Macro Elements in the Finite Element Analysis of Multi-Conductor Eddy-Current Problems

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***Abstract*—The use of macro elements which replace the multi-conductor winding in the time-stepping finite element analysis of electrical machines is presented in the paper. The effect of eddy-currents in the windings is considered by a traditional finite element formulation combined with the so called elimination of inner nodes to achieve fast solution. The elimination method is incorporated into time stepping analysis in such a way that the creation of the numerical macro element is performed only once in a whole time stepping analysis.**

***Index Terms*—Eddy currents, finite element methods, machine windings, macro elements, skin effect.**

I. INTRODUCTION

COMPUTATION of the magnetic field in electromagnetic devices—especially rotating electrical machines—usually excludes the effect of eddy currents in the multi-conductor windings. Eddy-current losses are frequently calculated from a magnetic field solution in which the effect of eddy currents is not considered. There are methods which take the effect of eddy currents on the magnetic field into account in multi-conductor windings but all those approaches have serious drawbacks. Analytical formulations lead to very complicated models while numerical analysis leads to enormously large problem size for many conductors or in small penetration depth cases [1]. The goal of this paper is to present an efficient method to consider the effect of eddy currents in multi-conductor windings accurately within electromagnetic field computation.

The paper suggests a novel combination of a method called: "the elimination of inner nodes" with the conventional finite element technique. Based on the magnetic linearity of the winding regions the creation of a numerical macro element is suggested. The Gauss elimination of the nodes inside the finite element model of the multi conductor winding resulted in significantly decreased problem size and in accelerated solution of the system of equations [2]. The linear variables eliminated from the winding region were excluded from the nonlinear iteration process [3]. In these previous works the elimination had to be performed in all time steps. The novelty in this paper is that it is enough to perform the elimination only once in a whole time stepping analysis and the multi-conductor winding region can be conveniently modeled by a macro element.

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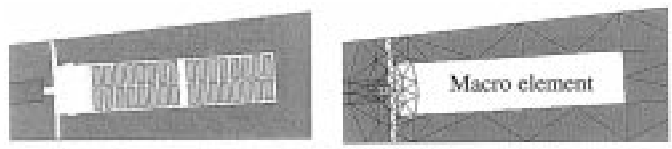

Fig. 1. A slot of an electrical machine and its macro element representation.

II. THE EDDY-CURRENT PROBLEM FORMULATION

The 2D finite element formulation to model eddy currents in this work is similar to [4]. The eddy-current effect in the windings is considered by adding an extra equation for each conductor. A single valued magnetization curve was used to model the nonlinearity of the iron. Finite element equations are coupled and solved together with the circuit equations.

$$\nabla \times (v \nabla \times \boldsymbol{A}) + \sigma \frac{\partial \boldsymbol{A}}{\partial t} - \left(\frac{1}{l_e} \sum_{j=1}^{Q_c} \sigma \eta_j^c u_j^c \right) \boldsymbol{e}_z = 0 \qquad (1)$$

$$u_m^s = Q_s \sum_{j=1}^{Q_c} \eta_{jm}^s u_j^c + L_b \frac{di_m^s}{dt} + R_b i_m^s; \qquad m = 1, \cdots, Q_m \qquad (2)$$

$$u_n^c = R_c \sum_{j=1}^{Q_c} \eta_{nj}^s i_j^s + R_c \int_{S_n} \sigma \frac{\partial \boldsymbol{A}}{\partial t} \cdot \boldsymbol{dS}_n; \qquad n = 1, \cdots, Q_c \qquad (3)$$

The three basic equations are the finite element equation (1) governing the electromagnetic field, the circuit equations (2) and extra equations (3) to account for eddy-currents in the stator conductors. The variables $\boldsymbol{A}$, u_m^s, i_m^s, u_n^c stand for the magnetic vector potential, the voltage and current of phase m, and the voltage on the conductor number n respectively. L_b, R_b stand for the inductance and DC resistance of one phase of the end-winding. Rc is the DC resistance of one conductor in the slot. Q_m, Q_c and Q_s are the number of phases, conductors and symmetry sectors respectively. Reluctivity is ν, and σ is conductivity. The η is a logical operator with the values: 0 and 1. The effective air gap length of the machine is l_e. S_n is the cross sectional area of the nth conductor. A generalized first order finite difference procedure is used for the time stepping method. Newton–Raphson method was used to handle the nonlinearity.

III. ELIMINATION OF INNER MODES

Due to the linearity of materials in the slot of an electrical machine, the variables associated with the nodes inside the slot can be eliminated from the system of equations for the duration of the nonlinear iteration process Fig. 1.

Variables describing the voltages of the conductors can similarly be eliminated. In the given example a system of equations (4) is formulated for Newton–Raphson method (5). The system matrix can be divided into a nonlinear sub matrix $\boldsymbol{A}_{11}$ depending on vector $\boldsymbol{u}$ representing nodes belonging to nonlinear elements and into linear sub matrices $\boldsymbol{A}_{12}$, $\boldsymbol{A}_{21}$, A_{22} independent from any of the variables. Performing Gauss elimination the number of variables and the size of the system matrix decreases (5), (6). It is enough to iterate with (6) and after reaching convergence linear variables in vector $\Delta\nu$ can be calculated in one step.

$$\begin{bmatrix} \boldsymbol{A}_{11}(\boldsymbol{u}) & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} \boldsymbol{S}_u \\ \boldsymbol{S}_v \end{bmatrix} \tag{4}$$

$$\begin{bmatrix} \boldsymbol{A}_{11} + \dfrac{\partial \boldsymbol{A}_{11}(\boldsymbol{u})}{\partial \boldsymbol{u}} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{u} \\ \Delta\boldsymbol{v} \end{bmatrix} = \begin{bmatrix} \boldsymbol{S}_u - \boldsymbol{A}_{11}\boldsymbol{u} - \boldsymbol{A}_{12}\boldsymbol{v} \\ \boldsymbol{S}_v - \boldsymbol{A}_{21}\boldsymbol{u} - \boldsymbol{A}_{22}\boldsymbol{v} \end{bmatrix} \tag{5}$$

$$\begin{aligned} &\left[\boldsymbol{A}_{11} + \frac{\partial \boldsymbol{A}_{11}(\boldsymbol{u})}{\partial \boldsymbol{u}} - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21}\right][\Delta\boldsymbol{u}] \\ &\quad = \left[\boldsymbol{S}_u - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{S}_v - (\boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21})\boldsymbol{u}\right] \end{aligned} \tag{6}$$

IV. Macro Elements in a Novel Time Stepping Formulation

In this section a time stepping technique is suggested by which it is enough to perform the matrix elimination only once in a whole time stepping analysis. It has been shown in [3] that the creation of the full eliminated system matrix is possible by the use of the eliminated linear part of the system matrix—as shown on the left hand side of (7)—and by adding the nonlinear components in each iteration step. It can be seen from (6) and (7) that inside a nonlinear iteration the residual can similarly be created using the stored eliminated linear part of the system matrix and by adding the nonlinear contributions in each iteration step.

$$\left[0 - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{12}\right][\boldsymbol{u}] = \left[\boldsymbol{S}_u - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21}\boldsymbol{S}_v\right] \tag{7}$$

The sources including the time dependent terms due to the eddy currents have to be updated on the right hand side of (6) in each time step. The equations (6) and (7) show it clearly that the linear part of the eliminated system matrix does not contain time dependent terms and so it is not necessary to perform the elimination of the system matrix in each time step. The real need for the elimination is to create the eliminated residual properly. The residual contains time dependent source terms which are also connected with the conductor nodes inside the eliminated region when modeling eddy currents. These time dependent terms represented by $\boldsymbol{S}_v$ have to be taken into account in order to consider the eddy current effect properly in the windings.

A. The Creation of a Macro Element

The elimination of the linear part of the system of equations in each time step can be avoided if we store the eliminating matrix for the source vector $\boldsymbol{S}_v$. The eliminated source term in the right hand side of (6) represented by: $\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{S}_v$ cannot be created using the stored linear part of the system matrix. If the eliminated $\boldsymbol{S}_v$ source term—representing the eddy current effect—is created by the stored elimination matrix the rest of the residual seen in (6) can easily be constructed, using the stored linear part of the system matrix (7), the nonlinear noneliminated components and the source terms connected with the noneliminated variables $\boldsymbol{S}_u$.

The macro element replacing the FEM model of the multiconductor winding in Fig. 1 can be fully represented by the eliminated linear part of the system matrix together with the eliminating matrix of the source terms.

TABLE I
NUMBER OF OPERATIONS REQUIRED FOR ELIMINATION

Number of nodes	Whole system	Residual	Ratio
873	14 341 730	76 435	187.63
1073	22 453 817	101 834	220.49
1257	38 284 012	143 078	267.57
1353	46 573 378	161 866	287.73
1589	83 125 897	247 528	335.82

B. Advantage of the Suggested Macro Element Representation

The number of operations to eliminate the whole system is significantly higher than the number operations needed for eliminating only the residual. In a practical application without renumbering the mesh for optimal elimination 14 341 730 operations were needed for the elimination of the whole system while only 76 435 operations were required for the elimination of the residual. The number of operations required for different meshes with increasing mesh density in the eliminated slot region are given in Table I and Fig. 2.

V. Optimization of the Mesh for Elimination

The aim of the elimination process is to create a macro element which is a function between the nodes on the boundary of the eliminated region. This function is represented by a full $N \times N$ matrix where N is the number of nodes on the boundary of the eliminated region. During the elimination the number of nonzeros vary in the originally sparse system matrix. Depending on how the elimination proceeds and which nodes are eliminated first we can obtain different amount of nonzero elements in our system matrix during the elimination. It could be low or high depending only on the sequence how the nodes are being eliminated from the macro element region. In an unlucky situation where the nodes closer to the boundary of the macro element are eliminated first we can arrive to a system of "islands." The "islands" are shown in Fig. 3.

All remaining nodes on the boundaries of the islands and on the boundary of the macro element (continuous lines) are connected with each other due to the elimination principle. By each node which has been removed we decrease the size of the system matrix, but the price to be paid for this is that a full matrix $M \times M$ has to be inserted into the system, where M is the number of nodes on the boundary of the eliminated region. In this case M is the number of nodes on the boundaries of the islands plus those on the original boundary of the macro element. Such situation can result in an enormously high number

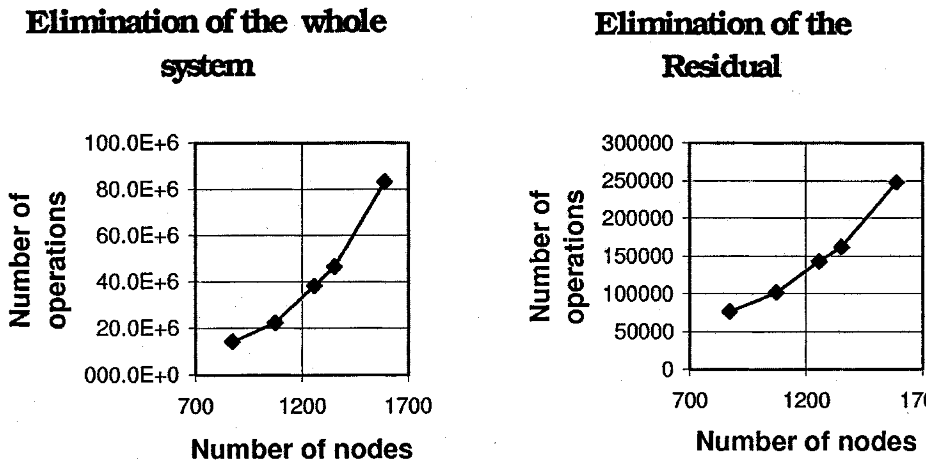

Fig. 2. Number of operations required for the elimination of the whole system and the residual versus the node number.

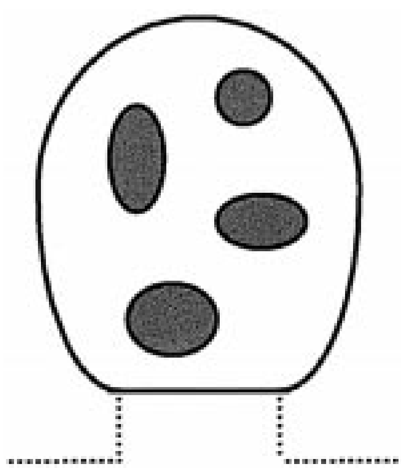

Fig. 3. Islands inside the macro element during elimination. All nodes on the boundaries of the islands and on the boundary of the macro element are connected with each other. This can result in an enormously high number of nonzeros in the system during the elimination.

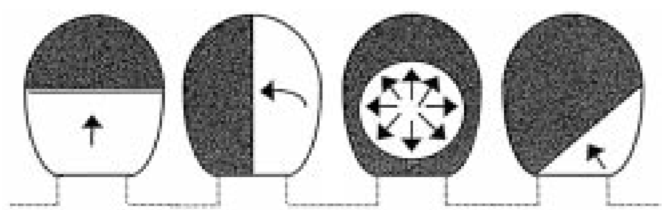

Fig. 4. Four possible selections of the elimination sequence.

TABLE II
NUMBER OF OPERATIONS REQUIRED FOR OPTIMIZED ELIMINATION

Number of nodes	Whole system	Residual	Ratio
873	931 242	19 673	47.33
1073	1 502 508	29 066	51.69
1257	2 302 181	40 277	57.16
1353	2 813 630	46 716	60.23
1589	3 964 263	61 752	64.20

of nonzeros during the elimination. The increase of the number of nonzeros in the system during the elimination can slow down the elimination process significantly. In order to avoid the appearance of the "islands" a controlled elimination process can be employed. By determining the sequence in which the nodes should be eliminated we can avoid the creation of islands. Fig. 4 represents different possible procedures of the elimination for a stator slot.

The four simple selections of elimination sequences are:

- a, Select elimination sequence of nodes according to their distance from the air gap. e.g.: Those closest to the air gap are eliminated first.
- b, Select elimination sequence of nodes according to their angular position. e.g.: Eliminate the nodes with respect to their angular position counter clockwise.
- c, Start the elimination from the inside of the macro element region, by selecting nodes for elimination in an extending "bubble" fashion.
- d, Perform the elimination to any direction which allows a continuous growth of the macro element and avoids the creation of islands. e.g.: If the mesh in the stator slot is numbered such a way that following that order satisfies the above mentioned conditions (continuous growth, no islands), the elimination can be performed in that order of the nodes and no extra efforts to rearrange the nodes is needed.

VI. THE EFFECT OF MESH OPTIMIZATION ON THE ELIMINATION

The a, variant presented in Fig. 4. was applied for the optimized elimination. Nodes were eliminated in a sequence determined by their distance from the air gap. Those closer to the air gap were eliminated first. The same meshes were used as in Table I. but this time these were optimized for elimination in a simple manner. It was accomplished by renumbering the nodes inside the macro element with respect to their distance from the air gap. Nodes with higher distance become in the numbering ahead of the nodes with smaller distance. Than the elimination was performed backward according to node numbering. The effect of the mesh optimization on the elimination is presented by Table II.

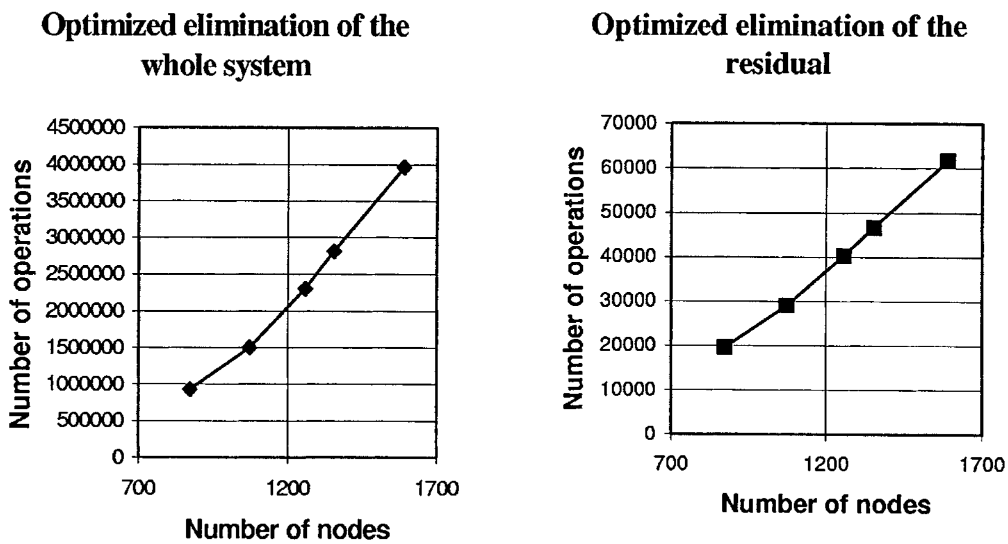

Fig. 5. Number of operations needed for the optimized eliminations versus the number of nodes.

These results are expressed in graphical form in Fig. 5. The optimized and the nonoptimized eliminations are clearly different with respect to the number of required operations. It is clear that the optimized elimination process accelerates the creation of the macro element significantly.

VII. Discussion

The goal of the macro element approach is to accelerate and simplify the solution of the eddy current problem in the multi-conductor windings of electrical machines. It has been shown in previous papers that the eliminated winding region results in a reduced system of equation which requires much smaller computation effort to solve than the original system, if certain conditions are met. These conditions were:

1) High number of eliminated nodes (variables)
2) Low number of nodes on the boundary of the eliminated region (macro element)

The computation time does not consists only of the solution of the system of equations, but also a significant amount of time is used for the construction of them when using finite element analysis. The total time used for the macro element approach can be divided to the following parts:

1) Gauss elimination of the macro element regions, for the creation of the macro elements. It should be performed only once in a time stepping analysis, or more exactly once after the winding region is constructed. The macro element can be saved and loaded unchanged when the same winding mesh is inserted into a different machine model. The elimination can take a long time, but it only has to be performed once, and its significance is decreased by creating the macro element to be time independent.
2) Elimination of the right hand side vector—due to time dependent sources—in each time step once. It takes much less time than the elimination of the whole system, but it is a time factor which does not appear without macro elements.
3) Construction of the system outside the macro element, or in other words the construction of the nonlinear regions. It takes the same amount of time with or without macro elements.
4) Solution of the system of equations in each iteration step. It can take much less time when using macro elements. For a huge multi-conductor winding mesh the difference can be very large. Depending on the nonlinear behavior of the problem this step can become an important time factor. The "more nonlinear" the problem, the more iteration steps are needed.
5) Recovery of the eliminated variables when the iteration converged. This step is also unique for the macro element approach and does not appear in the classical finite element analysis. This step should be performed once in each time step in order to model the time derivative and so the eddy currents properly.

Studying these steps one fact can be seen clearly. In case of high number of time steps in nonlinear problems—like electromagnetic devices with iron cores and with a huge finite element model for the multi-conductor winding in them, to consider eddy currents—the method promises huge improvement over the classical approach. It has been experienced that even with a few steps of nonlinear iteration in each time step the method is superior over the classical approach if the winding mesh is dense enough.

One can conclude, that the macro element method is greatly advantageous if the problem is nonlinear, nonsinusoidal, and complicated from the eddy current point of view (multi conductor). In that case there is little sense to apply boundary element method combined with Fourier analysis. The problem can be conveniently formulated by the finite element method and—as it was discussed before—macro elements should be useful to reduce computational needs.

VIII. Conclusions

The suggested macro element representation allows us to construct the eliminated system of equations very conveniently and fast, and to utilize numerical macro elements in the finite element computation of eddy currents in multi-conductor windings of electrical machines. One needs to perform the elimination of the system matrix only once during a whole time stepping analysis. The macro element is fully represented by the eliminated linear part of the system matrix together with the eliminating matrix of the source terms. The study has shown that the number of operations required for the suggested technique—which is the same as the number of operations required to eliminate the source vector—are much less then for the elimination of the whole system.

It has also been shown that the optimized elimination process allows much more efficient creation of the macro element than the elimination without optimization. The number of operations needed for the elimination in the optimized case is significantly lower, in one case it appeared to be 16 times lower.

It is important to observe that the exponential behavior of the number of operations vs. the node number has changed to linear dependency when the elimination has been optimized. This means that larger problems only require proportionally larger amount of computation effort.

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