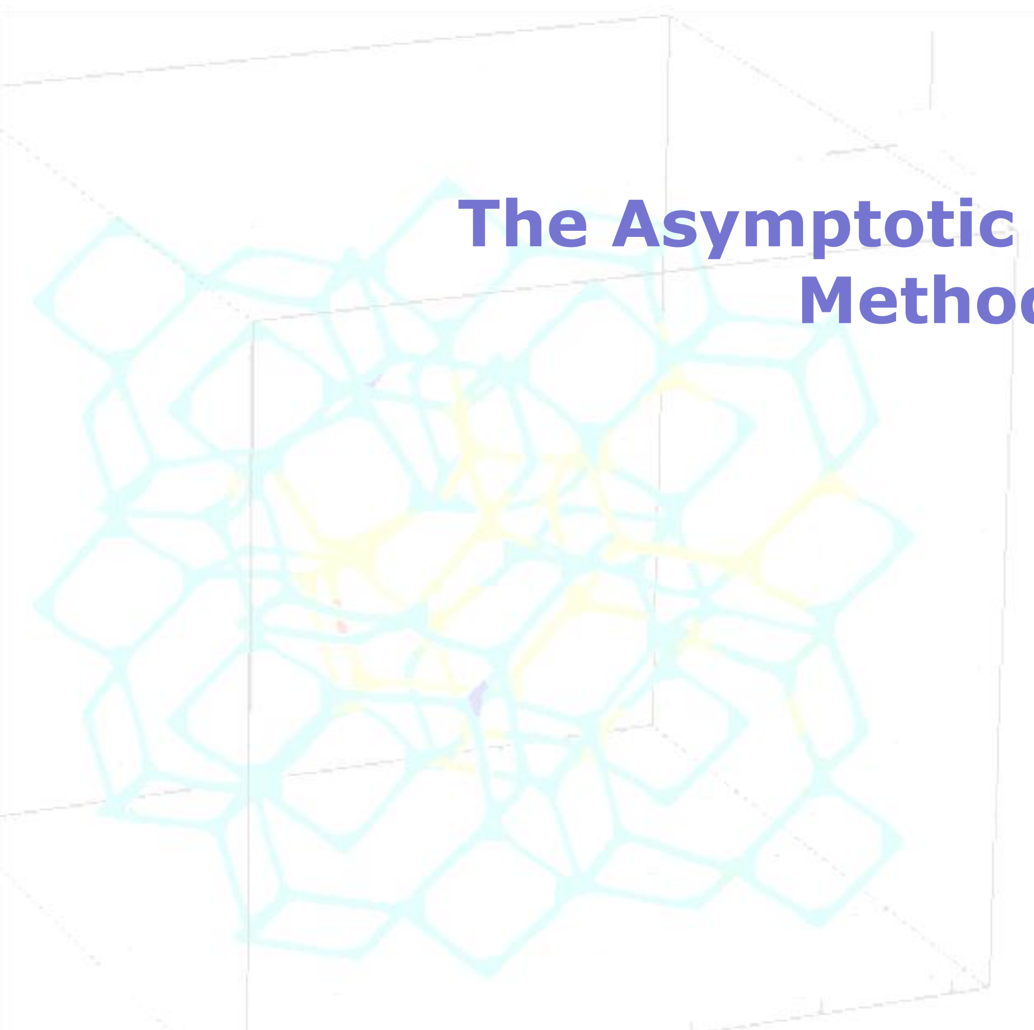
Méthodes de résolution alternatives des problèmes non linéaires en mécanique

Plan

- La méthode asymptotique numérique
- Méthodes de réduction de modèle basées sur la POD



The Asymptotic Numerical Method

Classical approaches for solving nonlinear problems in mechanics

- Finite Element method + Newton-Raphson iterative procedure

Weak form of PDE:

$$\text{Find } u \in \vartheta, \vartheta = \left\{ u / u = \bar{u} \text{ on } \partial\Omega_u, u \in W^{1,4}(\Omega) \right\} \text{ such that:}$$

$$\int_\Omega S_{ij}(u)\delta E_{ij}(u,\delta u)d\Omega = \lambda \int_{\partial\Omega_F} F_i \delta u_i d\Gamma$$

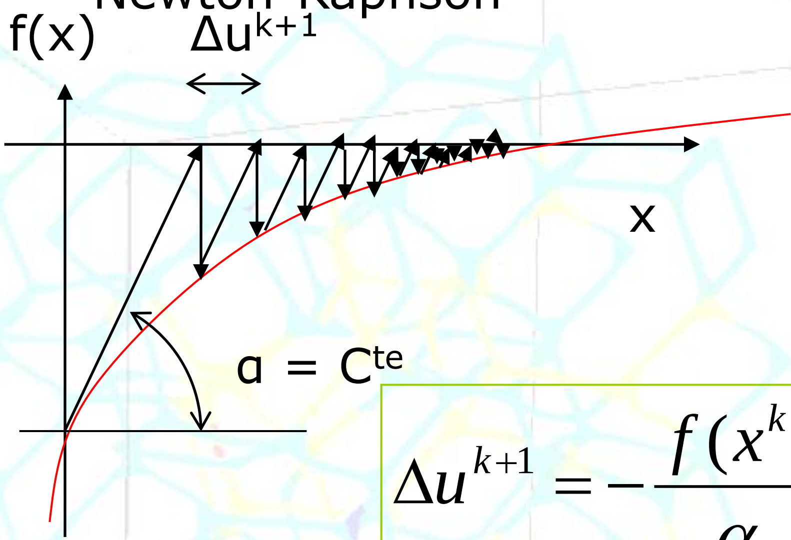
$$\forall \delta u \in \vartheta_0, \vartheta_0 = \left\{ u / u = 0 \text{ on } \partial\Omega_u, u \in W^{1,4}(\Omega) \right\}$$

$$\delta E_{ij}(u,\delta u) = \frac{1}{2}\left(\frac{\partial(\delta u_i)}{\partial X_j} + \frac{\partial(\delta u_j)}{\partial X_i} + \frac{\partial(\delta u_j)}{\partial X_i}\frac{\partial(u_i)}{\partial X_j} + \frac{\partial(\delta u_i)}{\partial X_j}\frac{\partial(u_j)}{\partial X_i} \right)$$

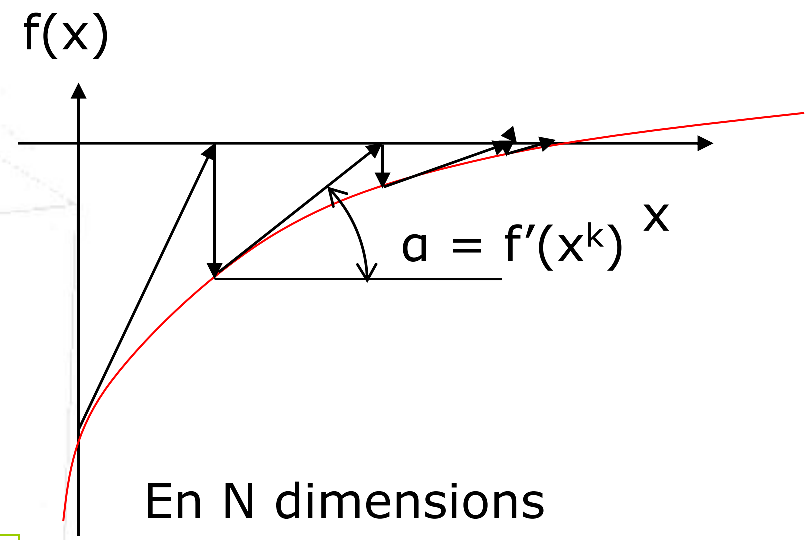
Méthodes classiques pour la résolution itérative des équations non-linéaires

- Méthode de point fixe

- Newton-Raphson



$$\Delta u^{k+1} = -\frac{f(x^k)}{\alpha}$$



En N dimensions

$$\mathbf{K_t}(\mathbf{u}^k)\mathbf{\Delta u}^{k+1} = -\mathbf{R}(\mathbf{u}^k)$$

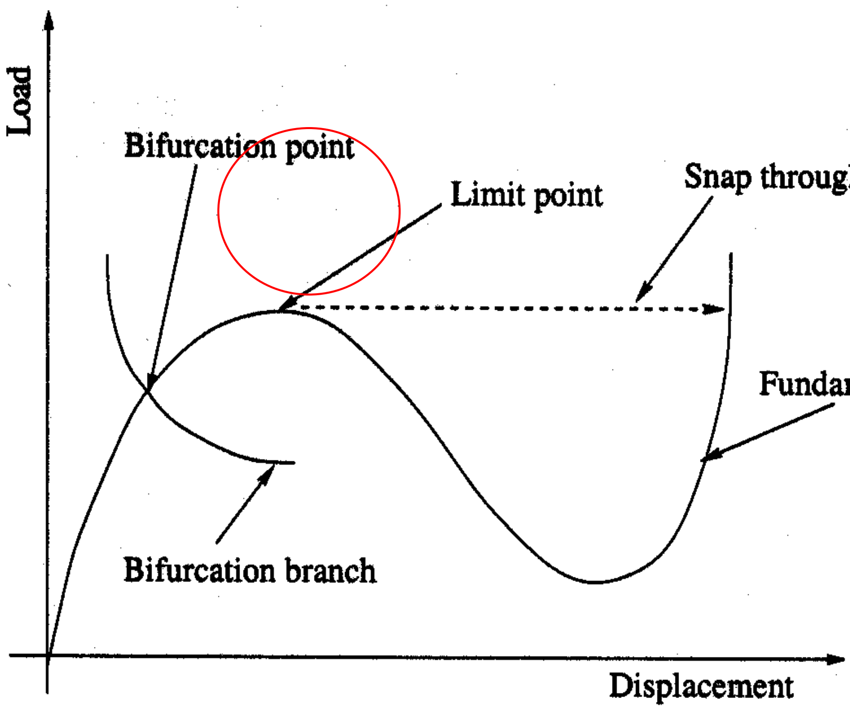
$$\mathbf{u}^{k+1} = \mathbf{u}^k + \mathbf{\Delta u}^{k+1}$$

Issues with problems involving instabilities and bifuraction solutions

Divergence of Newton-Raphson algorithms near limit points

Solutions : arc-length control

Difficulties : choice of numerical control parameters, small iterations, detection of bifurcation points…



J YVONNET 2006

A typical (one scale) mechanical nonlinear problem: elasticity with geometric nonlinearities: nonlinear partial differential equation

$$\boxed{\frac{\partial P_{ij}(u(X))}{\partial X_j} = 0 \quad \text{in } \Omega}$$

P: 1st Piola Kirchhoff stress tensor

$$P_{ij}(u) = F_{ik}(u) S_{kj}(u)$$

$$F_{ij}(u) = \delta_{ij} + \frac{\partial u_i}{\partial X_j}$$

$$S_{ij}(u) = C_{ijkl} E_{kl}(u)$$

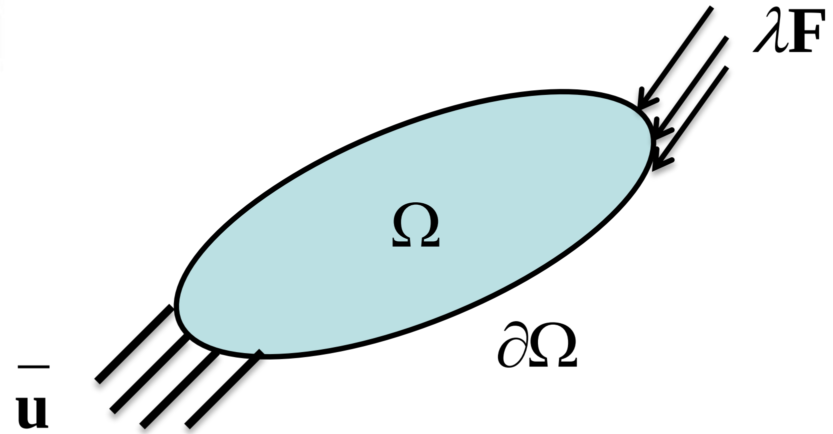
$$E_{ij}(u) = \frac{1}{2}\left(F_{ki}(u) F_{kj}(u) - \delta_{ij}\right)$$

$$P_{ij} N_j = \lambda F_i \quad \text{on } \partial\Omega_{\mathrm{F}}$$

$$u_i = \overline{u}_i \quad \text{on } \partial\Omega_{\mathrm{u}}$$

Boundary conditions

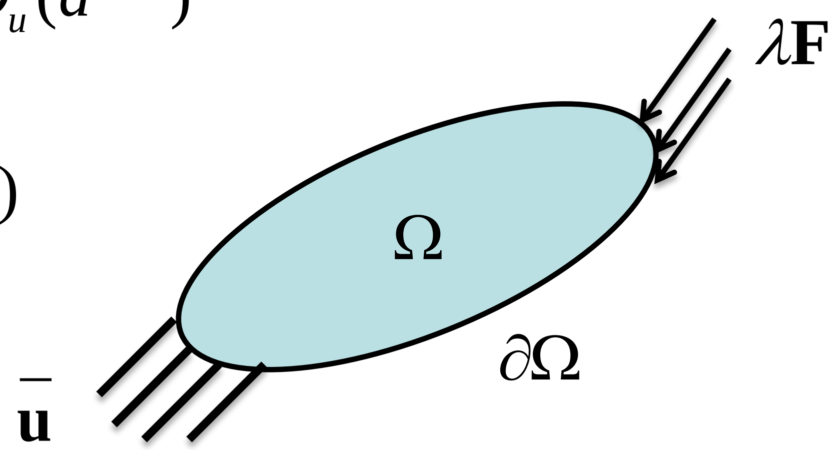
$$\partial\Omega = \partial\Omega_{\mathrm{F}} \cup \partial\Omega_{\mathrm{u}}, \quad \partial\Omega_{\mathrm{F}} \cap \partial\Omega_{\mathrm{u}} = \varnothing$$



The Asymptotic Numerical Method [Damil, Potier-Ferry, Cochelin 90-94….]

- Perturbation method (Taylor expansion) around a known solution u_0, λ_0
- Development parameter a: related to the system evolution (in a mechanical problem: amplitude of applied force)
- Continuation procedure

$$\begin{cases} u(x,a) = u_0(x) + \sum_{i=1}^{N} \frac{a^i}{i!} \frac{\partial^i u(x,a)}{\partial a^i} + O_u(a^{N+1}) \\ \lambda(a) = \lambda_0 + \sum_{i=1}^{N} \frac{a^i}{i!} \frac{\partial^i \lambda(a)}{\partial a^i} + O_\lambda(a^{N+1}) \end{cases}$$



High order terms: not known explicitely in general

$$\begin{cases} u(x,a) = u_0(x) + \sum_{i=1}^{N} \frac{a^i}{i!} \frac{\partial^i u(x,a)}{\partial a^i} \\ \lambda(a) = \lambda_0 + \sum_{i=1}^{N} \frac{a^i}{i!} \frac{\partial^i \lambda(a)}{\partial a^i} \end{cases}$$

Defined as unknown fields u_i and λ_i and computed numerically (via the Finite Element method)

$$\begin{cases} u(x) = u_0(x) + \sum_{i=1}^{N} a^i u_i(x) \\ \lambda = \lambda_0 + \sum_{i=1}^{N} a^i \lambda_i \end{cases}$$

A simple illustration

Example: Nonlinear PDE

$$\frac{\partial^2 u}{\partial x^2} + u^2 = \lambda \qquad \text{+b.c}$$

$$\lambda = a$$

$$u(a) = u_0 + a u_1 + a^2 u_2 + \ldots + a^N u_N$$

$$\lambda(a) = \lambda_0 + a\lambda_1 + a^2 \lambda_2 + \ldots + a^N \lambda_N$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u_0}{\partial x^2} + a\,\frac{\partial^2 u_1}{\partial x^2} + a^2\,\frac{\partial^2 u_1}{\partial x^2} + \ldots + a^N\,\frac{\partial^2 u_N}{\partial x^2}$$

Exemple: EDP non linéaire

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + u^2 = \lambda \\ \lambda = a \end{cases}$$

$$\begin{cases}
\dfrac{\partial^2 u_0}{\partial x^2} + a\dfrac{\partial^2 u_1}{\partial x^2} + a^2\dfrac{\partial^2 u_2}{\partial x^2} + \ldots + a^N \dfrac{\partial^2 u_N}{\partial x^2} \\
+ u_0^2 + a u_0 u_1 + a^2 u_0 u_2 + a^3 u_0 u_3 + \ldots + a^N u_0 u_N \\
\qquad + a u_1 u_0 + a^2 u_1 u_1 + a^3 u_1 u_2 + \ldots + \\
\qquad\qquad + a^2 u_2 u_0 + a^3 u_2 u_1 + \ldots \\
\qquad\qquad\qquad + a^3 u_3 u_0 + \ldots \\
\qquad\qquad\qquad\qquad + \ldots \\
= \lambda_0 + a\lambda_1 + a^2\lambda_2 + \ldots + a^N \lambda_N \\
\lambda_0 + a\lambda_1 + a^2\lambda_2 + \ldots + a^N \lambda_N = a
\end{cases}$$

Identification des même puissances de « *a* »:

$$\left\{\begin{array}{l}\boxed{\dfrac{\partial^2 u_0}{\partial x^2}} + a\dfrac{\partial^2 u_1}{\partial x^2} + a^2\dfrac{\partial^2 u_2}{\partial x^2} + \ldots + a^N\dfrac{\partial^2 u_N}{\partial x^2} \\ \boxed{+u_0^2} + au_0u_1 + a^2u_0u_2 + a^3u_0u_3 + \ldots + a^N u_0 u_N \\ \qquad + au_1u_0 + a^2u_1u_1 + a^3u_1u_2 + \ldots + \\ \qquad\qquad + a^2u_2u_0 + a^3u_2u_1 + \ldots \\ \qquad\qquad\qquad + a^3u_3u_0 + \ldots \\ \qquad\qquad\qquad\qquad + \ldots \\ = \boxed{\lambda_0} + a\lambda_1 + a^2\lambda_2 + \ldots + a^N\lambda_N \\ \boxed{\lambda_0} + a\lambda_1 + a^2\lambda_2 + \ldots + a^N\lambda_N = a\end{array}\right.$$

Ordre 0

$$\boxed{\begin{array}{l}\dfrac{\partial^2 u_0}{\partial x^2} + u_0^{\,2} = \lambda_0 \\ \lambda_0 = 0\end{array}}$$

Ordre 0

$$\boxed{\begin{aligned} &\frac{\partial^2 u_0}{\partial x^2} + {u_0}^2 = \lambda_0 \\ &\lambda_0 = 0 \end{aligned}}$$

On suppose u_0 connue, exemple:
$u_0 = 0$

$$\boxed{\lambda_0 = 0}$$

Identification des même puissances de a:

$$\left\{\begin{aligned} & \frac{\partial^2 u_0}{\partial x^2} + a\boxed{\frac{\partial^2 u_1}{\partial x^2}} + a^2 \frac{\partial^2 u_2}{\partial x^2}+\ldots + a^N \frac{\partial^2 u_N}{\partial x^2} \\ & + u_0^2 + a\boxed{u_0u_1} + a^2u_0u_2 + a^3u_0u_3+\ldots+a^Nu_0u_N \\ & + a\boxed{u_1u_0} + a^2u_1u_1 + a^3 u_1u_2 + \ldots + \\ & + a^2u_2u_0 + a^3u_2u_1+\ldots \\ & +a^3u_3u_0+\ldots \\ & +\ldots \\ & = \lambda_0 + a\boxed{\lambda_1} + a^2\lambda_2+\ldots+a^N\lambda_N \\ & \lambda_0 + a\boxed{\lambda_1} + a^2\lambda_2 + \ldots + a^N\lambda_N = a.\boxed{1}\end{aligned}\right.$$

Ordre 1

$$\boxed{\begin{aligned} & \frac{\partial^2 u_1}{\partial x^2} + 2u_0u_1 = \lambda_1 \\ & \lambda_1 = 1 \end{aligned}}$$

Ordre 1

Ordre 1

$$\boxed{\begin{gathered}\frac{\partial^2 u_1}{\partial x^2} + 2u_0 u_1 = \lambda_1 \\ \lambda_1 = 1\end{gathered}}$$

$$\frac{\partial^2 u_1}{\partial x^2} = \lambda_1$$

$$\lambda_1 = 1$$

On trouve u_1 $\lambda_1 = 1$

Identification des même puissances de a:

$$
\left\{
\begin{aligned}
&\frac{\partial^2 u_0}{\partial x^2} + a\frac{\partial^2 u_1}{\partial x^2} + a^2 \boxed{\frac{\partial^2 u_2}{\partial x^2}} + \ldots + a^N \frac{\partial^2 u_N}{\partial x^2} \\
&+ u_0^2 + a u_0 u_1 + a^2 \boxed{u_0 u_2} + a^3 u_0 u_3 + \ldots + a^N u_0 u_N \\
&\qquad + a u_1 u_0 + a^2 \boxed{u_1 u_1} + a^3 u_1 u_2 + \ldots + \\
&\qquad\qquad + a^2 \boxed{u_2 u_0} + a^3 u_2 u_1 + \ldots \\
&\qquad\qquad\qquad + a^3 u_3 u_0 + \ldots \\
&\qquad\qquad\qquad\qquad + \ldots \\
&= \lambda_0 + a\lambda_1 + a^2 \boxed{\lambda_2} + \ldots + a^N \lambda_N \\
&\lambda_0 + a\lambda_1 + a^2 \boxed{\lambda_2} + \ldots + a^N \lambda_N = a.1
\end{aligned}
\right.
$$

Ordre 2

$$
\boxed{
\begin{aligned}
&\frac{\partial^2 u_2}{\partial x^2} + 2u_0 u_2 + {u_1}^2 = \lambda_2 \\
&\lambda_2 = 0
\end{aligned}
}
$$

Ordre 2

$$\boxed{\begin{aligned}&\frac{\partial^2 u_2}{\partial x^2} + 2u_0 u_2 + u_1{}^2 = \lambda_2\\ &\lambda_2 = 0\end{aligned}}$$

$$\frac{\partial^2 u_2}{\partial x^2} = -u_1{}^2$$

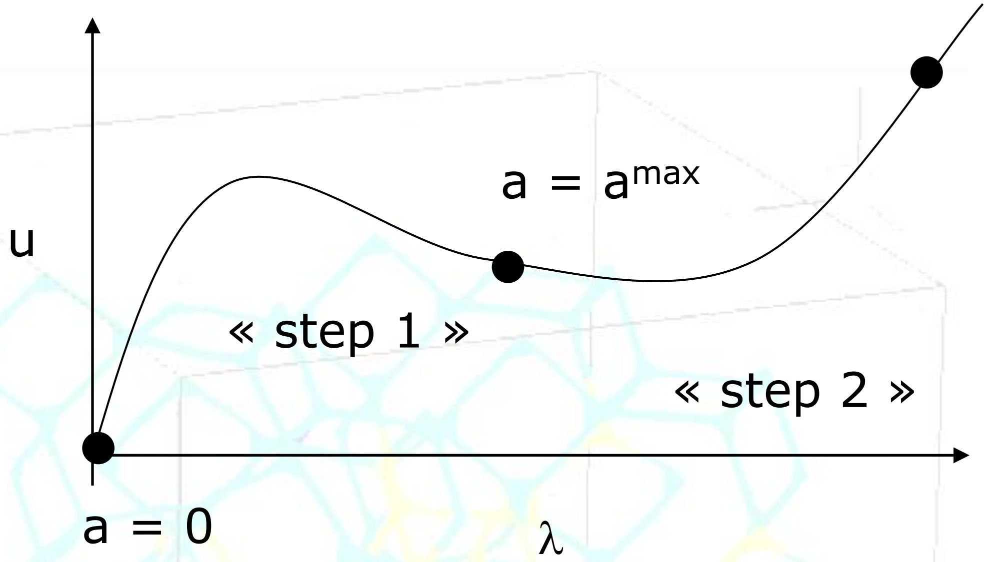
$$\lambda_2 = 0$$

On trouve u_2

$$\lambda_1 = 1$$

Etc...

The series has a radius of convergence: the solution is defined along « steps » through a continuation method



Continuation method

Given N, a tolerance ϵ is chosen such that

$$\epsilon = \frac{\|\mathbf{u}^{n+1}(N, a_{max}) - \mathbf{u}^{n+1}(N-1, a_{max})\|}{\|\mathbf{u}^{n+1}(N, a_{max}) - \mathbf{u}^{n}\|}$$

A more involved example: elasticity with geometrical nonlinearity

$$\int_\Omega {}^t\mathbf{P} : \delta\,\mathbf{F}\, d\Omega = \lambda\ \mathscr{P}_{ext}\left(\delta\,\mathbf{u}\right)$$

Constitutive law:

$$\mathbf{S} = \mathbb{C} : \gamma$$

Kinematic relations:

$$\mathbf{P} = \mathbf{F}\cdot\mathbf{S}$$

$$\gamma = \tfrac{1}{2}\left({}^t\mathbf{F}\cdot\mathbf{F} - \mathbf{I}\right)$$

$$\mathbf{F} = \mathbf{I} + \nabla\,\mathbf{u}$$

$$\left\{\begin{array}{c}\mathbf{U}(a)\\ \lambda(a)\end{array}\right\} = \left\{\begin{array}{c}\mathbf{U}_0\\ \lambda_0\end{array}\right\} + \sum_{i=1}^{N} a^i \left\{\begin{array}{c}\mathbf{U}_i\\ \lambda_i\end{array}\right\}$$

$$\mathbf{U} = \{\mathbf{P}, \mathbf{F}, \mathbf{S}, \gamma, \mathbf{u}\}$$

$$\Rightarrow$$

$$\left\{\begin{array}{l}
\displaystyle\int_\Omega \left(\sum_{i=0}^{N} a^i\, {}^t\mathbf{P}_i : \delta\,\mathbf{F}\right) d\Omega = \sum_{i=0}^{N} a^i \lambda_i\ \mathscr{P}_{ext}\left(\delta\,\mathbf{u}\right)\\
\displaystyle\sum_{i=0}^{N} a^i \mathbf{S}_i = \mathbb{C} : \sum_{i=0}^{N} a^i \gamma_i\\
\displaystyle\sum_{i=0}^{N} a^i\, \mathbf{P}_i = \left(\sum_{i=0}^{N} a^i \mathbf{F}_i\right)\cdot\left(\sum_{i=0}^{N} a^i \mathbf{S}_i\right)\\
\displaystyle\sum_{i=0}^{N} a^i \gamma_i = \frac{1}{2}\left[\left(\sum_{i=0}^{N} a^i\, {}^t\mathbf{F}_i\right)\cdot\left(\sum_{i=0}^{N} a^i \mathbf{F}_i\right) - \mathbf{I}\right]\\
\displaystyle\sum_{i=0}^{N} a^i \mathbf{F}_i = \mathbf{I} + \nabla\left(\sum_{i=0}^{N} a^i \mathbf{u}_i\right)
\end{array}\right.$$

Grouping terms with same exponent produces at order k

$$\begin{cases} \int_\Omega {}^t\mathbf{P}_k : \delta\,\mathbf{F}\, d\Omega = \lambda_k\, \mathscr{P}_{ext}(\delta\,\mathbf{u}) \\ \mathbf{S}_k = \mathbb{C} : \gamma_k \\ \mathbf{P}_k = \mathbf{F}_0 \cdot \mathbf{S}_k + \mathbf{F}_k \cdot \mathbf{S}_0 + \sum_{r=1}^{k-1} \mathbf{F}_{k-r} \cdot \mathbf{S}_r \\ \gamma_k = \frac{1}{2}\left({}^t\mathbf{F}_0 \cdot \mathbf{F}_k + {}^t\mathbf{F}_k \cdot \mathbf{F}_0 + \sum_{r=1}^{k-1} {}^t\mathbf{F}_{k-r} \cdot \mathbf{F}_r \right) \\ \mathbf{F}_k = \nabla\, \mathbf{u}_k \end{cases}$$

Sequence of linear problems with same linear operator solved with FEM

$$\boxed{\mathscr{L}(\mathbf{u}_k, \delta\mathbf{u}) = \lambda_k\, \mathscr{P}_{ext}(\delta\,\mathbf{u}) + \mathscr{F}_k^{nl}(\delta\mathbf{u})}$$

Order k

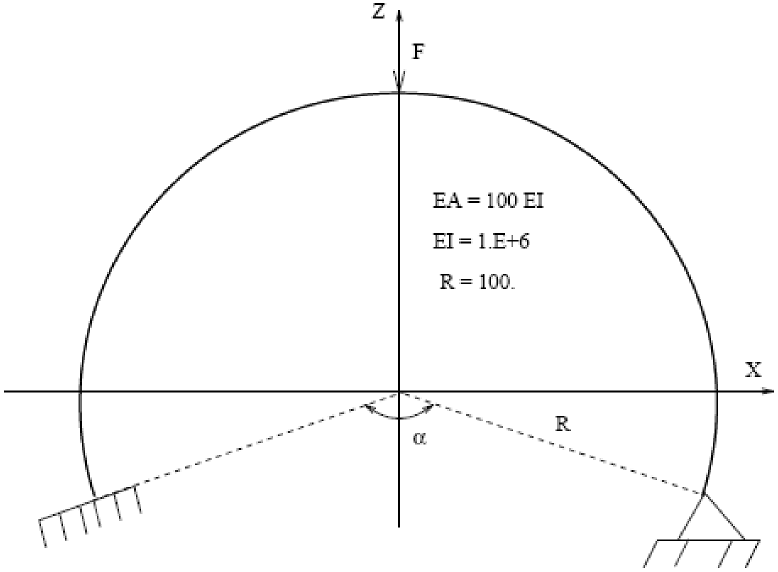
$$\mathscr{L}(\mathbf{u}_k, \delta\mathbf{u}) = \int_\Omega {}^t\nabla\,\mathbf{u}_k : \mathbb{H} : \nabla\,\delta\mathbf{u}\, d\Omega\ ,$$

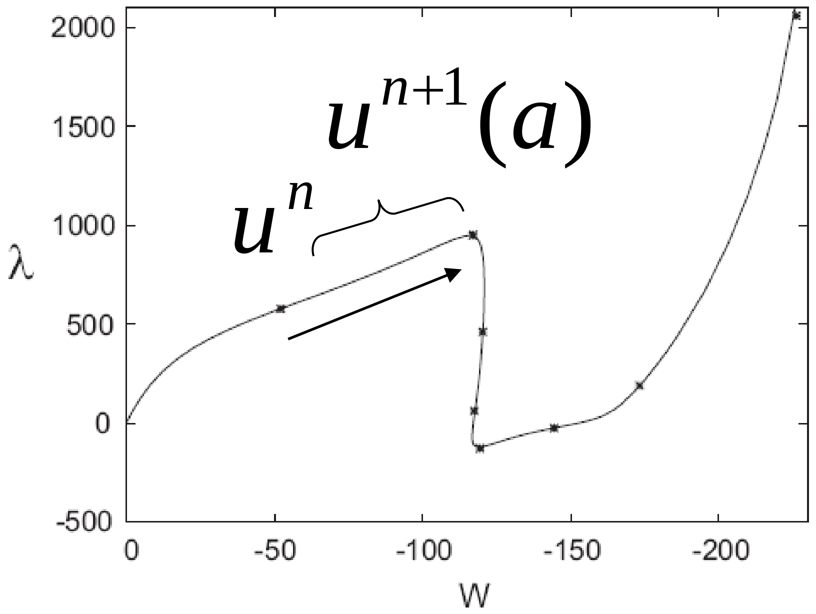
$$\mathscr{F}_k^{nl}(\delta\mathbf{u}) = -\int_\Omega {}^t\mathbf{P}_k^{nl} : \nabla\,\delta\mathbf{u}\;\; d\Omega \qquad 1 < k \leq N$$

Application to elastic buckling

$$u^{n+1}(a) = u^n + au_1 + a^2u_2 + \ldots + a^p u_p$$

$$\lambda^{n+1}(a) = \lambda^n + a\lambda_1 + a^2\lambda_2 + \ldots + a^p \lambda_p$$

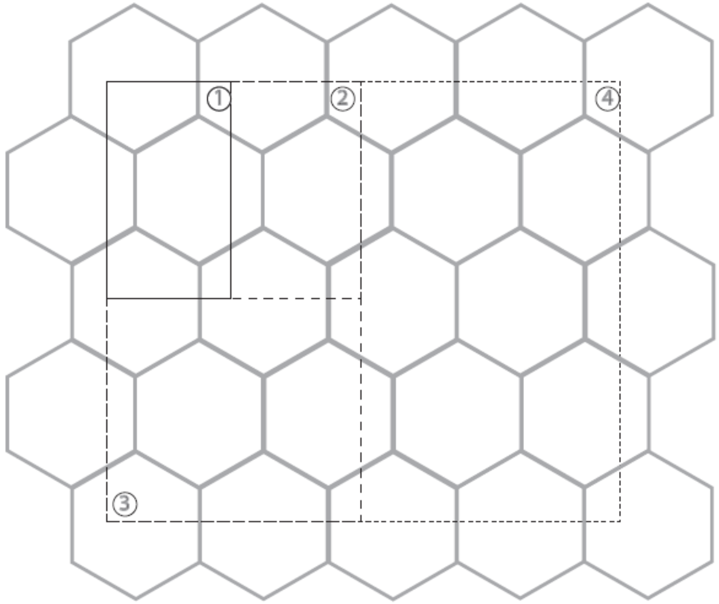




- Piece-wise continuous approximation of the solution with respect to the development parameter a
- remove difficulties related to limit points

Méthode asymtotique numérique (MAN)

Exemples : flambement de microstructures cellulaires en compression



[Yvonnet, Zahrouni, Potier-Ferry, Computer Methods in Applied Mechanics and Enginnering, 2008]

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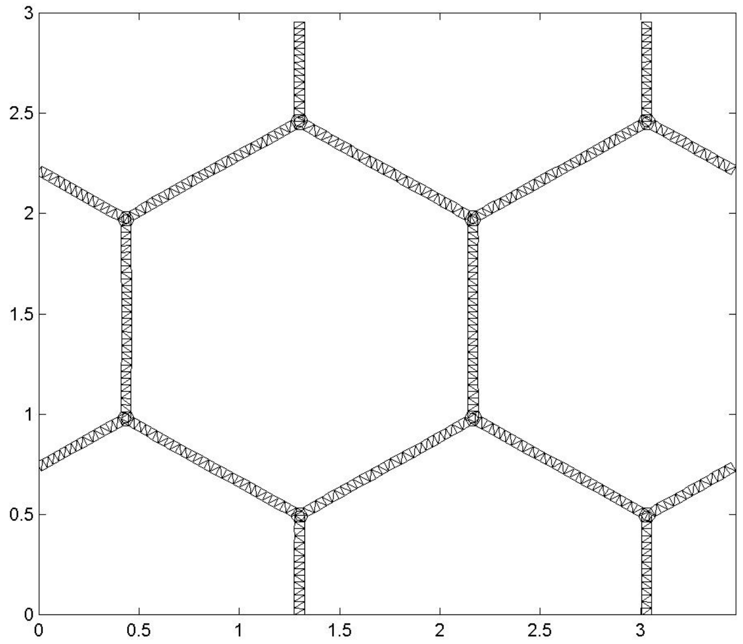

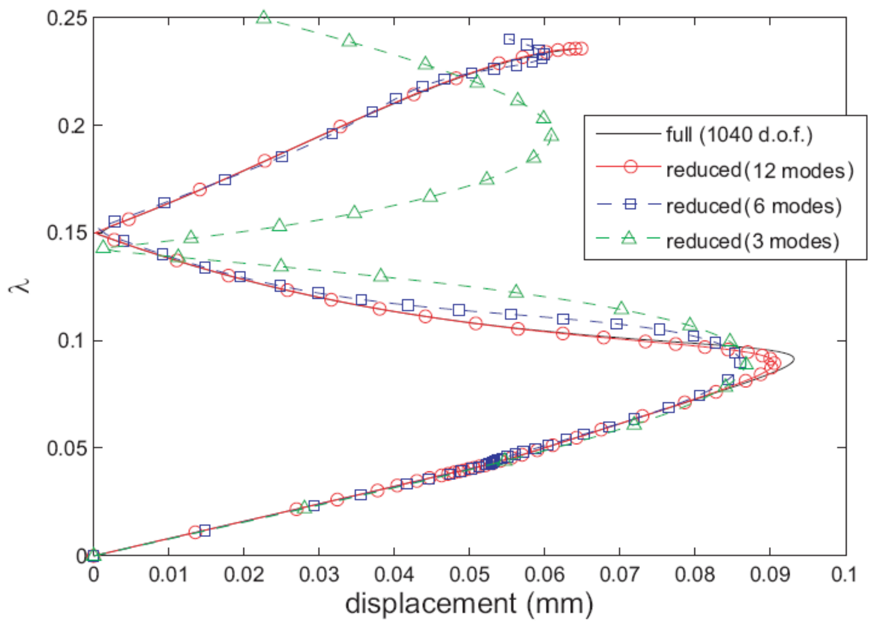

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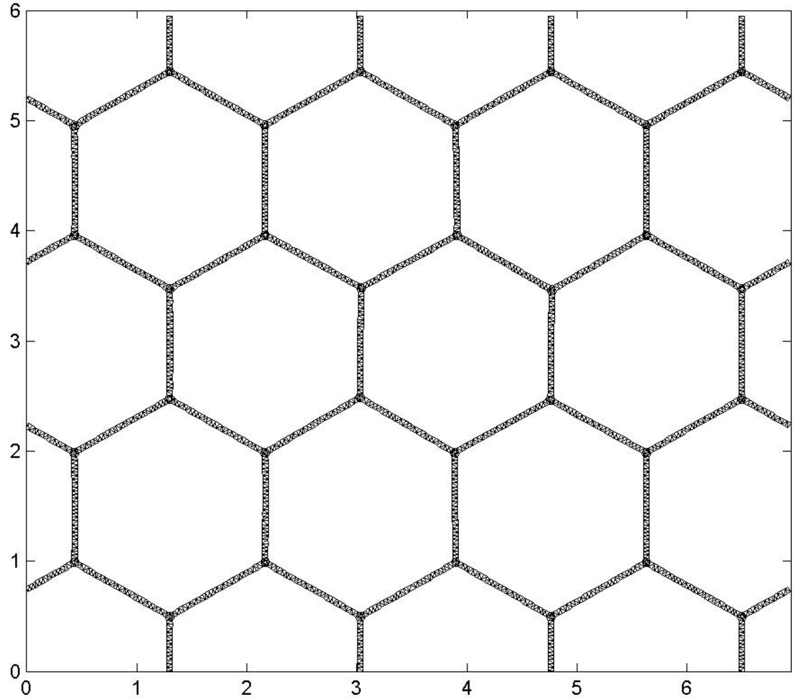

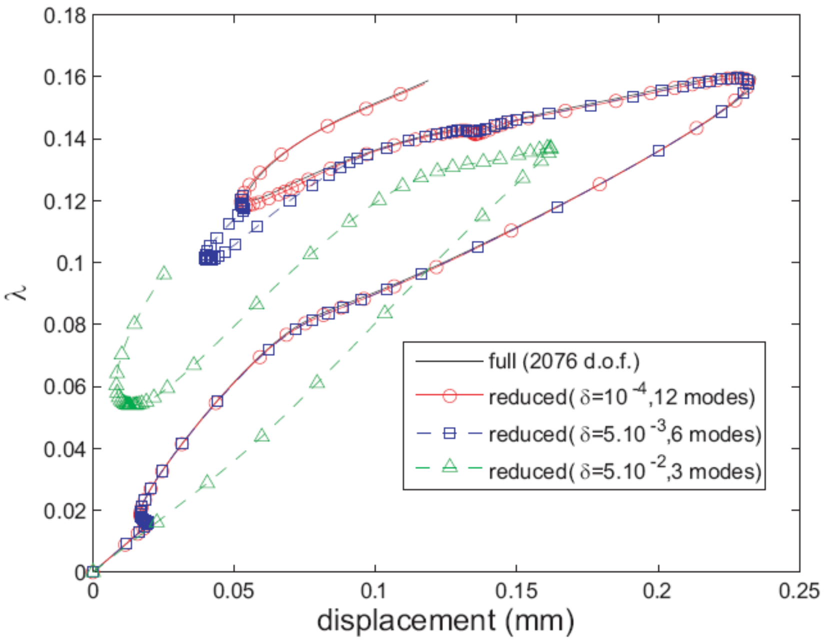

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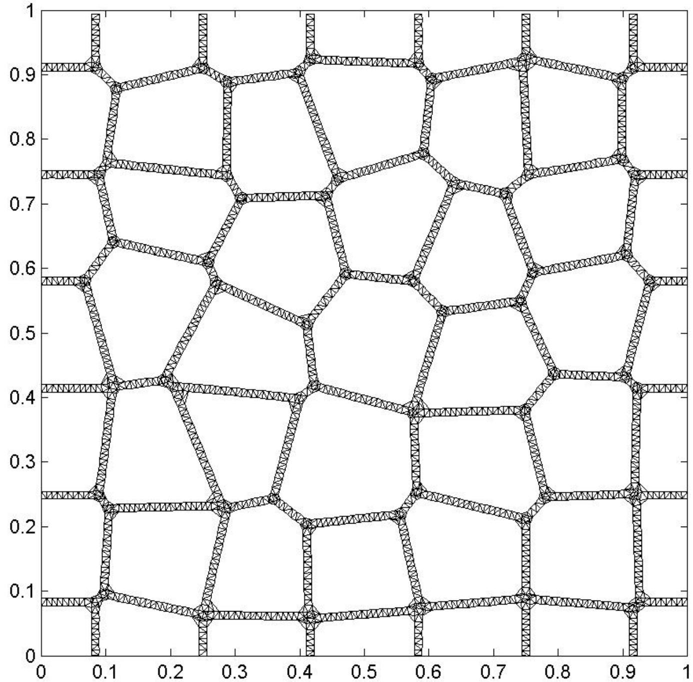

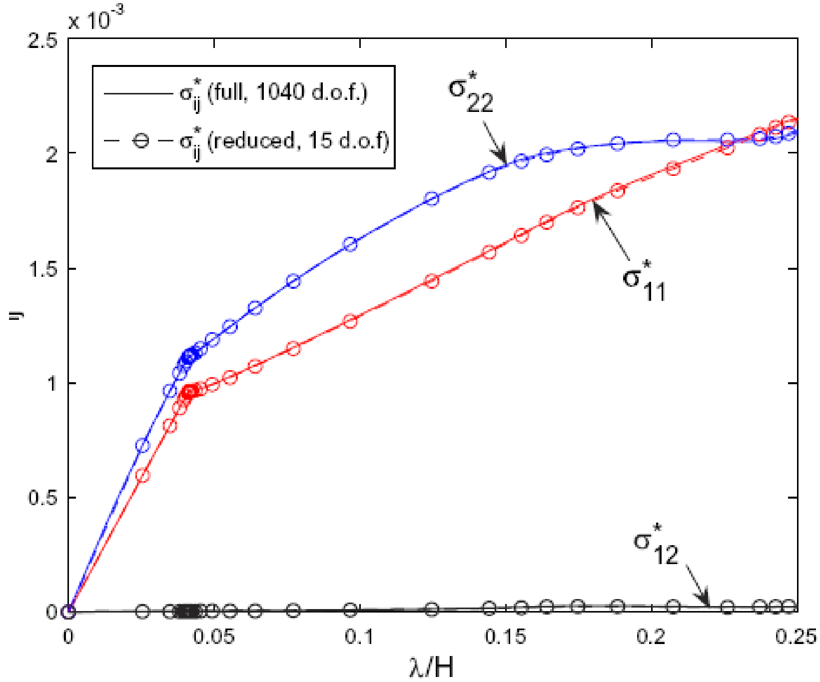

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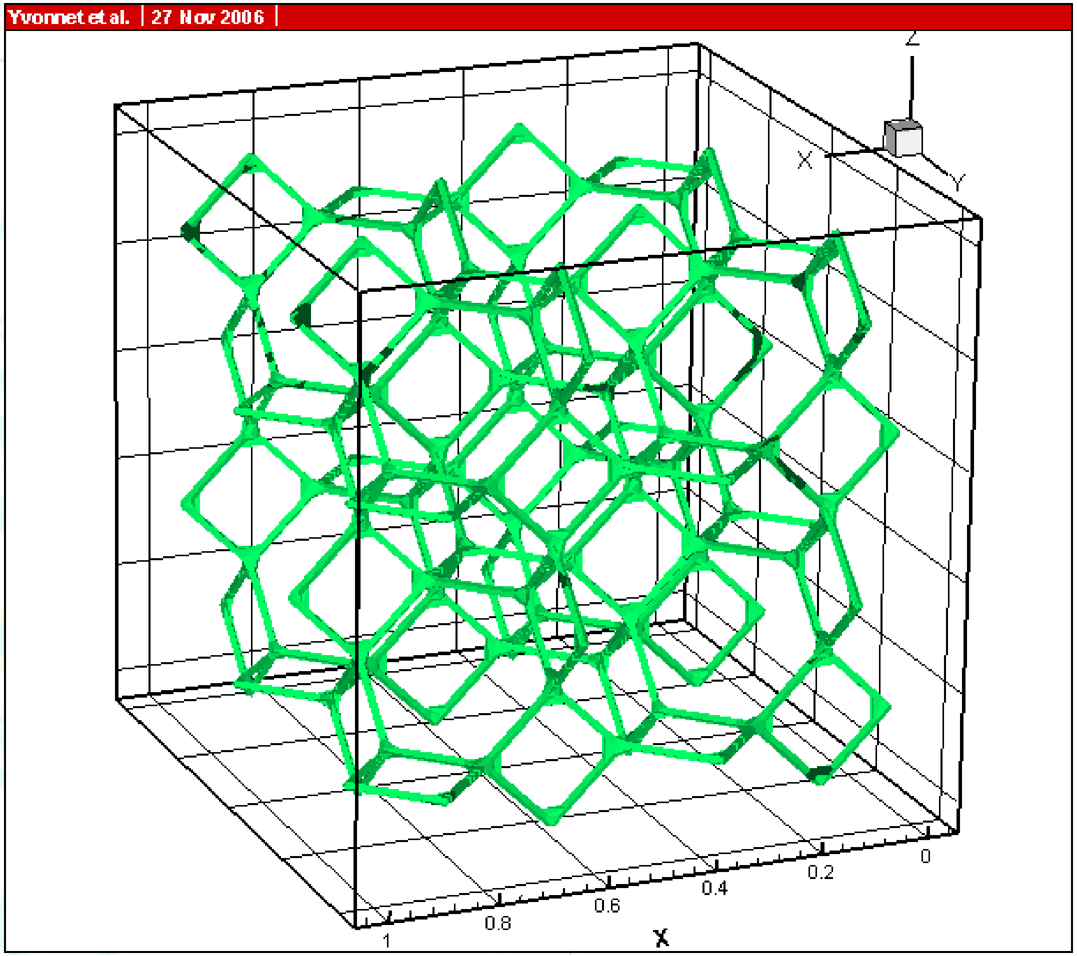

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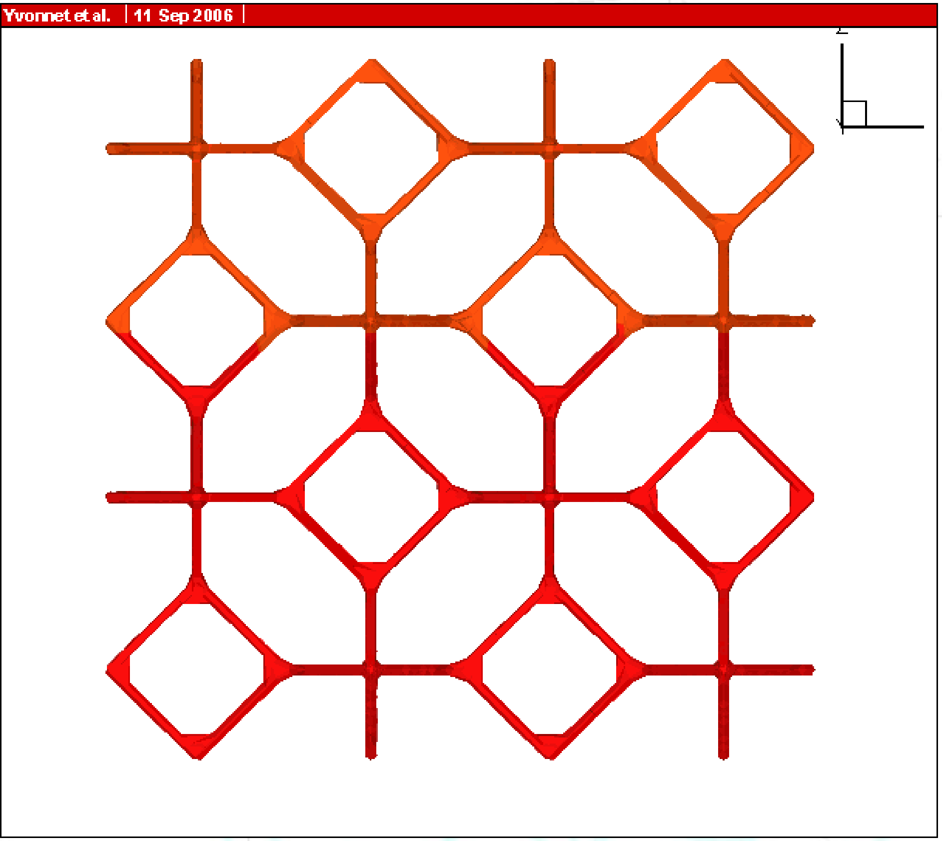
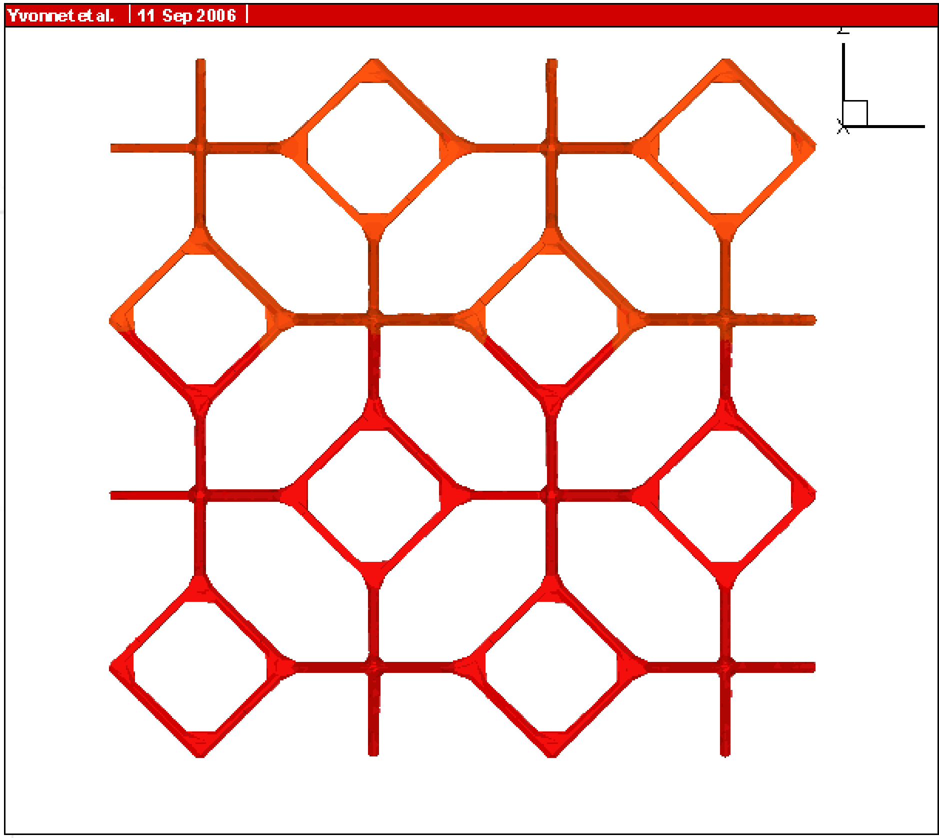




Méthodes de réduction de modèles de type POD

Méthodes de réduction de modèles

Systèmes de grandes dimensions (nombre important de d.d.l.) :

- Matrice de grandes tailles à décomposer (inverser)
- Assemblage des matrices coûteux, place mémoire importante

Enjeux des méthodes de réduction de modèle : transformer un système d'équations de taille NxN en un système de « taille réduite » MxM avec M<<N

Applications : séries de problèmes non-linaires « similaires » de taille importante (Optimisation, calcul multi-échelles…)

Méthode de réduction de modèles de type POD

<u>POD (Proper orthognal decomposition)</u>

Approximation de la solution Éléments Finis par projection dans un sous-espace de taille

$$\mathbf{q}^R(t) = \phi_0 + \sum_{m=1}^{M} \phi_m \xi_m(t)$$

Variables arbitraires

Fonctions de base globales

Soit H un espace de Hilbert d$\Re^N$s Rn muni du produit $\langle \cdot, \cdot \rangle$laire XX et de l $\|\psi\| = \sqrt{\langle \psi, \psi \rangle}$ X .

Fonctions de base définies par le problème de minimisation

$$\underset{\boldsymbol{\phi}}{\text{MIN}} \int_0^T \left\| \mathbf{u}(t) - \boldsymbol{\phi}\boldsymbol{\phi}^T \mathbf{u}(t) \right\|^2 dt$$

Sous la contrainte $\left\langle \boldsymbol{\phi}_i, \boldsymbol{\phi}_j \right\rangle = \delta_{ij}$

Méthode de réduction de modèles de type POD

La minimisation conduit, après discrétisation temporelle du problème aux valeurs propres [Liang 2002,etc…]

$$\boxed{\mathbf{Q}\boldsymbol{\phi}_i = \lambda_i \boldsymbol{\phi}_i}$$

$$\boldsymbol{\phi}_0 = \bar{\mathbf{q}} = \frac{1}{S}\sum_{i=1}^{S} \mathbf{q}(t_i)$$

$$\mathbf{Q} = \frac{1}{S}\mathbf{U}\mathbf{U}^T$$

$$\mathbf{U} = \{\mathbf{q}(t_1) - \bar{\mathbf{q}}, \mathbf{q}(t_2) - \bar{\mathbf{q}}, \ldots, \mathbf{q}(t_S) - \bar{\mathbf{q}}\}$$

On cherche à obtenir un sous-espace de dimension M<<N

Erreur liée à la troncature de la base p $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_M \geq \ldots \geq \lambda_{DN} \geq 0.$

$$\epsilon(M) = \sum_{i=1}^{S} \left\| \mathbf{q}(\mathbf{x}, t_i) - \mathbf{q}^R(\mathbf{x}, t_i) \right\| = \left(\sum_{i=M+1}^{DN} \lambda_i \right)^{1/2}$$

Critère : $\dfrac{\left(\sum_{i=M+1}^{DN} \lambda_i\right)^{1/2}}{\left(\sum_{i=1}^{DN} \lambda_i\right)^{1/2}} < \delta$

Méthode de réduction de modèles de type POD

$$\mathbf{K}_{\mu}^{k}\boldsymbol{\Delta}\mathbf{q}^{k+1} = \mathbf{f}_{ext(\mu)} - \mathbf{f}_{int}^{k}(\mathbf{q}_{\mu}^{k})$$

Incréments de déplacements projetés dans la base réduite

$$\boldsymbol{\Delta}\mathbf{q}^{k+1} = \sum_{m=1}^{M} \phi_m \Delta\xi_m^{k+1}$$

$$\boldsymbol{\Phi} = \{\phi_1, \phi_2, \dots, \phi_M\}$$

$$\boldsymbol{\xi} = \{\xi_1, \xi_2, \dots, \xi_M\}$$

$\Downarrow$

Problème réduit linéarisé

M<< N

$$\boxed{\boldsymbol{\Phi}^{T}\mathbf{K}_{\mu}^{k}\boldsymbol{\Phi}}\boldsymbol{\Delta}\boldsymbol{\xi}^{k+1} = \boldsymbol{\Phi}^{T}\left[\mathbf{f}_{ext(\mu)} - \mathbf{f}_{int}^{k}(\mathbf{q}_{\mu})\right]$$

[M x M]

Actualisation des variables réduites

$$\boldsymbol{\xi}^{k+1} = \boldsymbol{\xi}^{k} + \boldsymbol{\Delta}\boldsymbol{\xi}^{k+1}$$

Méthode de réduction de modèles de type POD

Exemple d'application : hyperelasticicité non-linéaire en grandes transformations

$$\nabla \cdot \bar{\mathbf{P}} + \bar{\mathbf{B}} = 0 \quad and \quad \bar{\mathbf{P}}\bar{\mathbf{F}}^T = \left(\bar{\mathbf{P}}\bar{\mathbf{F}}^T\right)^T \qquad in \quad \Omega_0$$

$$\mathbf{P} = \frac{\partial \Psi(\mathbf{F})}{\partial \mathbf{F}} \qquad \Psi = c(J-1)^2 - d\ log(J) + c_1(I_1 - 3) + c_2(I_2 - 3)$$

Matériau hyperelastique de type Mooney-Rivlin compressible

$$D_{\Delta \mathbf{u}} \delta W_{int}(\mathbf{u}, \delta \mathbf{u}) = \int_{\Omega_0} \Big[\nabla_X(\delta \mathbf{u}) : \nabla_X(\Delta \mathbf{u}) \bar{\mathbf{S}}$$

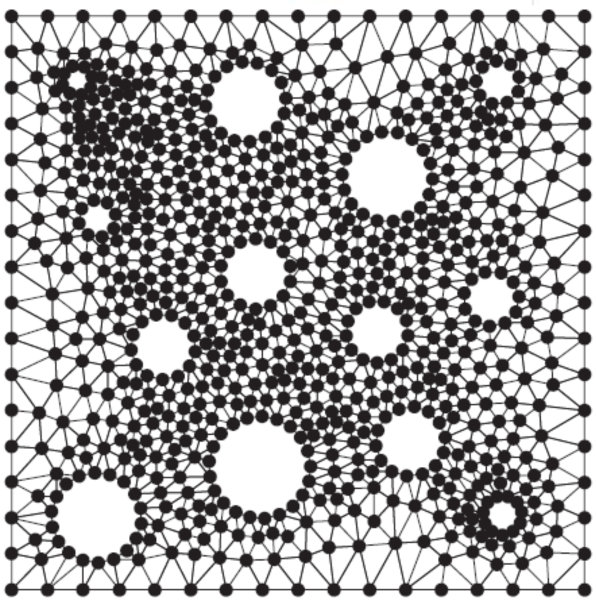
$$+ \bar{\mathbf{F}}^T \nabla_X(\delta \mathbf{u}) : \bar{\mathbf{C}}^e : \bar{\mathbf{F}}^T \nabla_X(\Delta \mathbf{u}) \Big] d\Omega$$

Problème tangent

Méthode de réduction de modèles de type POD

Exemple d'application : hyperelasticité non-linéaire en grandes transformations

Conditions aux limites : $\mathbf{u}=[\mathbf{F}-\mathbf{1}]\mathbf{X}$



Réponse d'un VER sous sollicitations complexes

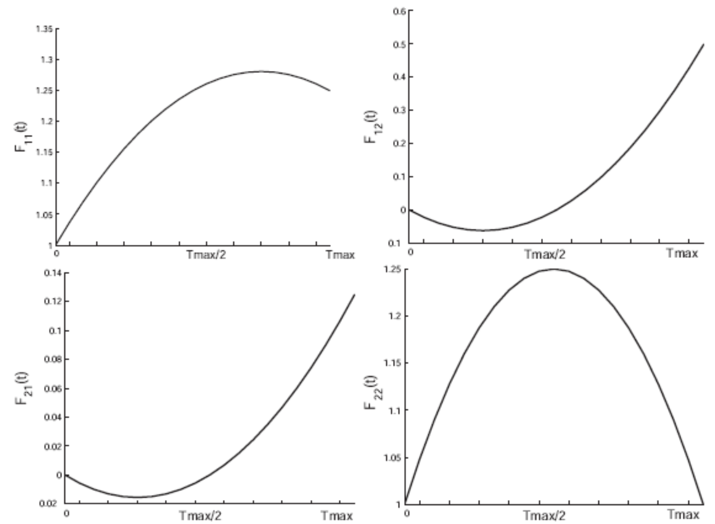
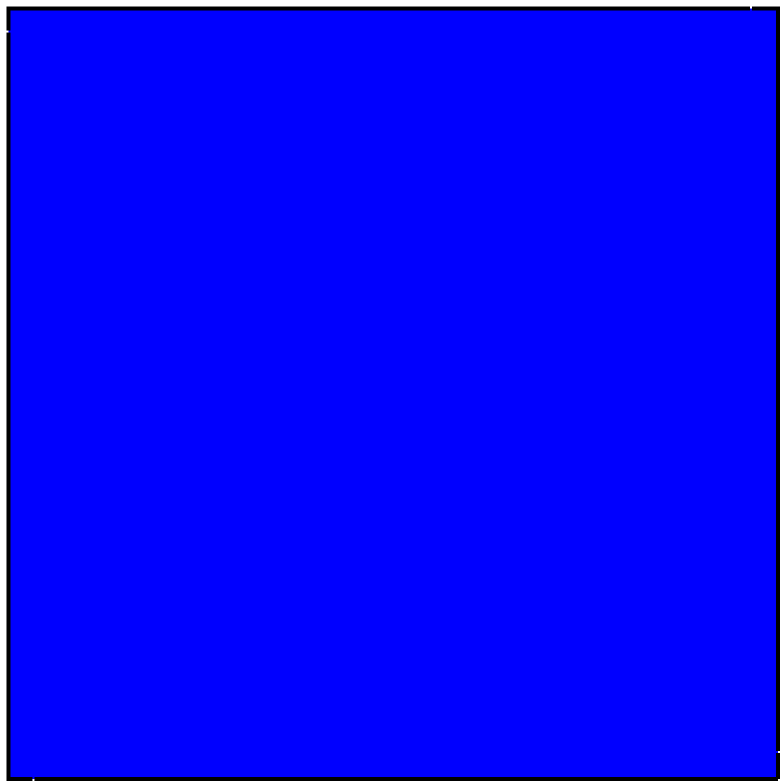


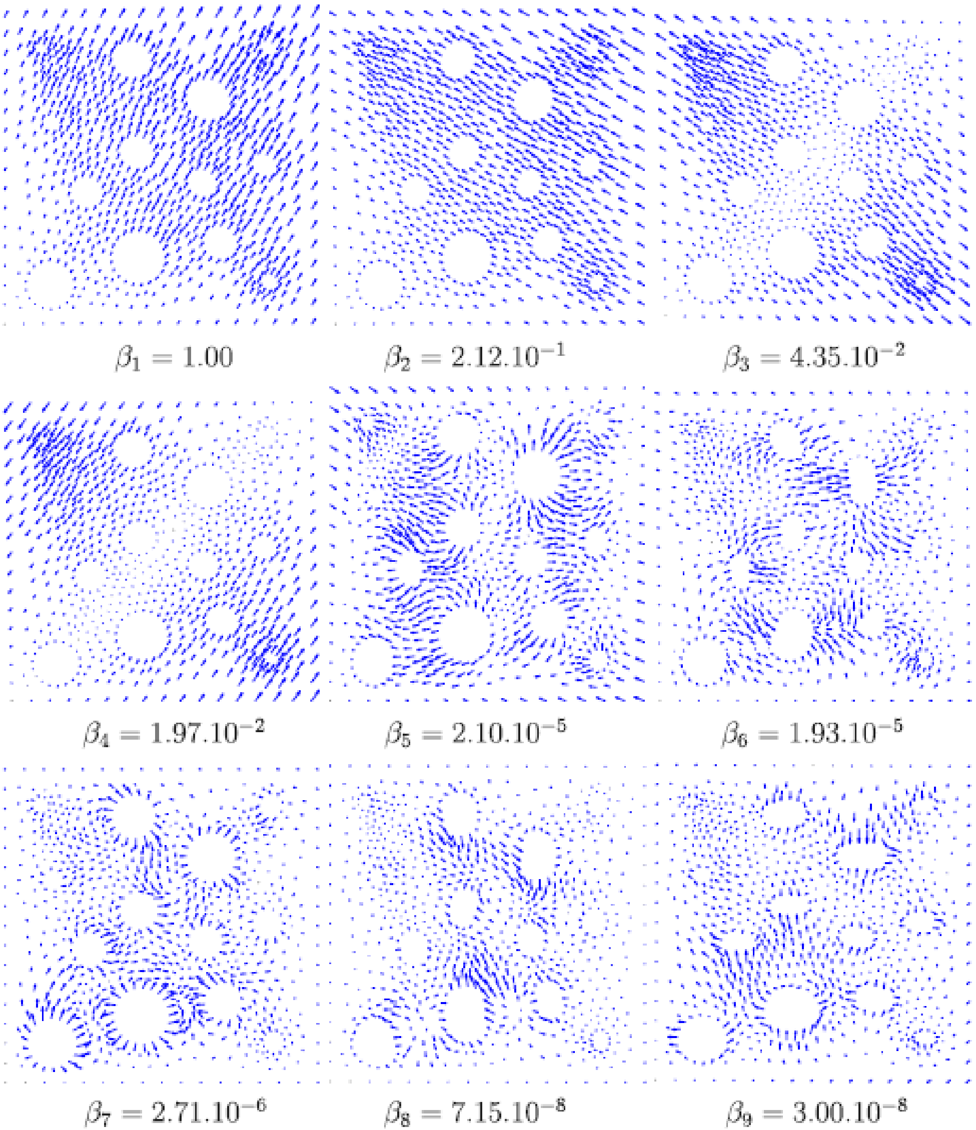
Fig. 5. Values of F_{11}, F_{12}, F_{21} and F_{22} along the simulation.

Évolution des composantes de **F**

[Yvonnet & He 2007]



Méthode de réduction de modèles de

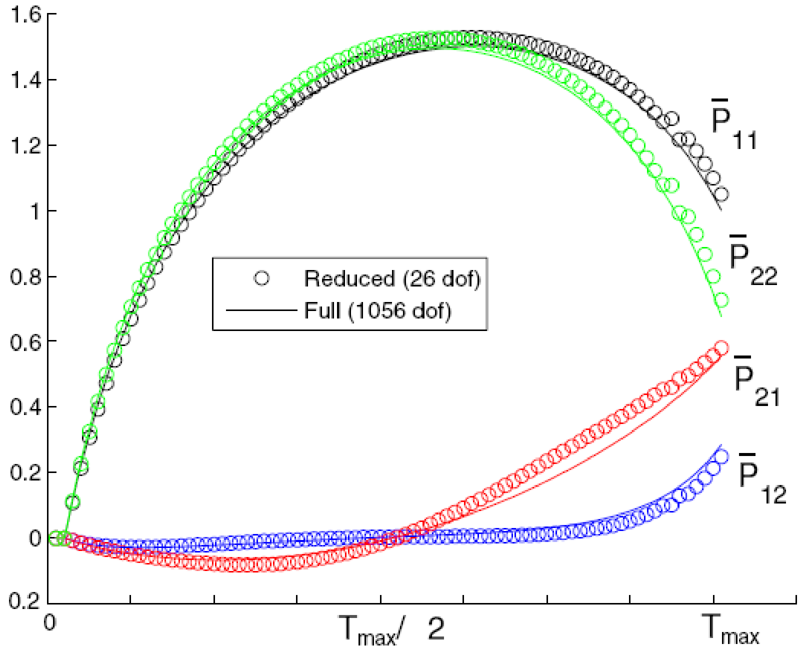


9 premiers modes

26 modes retenus pour 1056 ddl

[Yvonnet & He, Computational Physics (2007)]

Méthode de réduction de modèles de type POD



Comparaison entre solution complète (1056 ddl) et solution par modèle réduit (26 ddl)

[Yvonnet & He, Computational Physics (2007)]

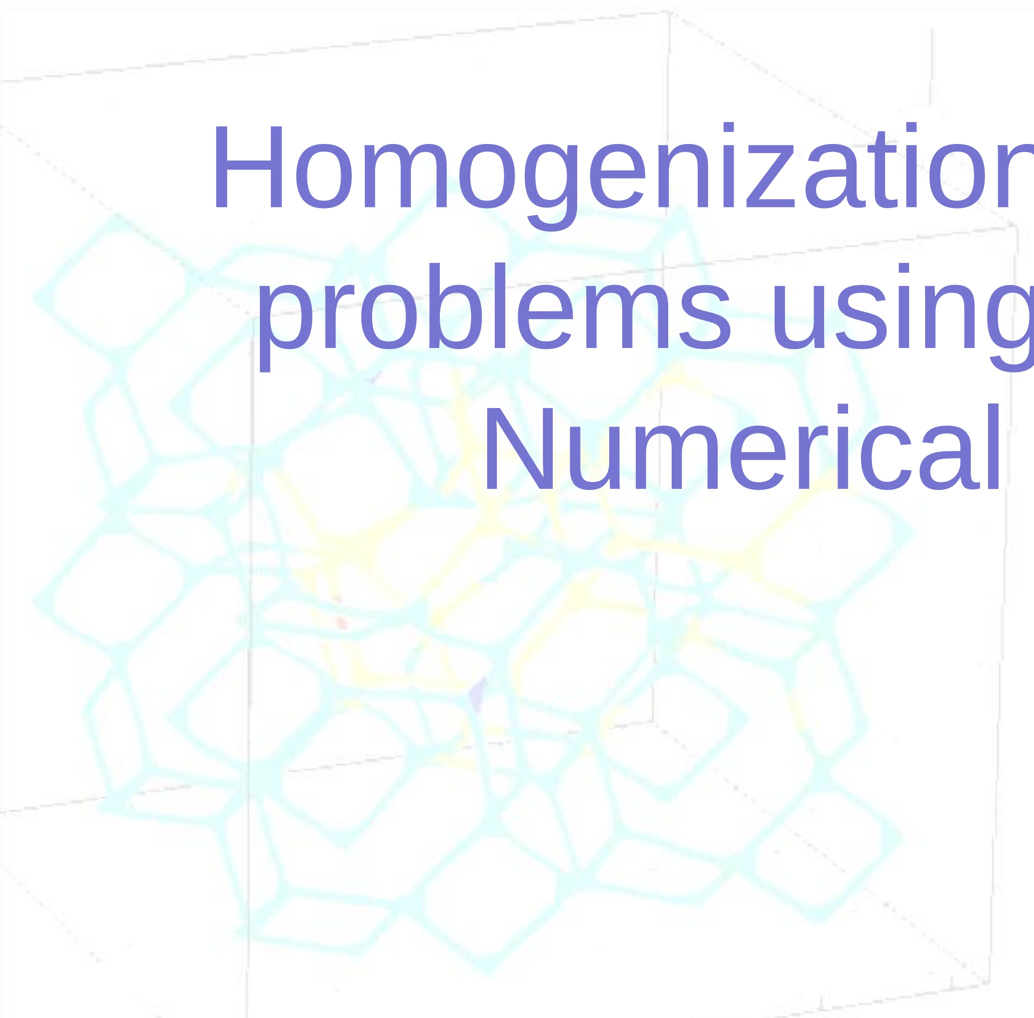
Méthode de réduction de modèle

- Un problème complet doit être résolu pour détecter les corrélations entre modes et construire la base réduite
- Applications: problèmes à plusieurs paramètres
- Optimisation
- Méthodes multi-échelles

Conclusion

Pourquoi des méthodes alternatives à la méthode de Newton-Raphson ?

- Problèmes avec points limites → MAN
- Problèmes de grandes taille → Réduction de modèle



Homogenization of nonlinear problems using Asymptotic Numerical Method

A non linear homogenization method with posible instabilities: the multiscale MAN

The problem is formulated at two scales: both microscopic and macroscopic variables are expanded in finite series (ANM)

- Nonlinear problems (micro and macro) result in sequences of linear problems
- Superposition principle is applied on each linear problem;
- A localization tensor can be constructed for each step of the asymptotic expansion

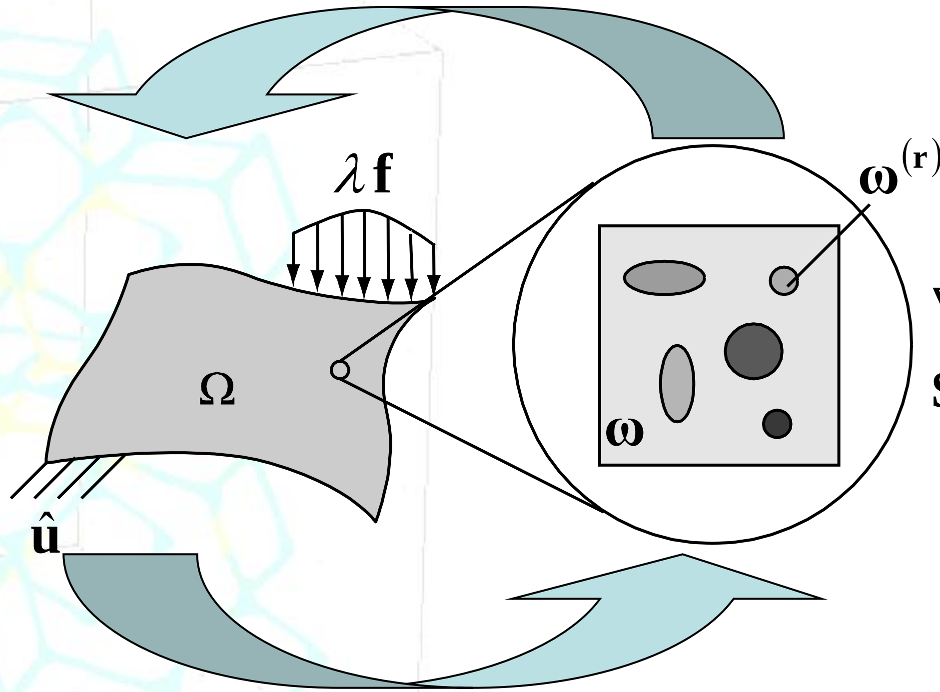
Multiscale elasticity problem with geometrical/material nonlinearities

Multiscale FEM [Renard et al. 1987, Feyel 1998, Smit 1998, Geers and Kouznetsova], Asymptotic Homogenization [Hou and Wu 1997, Terada et al. 2000]

$$\overline{\mathbf{P}} = \frac{1}{|\boldsymbol{\omega}|}\int_{\boldsymbol{\omega}} \mathbf{P}\, d\boldsymbol{\omega}$$

$$\nabla.\overline{\mathbf{P}} = 0 \qquad dans \ \ \Omega$$

$$\begin{cases} \overline{\mathbf{P}}.\overline{\mathbf{N}} = \lambda \mathbf{f} \ \ sur\, \partial\Omega_t \\ \overline{\mathbf{u}} = \hat{\mathbf{u}} \qquad sur \ \partial\Omega_u \end{cases}$$



$$\nabla.\mathbf{P} = 0 \ \ dans \ \ \boldsymbol{\omega}$$

$$\mathbf{S} = \mathbf{C}^{(r)} : \boldsymbol{\gamma}$$

$$\overline{\mathbf{F}} = \frac{1}{|\boldsymbol{\omega}|}\int_{\boldsymbol{\omega}} \mathbf{F}\, d\boldsymbol{\omega}$$

Boundary conditions applied on the RVE:

$$\boxed{\mathbf{u}^{+} - \mathbf{u}^{-} = (\overline{\mathbf{F}} - \mathbf{I}).\left(\mathbf{X}^{+} - \mathbf{X}^{-}\right)}$$

Multiscale ANM

$$\left\{ \begin{array}{c} \boldsymbol{\Lambda}(a) \\ \lambda(a) \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{\Lambda}_0 \\ \lambda_0 \end{array} \right\} + \sum_{p=1}^{N} a^p \left\{ \begin{array}{c} \boldsymbol{\Lambda}_p \\ \lambda_p \end{array} \right\}$$

Series expansions of both microscopic and macroscopic variables

$$\boldsymbol{\Lambda} = (\bar{\mathbf{u}}, \overline{\mathbf{P}}, \mathbf{u}, \mathbf{P}, \ldots)$$

Macroscopic problem

Weak form related to the macroscopic problem

$$\int_{\Omega} {}^{t}\overline{\mathbf{P}} : \delta \overline{\mathbf{F}}\, d\Omega = \lambda \int_{\partial\Omega_t} \mathbf{f} . \delta \overline{\mathbf{u}}\, d\Gamma$$

MAN expansion: problem at order p

$$\int_{\Omega} {}^{t}\overline{\mathbf{P}}_p : \delta \overline{\mathbf{F}}\, d\Omega = \lambda_p \int_{\partial\Omega_t} \mathbf{f} . \delta \overline{\mathbf{u}}\, d\Gamma \qquad \forall p = 1 \ldots N$$

The constitutive law is not known explicitely at this scale, but known implicitely via the microscopic scale

Microscopic problem

Equations related to the microscopic problem

$$
\begin{cases}
\int_{\boldsymbol{\omega}} {}^{t}\mathbf{P} : \delta \mathbf{F} \, d\boldsymbol{\omega} = 0 \\
\mathbf{S} = \mathbf{C}^{(r)} : \boldsymbol{\gamma} \\
\mathbf{P} = \mathbf{F} . \mathbf{S} \\
\boldsymbol{\gamma} = \frac{1}{2}\left({}^{\mathbf{t}}\mathbf{F} . \mathbf{F} - \mathbf{I}\right) \\
\mathbf{F} = \nabla \mathbf{u} + \mathbf{I}
\end{cases}
\qquad \textit{in } \boldsymbol{\omega}
$$

$$
+ C.L. : \mathbf{u}^{+} - \mathbf{u}^{-} = (\overline{\mathbf{F}} - \mathbf{I}) . \left(\mathbf{X}^{+} - \mathbf{X}^{-}\right) \qquad \textit{on } \partial\boldsymbol{\omega}
$$

Microscopic problem

Introducing asymptotic expansion

Order *1*:

$$\begin{cases} \int_{\boldsymbol{\omega}} {}^{t}\mathbf{P}_1 : \delta\mathbf{F}\, d\boldsymbol{\omega} = 0 \\ \mathbf{S}_1 = \mathbf{C}^{(r)} : \boldsymbol{\gamma}_1 \\ \mathbf{P}_1 = \mathbf{F}_0.\mathbf{S}_1 + \mathbf{F}_1.\mathbf{S}_0 & \text{in } \boldsymbol{\omega} \\ \boldsymbol{\gamma}_1 = \frac{1}{2}\left({}^{\mathbf{t}}\mathbf{F}_0.\mathbf{F}_1 + {}^{\mathbf{t}}\mathbf{F}_1.\mathbf{F}_0\right) \\ \mathbf{F}_1 = \nabla\mathbf{u}_1 \end{cases}$$

$$+ C.L.: \mathbf{u}_1^+ - \mathbf{u}_1^- = \overline{\mathbf{F}}_1.\left(\mathbf{X}^+ - \mathbf{X}^-\right) \quad on\ \partial\boldsymbol{\omega}$$

Order *p*:

$$\begin{cases} \int_{\boldsymbol{\omega}} {}^{t}\mathbf{P}_p : \delta\mathbf{F}\, d\boldsymbol{\omega} = 0 \\ \mathbf{S}_p = \mathbf{C}^{(r)} : \boldsymbol{\gamma}_p \\ \mathbf{P}_p = \mathbf{F}_0.\mathbf{S}_p + \mathbf{F}_p.\mathbf{S}_0 + \sum_{r=1}^{p} \mathbf{F}_r.\mathbf{S}_{p-r} & \text{in } \boldsymbol{\omega} \\ \boldsymbol{\gamma}_p = \frac{1}{2}\left({}^{\mathbf{t}}\mathbf{F}_0.\mathbf{F}_p + {}^{\mathbf{t}}\mathbf{F}_p.\mathbf{F}_0 + \sum_{r=1}^{p} {}^{\mathbf{t}}\mathbf{F}_r.\mathbf{F}_{p-r}\right) \\ \mathbf{F}_p = \nabla\mathbf{u}_\mathbf{p} \end{cases}$$

$$+ C.L.: \mathbf{u}^+{}_\mathbf{p} - \mathbf{u}^-{}_\mathbf{p} = \overline{\mathbf{F}}_\mathbf{p}.\left(\mathbf{X}^+ - \mathbf{X}^-\right) \qquad on\ \partial\boldsymbol{\omega}$$

Constructing the macroscopic constitutive law at order p

Order *1*: $$\mathbf{u}_1 = \mathbf{A} : \overline{\mathbf{F}}_1 \qquad A_{ijk} = \tilde{u}_i^{jk}$$

A is a third order tensor. Asymptotic development of gradient tensor F yields: :

Localization tensor

$$\nabla \mathbf{u}_1 = \mathbf{u}_{1,X} = \mathbf{F}_1 = \boxed{\mathbf{A}_{,X}} : \overline{\mathbf{F}}_1$$

$A_{,X}$ is a fourth-order tensor.

At order p we obtain

$$\mathbf{u}_p = \mathbf{A} : \overline{\mathbf{F}}_p + \mathbf{u}_p^{nl}$$

The vector u_p^{nl} is obtained by

$$L_t(\mathbf{u}_\mathbf{p}^{\mathbf{nl}}, \delta \mathbf{u}) = F_\mathbf{p}^{\mathbf{nl}}(\delta \mathbf{u}) \qquad in \;\; \boldsymbol{\omega}$$

Then

$$\mathbf{F}_p = \mathbf{A}_{,X} : \overline{\mathbf{F}}_p + \mathbf{u}_{p,X}^{nl}$$

[Nezamabadi et al. 2009, Comput. Meth. Appl. Mech. Engng.]

Constructing the macroscopic constitutive law at order p

At order p we have:

$$\mathbf{P}_p = \mathbf{L} : \overline{\mathbf{F}}_p + \mathbf{P}_p^{nl}$$

The stress is obtained through:

$$\overline{\mathbf{P}}_p = \langle \mathbf{L} \rangle : \overline{\mathbf{F}}_p + \langle \mathbf{P}_p^{nl} \rangle$$

Introduction into the p-th order macroscopic problem

$$\int_\Omega {}^t\overline{\mathbf{P}}_p : \delta \overline{\mathbf{F}} \, d\Omega = \lambda_p \int_{\partial\Omega_t} \mathbf{f} . \delta \overline{\mathbf{u}} \, d\Gamma$$

$$\int_\Omega {}^t\overline{\mathbf{F}}_p : \langle \mathbf{L} \rangle : \delta \overline{\mathbf{F}} \, d\Omega = \lambda_p \int_{\partial\Omega_t} \mathbf{f} . \delta \overline{\mathbf{u}} \, d\Gamma - \int_\Omega {}^t\langle \mathbf{P}_p^{nl} \rangle : \delta \overline{\mathbf{F}} \, d\Omega$$

Macroscopic problem at order p

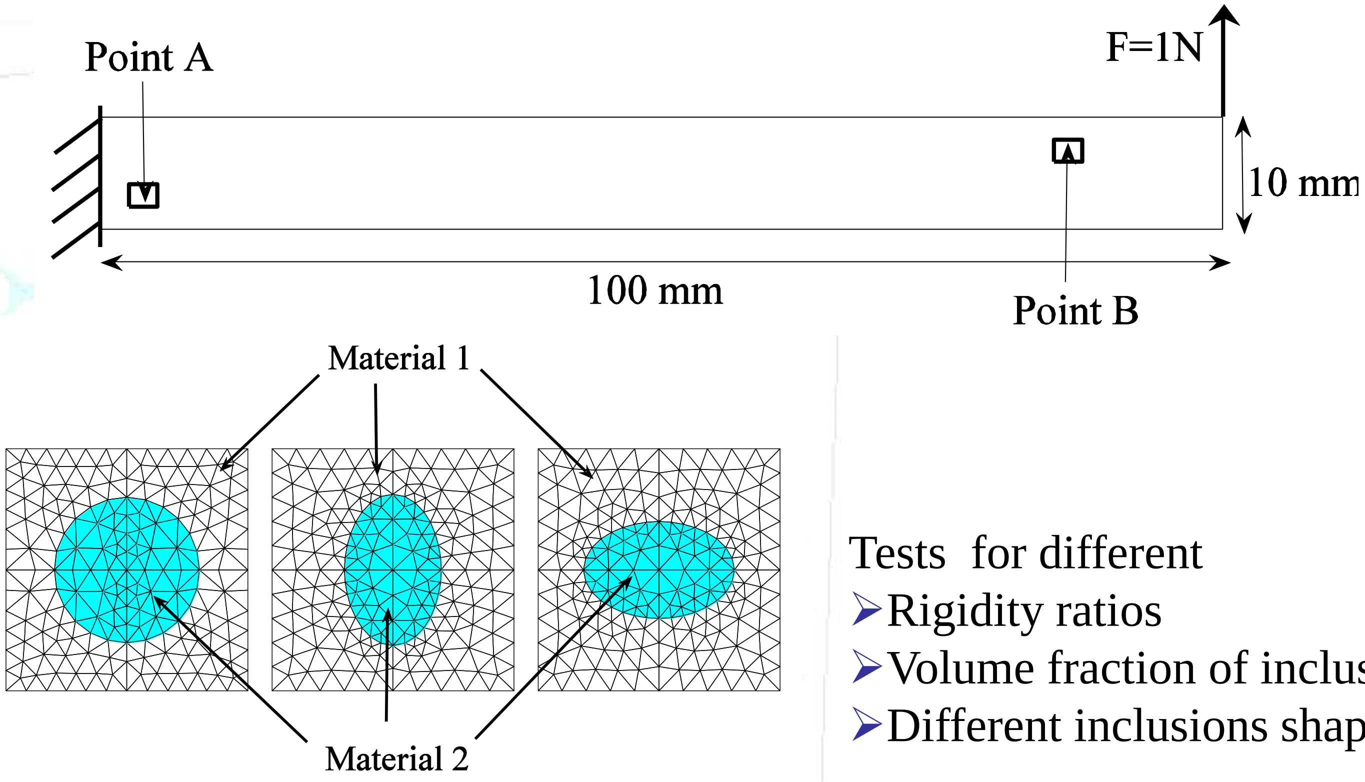
Numerical examples

In all examples:

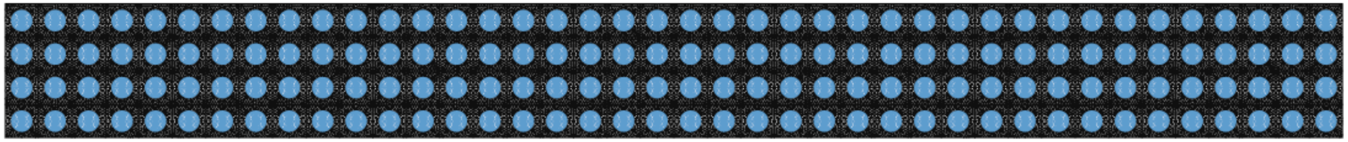
ANM parameters: N = 15 $\delta = 10^{-6}$

Poisson's coefficient $\nu = 0{,}3$

validation of the Multiscale ANM procedure: heterogeneous nonlinear elastic beam

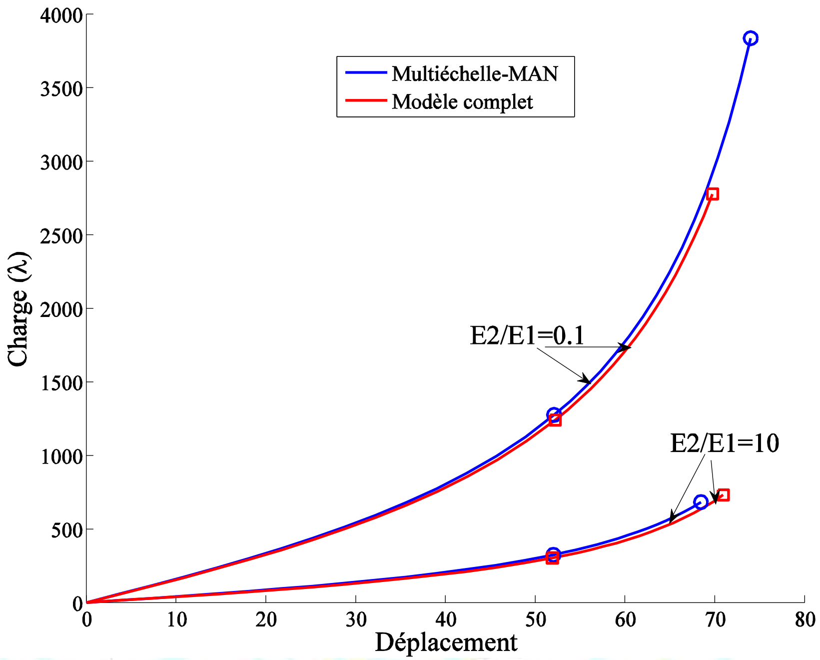


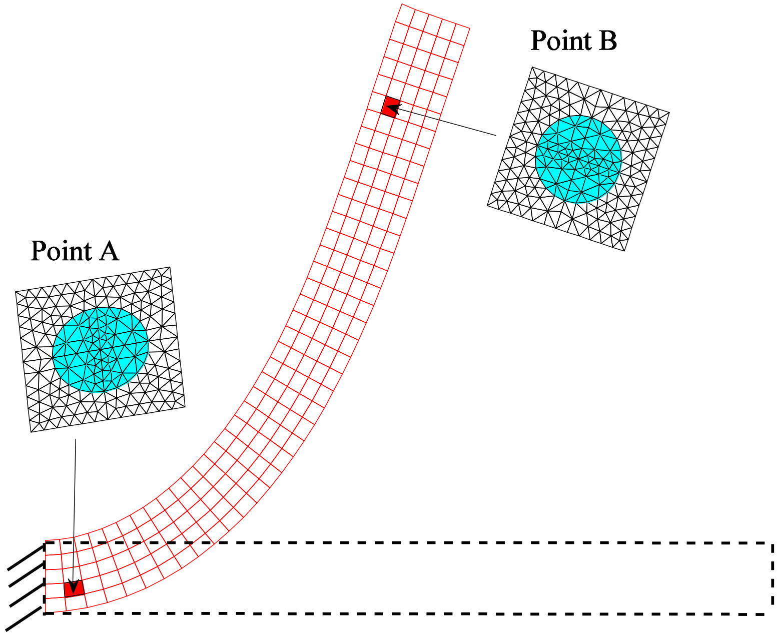
Tests for different
- Rigidity ratios
- Volume fraction of inclusions;
- Different inclusions shapes



Complete mesh for comparison with the homogenized model

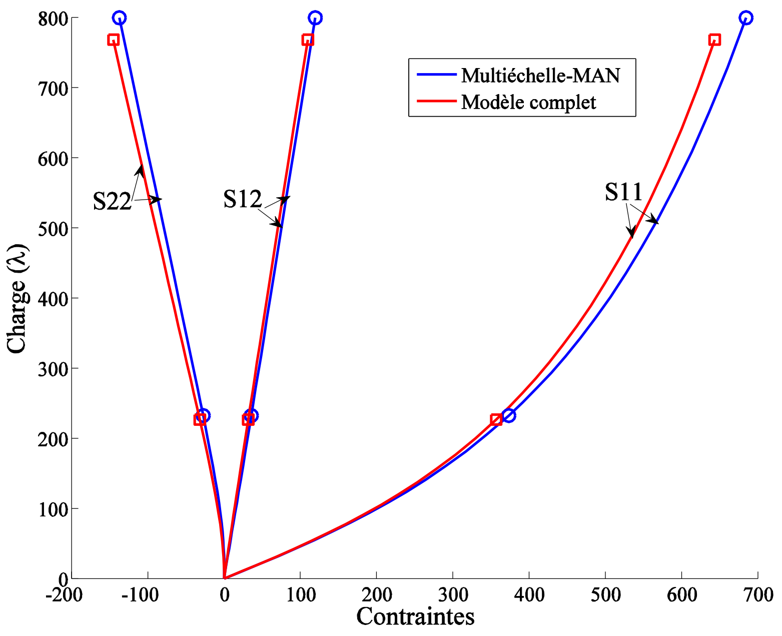
heterogeneous nonlinear elastic beam



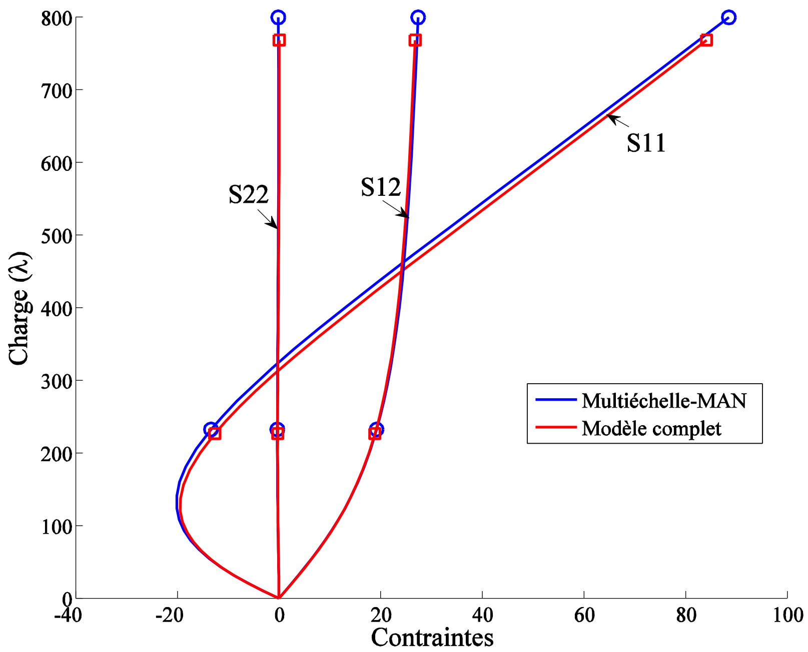


Circular inclusion

heterogeneous nonlinear elastic beam



Point A



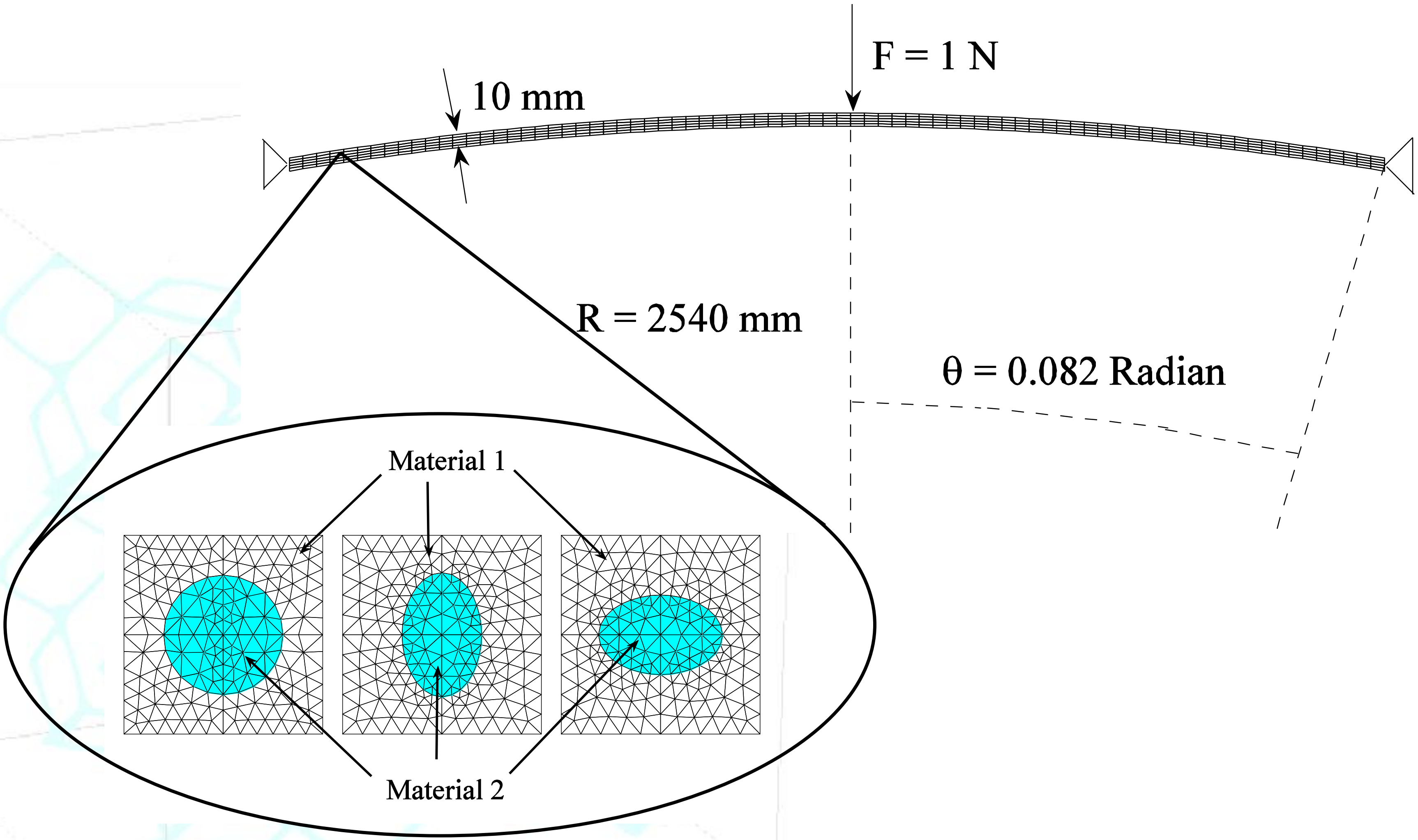
Point B

Circular inclusion, rigidity ratio $E_2/E_1 = 10$

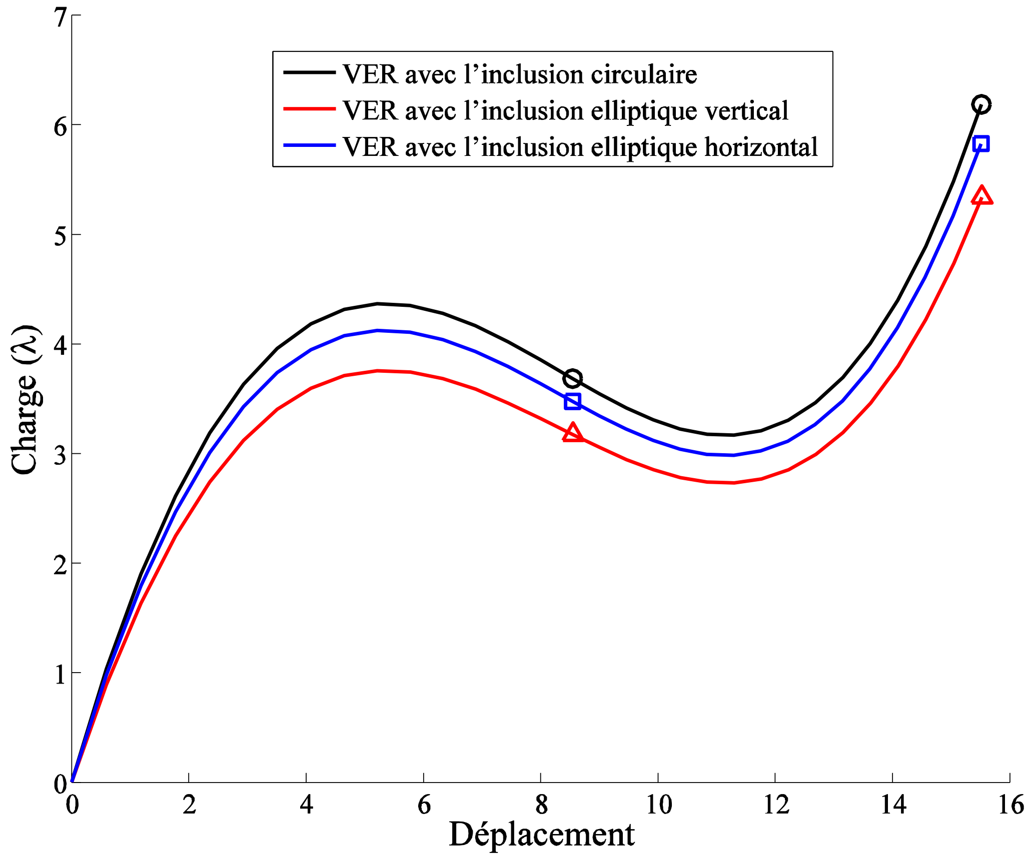
Heterogeneous nonlinear problems with buckling

- **Macroscopic buckling: curved roof**
- **Microscopic buckling: cellular microstructure**

A curved roof with heterogeneous microstructure

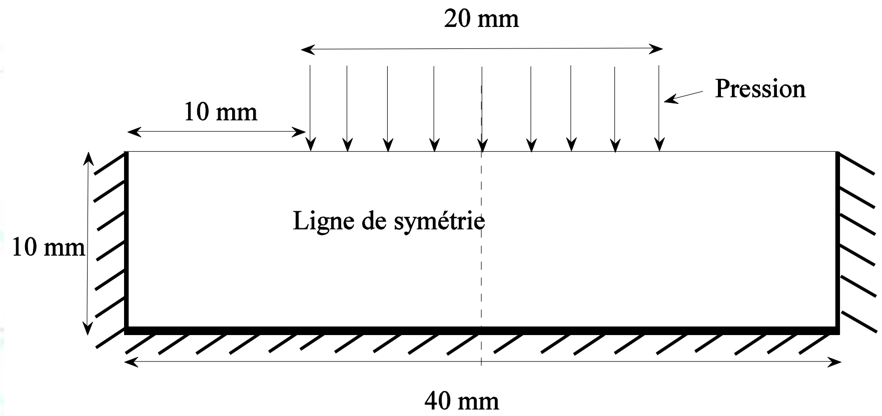


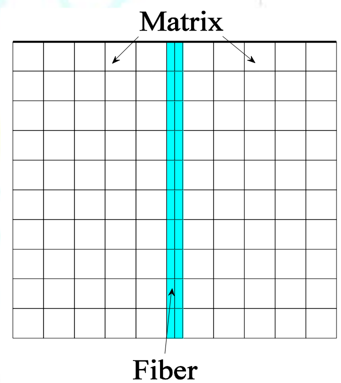
A curved roof with heterogeneous microstructure



$E_2/E_1=10$

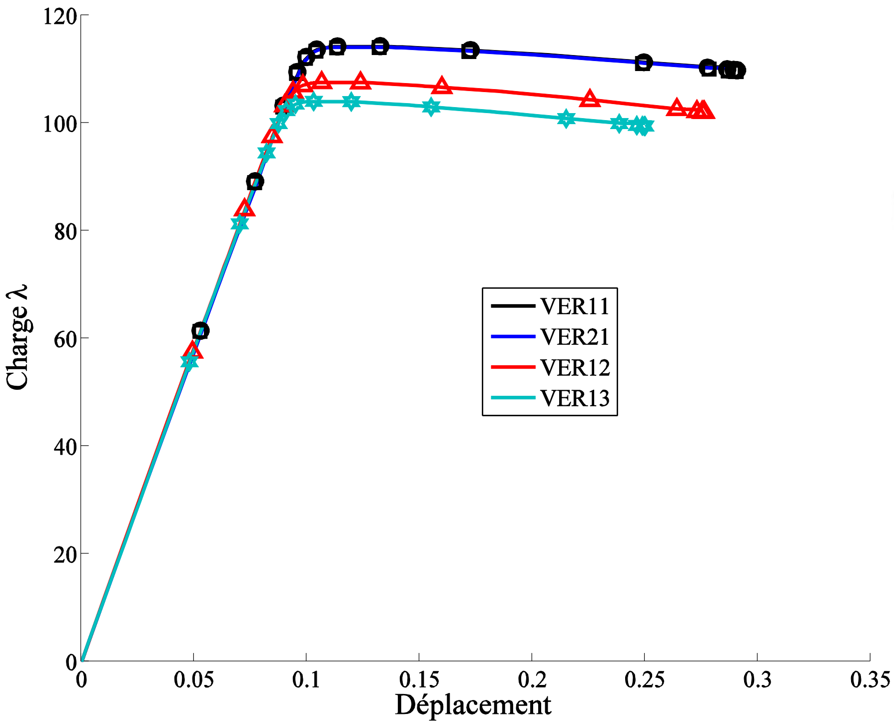
Composite structure with long fibers in compression

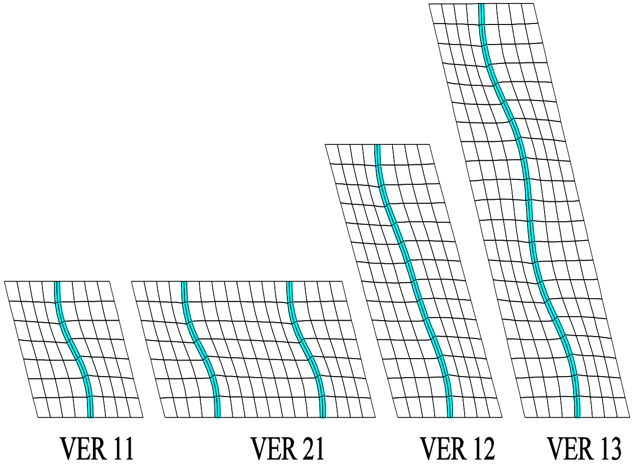


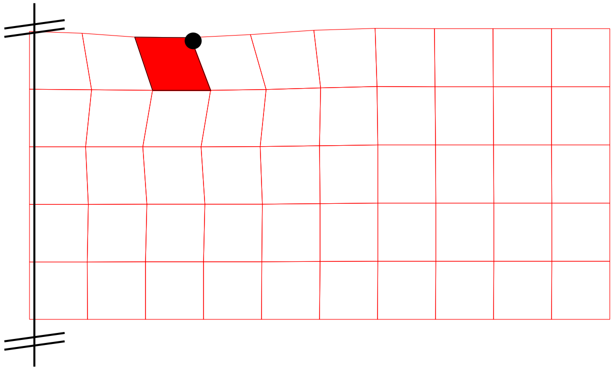


Rigidity ratio E_{Fiber} */* E_{Matrix} *=1000)*

Composite structure with long fibers in compression

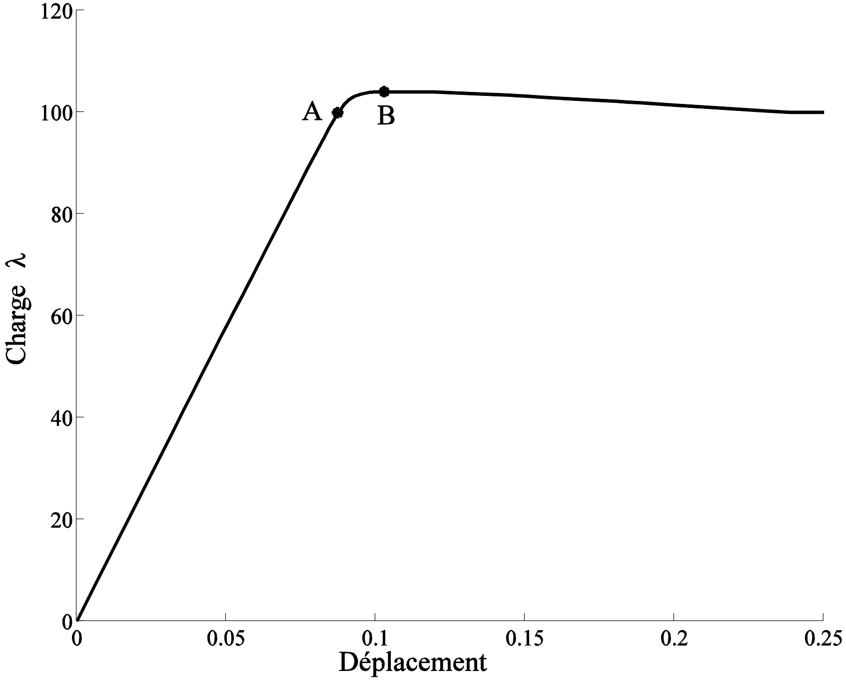




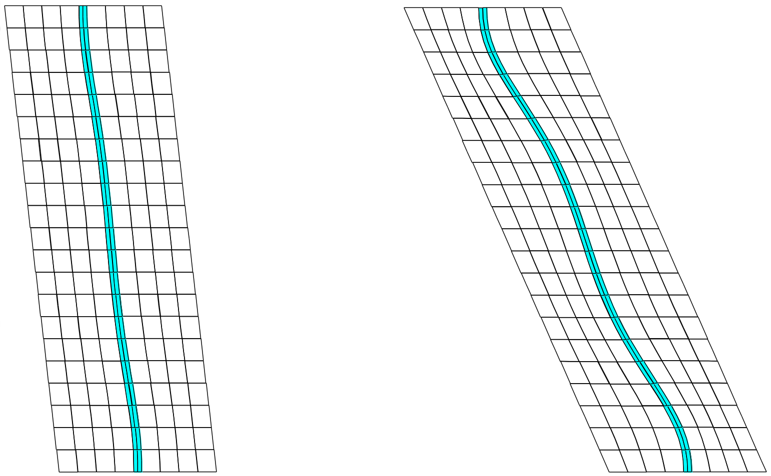


Choice of the RVE: numerical convergence tests

Composite structure with long fibers in compression



Macroscopic response



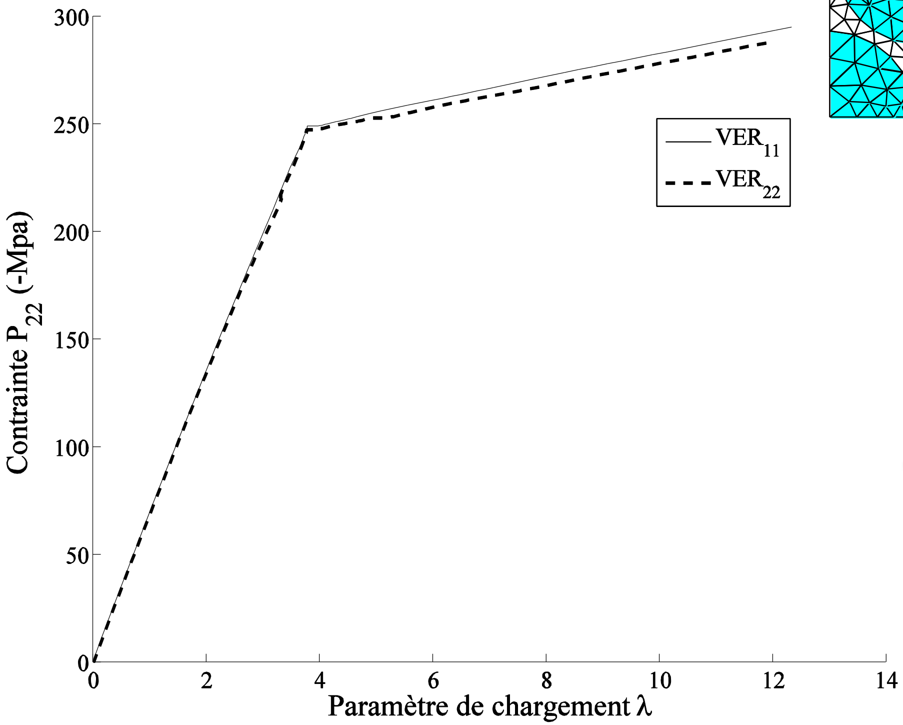
Point A

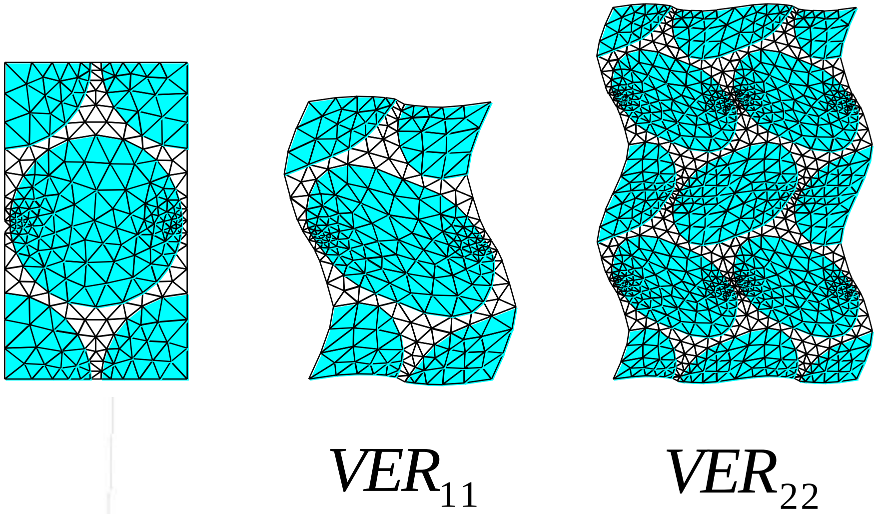
Point B

RVE deformation for macroscopic loads A and B

Composite structure with cellular microstructure in compression

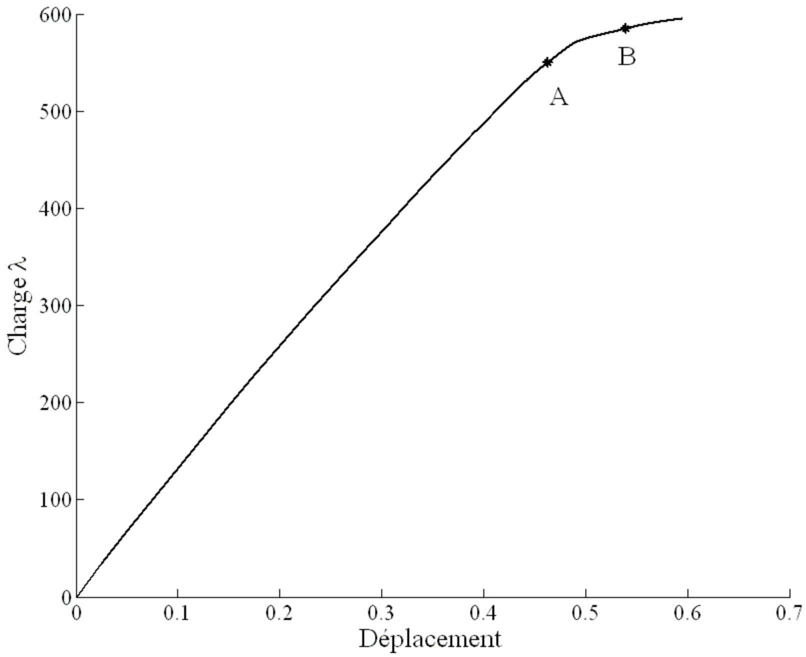
Cellular microstructure



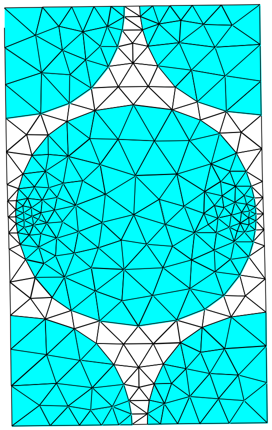


Microscopic response
(convergende tests)

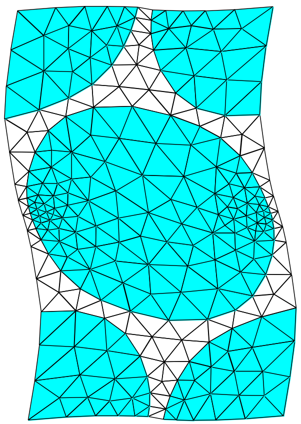
Composite structure with cellular microstructure in compression



Macroscopic response (structure)



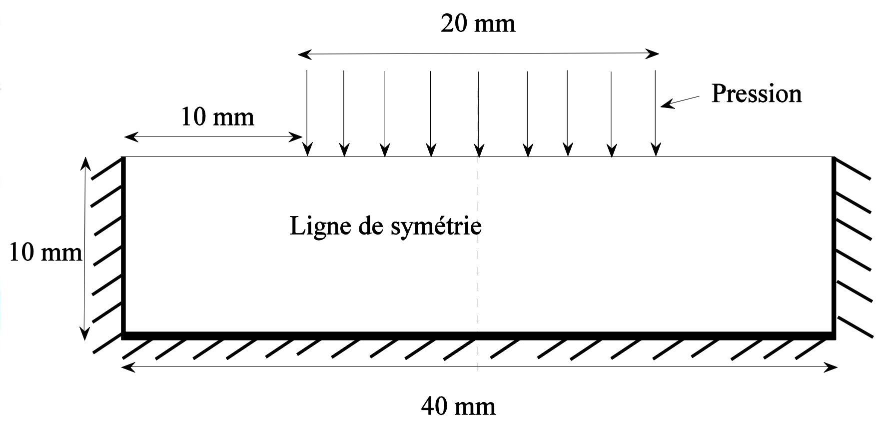
Point A

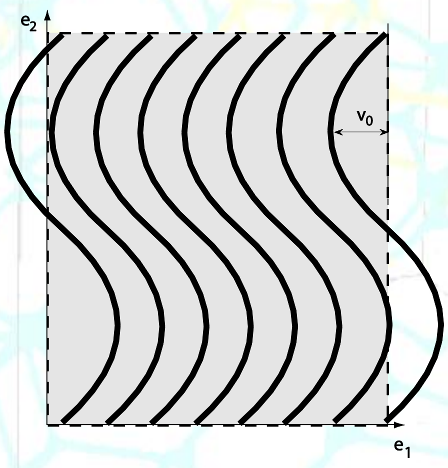


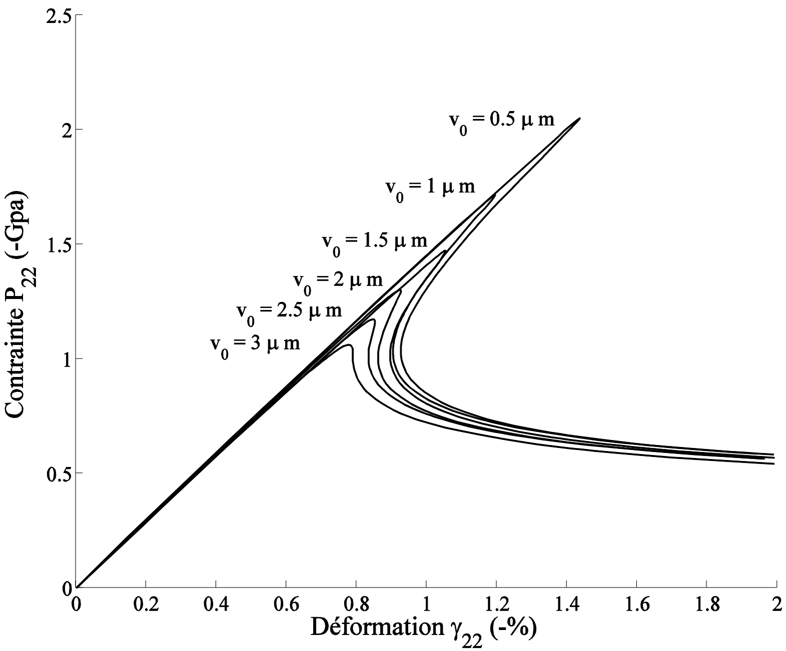
Point B

RVE deformation for macroscopic loads A and B

Local buckling of composite with elastoplastic long fibers

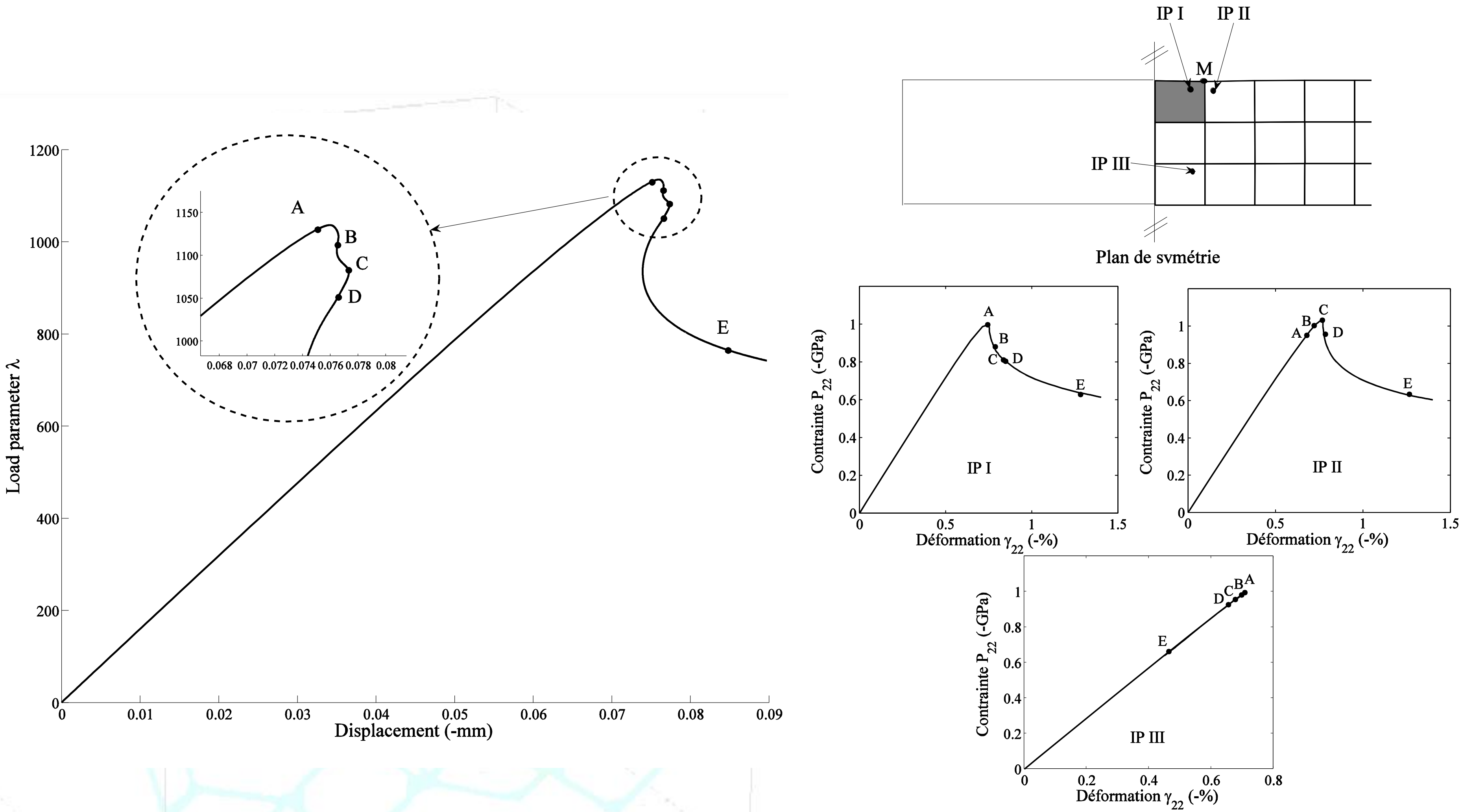






[Nezamabadi et al. In press, International Journal for Multiscale Computational Engineering 2010**]**

Local buckling of composite with elastoplastic long fibers



Conclusion

The ANM is an alternative to classical FEM/Newton-Raphon algorithms for solving nonlinear problems

Removes issues of NR when bifurcation and limit points occur

In some cases siginficantly **reduces computational costs**

(other advantages – not presented here – construction of bifurcation indicators)

Nonlinear problems are reformulated into a **sequence of linear problems**: advantages for homogenization problem: the superposition principle can be applied – well-known procedures for **linear homogenization can be applied at each order**

Instabilities at several scales can be handled

Thank you for your attention !

References

Nezamabadi S., Zahrouni H., Yvonnet J., Potier-Ferry M., A multiscale finite element approach for buckling analysis of elastoplastic long fibre composites, accepted in International Journal for Multiscale Computational Engineering, April 2009.

Nezamabadi S., Yvonnet J., Zahrouni H., Potier-Ferry M., A multilevel computational strategy for handling microscopic and macroscopic instabilities, Computer Methods in Applied Mechanics and Engineering, 198:2099-2110 (2009).

Yvonnet J., Zahrouni H., Potier-Ferry M., A model reduction method for the post-buckling analysis of cellular microstructures, Computer Methods in Applied Mechanics and Engineering, 197, 265-280 (2007).

More references about previous papers on ANM can be found in these references