Problem Set 7 — Solutions

$\boxed{1}$ Parametrize ∂D by $\vec{c}_1 : [0, 2\pi] \longrightarrow \mathbb{R}^2$ and $\vec{c}_2 : [0,2\pi] \longrightarrow \mathbb{R}^2$

$$t \longmapsto (b\cos t,\, b\sin t) \qquad\qquad t \longmapsto (a\cos t,\, -a\sin t).$$

Then
$$\begin{aligned}
\int_{\partial D} P\,dx + Q\,dy &= \int_{\vec{c}_1} P\,dx + Q\,dy + \int_{\vec{c}_2} P\,dx + Q\,dy \\
&= \int_0^{2\pi} \left((2b^3\cos^3 t - b^3\sin^3 t)(-b\sin t) + (b^3\cos^3 t + b^3 \sin^3 t)(b\cos t) \right) dt \\
&\quad + \int_0^{2\pi} \left((2a^3\cos^3 t + a^3\sin^3 t)(-a\sin t) + (a^3\cos^3 t - a^3\sin^3 t)(-a\cos t) \right) dt \\
&= b^4 \int_0^{2\pi} (-2\cos^3 t \sin t + \sin^4 t + \cos^4 t + \sin^3 t\cos t)\,dt \\
&\quad + a^4 \int_0^{2\pi} (-2\cos^3 t\sin t - \sin^4 t - \cos^4 t + \sin^3 t \cos t)\,dt \\
&= b^4 \int_0^{2\pi} (\sin^4 t + \cos^4 t)\,dt - a^4 \int_0^{2\pi} (\sin^4 t + \cos^4 t)\,dt \\
&= (b^4 - a^4) \int_0^{2\pi} \left((\sin^2 t + \cos^2 t)^2 - 2\sin^2 t\cos^2 t \right) dt \\
&= (b^4 - a^4) \int_0^{2\pi} \left(1 - \tfrac{1}{2}\sin^2 2t\right) dt \\
&= (b^4 - a^4)\left(2\pi - \tfrac{1}{2}\int_0^{2\pi} \sin^2 2t\,dt\right) \\
&= (b^4 - a^4)\left(2\pi - \tfrac{1}{4}\int_0^{2\pi} (1 - \cos 4t)\,dt\right) \\
&= (b^4 - a^4)\left(2\pi - \tfrac{1}{4}2\pi - 0\right) \\
&= \frac{3\pi}{2}(b^4 - a^4)
\end{aligned}$$

Also,
$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_D (3x^2 + 3y^2)\,dx\,dy = \int_a^b \int_0^{2\pi} (3r^2)\, r\,dr\,d\theta = 6\pi \int_a^b r^3\,dr$$

$$= 6\pi \, \frac{1}{4} r^4 \Big|_a^b = \frac{3\pi}{2}(b^4 - a^4)$$

$\boxed{2}$ By Green's theorem, $\int_{\partial D} \frac{\partial f}{\partial y} dx - \frac{\partial f}{\partial x} dy = \iint_D \left(-\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right) dx\,dy = -\iint_D 0 \; dx\,dy = 0.$

$\boxed{3}$ Apply change of coordinates: $x = a\cos\theta$ $\quad \theta \in [0, 2\pi]$

$y = b\sin\theta$

$$A = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} \left(a\cos\theta\, b\cos\theta + b\sin\theta\, a\sin\theta\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab \; d\theta$$

$$= ab\pi$$