cartesian cells we conclude that these composites themselves coincide, showing that the composite $\psi \circ (\phi_1, \ldots, \phi_n)$ is preserved by $(-)_*$.

To prove that $(-)_*$ is part of an equivalence it suffices by Proposition 3.8 to show that it is full, faithful and essentially surjective. That it is essentially surjective and full and faithful on vertical morphisms is clear; we have to show that it is locally full and faithful, that is full and faithful on cells. To see this we denote, for each $j\colon A \nrightarrow B$ in $(Q \circ V)_*(\mathcal{K})$, by η_j the weakly cocartesian cell that corresponds to ε_j as in Lemma 5.4, so that the pair (ε_j, η_j) satisfies the companion identities. To show faithfulness, consider cells ψ and $\chi\colon \underline{j} \Rightarrow \underline{k}$ in $(Q \circ V)_*(\mathcal{K})$ such that $\psi_* = \chi_*$. It follows that the left-hand sides of (7) coincide for $\phi = \psi$ and $\phi = \chi$ so that, by precomposing both with the cells $\eta_{j_1}, \ldots, \eta_{j_n}$, $\psi = \chi$ follows from the vertical companion identities. To show fullness on unary cells, consider $\psi\colon (j_{1*}, \ldots, j_{n*}) \Rightarrow k_*$ in $(Q \circ V)_*(\mathcal{K})$. We claim that the composite

$$\phi := \varepsilon_k \circ \psi \circ (\eta'_{j_1}, \ldots, \eta'_{j_n}),$$

where $\eta'_{j_i} := \eta_{j_i} \circ j_{i'} \circ \cdots \circ j_1$ for each $i = 1, \ldots, n$, is mapped to ψ by $(-)_*$. Indeed, plugging ϕ into the left-hand side of (7) we find $\varepsilon_k \circ \psi = \varepsilon_k \circ \phi_*$ by using the horizontal companion identities, so that $\psi = \phi_*$ follows. The case of a nullary cell $\psi\colon (j_{1*}, \ldots, j_{n*}) \Rightarrow C$ is similar: simply take $\phi := \psi \circ (\eta'_{j_1}, \ldots, \eta'_{j_n})$ instead.

We now turn to proving that the functors $(-)_*$ combine to form a pseudonatural transformation $(Q \circ V)_* \Rightarrow \mathsf{Rep}$ of strict 2-endofunctors on $\mathsf{AugVirtDblCat}$. We have to supply an invertible transformation ν_F as on the left below, for each functor $F\colon \mathcal{K} \to \mathcal{L}$ of augmented virtual double categories. We take ν_F to consist of identities $(\nu_F)_A = \mathrm{id}_{FA}$ on objects and, for each $j\colon A \nrightarrow B$ in $(Q \circ V)_*(\mathcal{K})$, the unique factorisation $(\nu_F)_j\colon F(j_*) \Rightarrow (Fj)_*$ as on the right below. The latter is invertible since $F\varepsilon_j$, on the left-hand side, is cartesian by Corollary 5.5.

We have to show that the components of ν_F are natural with respect to the cells of $(Q \circ V)_*(\mathcal{K})$, in the sense of Definition 3.2. We will do so in the case of a unary cell $\phi\colon (j_1, \ldots, j_n) \Rightarrow k$; the case of nullary cells is similar. Consider the following equation, where $\varepsilon'_{Fj_i} := Fg \circ Fj_n \circ \cdots \circ Fj_{i+1} \circ \varepsilon_{Fj_i}$ and $\varepsilon'_{j_i} = g \circ j_n \circ \cdots \circ j_{i+1} \circ \varepsilon_{j_i}$ for each $i = 1, \ldots, n$, as in the left-hand side of (7). The identities follow from (7) for $F\phi$, the identity above, F preserves composition, the F-image of (7) for ϕ and the identity above again. Since factorisations through ε_{Fk} in the left and right-hand side below are unique the naturality of the components of ν_F with respect to ϕ follows. This completes the definition of the transformation ν_F.

$$\varepsilon_{Fk} \circ (F\phi)_* \circ \big((\nu_F)_{j_1}, \ldots, (\nu_F)_{j_n}\big) = (F\phi \odot \varepsilon'_{Fj_1} \odot \cdots \odot \varepsilon'_{Fj_n}) \circ \big((\nu_F)_{j_1}, \ldots, (\nu_F)_{j_n}\big)$$

$$= F\phi \odot F\varepsilon'_{j_1} \cdots \odot F\varepsilon'_{j_n} = F(\phi \odot \varepsilon'_{j_1} \odot \cdots \odot \varepsilon'_{j_n}) = F(\varepsilon_k \circ \phi_*) = \varepsilon_{Fk} \circ (\nu_F)_k \circ F(\phi_*)$$

Finally we have to show that the transformations ν_F are natural with respect to the transformations $\xi\colon F \Rightarrow G$ in AugVirtDblCat, and that they are compatible with compositions and identities, that is $\nu_{\mathrm{id}} = \mathrm{id}$ and $\nu_G F \circ G\nu_F = \nu_{G\circ F}$. The latter is a direct consequence of the uniqueness of the components of ν. To prove the former we have to show that $(\nu_G)_j \circ \xi_{(j_*)} = (\xi_j)_* \circ (\nu_F)_j$, for each $j\colon A \nrightarrow B$ in $(Q \circ V)_*(\mathcal{K})$. To do so consider the equation

$$\varepsilon_{Gj} \circ (\nu_G)_j \circ \xi_{(j_*)} = G\varepsilon_j \circ \xi_{(j_*)} = \xi_B \circ F\varepsilon_j = \xi_B \circ \varepsilon_{Fj} \circ (\nu_F)_j = \varepsilon_{Gj} \circ (\xi_j)_* \circ (\nu_F)_j,$$

where we have used the defining identities for $(\nu_G)_j$ and $(\nu_F)_j$, the naturality of ξ, identity (7) for ξ_j, and the fact that the latter is simply the quintet given by the naturality square $Gj \circ \xi_A = \xi_B \circ Fj$; see the proof of Proposition 6.4. Using the cartesianess of ε_{Gj} we conclude that $(\nu_G)_j \circ \xi_{(j_*)} = (\xi_j)_* \circ (\nu_F)_j$, proving the naturality of the transformations ν_F. This concludes the proof. ∎

7. Composition of horizontal morphisms

We now turn to compositions of horizontal morphisms in augmented virtual double categories. Analogous to the case of virtual double categories (see Section 2 of [DPP06] or Section 5 of [CS10]) such composites are defined by horizontal 'cocartesian cells', whose universal property strengthens that of horizontal weakly cocartesian cell (Definition 4.1), as in the following definition. Generalising the latter notions in the obvious way, it also defines (weakly) cocartesian *paths* of cells that are not necessarily horizontal.

7.1. DEFINITION. *A path of cells $\underline{\phi} = (\phi_1, \ldots, \phi_n)$, as in the right-hand side below, is called* weakly cocartesian *if any cell χ, as on the left-hand side, factors uniquely through $\underline{\phi}$ as shown.*
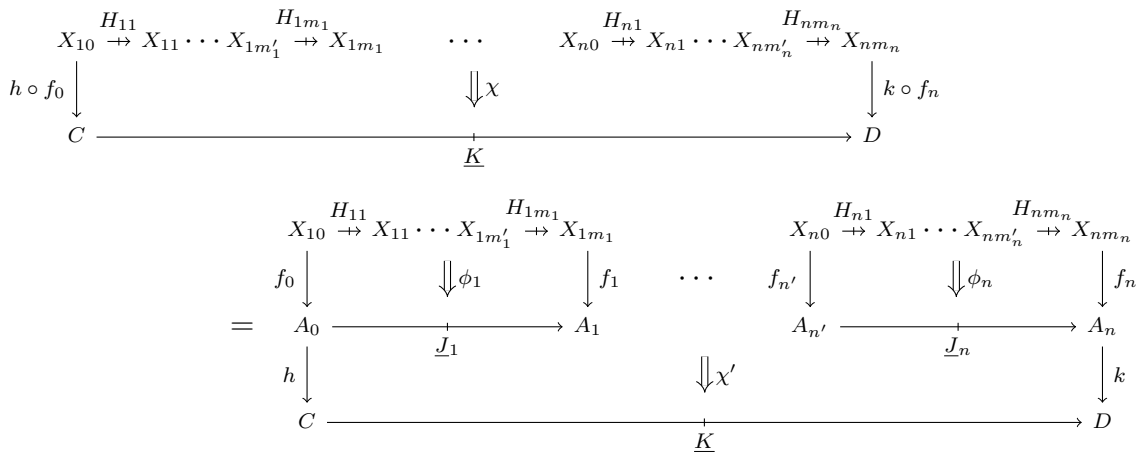
A weakly cocartesian path $\underline{\phi} = (\phi_1, \ldots, \phi_n)$ is called cocartesian *if any path of the form*

below, where $p, q \geq 1$, is weakly cocartesian.

$$
\begin{array}{ccccccc}
X_0' \xrightarrow{H_1'} X_1' & & X_{p''}' \xrightarrow{H_{p'}'} X_{p'}' \xrightarrow{H_p'(\mathrm{id},\, f_0)} X_{10} \xrightarrow{H_{11}} X_{11} \cdots X_{1m_1'} \xrightarrow{H_{1m_1}} X_{1m_1} \\
\| \quad \| \cdots & \| \quad \| \quad \mathrm{cart} \quad \downarrow f_0 \quad \Downarrow \phi_1 \quad \downarrow f_1 \\
X_0' \xrightarrow{H_1'} X_1' & & X_{p''}' \xrightarrow{H_{p'}'} X_{p'}' \xrightarrow{H_p'} A_0 \xrightarrow{\underline{J_1}} A_1
\end{array}
$$

$$
\begin{array}{ccccccc}
X_{n0} \xrightarrow{H_{n1}} X_{n1} \cdots X_{nm_n'} \xrightarrow{H_{nm_n}} X_{nm_n} \xrightarrow{H_1''(f_n,\, \mathrm{id})} X_1'' \xrightarrow{H_2''} X_2'' & & X_{q'}'' \xrightarrow{H_q''} X_q'' \\
\cdots \downarrow f_{n'} \quad \Downarrow \phi_n \quad f_n \downarrow \quad \mathrm{cart} \quad \| \quad \| \cdots \| \quad \| \\
A_{n'} \xrightarrow{\underline{J_n}} A_n \xrightarrow{H_1''} X_1'' \xrightarrow{H_2''} X_2'' & & X_{q'}'' \xrightarrow{H_q''} X_q''
\end{array}
$$

Notice that cocartesianness of the path $\underline{\phi}$ depends on the existence of restrictions along f_0 and f_n. If no restrictions along f_0 and $\overline{f_n}$ exist then $\underline{\phi}$ is cocartesian precisely if it is weakly cocartesian.

Given a cocartesian horizontal cell of the form

$$
\begin{array}{ccc}
X_0 \xrightarrow{H_1} X_1 \cdots X_{n'} \xrightarrow{H_n} X_n \\
\| \qquad \Downarrow \phi \qquad \| \\
X_0 \xrightarrow[J]{} X_n
\end{array}
\tag{8}
$$

we call, in the case that $n \geq 1$, J the *(horizontal) composite* of $(H_1, \ldots, H_n)$ and write $(H_1 \odot \cdots \odot H_n) := J$. If $n = 0$ then, by Lemma 5.9, $\phi \colon X_0 \Rightarrow J$ corresponds to a horizontal cartesian cell $\psi \colon J \Rightarrow X_0$ that defines J as the (horizontal) unit $I_{X_0} = J$ of X_0, in the sense of Section 4. Conversely, in Lemma 7.6 below we will see that any horizontal cartesian cell $J \Rightarrow X_0$ corresponds to a horizontal cocartesian cell $X_0 \Rightarrow J$.

By their universal property any two cocartesian horizontal cells defining the same composite or unit factor through each other as invertible horizontal cells. The same property also ensures that composites of composites and units are associative and unital up to isomorphisms, as we will see after Lemma 7.7 below. Like weakly cocartesian cells, in diagrams we denote single cocartesian cells simply by "cocart".

Recall from the discussion following Definition 4.1 that, when restricting its universal property to unary cells, the notion of weakly cocartesian cell in augmented virtual double categories coincides with the corresponding notion for virtual double categories. From this it follows that the notions of horizontal composite and horizontal unit likewise restrict to the corresponding notions for virtual double categories considered in Section 2 of [DPP06] or Section 5 of [CS10].

Notice that the concatenation $\underline{\phi} \frown \underline{\psi}$ of two cocartesian paths $\underline{\phi} = (\phi_1, \ldots, \phi_n)$ and $\underline{\psi} = (\psi_1, \ldots, \psi_m)$ is again cocartesian whenever the common vertical target of ϕ_n and vertical source of ψ_1 is an identity morphism. That this is not true in general is shown in Example 8.4 below.

7.2. EXAMPLE. In Example 4.18 we characterised weakly cocartesian cells ϕ in $\mathcal{V}$-Prof (Example 2.4) as in (8), but with $X_0 = I = X_n$ the unit $\mathcal{V}$-category, as those defining J as the coend $\int^{u_1 \in X_1} \cdots \int^{u_{n'} \in X_{n'}} H_1(*, u_1) \otimes \cdots \otimes H_n(u_{n'}, *)$. Such a weakly cocartesian cell ϕ is cocartesian if and only if its coend is preserved by all functors $x \otimes -$ and $- \otimes x$, for all $x \in \mathcal{V}$.

7.3. EXAMPLE. In $\mathsf{Span}(\mathcal{E})$ (Example 2.9) all weakly cocartesian cells, as characterised in Example 4.19, are cocartesian. Thus $\mathsf{Span}(\mathcal{E})$ has all horizontal composites, besides having all horizontal units (see Example 4.3).

7.4. EXAMPLE. In Example 4.20 we saw that for a cell ϕ as in (8) to be weakly cocartesian in $\mathsf{Rel}(\mathcal{E})$ it suffices that its underlying morphism $\phi \colon H_1 \times_{X_1} \cdots \times_{X_{n'}} H_n \to J$ is a strong epimorphism. In that case ϕ is cocartesian as soon as all pullbacks of ϕ are again strong epimorphisms. In particular $\mathsf{Rel}(\mathcal{E})$ has all horizontal composites and units whenever $\mathcal{E}$ is *regular*, in the sense of Section 3 of [CKS84].

7.5. EXAMPLE. If $\mathcal{E}$ has reflexive coequalisers preserved by pullback then the augmented virtual double category $\mathsf{Prof}(\mathcal{E})$ of profunctors internal to $\mathcal{E}$ (Example 2.9) has all horizontal composites. The composite of internal profunctors is an "internal coend". That $\mathsf{Prof}(\mathcal{E})$ has all horizontal units follows from Example 4.8.

Next let $\mathcal{K}$ be a finitely complete 2-category that has reflexive coequalisers preserved by pullback. Since the embedding $\mathsf{spFib}(\mathcal{K}) \hookrightarrow \mathsf{Prof}(\mathcal{K}_0)$ (Example 2.11) is surjective on horizontal morphisms as well as locally full and faithful, it follows from Lemma 9.4 below that $\mathsf{spFib}(\mathcal{K})$ too has all units and composites.

The following lemma shows that in conditions (b) and (e) of Lemma 5.9 'weakly cocartesian' can be replaced by 'cocartesian'.

7.6. LEMMA. *Let (ψ, ϕ) be a pair of cells that satisfies identities* (A) *and* (J) *of Lemma 5.9. Then, both ψ and ϕ are cocartesian.*

PROOF (SKETCH). Identities (A) and (J) imply that the unique factorisation of a cell χ through a path of the form $(\mathrm{id}_{H_1'}, \ldots, \mathrm{id}_{H_p'}, \phi, \mathrm{id}_{H_1''}, \ldots, \mathrm{id}_{H_q''})$, as in Definition 7.1, is given by $\chi' = \chi \circ (\mathrm{id}_{H_1'}, \ldots, \mathrm{id}_{H_p'}, \psi, \mathrm{id}_{H_1''}, \ldots, \mathrm{id}_{H_q''})$. Likewise factorisations through $(\mathrm{id}_{H_1'}, \ldots, \mathrm{id}_{H_p'}, \psi, \mathrm{id}_{H_1''}, \ldots, \mathrm{id}_{H_q''})$ are given by composing with ϕ. ∎

(Weakly) cocartesian paths, like cartesian cells, satisfy a pasting lemma as follows. For a proof of the second assertion use the pasting lemma for cartesian cells (Lemma 4.15).

7.7. LEMMA. [Pasting lemma] *Consider the configuration of unary cells below, where the vertical source of ψ_1 is denoted by $h_0 \colon A_{10} \to C_0$ and the vertical target of ψ_n by $h_n \colon A_{nm_n} \to C_n$.*
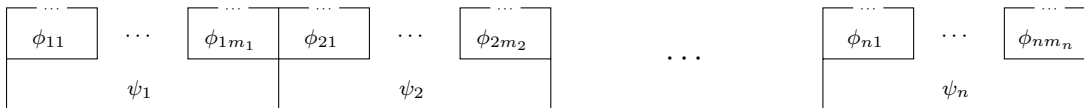
Firstly assume that the path $(\phi_{11}, \ldots, \phi_{nm_n})$ *is weakly cocartesian. Then the path of composites* $\big(\psi_1 \circ (\phi_{11}, \ldots, \phi_{1m_1}), \ldots, \psi_n \circ (\phi_{n1}, \ldots, \phi_{nm_n})\big)$ *is weakly cocartesian if and only if* $(\psi_1, \ldots, \psi_n)$ *is so.*

Secondly assume that the path $(\phi_{11}, \ldots, \phi_{nm_n})$ *is cocartesian and that, for any horizontal morphisms* $K' \colon C' \twoheadrightarrow C_0$ *and* $K'' \colon C_n \twoheadrightarrow C''$, *the restrictions* $K'(\mathrm{id}, h_0)$ *and* $K''(h_n, \mathrm{id})$ *exist. Then the path* $\big(\psi_1 \circ (\phi_{11}, \ldots, \phi_{1m_1}), \ldots, \psi_n \circ (\phi_{n1}, \ldots, \phi_{nm_n})\big)$ *is cocartesian if and only if* $(\psi_1, \ldots, \psi_n)$ *is so.*

Applying the pasting lemma to compositions $\psi \circ (\phi_1, \ldots, \phi_n)$ of horizontal cells shows that the collection of horizontal composites and units in an augmented virtual double category is coherent as follows. Let $(\underline{J}_1, \ldots, \underline{J}_n)$ be a path of paths $\underline{J}_i = (J_{i1}, \ldots, J_{im_i})$ of horizontal morphisms. If all composites $(J_{11} \odot \cdots \odot J_{1m_1})$, $\ldots$, $(J_{n1} \odot \cdots \odot J_{nm_n})$ exist then the composite $(J_{11} \odot \cdots \odot J_{nm_n})$ of the concatenation $J_{11} \frown \cdots \frown J_{nm_n}$ exists if and only if

$$\big((J_{11} \odot \cdots \odot J_{1m_1}) \odot \cdots \odot (J_{n1} \odot \cdots \odot J_{nm_n})\big)$$

does, in which case they are canonically isomorphic. Notice that this also includes isomorphisms of the form $(I_A \odot J) \cong J \cong (J \odot I_B)$, for any $J \colon A \twoheadrightarrow B$, and similar.

Remember that any functor between unital augmented virtual double categories preserves horizontal units by Corollary 5.5. We follow [CS10] in calling a functor $F \colon \mathcal{K} \to \mathcal{L}$ *strong* if it preserves horizontal composites too, that is its image of any horizontal cocartesian cell in $\mathcal{K}$ is cocartesian in $\mathcal{L}$.

To complete the picture we now briefly describe the classical notion of 'pseudo double category' as introduced by Grandis and Paré in the Appendix to [GP99]; see also Section 2 of [Shu08]. In our terms, a *pseudo double category* is a virtual double category that contains $(1,1)$-ary cells only and which is equipped with a horizontal composition

$$A \xrightarrow{J} B \xrightarrow{H} E \qquad \mapsto \qquad A \xrightarrow{J \odot H} E;$$

$$
\begin{array}{ccc}
A \xrightarrow{J} B \xrightarrow{H} E & & A \xrightarrow{J \odot H} E \\
f \downarrow \quad \Downarrow \phi \quad \downarrow g \quad \Downarrow \psi \quad \downarrow h & \mapsto & f \downarrow \quad \Downarrow \phi \odot \psi \quad \downarrow h \\
C \xrightarrow[K]{} D \xrightarrow[L]{} F & & C \xrightarrow[K \odot L]{} F,
\end{array}
$$

as well as horizontal units $I_A \colon A \twoheadrightarrow A$; $I_f \colon I_A \Rightarrow I_C$. These come with horizontal coherence cells of the forms $(J \odot H) \odot M \cong J \odot (H \odot M)$, $I_A \odot J \cong J$ and $J \odot I_B \cong J$ which satisfy the usual coherence axioms, analogous to those for a monoidal category or bicategory; see e.g. Section VII.1 of [ML98]. A pseudo double category with identity cells as coherence cells is called a *strict* double category.

Any pseudo double category gives rise to a virtual double category with the same objects and morphisms, whose cells $(J_1, \ldots, J_n) \Rightarrow K$ correspond to cells $J_1 \odot \cdots \odot J_n \Rightarrow K$ in the double category. The following result, which is Proposition 2.8 of [DPP06] and Theorem 5.2 of [CS10], characterises the virtual double categories obtained in this way.

7.8. PROPOSITION. [MacG. Dawson, Paré and Pronk] *A virtual double category is induced by a pseudo double category if and only if it has all horizontal composites and units.*

In Theorem 10.1 below we will see that in the presence of horizontal units the notions of augmented virtual double category and virtual double category coincide. In view of this and the proposition above, by a *pseudo double category* we shall mean either an (augmented) virtual double category that has all horizontal composites and units or, equivalently, a pseudo double category in the classical sense. Following [CS10], by an *equipment* we shall mean a double category that has all restrictions. Table 4.1 includes most of the double categories and equipments considered in this paper.

7.9. EXAMPLE. The augmented virtual double category $Q(\mathcal{C})$ of quintets in a 2-category $\mathcal{C}$ (Example 6.3) clearly is a strict double category: the composite $(j \odot k)$ of two horizontal morphisms in $Q(\mathcal{C})$ is simply given by their composite $k \circ j$ in $\mathcal{C}$.

In closing this section we consider augmented virtual double categories $\mathcal{K}$ that have all weakly cocartesian paths of $(0, 1)$-ary cells as in the definition below. In such $\mathcal{K}$ general cells $(J_1, \ldots, J_n) \Rightarrow K$ can be identified with cells into $\underline{K}$ with empty horizontal source. This can be used to show that certain notions in $\mathcal{K}$ are equivalent to the corresponding notions in the vertical 2-category $V(\mathcal{K})$. The proposition below asserts such equivalences for the notions of full and faithful morphism and absolute left lifting; for the case of pointwise Kan extension see Section 5.5 of [Kou14] and Section 4.6 of [Kou15].

7.10. DEFINITION. *An augmented virtual double category is said to* have all weakly cocartesian paths of $(0, 1)$-ary cells *if, for every path $\underline{J} = (J_1, \ldots, J_n) \colon A_0 \nrightarrow A_n$ of horizontal morphisms, there exists a weakly cocartesian path $\underline{\phi} = (\phi_1, \ldots, \phi_n)$ of $(0, 1)$-ary cells*

$$
\begin{array}{c}
X \\
{}^{f_{i'}}\!\swarrow \;\; \Downarrow\phi_i \;\; \searrow^{f_i} \\
A_{i'} \xrightarrow[J_i]{} A_i.
\end{array}
$$

7.11. EXAMPLE. The *graph* $\langle \underline{J} \rangle$ of a path of Set'-profunctors $\underline{J} \colon A_0 \nrightarrow A_n$ in Set'-Prof (Example 2.4) is the category whose objects are tuples $\underline{u} = (x_0, u_1, x_1, \ldots, u_n, x_n)$ of, alternatingly, objects $x_i \in A_i$ and elements $u_i \in J_i(x_{i'}, x_i)$, while its morphisms $\underline{u} \to \underline{u}'$ are tuples $(s_1, \ldots, s_n)$ of morphisms $s_i \colon x_i \to x_i'$ in A_i such that $\lambda(s_{i'}, u_i') = \rho(u_i, s_i)$.

$$
\begin{array}{c}
\langle \underline{J} \rangle \\
{}^{p_{i'}}\!\swarrow \;\; \Downarrow\pi_i \;\; \searrow^{p_i} \\
A_{i'} \xrightarrow[J_i]{} A_i
\end{array}
$$

Writing $p_i \colon \langle \underline{J} \rangle \to A_i$ for the projections, consider the $(0, 1)$-ary cells π_i above, which map $\underline{u} = (x_0, u_1, x_1, \ldots, u_n, x_n)$ to $u_i \in J_i(x_{i'}, x_i)$. It is straightforward to check that

the path $(\pi_1, \ldots, \pi_n)$ is cocartesian. Cocartesian paths of $(0, 1)$-ary cells in Set-Prof and $(\mathsf{Set}, \mathsf{Set}')$-Prof (Example 2.6) can be obtained analogously.

Restricting to the case $n = 1$ the single cell $\pi_1 \colon \langle J_1 \rangle \Rightarrow J_1$ above is universal with respect to all $(0, 1)$-ary cells $\phi \colon X \Rightarrow J_1$, exhibiting $\langle J_1 \rangle$ as the 'tabulation' of J_1; see e.g. Section 4.5 of [Kou15]. There it is also shown that, in general, 'cocartesian tabulations' can be combined to obtain cocartesian paths $(\pi_1, \ldots, \pi_n)$, similar to the construction above.

7.12. PROPOSITION. *In an augmented virtual double category $\mathcal{K}$ that has all weakly co-cartesian paths of $(0, 1)$-ary cells the converses of Lemma 4.14 and Lemma 5.17 hold.*

PROOF (SKETCH). For any path $\underline{J} \colon A_0 \nrightarrow A_n$ in $\mathcal{K}$ consider a weakly cocartesian path $\underline{\phi} = (\phi_1, \ldots, \phi_n)$ of $(0, 1)$-ary cells, as in the definition above. Composing with $\underline{\phi}$ gives a bijection between nullary cells ψ with horizontal source $\underline{J}$ and vertical cells χ with source X and, in the case of both lemmas, the universal property for the cells ψ under consideration (defining a notion in $\mathcal{K}$) is equivalent to that for the vertical cells χ (defining the corresponding notion in $V(\mathcal{K})$) under this bijection.   ∎

8. Restrictions and extensions in terms of companions and conjoints

Here we make precise the fact that restrictions and extensions of a horizontal morphism can be obtained by composing it with companions and conjoints, as anticipated in the discussion following Definition 4.1.

We start with restrictions. In the setting of unital virtual equipments the 'only if'-part of the first assertion of the following lemma was proved as Theorem 7.16 of [CS10]; notice that here we do not have to assume the existence of horizontal units. In Lemma 9.7 below we will see that the composite of $f_* \frown \underline{K} \frown g^*$ considered below is in fact 'pointwise'.

8.1. LEMMA. *In an augmented virtual double category $\mathcal{K}$ assume that the companion $f_* \colon A \nrightarrow C$ and the conjoint $g^* \colon D \nrightarrow B$ exist. For each path $\underline{K} \colon C \nrightarrow D$ of length ≤ 1 the restriction $\underline{K}(f, g)$ exists if and only if the horizontal composite of the path $f_* \frown \underline{K} \frown g^*$ does, and in that case they are isomorphic.*

$$
\begin{array}{ccc}
A \xrightarrow{f_*} C \xrightarrow{K} D \xrightarrow{g^*} B & & A \xrightarrow{f_* \frown \underline{K} \frown g^*} B \\
\begin{array}{c} f \searrow \end{array} \begin{array}{c} \text{cart} \\ \| \end{array} \begin{array}{c} \text{cart} \\ \| \end{array} \begin{array}{c} / g \end{array} & = & \begin{array}{c} \| \end{array} \begin{array}{c} \Downarrow \phi \end{array} \begin{array}{c} \| \end{array} \\
C \xrightarrow[\underline{K}]{} D & & A \xrightarrow[J]{} B \\
& & f \downarrow \quad \Downarrow \psi \quad \downarrow g \\
& & C \xrightarrow[\underline{K}]{} D
\end{array}
$$

In detail, for a factorisation as above (where the empty cell is the vertical identity cell id_C if $\underline{K}$ is empty) the following are equivalent: ψ is cartesian; ϕ is cocartesian; the identity

below holds. Moreover in this case the path $(\mathrm{cocart}, \psi, \mathrm{cocart})$, *making up the top row of the left-hand side below, is cocartesian.*

$$
\begin{array}{c}
A \xrightarrow{\quad J \quad} B \\[6pt]
\text{(diagram)}
\end{array}
\qquad = \qquad
\begin{array}{c}
A \xrightarrow{\ J\ } C \\
\| \quad \Downarrow \mathrm{id}_J \quad \| \\
A \xrightarrow{\ J\ } C
\end{array}
$$

Analogous assertions hold for one-sided restrictions. In particular $K(f, \mathrm{id})$ *exists precisely if* $f_* \odot K$ *does, while* $K(\mathrm{id}, g)$ *exists if and only if* $K \odot g^*$ *does.*

PROOF. Assuming that the top identity above holds, it follows from the companion and conjoint identities (see Lemma 5.4 and its horizontal dual) that vertically precomposing the composite on the left-hand side of the bottom equation with ϕ again results in ϕ, while postcomposing it with ψ gives back ψ. Using the uniqueness of factorisations through (co)cartesian cells we conclude that either ψ or ϕ being (co)cartesian implies the bottom identity.

Conversely, assume that both identities above hold; we will prove that ψ is cartesian and that ϕ and $(\mathrm{cocart}, \psi, \mathrm{cocart})$ are cocartesian. For the first it suffices to show that the following assignment of cells is a bijection. Indeed the identities imply that its inverse can be given by $\chi \mapsto \phi \circ (\mathrm{cocart} \circ h, \chi, \mathrm{cocart} \circ k)$, where the weakly cocartesian cells define f_* and g^* respectively.

$$
\left\{
\begin{array}{c}
X_0 \xrightarrow{H_1} X_1 \cdots X_{n'} \xrightarrow{H_n} X_n \\
h\Big\downarrow \qquad \Downarrow \chi' \qquad \Big\downarrow k \\
A \xrightarrow{\quad J \quad} B
\end{array}
\right\}
\xrightarrow{\ \psi \, \circ \, - \ }
\left\{
\begin{array}{c}
X_0 \xrightarrow{H_1} X_1 \cdots X_{n'} \xrightarrow{H_n} X_n \\
f \circ h\Big\downarrow \qquad \Downarrow \chi \qquad \Big\downarrow g \circ k \\
C \xrightarrow{\quad \underline{K} \quad} D
\end{array}
\right\}
$$

Similarly that ϕ and $(\mathrm{cocart}, \psi, \mathrm{cocart})$ are cocartesian follows from the fact that, for any paths $\underline{J}' \colon A_0' \twoheadrightarrow A_p' = A$ and $\underline{J}'' \colon B = B_0' \twoheadrightarrow B_q'$, the assignments of cells

$$
\{\xi' \colon \underline{J}' \frown J \frown \underline{J}'' \to \underline{L}\}
\underset{-\, \circ \, (\underline{\mathrm{id}}, \mathrm{cocart}, \psi, \mathrm{cocart}, \underline{\mathrm{id}})}{\overset{-\, \circ \, (\underline{\mathrm{id}}, \phi, \underline{\mathrm{id}})}{\rightleftharpoons}}
\{\xi \colon \underline{J}' \frown f_* \frown \underline{K} \frown g^* \frown \underline{J}'' \to \underline{L}\}
$$

are inverses whenever both identities hold.   ∎

The remainder of this section consists of corollaries of the lemma above. The first of these shows that functors of augmented virtual double categories behave well with respect to restrictions along morphisms that admit companions/conjoints. This is a variation on the corresponding result for functors between double categories; see Proposition 6.8 of [Shu08].

8.2. COROLLARY. *Let $F \colon \mathcal{K} \to \mathcal{L}$ be a functor between augmented virtual double categories. Consider morphisms $f \colon A \to C$ and $g \colon B \to D$ in $\mathcal{K}$ and let $\underline{K} \colon C \nrightarrow D$ be a path of length ≤ 1. If the companion $f_* \colon A \nrightarrow C$ and the conjoint $g^* \colon D \nrightarrow B$ exist then F preserves both the cartesian cell defining the restriction $\underline{K}(f, g)$ as well as the cocartesian cell defining the horizontal composite of the path $f_* \frown \underline{K} \frown g^*$.*

Under the same conditions the cartesian cells defining the restrictions of the form $K(f, \mathrm{id})$ and $K(\mathrm{id}, g)$, as well as the cocartesian ones defining the horizontal composites of the form $(f_* \odot K)$ and $(K \odot g^*)$, are preserved by F.

PROOF. This follows from the fact that F preserves the identities of the previous lemma as well as the (weakly co)cartesian cells that define f_* and g^*; the latter by Corollary 5.5. ∎

8.3. COROLLARY. *Weakly cocartesian cells that define companions or conjoints, as in the discussion preceding Lemma 5.4, are cocartesian.*

PROOF. Let $f \colon A \to C$ be a vertical morphism. We will prove that any weakly cocartesian cell defining the companion f_*, as in the composite below, is cocartesian; the proof for the conjoint f^* is horizontally dual. By Definition 7.1 it suffices to prove that for any $K \colon C \nrightarrow D$ the path

$$
\begin{array}{ccc}
A & \xrightarrow{K(f, \mathrm{id})} & D \\
\Big\| {\scriptstyle \text{cocart}} \quad \Big\downarrow {\scriptstyle f} \quad \Downarrow {\scriptstyle \psi} & & \Big\| \\
A & \xrightarrow[f_*]{} C \xrightarrow[K]{} & D,
\end{array}
$$

where ψ is the cartesian cell defining $K(f, \mathrm{id})$, is cocartesian. But this follows directly from the second assertion of Lemma 8.1 for $K(f, \mathrm{id})$. ∎

8.4. EXAMPLE. To show that a path of cocartesian cells need not be cocartesian itself in general consider a morphism $f \colon A \to C$ such that f_*, f^* and $C(f, f)$ exist. We claim that the path

$$
\begin{array}{ccc}
 & A & \\
{\scriptstyle \text{cocart}} \Big\| \quad \Big\downarrow {\scriptstyle f} \quad \Big\| {\scriptstyle \text{cocart}} & & \\
A \xrightarrow[f_*]{} & C & \xrightarrow[f^*]{} A
\end{array}
$$

is weakly cocartesian only if f is full and faithful (Definition 4.12). To see this let the cocartesian cell $\phi \colon (f_*, f^*) \Rightarrow C(f, f)$ and the cartesian cell $\psi \colon C(f, f) \Rightarrow C$ be as in Lemma 8.1. If the path above is weakly cocartesian then so is the composite $\phi \circ (\text{cocart}, \text{cocart})$ by the pasting lemma for cocartesian paths (Lemma 7.7); that is cartesian too then follows from Lemma 5.9. Using the pasting lemma for cartesian cells (Lemma 4.15) it follows that $\psi \circ \phi \circ (\text{cocart}, \text{cocart})$ is cartesian which, by the first identity of Lemma 8.1 and the vertical companion and conjoint identities (Lemma 5.4), equals id_f. We conclude that f is full and faithful.

Together with the pasting lemma for cocartesian paths (Lemma 7.7) Corollary 8.3 allows us to describe extensions along vertical morphisms in terms of compositions with their companions and conjoints as follows; this is a variation on the corresponding result for unital virtual equipments Theorem 7.20 of [CS10].

8.5. COROLLARY. *Any composite of the form below is cocartesian, so that it defines J as the extension of $(H_1, \ldots, H_n)$ along h. Cocartesian cells that define extensions on the right or that define two-sided extensions can be constructed analogously.*

$$
\begin{array}{c}
X_0 \xrightarrow{H_1} X_1 \qquad X_{n'} \xrightarrow{H_n} X_n \\
\left.\begin{array}{c} h \\ \swarrow \\ \text{cocart} \end{array}\right\| \qquad \| \qquad \cdots \qquad \| \qquad \| \\
A \xrightarrow[h^*]{} X_0 \xrightarrow{H_1} X_1 \qquad X_{n'} \xrightarrow{H_n} X_n \\
\| \qquad \qquad \text{cocart} \qquad \qquad \| \\
A \xrightarrow[\quad J \quad]{} X_n
\end{array}
$$
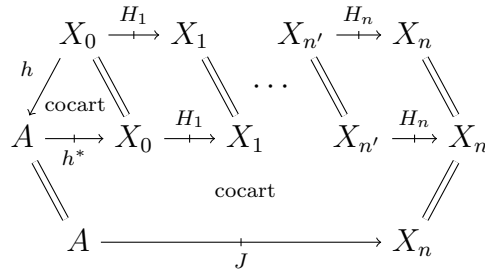
9. Pointwise horizontal composites

Consider a path $(H_1, \ldots, H_n): X_0 \nrightarrow X_n$ in the augmented virtual double category $\mathcal{V}\text{-Prof}$ of $\mathcal{V}$-profunctors (Example 2.4). In Example 7.2 we have seen that, in the special case where $X_0 = I = X_n$ is the unit $\mathcal{V}$-category, the horizontal composite $(H_1 \odot \cdots \odot H_n)$ is given by the coend $\int^{u_1 \in X_1} \cdots \int^{u_{n'} \in X_{n'}} H_1(*, u_1) \otimes \cdots \otimes H_n(u_{n'}, *)$, provided that it is preserved by the monoidal product $\otimes$ of $\mathcal{V}$ on both sides. Recall that in the general case, where X_0 and X_n are any $\mathcal{V}$-categories, the composite $(H_1 \odot \cdots \odot H_n)$ can be built up "pointwise" from such coends, by taking

$$
(H_1 \odot \cdots \odot H_n)(x, y) = \int^{u_1 \in X_1} \cdots \int^{u_{n'} \in X_{n'}} H_1(x, u_1) \otimes \cdots \otimes H_n(u_{n'}, y)
$$

for each pair $x \in X_0$ and $y \in X_n$. The definition of 'pointwise horizontal composite' below formalises the pointwise character of this composite inside an augmented virtual double category; informally it captures that "any restriction of a pointwise horizontal composite $(H_1 \odot \cdots \odot H_n)$ is again a horizontal composite". Pointwise horizontal composites are important in the study of "pointwise Kan extensions" in augmented virtual double categories; see Section 4 of [Kou15]. While the definition below is stated in terms of a path $(\phi_1, \ldots, \phi_n)$ of unary cells we will mostly apply it to single horizontal cocartesian cells $\phi_1: (H_1, \ldots, H_n) \Rightarrow (H_1 \odot \cdots \odot H_n)$.

9.1. DEFINITION. *Consider a path $\underline{\phi} = (\phi_1, \ldots, \phi_n)$ of unary cells whose last cell ϕ_n has non-empty horizontal source and trivial vertical target, as in the composite on the left-hand side below. Let $f: Y \to X_{nm_n}$ be any morphism such that both restrictions $H_{nm_n}(\text{id}, f)$ and $J_n(\text{id}, f)$ exist.*

We call $\underline{\phi}$ right pointwise cocartesian with respect to f if the path $(\phi_1, \ldots, \phi'_n)$ is cocartesian, where ϕ'_n is the unique factorisation as below. We call $\underline{\phi}$ right pointwise cocartesian if it is right pointwise cocartesian with respect to all such morphisms f.

$$
\begin{array}{ccc}
X_{n0} \xrightarrow{H_{n1}} X_{n1} \quad X_{n(m_n)'} \xrightarrow{H_{nm_n}(\mathrm{id}, f)} Y & & X_{n0} \xrightarrow{H_{n1}} X_{n1} \cdots X_{n(m_n)'} \xrightarrow{H_{nm_n}(\mathrm{id}, f)} Y \\
\end{array} \tag{9}
$$

The notion of left pointwise cocartesian *path is horizontally dual. A path that is both left and right pointwise cocartesian is called* pointwise cocartesian.

Notice that any right (or left) pointwise cocartesian path is cocartesian, by taking $f = \mathrm{id}_{X_{nm_n}}$ in the above. Conversely, in Lemma 9.8 below we will see that any cocartesian path is pointwise with respect to morphisms f that admit conjoints. A single horizontal cocartesian cell $\phi: (H_1, \ldots, H_n) \Rightarrow J$ is called *pointwise cocartesian* whenever the singleton path (ϕ) is pointwise cocartesian; in that case we call J the *pointwise composite* of $(H_1, \ldots, H_n)$.

9.2. EXAMPLE. Let $(H_1, \ldots, H_n): X_0 \nrightarrow X_n$ be a path of $\mathcal{V}$-profunctors. As anticipated in the introduction to this section, a horizontal cell $\phi: (H_1, \ldots, H_n) \Rightarrow J$ is pointwise cocartesian in $\mathcal{V}$-Prof (Example 2.4) if and only if, for all pairs $x \in X_0$ and $y \in X_n$, the components $\phi: H_1(x, u_1) \otimes \cdots H_n(u_{n'}, y) \to J(x, y)$ define $J(x, y)$ as the coend

$$
J(x, y) = \int^{u_1 \in X_1} \cdots \int^{u_{n'} \in X_{n'}} H_1(x, u_1) \otimes \cdots \otimes H_n(u_{n'}, y)
$$

which is preserved by the monoidal product $\otimes$ of $\mathcal{V}$ on both sides. The 'only if'-part follows from applying Example 7.2 to the restrictions of ϕ along $\mathcal{V}$-functors of the form $x: I \to X_0$ and $y: I \to X_n$. The 'if'-part follows from the "functoriality of coends", dual to that of ends as described in Section 2.1 of [Kel82]. We conclude that $\mathcal{V}$-Prof is a pseudo double category whenever $\mathcal{V}$ has large colimits that are preserved by $\otimes$ on both sides.

Now let $\mathcal{V} \subset \mathcal{V}'$ be a universe enlargement as in Example 2.7. Here $\mathcal{V}'$ is large cocomplete and closed, so that $\mathcal{V}'$-Prof is a pseudo double category by the above. Since the embedding $(\mathcal{V}, \mathcal{V}')$-Prof $\hookrightarrow \mathcal{V}'$-Prof preserves cartesian cells, Lemma 9.4 below implies that the pointwise composite $(H_1 \odot \cdots \odot H_n)$ exists in $(\mathcal{V}, \mathcal{V}')$-Prof whenever the coends above, which exist in $\mathcal{V}'$, are isomorphic to $\mathcal{V}$-objects.

9.3. EXAMPLE. Let $J: A \nrightarrow B$ and $H: B \nrightarrow E$ be small $\mathcal{V}$-profunctors between (possibly large) $\mathcal{V}$-categories; see Example 2.8. If the monoidal product $\otimes$ of $\mathcal{V}$ preserves colimits

(large ones, if B is large) on both sides then, as we will show, the composite $J \odot H$ can be computed as the family of small colimits

$$(J \odot H)(x, z) = \int^{y' \in B_z} J(x, y') \otimes H(y', z)$$

where $B_z \subseteq B$ are the small sub-$\mathcal{V}$-categories that exhibit H as small (see Example 2.8). Moreover these colimits, if they exist, form a small $\mathcal{V}$-profunctor which forms the pointwise composite of J and H in $\mathcal{V}$-sProf. We conclude that $\mathcal{V}$-sProf, which has horizontal units by Example 4.7, is a pseudo double category whenever $\mathcal{V}$ is small cocomplete and $\otimes$ preserves large colimits on both sides.

To see the above choose any universe enlargement $\mathcal{V} \subset \mathcal{V}'$ (Example 2.7). By the previous example the pointwise composite $(J \odot H)$ exists in $\mathcal{V}'$-Prof: it is defined by the coends on the left below. The cascade of isomorphisms below shows that $(J \odot H)$ can be computed as above. Here we have used the smallness of H, the assumption that $\otimes$ preserves large colimits on both sides, the "interchange of coends" theorem (see e.g. Formula 2.9 of [Kel82]), while the last isomorphism follows from the enriched Yoneda's lemma, see e.g. Formula 3.71 of [Kel82].

$$(J \odot H)(x, z) = \int^{y \in B} J(x, y) \otimes H(y, z) \cong \int^{y \in B} J(x, y) \otimes \left(\int^{y' \in B_z} B(y, y') \otimes H(y', z) \right)$$

$$\cong \int^{y' \in B_z} \left(\int^{y \in B} J(x, y) \otimes B(y, y') \right) \otimes H(y', z) \cong \int^{y' \in B_z} J(x, y') \otimes H(y', z)$$

Now assume that the small colimit above (and thus all colimits above) exists in $\mathcal{V}$. To see that, in this case, $(J \odot H)$ is again a small $\mathcal{V}$-profunctor take, for each $z \in E$, $A_z \subseteq A$ to be the smallest full sub-$\mathcal{V}$-category containing all A_y, where y ranges over the objects of B_z. Then A_z is small and we have

$$\int^{x' \in A_z} A(x, x') \otimes (J \odot H)(x', z) = \int^{x' \in A_z} A(x, x') \otimes \left(\int^{y \in B} J(x', y) \otimes H(y, z) \right)$$

$$\cong \int^{y \in B} \left(\int^{x' \in A_z} A(x, x') \otimes J(x', y) \right) \otimes H(y, z') \cong \int^{y \in B} J(x, y) \otimes H(y, z) = (J \odot H)(x, z),$$

which shows that $(J \odot H)$ is small. For the second isomorphism here recall from Example 2.8 that each $J(-, y)$ is a left Kan extension along $A_y \subseteq A$: the isomorphism follows from the fact that the latter factors as a Kan extension along $A_z \subseteq A$ as a consequence of the "pasting lemma" for Kan extensions, see e.g. Theorem 4.47 of [Kel82]. We can now conclude that $(J \odot H)$, as defined above, exists in $\mathcal{V}$-sProf; that it forms the pointwise composite of J and H there follows from applying the lemma below to the locally full

embedding $\mathcal{V}\text{-sProf} \hookrightarrow \mathcal{V}'\text{-Prof}$ which, as follows from Example 4.7, preserves cartesian cells.

Besides reflecting restrictions (Lemma 4.5) locally full and faithful functors reflect horizontal composites.

9.4. LEMMA. *Any locally full and faithful functor* $F\colon \mathcal{K} \to \mathcal{L}$ *(Definition 3.6) reflects (weakly) cocartesian paths, that is a path* $(\phi_1, \ldots, \phi_n) \in \mathcal{K}$ *is (weakly) cocartesian whenever its image* $(F\phi_1, \ldots, F\phi_n)$ *is (weakly) cocartesian in* $\mathcal{L}$.

If moreover F preserves cartesian cells then it reflects (right/left) pointwise cocartesian paths as well.

Pointwise cocartesian paths are coherent in the following sense.

9.5. LEMMA. *If the path* $\underline{\phi} = (\phi_1, \ldots, \phi_n)$ *is right pointwise cocartesian then any path of the form* $(\phi_1, \ldots, \phi'_n)$, *as in Definition 9.1, is again right pointwise cocartesian. An analogous result holds for (left) pointwise cocartesian paths.*

PROOF. Let $f\colon Y \to X_{nm_n}$ be as in Definition 9.1; that is $H_{nm_n}(\text{id}, f)$ and $J_n(\text{id}, f)$ exist. Let $g\colon Z \to Y$ be any morphism such that $H_{nm_n}(\text{id}, f)(\text{id}, g) \cong H_{nm_n}(\text{id}, f \circ g)$ and $J_n(\text{id}, f)(\text{id}, g) \cong J_n(\text{id}, f \circ g)$ exist, where the isomorphisms follow from the pasting lemma for cartesian cells (Lemma 4.15). Consider the unique factorisation ϕ''_n in $\phi'_n \circ (\text{id}, \ldots, \text{id}, \text{cart}) = \text{cart} \circ \phi''_n$, as in Definition 9.1 but for ϕ', where the cartesian cells define $H_{nm_n}(\text{id}, f)(\text{id}, h)$ and $J_n(\text{id}, f)(\text{id}, h)$ respectively; we have to show that $(\phi_1, \ldots, \phi''_n)$ is cocartesian. To see this consider the following equation where, in each composite, the bottom cartesian cell (denoted 'c') defines a restriction along f and the top cartesian cell (also denoted 'c') defines a restriction along g, and where the identities follow from the definitions of ϕ''_n and ϕ'_n respectively.

The composites of cartesian cells in the left-hand and right-hand sides above are again cartesian by the pasting lemma, so that $(\phi_1, \ldots, \phi''_n)$ is cocartesian because $(\phi_1, \ldots, \phi_n)$ is right pointwise cocartesian. This concludes the proof. ∎
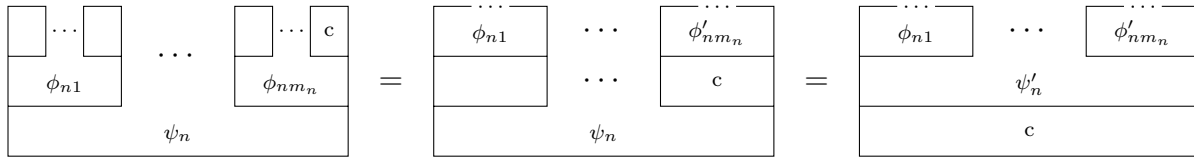
The pasting lemma for cocartesian paths (Lemma 7.7) induces one for pointwise cocartesian paths as follows.

9.6. LEMMA. [Pasting lemma] *Consider a configuration of unary cells below such that both vertical targets of* ϕ_{nm_n} *and* ψ_n *are the identity morphism on some object X. Assume that the path* $(\phi_{11}, \ldots, \phi_{nm_n})$ *is right pointwise cocartesian with respect to a morphism* $f\colon Y \to X$. *The path* $\big(\psi_1 \circ (\phi_{11}, \ldots, \phi_{1m_1}), \ldots, \psi_n \circ (\phi_{n1}, \ldots, \phi_{nm_n})\big)$ *is right pointwise*

cocartesian with respect to f if and only if $(\psi_1, \ldots, \psi_n)$ is so. An analogous result holds for (left) pointwise cocartesian paths.
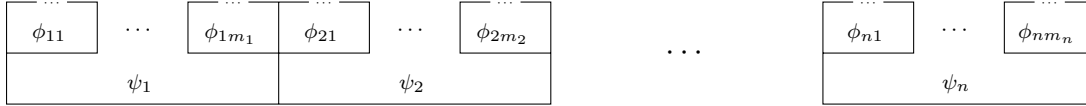
PROOF. Notice that, assuming that two of the three paths referred to in the statement are right pointwise cocartesian with respect to f, the following restrictions along f exist: those of the horizontal targets of ψ_n and ϕ_{nm_n} as well as that of the last morphism in the horizontal source of ϕ_{nm_n}. Using these restrictions we obtain factorisations ψ'_n and ϕ'_{nm_n}, as in Definition 9.1, such that the following equation holds, where 'c' denotes any cartesian cell defining one of the restrictions along f.

The above equation implies that the unique factorisation $[\psi_n \circ (\phi_1, \ldots, \phi_n)]'$ corresponding to $\psi_n \circ (\phi_{n1}, \ldots, \phi_{nm_n})$, as in Definition 9.1 and with respect to f, coincides with $\psi'_n \circ (\phi_{n1}, \ldots, \phi'_{nm_n})$. By assumption $(\phi_{11}, \ldots, \phi'_{nm_n})$ is cocartesian so that the proof follows by the pasting lemma (Lemma 7.7): the cocartesianness of $(\psi_1, \ldots, \psi'_n)$ is equivalent to that of $\big(\psi_1 \circ (\phi_{11}, \ldots, \phi_{1m_1}), \ldots, [\psi_n \circ (\phi_1, \ldots, \phi_n)]'\big)$. ∎

Pointwise cocartesian cells can be obtained from the following lemmas.

9.7. LEMMA. *Let (ψ, ϕ) be a pair of cells that satisfies both identities of Lemma 8.1. The cocartesian cell ϕ is pointwise cocartesian.*

PROOF. We will show that ϕ is right pointwise cocartesian; a horizontally dual argument shows that ϕ is left pointwise cocartesian too. Let $p\colon Y \to B$ be any morphism such that $g^*(\mathrm{id}, p) \cong (g \circ p)^*$ (see Lemma 5.11) and $J(\mathrm{id}, p)$ exist. Let $\phi'\colon f_* ^\frown \underline{K} ^\frown (g \circ p)^* \Rightarrow J(\mathrm{id}, p)$ be the unique factorisation in $\phi \circ (\mathrm{id}, \mathrm{id}, \mathrm{cart}) = \mathrm{cart} \circ \phi'$, as in Definition 9.1, where the cartesian cells define the restrictions along p. We have to show that ϕ' is cocartesian. To see this compose the first identity of Lemma 8.1 with the cartesian cell defining $g^*(\mathrm{id}, p)$, giving the first identity in the equation below. The second identity follows from the definition of ϕ'.
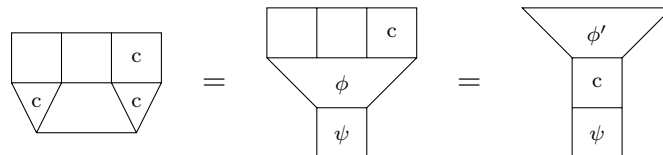
Since the composite of cartesian cells in the left-hand side defines the companion of $g \circ p$, the equation above is of the same form as the first identity of Lemma 8.1. Moreover by

the pasting lemma (Lemma 4.15) the composite of the bottom two cells in the right-hand side is cartesian, so that ϕ' is cocartesian by Lemma 8.1. ■

9.8. LEMMA. *Consider the path* $\underline{\phi} = (\phi_1, \ldots, \phi_n)$ *and the morphism* $f\colon Y \to X_{nm_n}$ *of Definition 9.1. If the conjoint* f^* *exists and* $\underline{\phi}$ *is cocartesian then* $\underline{\phi}$ *is right pointwise cocartesian with respect to* f. *An analogous result holds for (left) pointwise cocartesianness.*

Consequently in an augmented virtual equipment (Definition 4.10) the horizontal composite of a path $(H_1, \ldots, H_n)\colon X_0 \nrightarrow X_n$ *is pointwise whenever* X_0 *and* X_n *are unital.*

PROOF. As in Definition 9.1 we assume that $H_{nm_n}(\mathrm{id}, f)$ and $J(\mathrm{id}, f)$ exist. By Lemma 8.1 we have $H_{nm_n}(\mathrm{id}, f) \cong H_{nm_n} \odot f^*$ and $J(\mathrm{id}, f) \cong J \odot f^*$ such that each pair of cartesian and cocartesian cells, defining the restriction and the composite, satisfy the identities of that lemma. Let ϕ'_n be the factorisation as in Definition 9.1; we have to show that $(\phi_1, \ldots, \phi'_n)$ is cocartesian. To see this consider the following equation of composites, where the cartesian cells defining $H_{nm_n}(\mathrm{id}, f)$, $J(\mathrm{id}, f)$ and f^* are denoted 'c' and the cocartesian cells defining $H_{nm_n} \odot f^*$ and $J \odot f^*$ are denoted 'cc'. The identities follow from the definition of ϕ'_n and the first identity of Lemma 8.1.

We conclude that $\phi'_n \circ (\mathrm{id}, \ldots, \mathrm{id}, \mathrm{cocart}) = \mathrm{cocart} \circ (\phi_n, \mathrm{id})$, by the uniqueness of factorisations through cartesian cells. It then follows from the pasting lemma (Lemma 7.7) that $\underline{\phi}$ being cocartesian implies that $(\phi_1, \ldots, \mathrm{cocart} \circ (\phi_n, \mathrm{id})) = (\phi_1, \ldots, \phi'_n \circ (\mathrm{id}, \ldots, \mathrm{id}, \mathrm{cocart}))$ is cocartesian which in turn means that $(\phi_1, \ldots, \phi'_n)$ is cocartesian. This proves the first assertion. The final assertion follows by recalling from Corollary 4.16 that, in an augmented virtual equipment, all morphisms into unital objects X_0 and X_n admit companions and conjoints. ■

10. The equivalence of unital augmented virtual double categories and unital virtual double categories

In this last section we will show that the notions of augmented virtual double category and virtual double category are equivalent when all horizontal units exist. We denote by VirtDblCat$_\mathrm{u}$ the locally full sub-2-category of VirtDblCat consisting of virtual double categories that have all horizontal units, *normal* functors—that preserve the cocartesian cells defining horizontal units—, and all transformations between them. Likewise AugVirtDblCat$_\mathrm{u}$ ⊂ AugVirtDblCat denotes the full sub-2-category generated by the augmented virtual double categories that have all horizontal units. Remember that any functor of augmented virtual double categories preserves horizontal units (Corollary 5.5).

Recall the strict 2-functor $U\colon \mathsf{AugVirtDblCat} \to \mathsf{VirtDblCat}$ (Proposition 3.3) that maps any augmented virtual double category $\mathcal{K}$ to the underlying virtual double category $U(\mathcal{K})$ consisting of the unary cells of $\mathcal{K}$. Clearly unary cocartesian cells in $\mathcal{K}$ are again cocartesian in $U(\mathcal{K})$ so that U restricts to a strict 2-functor $U\colon \mathsf{AugVirtDblCat}_{\mathrm{u}} \to \mathsf{VirtDblCat}_{\mathrm{u}}$. The theorem of this section proves that the latter 2-functor, together with the assignment $\mathcal{K} \mapsto N(\mathcal{K})$ of Example 1.7, extends to a 2-equivalence $\mathsf{AugVirtDblCat}_{\mathrm{u}} \simeq \mathsf{VirtDblCat}_{\mathrm{u}}$.

In order to make the distinction between cells of $N(\mathcal{K})$ and those of $\mathcal{K}$ clear, in this section only we will place a bar over those of $N(\mathcal{K})$. Thus $N(\mathcal{K})$ has the same objects and morphisms as $\mathcal{K}$ while each unary cell $\bar\phi$ of $N(\mathcal{K})$ corresponds to a cell ϕ in $\mathcal{K}$ and each nullary cell $\bar\psi$ of $N(\mathcal{K})$, of the shape on the left below, corresponds to a unary cell ψ in $\mathcal{K}$ as on the right, where I_C is the chosen horizontal unit for C.

$$
\begin{array}{ccc}
A_0 \xrightarrow{J_1} A_1 \ \cdots \ A_{n'} \xrightarrow{J_n} A_n & \qquad\qquad & A_0 \xrightarrow{J_1} A_1 \ \cdots \ A_{n'} \xrightarrow{J_n} A_n \\
f \searrow \quad \Downarrow\bar\psi \quad \swarrow g & & f\downarrow \qquad \Downarrow\psi \qquad \downarrow g \\
C & & C \xrightarrow{\ I_C\ } C
\end{array}
$$

Recall that for each object $C \in \mathcal{K}$ we denote by $\eta_C\colon C \Rightarrow I_C$ the cocartesian cell in $\mathcal{K}$ that defines the chosen horizontal unit $I_C\colon C \nrightarrow C$. Using the bar notation, composition in $N(\mathcal{K})$ is defined as

$$\bar\chi \circ (\bar\xi_1, \ldots, \bar\xi_n) := \overline{\chi' \circ (\xi_1, \ldots, \xi_n)} \tag{10}$$

where χ' is the unique factorisation of χ through the cocartesian path of cells $(\eta_{\bar\xi_1}, \ldots, \eta_{\bar\xi_n})$ in $\mathcal{K}$, where $\eta_{\bar\xi_i} := \eta_{C_{i'}}$ if $\bar\xi_i$ is nullary with horizontal target $C_{i'}$ and $\eta_{\bar\xi_i} := \mathrm{id}_{K_i}$ if $\bar\xi_i$ is unary with horizontal target $K_i\colon C_{i'} \nrightarrow C_i$. The identity cells in $N(\mathcal{K})$, for morphisms $J\colon A \nrightarrow B$ and $f\colon A \to C$, are given by

$$\mathrm{id}_J := \overline{\mathrm{id}_J} \qquad\text{and}\qquad \mathrm{id}_f := \overline{\eta_C \circ f}.$$

10.1. THEOREM. *The strict 2-functor $U\colon \mathsf{AugVirtDblCat}_{\mathrm{u}} \to \mathsf{VirtDblCat}_{\mathrm{u}}$ together with the assignment $\mathcal{K} \mapsto N(\mathcal{K})$ (Example 1.7), both as recalled above, extend to a strict 2-equivalence $\mathsf{AugVirtDblCat}_{\mathrm{u}} \simeq \mathsf{VirtDblCat}_{\mathrm{u}}$.*

PROOF. That the composition for $N(\mathcal{K})$ as defined above satisfies the associativity and unit axioms is a straightforward consequence of those axioms in $\mathcal{K}$, combined with the uniqueness of the factorisations χ' in (10).

To show that $N(\mathcal{K})$ has all horizontal units let A be any object in $N(\mathcal{K})$; we claim that $\bar\eta_A\colon A \Rightarrow I_A$ defines I_A as the horizontal unit of A in $N(\mathcal{K})$. To see this, consider the identity of I_A as a nullary cell $\overline{\mathrm{id}_{I_A}}\colon I_A \Rightarrow A$ in $N(\mathcal{K})$; we will show that $\bar\eta_A$ and $\overline{\mathrm{id}_{I_A}}$ satisfy the horizontal unit identities of Lemma 5.9. Indeed, we have $\overline{\mathrm{id}_{I_A}} \circ \bar\eta_A = \overline{(\mathrm{id}_{I_A} \circ \eta_A)} = \bar\eta_A = \mathrm{id}_A$ (the identity cell of A in $N(\mathcal{K})$). On the other hand we have

$$\bar\eta_A \circ \overline{\mathrm{id}_{I_A}} = \overline{\eta_A' \circ \mathrm{id}_{I_A}} = \overline{\mathrm{id}_{I_A} \circ \mathrm{id}_{I_A}} = \overline{\mathrm{id}_{I_A}} = \mathrm{id}_{I_A},$$

where the right-hand side is the identity cell of I_A in $N(\mathcal{K})$ and where $\eta'_A = \mathrm{id}_{I_A}$ is the unique factorisation of η_A through $\eta_{\overline{\mathrm{id}_{I_A}}} = \eta_A$.

We conclude that $N(\mathcal{K})$ forms a well-defined augmented virtual double category that has all horizontal units. Next we extend the assignment $\mathcal{K} \mapsto N(\mathcal{K})$ to a strict 2-functor $N\colon \mathsf{VirtDblCat_u} \to \mathsf{AugVirtDblCat_u}$. For the action of N on morphisms consider a normal functor $F\colon \mathcal{K} \to \mathcal{L}$ between unital virtual double categories. Since F preserves the co-cartesian cells η_A of $\mathcal{K}$ we can obtain, for each object $A \in K$, an invertible horizontal cell $(F_I)_A\colon FI_A \Rightarrow I_{FA}$ in $\mathcal{L}$ that is the unique factorisation in

$$
\begin{array}{ccc}
& & FA \\
FA & & \Vert\!\Downarrow^{F\eta_A}\!\Vert \\
\Vert\!\Downarrow^{\eta_{FA}}\!\Vert & = & FA \xrightarrow[FI_A]{} FA \\
FA \xrightarrow[I_{FA}]{} FA & & \Vert\;\Downarrow^{(F_I)_A}\;\Vert \\
& & FA \xrightarrow[I_{FA}]{} FA
\end{array}
\qquad (11)
$$

We define $NF\colon N(\mathcal{K}) \to N(\mathcal{L})$ as follows. On objects and morphisms it simply acts as F does. To define its action on cells we first define, for each $\bar{\xi}$ in $N(\mathcal{K})$, the cell $\delta_{\bar{\xi}}$ in $\mathcal{L}$ by $\delta_{\bar{\xi}} := (F_I)_C$ if $\bar{\xi}$ is nullary with horizontal target C, and $\delta_{\bar{\xi}} := \mathrm{id}_{FK}$ if $\bar{\xi}$ is unary with horizontal target $K\colon C \nrightarrow D$; we then set $(NF)(\bar{\xi}) := \overline{(\delta_{\bar{\xi}} \circ F\xi)}$. That this assignment preserves identity cells is easily checked; that it preserves any composition $\bar{\chi} \circ (\bar{\xi}_1, \ldots, \bar{\xi}_n)$ in $N(\mathcal{K})$, as in (10), is shown by

$$
\begin{aligned}
(NF)&(\bar{\chi}) \circ \big((NF)(\bar{\xi}_1), \ldots, (NF)(\bar{\xi}_n)\big) \\
&= \overline{\delta_{\bar{\chi}} \circ F\chi} \circ \big(\overline{\delta_{\bar{\xi}_1} \circ F\xi_i}, \ldots, \overline{\delta_{\bar{\xi}_n} \circ F\xi_n}\big) \\
&= \overline{\delta_{\bar{\chi}} \circ (F\chi)' \circ (\delta_{\bar{\xi}_1} \circ F\xi_1, \ldots, \delta_{\bar{\xi}_n} \circ F\xi_n)} \\
&= \overline{\delta_{\bar{\chi}} \circ F(\chi') \circ (F\xi_1, \ldots, F\xi_n)} = \overline{\delta_{\bar{\chi}} \circ F\big(\chi' \circ (\xi_1, \ldots, \xi_n)\big)} \\
&= (NF)\big(\overline{\chi' \circ (\xi_1, \ldots, \xi_n)}\big) = (NF)\big(\bar{\chi} \circ (\bar{\xi}_1, \ldots, \bar{\xi}_n)\big),
\end{aligned}
$$

where the third identity is shown as follows. The cells $(F\chi)'$ and χ', on either side, are the factorisations in $F\chi = (F\chi)' \circ (\eta_{(NF)(\bar{\xi}_1)}, \ldots, \eta_{(NF)(\bar{\xi}_n)})$ and $\chi = \chi' \circ (\eta_{\bar{\xi}_1}, \ldots, \eta_{\bar{\xi}_n})$ respectively. The identity follows from the fact that

$$
\begin{aligned}
(F\chi)' \circ (\delta_{\bar{\xi}_1}, \ldots, \delta_{\bar{\xi}_n}) \circ (F\eta_{\bar{\xi}_1}, \ldots, F\eta_{\bar{\xi}_n}) &= (F\chi)' \circ \big(\eta_{(NF)(\bar{\xi}_1)}, \ldots, \eta_{(NF)(\bar{\xi}_n)}\big) \\
&= F\chi = F(\chi') \circ (F\eta_{\bar{\xi}_1}, \ldots, F\eta_{\bar{\xi}_n})
\end{aligned}
$$

together with the uniqueness of factorisations through the path $(F\eta_{\bar{\xi}_1}, \ldots, F\eta_{\bar{\xi}_n})$, which is cocartesian because F is normal. This concludes the definition of N on morphisms.

Next consider a transformation $\zeta\colon F \Rightarrow G$ between normal functors F and $G\colon \mathcal{K} \to \mathcal{L}$ of unital virtual double categories. We claim that the components of ζ again form a transformation $NF \Rightarrow NG$, which we take to be the image $N\zeta$. For instance, that

the components of ζ are natural with respect to a nullary cell $\bar{\psi} \colon \underline{J} \Rightarrow C$ in $N(\mathcal{K})$, with non-empty horizontal source $\underline{J}$, is shown below, where $(\eta_{GC} \circ \zeta_C)'$ is the unique factorisation of $\eta_{GC} \circ \zeta_C$ through $\eta_{(NF)(\bar{\psi})} = \eta_{FC}$.

$$
\begin{aligned}
(NG)(\bar{\psi}) \circ (\bar{\zeta}_{J_1}, \ldots, \bar{\zeta}_{J_n}) &= \overline{(G_I)_C \circ G\psi} \circ (\bar{\zeta}_{J_1}, \ldots, \bar{\zeta}_{J_n}) \\
&= \overline{(G_I)_C \circ G\psi \circ (\zeta_{J_1}, \ldots, \zeta_{J_n})} = \overline{(G_I)_C \circ \zeta_{I_C} \circ F\psi} \\
&= \overline{(\eta_{GC} \circ \zeta_C)' \circ (F_I)_C \circ F\psi} = \zeta_C \circ (NF)(\bar{\psi})
\end{aligned}
$$

Here the last identity follows from the definition of id_{ζ_C} in $N(\mathcal{L})$ while the penultimate identity follows from

$$
\begin{aligned}
(G_I)_C \circ \zeta_{I_C} \circ F\eta_C &= (G_I)_C \circ G\eta_C \circ \zeta_C = \eta_{GC} \circ \zeta_C \\
&= (\eta_{GC} \circ \zeta_C)' \circ \eta_{FC} = (\eta_{GC} \circ \zeta_C)' \circ (F_I)_C \circ F\eta_C,
\end{aligned}
$$

by using that $F\eta_C$ is cocartesian. Naturality of the components of ζ with respect to cells in $N(\mathcal{K})$ of other shapes can be shown similarly.

That the assignments $\mathcal{K} \mapsto N(\mathcal{K})$, $F \mapsto NF$ and $\zeta \mapsto N\zeta$ combine into a strict 2-functor $N \colon \mathsf{VirtDblCat_u} \to \mathsf{AugVirtDblCat_u}$ follows easily from the uniqueness of the factorisations (11). It is also clear that the obvious isomorphism $(U \circ N)(\mathcal{K}) \cong \mathcal{K}$ of virtual double categories extends to an isomorphism $U \circ N \cong \mathrm{id}$ of strict 2-endofunctors on $\mathsf{VirtDblCat}$. Thus it remains to construct an invertible 2-natural transformation $\tau \colon \mathrm{id} \xRightarrow{\cong} N \circ U$. Given a unital augmented virtual double category $\mathcal{K}$ we define the functor $\tau_{\mathcal{K}} \colon \mathcal{K} \to (N \circ U)(\mathcal{K})$ as follows. It is the identity on objects and morphisms, it is given by $\phi \mapsto \bar{\phi}$ on unary cells and by $\psi \mapsto \overline{\eta_C \circ \psi}$ on nullary cells $\psi \colon \underline{J} \Rightarrow C$. That these assignments preserve composites and identity cells is easily checked; that the family $\tau = (\tau_{\mathcal{K}})_{\mathcal{K}}$ is 2-natural is clear. Finally, the inverse functor $\tau^{-1} \colon (N \circ U)(\mathcal{K}) \to \mathcal{K}$ can be given as the identity on objects and morphisms, as $\bar{\phi} \mapsto \phi$ on unary cells and as $\bar{\psi} \mapsto \varepsilon_C \circ \psi$ on nullary cells $\bar{\psi} \colon \underline{J} \Rightarrow C$, where $\varepsilon_C \colon I_C \Rightarrow C$ is the nullary cartesian cell that corresponds to $\eta_C \colon C \Rightarrow I_C$ as in Lemma 5.9. This completes the proof. ∎

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