Problem Set 5 — Solutions

1 $\vec{T}_u = (2\cos v, 2\sin v, 1)$

$\vec{T}_v = (-2u\sin v, 2u\cos v, 0)$

$$\Rightarrow \vec{T}_u \times \vec{T}_v = (-2u\cos v, -2u\sin v, 4u)$$

$$\Rightarrow \|\vec{T}_u \times \vec{T}_v\| = \sqrt{4u^2\cos^2 v + 4u^2\sin^2 v + 16u^2}$$
$$= 2u\sqrt{1+4} = 2\sqrt{5}\,u$$

$$f(\vec{\Phi}(u,v)) = f(2u\cos v, 2u\sin v, v) = 2u\cos v + 2u\sin v$$

$$\iint_S f\,dS = \int_0^{\pi}\int_0^4 2u(\cos v + \sin v)2\sqrt{5}\,u\,du\,dv$$
$$= 4\sqrt{5}\int_0^{\pi}(\cos v + \sin v)\,dv\int_0^4 u^2\,du$$
$$= 4\sqrt{5}\left[\sin v - \cos v\right]_0^{\pi}\left[\tfrac{1}{3}u^3\right]_0^4 = 4\sqrt{5}(0-(-1)-0+1)\left(\tfrac{1}{3}64\right) = \frac{512\sqrt{5}}{3}$$

2 Parametrize S by $\vec{\Phi}:[0,2\pi]\times[0,\pi]\to\mathbb{R}^3$, $(\theta,\varphi)\mapsto(\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$.

$\vec{T}_\theta = (-\sin\theta\sin\varphi, \cos\theta\sin\varphi, 0)$

$\vec{T}_\varphi = (\cos\theta\cos\varphi, \sin\theta\cos\varphi, -\sin\varphi)$

$$\Rightarrow \vec{T}_\theta \times \vec{T}_\varphi = (-\cos\theta\sin^2\varphi, -\sin\theta\sin^2\varphi, -\sin\varphi\cos\varphi)$$

$$\|\vec{T}_\theta \times \vec{T}_\varphi\| = \sqrt{\cos^2\theta\sin^4\varphi + \sin^2\theta\sin^4\varphi + \sin^2\varphi\cos^2\varphi} = \sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi} = \sqrt{\sin^2\varphi}$$
$$= |\sin\varphi| = \sin\varphi$$

since $\varphi\in[0,\pi]$

$$\iint_S (x+y+z)\,dS = \int_0^{\pi}\int_0^{2\pi}(\cos\theta\sin\varphi + \sin\theta\sin\varphi + \cos\varphi)\sin\varphi\,d\theta\,d\varphi$$
$$= \int_0^{\pi}\int_0^{2\pi}(\cos\theta\sin^2\varphi + \sin\theta\sin^2\varphi + \cos\varphi\sin\varphi)\,d\theta\,d\varphi$$
$$= \int_0^{\pi}\left(\left[\sin\theta\right]_0^{2\pi}\sin^2\varphi - \left[\cos\theta\right]_0^{2\pi}\sin^2\varphi + 2\pi\cos\varphi\sin\varphi\right)d\varphi$$
$$= \underbrace{\int_0^{\pi}(0+0+\pi\sin 2\varphi)\,d\varphi = \int_0^{2\pi}\pi\sin u\,du}_{u=2\varphi\Rightarrow du=2d\varphi} = -\pi\left[\cos u\right]_0^{2\pi} = 0$$

$\boxed{3}$ $\vec{\Phi}: [0,2\pi]\times[0,1] \longrightarrow \mathbb{R}^3$, $(\theta, z) \mapsto (2\cos\theta, 2\sin\theta, z)$

$$\left.\begin{array}{l}\vec{T}_\theta = (-2\sin\theta,\ 2\cos\theta,\ 0)\\ \vec{T}_z = (0,\ \ 0,\ \ 1)\end{array}\right\} \vec{T}_\theta \times \vec{T}_z = (2\cos\theta, 2\sin\theta, 0)$$

Note at $\theta = 0$, $z = 0$, $\vec{T}_\theta \times \vec{T}_z\big|_{(0,0)} = (2,0,0)$, which points outward, hence $\vec{T}_\theta \times \vec{T}_z$ gives the desired orientation.

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^1\int_0^{2\pi} (2\cdot 2\cos\theta, -2\cdot 2\sin\theta, z^2)\cdot(2\cos\theta, 2\sin\theta, 0)\,d\theta\,dz$$

$$= \int_0^1\int_0^{2\pi} (8\cos^2\theta - 8\sin^2\theta)\,d\theta\,dz$$

$$= 8\int_0^{2\pi}(\cos^2\theta - \sin^2\theta)\,d\theta = 8\int_0^{2\pi}\cos 2\theta\,d\theta \underbrace{=}_{u=2\theta \Rightarrow du = 2d\theta} 8\int_0^{4\pi}\cos u\,du = 8\left[\sin u\right]_0^{4\pi} = 0$$

$\boxed{4}$ $\vec{\Phi}: [0,\pi]\times[0,2\pi] \longrightarrow \mathbb{R}^3$, $(\varphi,\theta) \mapsto (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$

$$\left.\begin{array}{l}\vec{T}_\varphi = (\cos\theta\cos\varphi,\ \sin\theta\cos\varphi,\ -\sin\varphi)\\ \vec{T}_\theta = (-\sin\theta\sin\varphi,\ \cos\theta\sin\varphi,\ 0)\end{array}\right\} \Rightarrow \vec{T}_\varphi \times \vec{T}_\theta = (\cos\theta\sin^2\varphi,\ \sin\theta\sin^2\varphi,\ \cos\varphi\sin\varphi)$$

$$\vec{V}(\vec{\Phi}(\varphi,\theta)) = (3\cos\theta\sin^3\varphi\sin^2\theta,\ 3\cos^2\theta\sin^3\varphi\sin\theta,\ \cos^3\varphi)$$

$$\vec{V}(\vec{\Phi}(\varphi,\theta))\cdot(\vec{T}_\varphi\times\vec{T}_\theta) = 3\cos^2\theta\sin^2\theta\sin^5\varphi + 3\cos^2\theta\sin^2\theta\sin^5\varphi + \cos^4\varphi\sin\varphi$$

$$= 6\cos^2\theta\sin^2\theta\sin^5\varphi + \cos^4\varphi\sin\varphi$$

$$\iint_S \vec{V}\cdot d\vec{S} = \int_0^\pi\int_0^{2\pi} 6\cos^2\theta\sin^2\theta\sin^5\varphi\,d\theta\,d\varphi + \int_0^\pi\int_0^{2\pi}\cos^4\varphi\sin\varphi\,d\theta\,d\varphi$$

$$= 6\underbrace{\int_0^\pi \sin^5\varphi\,d\varphi}_{A}\underbrace{\int_0^{2\pi}\cos^2\theta\sin^2\theta\,d\theta}_{B} + 2\pi\underbrace{\int_0^\pi\cos^4\varphi\sin\varphi\,d\varphi}_{C}$$

$$A = \int_0^\pi \sin^5\varphi\,d\varphi = \int_0^\pi \sin\varphi(1-\cos^2\varphi)^2\,d\varphi = \int_0^\pi \sin\varphi(1 - 2\cos^2\varphi + \cos^4\varphi)\,d\varphi$$

$$= \int_0^\pi \sin\varphi\,d\varphi - \int_0^\pi 2\cos^2\varphi\sin\varphi\,d\varphi + \int_0^\pi\cos^4\varphi\sin\varphi\,d\varphi = \left[-\cos\varphi\right]_0^\pi + \left[\tfrac{2}{3}\cos^3\varphi\right]_0^\pi - \left[\tfrac{1}{5}\cos^5\varphi\right]_0^\pi =$$

$$= 2 - \frac{4}{3} + \frac{2}{5} = \frac{30}{15} - \frac{20}{15} + \frac{6}{15} = \frac{16}{15}$$

$$B = \int_0^{2\pi} \cos^2\theta \sin^2\theta \, d\theta = \underbrace{\int_0^{2\pi} \frac{1}{4}\sin^2 2\theta \, d\theta = \frac{1}{4}\int_0^{2\pi} \frac{1}{2}(1-\cos 4\theta)\, d\theta}_{\cos 2\alpha = 1-2\sin^2\alpha} = \frac{1}{8}\left(2\pi - \int_0^{2\pi} \cos 4\theta \, d\theta\right)$$

$$= \frac{1}{8}\left(2\pi - \frac{1}{4}\sin 4\theta \Big|_0^{2\pi}\right) = \frac{\pi}{4} - 0$$

$$C = \int_0^{\pi} \cos^4\varphi \sin\varphi \, d\varphi = -\left[\frac{1}{5}\cos^5\varphi\right]_0^{\pi} = -\frac{1}{5}(-2) = \frac{2}{5}$$

$$\iint_S \vec{\nabla} \cdot d\vec{S} = 6AB + 2\pi C = 6 \cdot \frac{16}{15} \cdot \frac{\pi}{4} + 2\pi \cdot \frac{2}{5} = 2 \cdot \frac{4}{5} \cdot \pi + \frac{4\pi}{5} = \frac{12\pi}{5}$$