Problem Set 4 — Solutions

$\boxed{1}$ $\vec{\Phi}: \mathbb{R}^2 \to \mathbb{R}^3$, $(u,v) \mapsto (u, v, 3u^2+8uv) = (x(u,v), y(u,v), z(u,v))$

$$\left.\begin{array}{l} \vec{T}_u = (1, 0, 6u+8v) \\ \vec{T}_v = (0, 1, 8u) \end{array}\right\} \Rightarrow \vec{T}_u \times \vec{T}_v = (-6u-8v, -8u, 1)$$

Note $\vec{\Phi}(1,0) = (1,0,3)$. $\quad \vec{T}_u \times \vec{T}_v \Big|_{(1,0)} = (-6,-8,1)$

So equation of tangent plane is $-6(x-1) - 8(y-0) + (z-3) = 0$

or $-6x - 8y + z = -3$

$\boxed{2}\boxed{(a)}$ $$\left.\begin{array}{l} \vec{T}_u = (e^u \cos v, e^u \sin v, 0) \\ \vec{T}_v = (-e^u \sin v, e^u \cos v, 1) \end{array}\right\} \Rightarrow \vec{T}_u \times \vec{T}_v = (e^u \sin v, -e^u \cos v, e^{2u})$$

$\boxed{(b)}$ $$\left.\begin{array}{l} \vec{\Phi}(0, \frac{\pi}{2}) = (0, 1, \frac{\pi}{2}) \\ \vec{T}_u \times \vec{T}_v \Big|_{(0,\frac{\pi}{2})} = (1,0,1) \end{array}\right\} \text{equation of tangent plane is } (x-0) + 0(y-1) + (z - \tfrac{\pi}{2}) = 0$$

or $x + z = \frac{\pi}{2}$

$\boxed{(c)}$ $\|\vec{T}_u \times \vec{T}_v\| = \sqrt{e^{2u}\sin^2 v + e^{2u}\cos^2 v + e^{4u}} = e^u\sqrt{1+e^{2u}}$

$$A(S) = \iint_D \|\vec{T}_u \times \vec{T}_v\|\, du\, dv = \int_0^1\int_0^{\pi} e^u\sqrt{1+e^{2u}}\, dv\, du = \pi \int_0^1 e^u \sqrt{1+e^{2u}}\, du$$

Let $x = e^u \Rightarrow dx = e^u\, du$

$$A(S) = \pi \int_1^e \sqrt{1+x^2}\, dx = \pi \left(\frac{x}{2}\sqrt{x^2+1} + \frac{1}{2}\log\left|x + \sqrt{x^2+1}\right| \Bigg|_1^e \right)$$

$$= \pi\left(\frac{e}{2}\sqrt{e^2+1} + \frac{1}{2}\log\left|e + \sqrt{e^2+1}\right| - \frac{1}{2}\sqrt{2} + \frac{1}{2}\log\left|1+\sqrt{2}\right|\right)$$

$\boxed{3}$ $\left.\begin{array}{l}\vec{T}_u = (1,1,v) \\ \vec{T}_v = (-1,1,u)\end{array}\right\} \Rightarrow \vec{T}_u \times \vec{T}_v = (u-v, -u-v, 2)$

$$\Rightarrow \|\vec{T}_u \times \vec{T}_v\| = \sqrt{u^2 - 2uv + v^2 + u^2 + 2uv + v^2 + 4}$$

$$= \sqrt{2}\sqrt{u^2+v^2+2}$$

$$A(\vec{\Phi}(D)) = \iint_D \|\vec{T}_u \times \vec{T}_v\| \, du\, dv = \iint_D \sqrt{2}\sqrt{u^2+v^2+2}\, du\, dv$$

Change parameters: $u = r\cos\theta$, $v = r\sin\theta$ where $r \in [0,1]$, $\theta \in [0, 2\pi]$.

Then $u^2+v^2 = r^2$ and Jacobian is r, ie, $du\,dv = r\,dr\,d\theta$. So

$$A(\vec{\Phi}(D)) = \int_0^{2\pi}\int_0^1 \sqrt{2}\sqrt{r^2+2}\, r\, dr\, d\theta$$

Let $t = r^2+2 \Rightarrow \frac{1}{2}dt = r\,dr$

$$A(\vec{\Phi}(D)) = \int_0^{2\pi}\int_2^3 \sqrt{2}\sqrt{t}\,\frac{1}{2}\,dt\,d\theta = \frac{\sqrt{2}}{2}2\pi\int_2^3 \sqrt{t}\,dt = \sqrt{2}\pi\,\frac{2}{3}\,t^{3/2}\Big|_2^3$$

$$= \frac{2\sqrt{2}\pi}{3}\left(3^{3/2} - 2^{3/2}\right)$$