PROBLEM SET 8 — SOLUTIONS

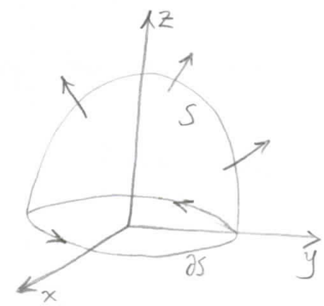
$\boxed{1}$ $\partial S = \{(x,y,0) \in \mathbb{R}^3 : x^2+y^2=1\}$

$$\nabla \times \vec{F} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{bmatrix} = (0-1)\vec{i} - (1-0)\vec{j} + (0-1)\vec{k} = -\vec{i} - \vec{j} - \vec{k}$$

Parametrize S by $\vec{\Phi} : [0, \frac{\pi}{2}] \times [0, 2\pi] \to \mathbb{R}^3$, $(\varphi, \theta) \mapsto (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi)$.

$$\left.\begin{aligned} \vec{T}_\varphi &= (\cos\theta \cos\varphi, \sin\theta \cos\varphi, -\sin\varphi) \\ \vec{T}_\theta &= (-\sin\theta \sin\varphi, \cos\theta \sin\varphi, 0) \end{aligned}\right\} \vec{T}_\varphi \times \vec{T}_\theta = (\cos\theta \sin^2\varphi, \sin\theta \sin^2\varphi, \sin\varphi \cos\varphi)$$

$$\begin{aligned} \iint_S (\nabla \times F) \cdot d\vec{S} &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} (-1,-1,-1) \cdot (\cos\theta \sin^2\varphi, \sin\theta \sin^2\varphi, \sin\varphi\cos\varphi)\, d\varphi\, d\theta \\ &= -\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left((\cos\theta + \sin\theta)\sin^2\varphi + \sin\varphi\cos\varphi\right) d\varphi\, d\theta \\ &= -2\pi \int_0^{\frac{\pi}{2}} \sin\varphi \cos\varphi\, d\varphi \\ &= -2\pi \tfrac{1}{2} \sin^2\varphi \Big|_0^{\frac{\pi}{2}} \\ &= -\pi \end{aligned}$$



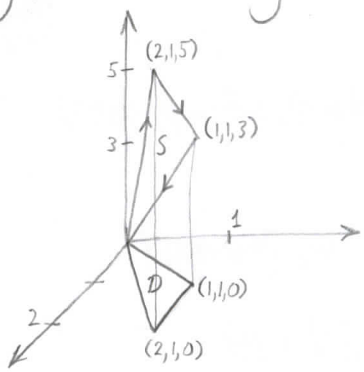
Parametrize ∂S by $\vec{c} : [0, 2\pi] \to \mathbb{R}^3$, $t \mapsto (\cos t, \sin t, 0)$.

Then $\vec{c}\,'(t) = (-\sin t, \cos t, 0)$.

$$\int_{\partial S} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} (y, z, x) \cdot \vec{c}\,'(t)\, dt = \int_0^{2\pi} (\sin t, 0, \cos t) \cdot (-\sin t, \cos t, 0)\, dt$$

$$= \int_0^{2\pi} (-\sin^2 t)\, dt = -\frac{1}{2}\int_0^{2\pi} (1 - \cos 2t)\, dt = -\frac{1}{2} 2\pi + 0 = -\pi$$

$\boxed{2}$ Since every surface integral of a vector field can be expressed as an integral of a scalar field, and since integrals over graphs of functions $g: \mathbb{R}^2 \to \mathbb{R}$ can be computed using
$\iint_S f \, dS = \iint_D \frac{f(x,y,g(x,y))}{\cos\theta} \, dx \, dy$, where θ is the angle between the normal vector to the surface and $\vec{k}$, we can apply this formula for our problem. Note that over the triangle, θ never changes.



Let S be the triangle spanned between the vertices $(0,0,0)$, $(1,1,3)$, and $(2,1,5)$. Then $\partial S = C$. Note that the orientation of C corresponds to the orientation of S as a graph. Also note that the region of the xy-plane below S (the shadow of S) is $D = \{(x,y,0) \in \mathbb{R}^3 : 0 \le y \le 1, \; y \le x \le 2y\}$, so D is the domain of a function $g: D \to \mathbb{R}$ whose graph is S. To determine g, since g must be linear in x and y, we have $g(x,y) = ax + by$.
Since $g(2,1) = 5$ and $g(1,1) = 3$, we have $\left.\begin{matrix} 2a + 1b = 5 \\ 1a + 1b = 3 \end{matrix}\right\} \Rightarrow a = 2 \Rightarrow b = 1$.

So $g: (x,y) \mapsto 2x + y$, and $S = \{(x,y,2x+y) \in \mathbb{R}^3 : (x,y) \in D\}$. Finally we need θ. A normal $\vec{n}$ to S can be given by $(2,1,5) \times (1,1,3) = (-2,-1,1)$. The z-coordinate is positive, so this gives the right orientation. Then

$$\theta = \frac{\vec{n} \cdot \vec{k}}{\|\vec{n}\| \, \|\vec{k}\|} = \frac{1}{\sqrt{6} \cdot 1} = \frac{1}{\sqrt{6}}.$$

Let $\vec{F}(x,y,z) = (xyz, xy, x)$. Then $\nabla \times \vec{F} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & xy & x \end{bmatrix} = (0-0)\vec{i} - (1-xy)\vec{j} + (y-xz)\vec{k}$.

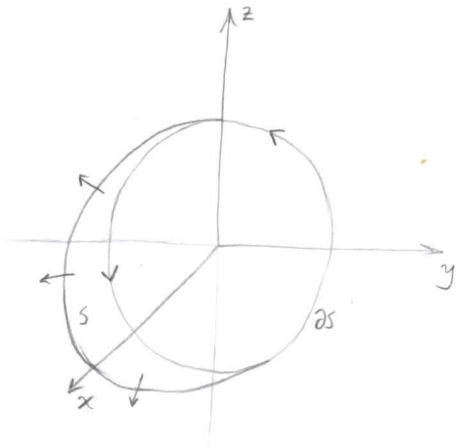
So $\int_C xyz \, dx + xy \, dy + x \, dz = \iint_S (\nabla \times \vec{F}) \cdot \frac{\vec{n}}{\|\vec{n}\|} \, dS = \iint_D (0, 1-xy, y - x(2x+y)) \cdot \sqrt{6}(-2,-1,1) \, dx \, dy$

$$= \sqrt{6} \int_0^1 \int_y^{2y} (xy - 1 + y - 2x^2 - xy) \, dx \, dy = \sqrt{6} \int_0^1 \left[-x + xy - \frac{2}{3}x^3 \right]_{x=y}^{x=2y} dy$$

$$= \sqrt{6} \int_0^1 \left(-2y + 2y^2 - \frac{2}{3} \cdot 8y^3 + y - y^2 + \frac{2}{3}y^3 \right) dy = \sqrt{6} \int_0^1 \left(-\frac{14}{3}y^3 + y^2 - y \right) dy$$

$$= \sqrt{6} \left(-\frac{14}{12}y^4 + \frac{1}{3}y^3 - \frac{1}{2}y^2 \right) \Big|_0^1 = \sqrt{6} \left(-\frac{7}{6} + \frac{1}{3} - \frac{1}{2} \right) = -\frac{4}{3}\sqrt{6}$$

$\boxed{3}$ Parametrize ∂S by $\vec{c}:[0,2\pi]\to\mathbb{R}^3$, $t\mapsto(0,\cos t,\sin t)$.
Then the orientation of $\vec{c}$ agrees with the outward orientation of S.



$$\iint_S (\nabla\times\vec{F})\cdot d\vec{S} = \int_{\partial S}\vec{F}\cdot d\vec{s} = \int_0^{2\pi}(x^3,-y^3,0)\cdot\vec{c}\,'(t)\,dt$$
$$= \int_0^{2\pi}(0,-\cos^3 t,0)\cdot(0,-\sin t,\cos t)\,dt$$
$$= \int_0^{2\pi}\cos^3 t\sin t\,dt$$
$$= -\cos^4 t\Big|_0^{2\pi} = 0$$

$\boxed{4}$ $\int_C \vec{v}\cdot d\vec{s} = \iint_S(\nabla\times\vec{v})\cdot d\vec{S} = \iint_S \vec{0}\cdot d\vec{S} = 0$

since $\vec{v}$ is constant, $\frac{\partial\vec{v}}{\partial x}=\frac{\partial\vec{v}}{\partial y}=\frac{\partial\vec{v}}{\partial z}=0$