Review Problems for Exam 2

(Note that all problems are odd-numbered problems from the textbook, so the answers are in the back of the book.)

Implicit differentiation

2.6.9 Find $\frac{dy}{dx}$ where $4\cos x \sin y = 1$.

2.6.11 Find $\frac{dy}{dx}$ where $\tan \frac{x}{y} = x + y$.

2.6.13 Find $\frac{dy}{dx}$ where $\sqrt{xy} = 1 + x^2 y$.

2.6.19 Find an equation of the tangent line to the ellipse $x^2 + xy + y^2 = 3$ at the point $(1, 1)$.

2.6.21 Find an equation of the tangent line to the curve $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$ (called a cardioid) at the point $(0, \frac{1}{2})$.

2.6.25 Find y'' where $x^3 + y^3 = 1$.

Related rates

2.7: Example 4 Car A is traveling west at 50 mi/hr and car B is traveling north at 60 mi/hr. Both are headed for the intersection of the two roads. At what rate are the cars approaching each ohter when car A is 0.3 mi from the intersection and car B is 0.4 mi from the intersection?

2.7.3 Each side of a square is increasing at a rate of 6 cm/s. At what rate is the area of the square increasing when the area of the square is 16 cm^2?

2.7.5 A cylindrical tank with radius 5 m is being filled with water at a rate of 3 m^3/min. How fast is the height of the water increasing?

2.7.11 If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

2.7.13 A plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/hr passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 mi away from the station.

2.7.15 Two cars start moving from the same point. One travels south at 60 mi/hr and the other travels west at 25 mi/hr. At what rate is the distance between the cars increasing two hours later?

2.7.25 A trough is 10 ft long and its ends have the shape of isosceles triangles that are 3 ft across the top and have a height of 1 ft. If the trough is being filled with water at a rate of 12 ft^3/min, how fast is the water level rising when the water is 6 inches deep?

Linear approximation and differentials

2.8.7 Verify that $1 + \frac{1}{2}x$ is the linear approximation to $\sqrt[4]{1+2x}$ at $a = 0$.

2.8.9 Verify that $1 - 8x$ is the linear approximation to $\frac{1}{(1+2x)^4}$ at $a = 0$.

2.8.17(a) Find the differential dy where $y = \tan \sqrt{t}$.

2.8.17(b) Find the differential dy where $y = \frac{1-v^2}{1+v^2}$.

2.8.21 The edge of a cube was found to be 30 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum possible error in computing the volume of the cube and the surface area of the cube.

2.8.23 The circumference of a sphere was measured to be 84 cm with a possible error of 0.5 cm. Use differentials to estimate the maximum possible error in computing the volume of the sphere and the surface area of the sphere.

Maximum and minimum values

3.1.7 Sketch the graph of a single continuous function with domain $[1, 5]$, an absolute minimum at 2, an absolute maximum at 3, and a local minimum at 4.

3.1.9 Sketch the graph of a single continuous function with domain $[1, 5]$, an absolute maximum at 5, an absolute minimum at 2, a local maximum at 3, and local minimums at 2 and 4.

3.1.11(b) Sketch the graph of a single function that has a local maximum at 2 and is continuous but not differentiable at 2.

3.1.39 Find the absolute maximum and absolute minimum values of $f(x) = 2x^2 - 3x^2 - 12x + 1$ on the interval $[-2, 3]$.

3.1.41 Find the absolute maximum and absolute minimum values of $f(x) = 3x^4 - 4x^3 - 12x + 1$ on the interval $[-2, 3]$.

3.1.43 Find the absolute maximum and absolute minimum values of $f(t) = t\sqrt{4 - t^2}$ on the interval $[-1, 2]$.

Mean value theorem

3.2.17 Use the mean value theorem to show that the equation $2x + \cos x = 0$ has exactly one real root. (Since the solution is not in the back of the book, I put the solution at the end of this pdf.)

3.2.19 Use the mean value theorem to show that the equation $x^3 - 15x + c = 0$ (where c is a constant) has at most one real root in the interval $[-2, 2]$. (Since the solution is not in the back of the book, I put the solution at the end of this pdf.)

3.2.23 If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ possibly be?

Ch. 3 review: 24 Suppose that f is continuous on $[0, 4]$, $f(0) = 1$, and $2 \leq f'(x) \leq 5$ for all x in $(0, 4)$. Show that $9 \leq f(4) \leq 21$. (Since this problem is even-numbered, I put the solution at the end of this pdf.)

Ch. 3 review: 25 By applying the Mean Value Theorem to the function $f(x) = x^{\frac{1}{5}}$ on the interval $[32, 33]$, show that $2 < \sqrt[5]{33} < 2.0125$.

Derivatives and the shapes of graphs (increasing/decreasing and concave up/down)

For each of the following functions, find the intervals on which they are increasing/decreasing and concave up/down, and determine their local minimums, local maximums, and inflection points.

3.3.1 $f(x) = 2x^2 + 3x^2 - 36x$

3.3.3 $f(x) = x^4 - 2x^2 + 3$

3.3.23 $f(x) = 2 + 2x^2 - x^4$

3.3.27 $F(x) = x\sqrt{6 - x}$

3.3.29 $f(\theta) = 2\cos\theta + \cos^2\theta$ where the domain of f is restricted to $[0, 2\pi]$

For each of the following set of criteria, sketch a graph of a function satisfying them.

3.3.15

- $f'(0) = f'(2) = f'(4) = 0$
- $f'(x) > 0$ if $x < 0$ or $2 < x < 4$
- $f'(x) < 0$ if $0 < x < 2$ or $x > 4$
- $f''(x) > 0$ if $1 < x < 3$
- $f''(x) < 0$ if $x < 1$ or $x > 3$.

3.3.17

- $f'(x) > 0$ if $|x| < 2$
- $f'(x) < 0$ if $|x| > 2$
- $f'(-2) = 0$
- $\lim_{x \to 2} |f'(x)| = \infty$
- $f''(x) > 0$ if $x \neq 2$

Optimization problems

3.5.3 Find two positive numbers whose product is 100 and whose sum is a minimum.

3.5.7 Find the dimensions of a rectangle with perimeter 100 whoser area is as large as possible.

3.5.11 If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

3.5.13(a) Show that of all the rectangles with a given area, the one with the smallest perimeter is a square.

3.5.13(b) Show that of all the rectangles with a given perimeter, the one with the greatest area is a square.

3.5.15 Find the point on the line $y = 2x + 3$ that is closest to the origin.

3.5.21 Find the dimension of the isosceles tringle of largest area that can be inscribed in a circle of fixed radius r.

3.5.23 A cylinder is inscribed in a sphere of a fixed radius r. Find the largest possible volume of such a cylinder.

A few solutions

3.2.17 *Use the mean value theorem to show that the equation $2x + \cos x = 0$ has exactly one real root.*

Let $f(x) = 2x + \cos x$. Note that

$$f(0) = 0 + \cos 0 = 0 + 1 = 1 > 0$$

and

$$f(-100) = -200 + \cos(-100) < 0$$

since $-1 \leq \cos\theta \leq 1$ for any θ. Since f is continuous, the Intermediate Value Theorem shows that there is some r_1 such that $f(r_1) = 0$, i.e., f has at least one real root. To show that there is exactly one real root, suppose by way of contradiction that f has two real roots r_1, r_2, where $r_1 < r_2$. Since f is differentiable, the Mean Value Theorem shows that there is some number $c \in (r_1, r_2)$ such that

$$f'(c) = \frac{f(r_2) - f(r_1)}{r_2 - r_1} = \frac{0 - 0}{r_2 - r_1} = 0.$$

But

$$f'(x) = 2 + \sin x,$$

which is always greater than 0 since $-1 \leq \sin x \leq 1$ for any x. This contradicts $f'(c) = 0$, so our assumption that there are two real roots must be wrong. So there must be exactly one real root.

3.2.19 *Use the mean value theorem to show that the equation $x^3 - 15x + c = 0$ (where c is a constant) has at most one real root in the interval $[-2, 2]$.*

Let $f(x) = x^3 - 15x + c$. Suppose by way of contradiction that f has two real roots $r_1, r_2 \in [-2, 2]$, where $r_1 < r_2$. Since f is differentiable, the Mean Value Theorem shows that there is some number $x_0 \in (r_1, r_2)$ such that

$$f'(x_0) = \frac{f(r_2) - f(r_1)}{r_2 - r_1} = \frac{0 - 0}{r_2 - r_1} = 0.$$

Note

$$f'(x) = 3x^2 - 15.$$

Since we're restricting to our attention to the interval $[-2, 2]$, we have

$$\begin{aligned} -2 &\le x \le 2 \\ 0 &\le x^2 \le 4 \\ 0 &\le 3x^2 \le 12 \\ -15 &\le 3x^2 - 15 \le -27 \\ -15 &\le f'(x) \le -27. \end{aligned}$$

This contradicts $f'(x_0) = 0$, so our assumption that there are two real roots must be wrong.

Ch. 3 review: 24 *Suppose that f is continuous on $[0, 4]$, $f(0) = 1$, and $2 \le f'(x) \le 5$ for all x in $(0, 4)$. Show that $9 \le f(4) \le 21$.*

We are told that f is continuous on $[0, 4]$ and differentiable on $(0, 4)$, so we can apply the Mean Value Theorem. So there is some number $c \in (0, 4)$ such that $f'(c) = \frac{f(4) - f(0)}{4 - 0} = \frac{f(4) - 1}{4}$. Since $2 \le f'(x) \le 5$ for all x in $(0, 4)$, we know that this inequality holds for c, i.e.,

$$\begin{aligned} 2 &\le f'(c) \le 5 \\ 2 &\le \frac{f(4) - 1}{4} \le 5 \\ 8 &\le f(4) - 1 \le 20 \\ 9 &\le f(4) \le 21. \end{aligned}$$

This is what we sought to prove.

3.5.13(a) *Show that of all the rectangles with a given area, the one with the smallest perimeter is a square.*

Let R be a rectangle of constant area A, height h, and width w. Then $A = hw$, so $h = \frac{A}{w}$. Let P be the perimeter of the square. Then $P = 2h + 2w = 2\frac{A}{w} + 2w$. We want to minimize $P(w)$. Since

$$P'(w) = -2Aw^{-2} + 2,$$

$P'(w) = 0$ if and only if

$$\begin{aligned} 2Aw^{-2} &= 2 \\ Aw^{-2} &= 1 \\ w &= \sqrt{A}. \end{aligned}$$

So $P(w)$ has a critical point at $w = \sqrt{A}$. Since $P''(w) = 6Aw^{-3}$, we have $P''(\sqrt{A}) = 6A(\sqrt{A})^{-3} > 0$, so P is concave up at $w = \sqrt{A}$. So P has a local minimum at $w = \sqrt{A}$. But we must show that P has an *absolute* minimum at $w = \sqrt{A}$. Since $P''(\sqrt{A}) = 6Aw^{-3} > 0$ for *all* $w > 0$ (and since the problem doesn't make sense for $w \leq 0$), the local minimum must be an absolute minimum. So the perimeter is minimized when $w = \sqrt{A}$. If $w = \sqrt{A}$, then $h = \frac{A}{\sqrt{A}} = \sqrt{A}$, i.e., we have a square.