Problem Set 9 — Solutions

$\boxed{1}$

$$\left.\begin{aligned}\frac{\partial Q}{\partial x} &= \frac{\partial}{\partial x}(2xy) = 2y \\ \frac{\partial P}{\partial y} &= \frac{\partial}{\partial y}(x^2+y^2) = 2y\end{aligned}\right\} \Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \Rightarrow \vec{F} \text{ is conservative}$$

Suppose $\nabla f = \vec{F}$. Then

$$\left.\begin{aligned}\frac{\partial f}{\partial x} &= x^2+y^2 \Rightarrow f(x,y) = \tfrac{1}{3}x^3 + xy^2 + g(y) \\ \frac{\partial f}{\partial y} &= 2xy \Rightarrow f(x,y) = xy^2 + h(x)\end{aligned}\right\} \Rightarrow h(x) = \tfrac{1}{3}x^3 + g(y) \Rightarrow g(y) = 0$$

$$\Rightarrow f(x,y) = \tfrac{1}{3}x^3 + xy^2$$

$\boxed{2}$

$$\nabla\times\vec{F} = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin y & e^x\cos y & z^2 \end{bmatrix} = (0-0)\vec{i} - (0-0)\vec{j} + (e^x\cos y - e^x\cos y)\vec{k} = \vec{0}$$

$\Rightarrow \vec{F}$ is conservative

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(e^x \sin y) + \frac{\partial}{\partial y}(e^x\cos y) + \frac{\partial}{\partial z}(z^2) = e^x\sin y - e^x \sin y + 2z = 2z \neq 0$$

$\Rightarrow \nexists\, \vec{G}$ s.t. $\nabla\times\vec{G} = \vec{F}$

$\boxed{3}$ Suppose $\nabla f = \vec{F}$. Then

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2xyz + \sin x \Rightarrow f(x,y,z) = x^2yz - \cos x + g_1(y,z) \\ \frac{\partial f}{\partial y} &= x^2 z \qquad\quad \Rightarrow f(x,y,z) = x^2yz + g_2(x,z) \end{aligned}\right\} \begin{aligned} &\Rightarrow -\cos x + g_1(y,z) = g_2(x,z) \\ &\Rightarrow g_1(y,z) \text{ cannot be a function of } y. \end{aligned}$$

$$\frac{\partial f}{\partial z} = x^2 y \qquad\quad \Rightarrow f(x,y,z) = x^2yz + g_3(x,y)$$

Similarly, we get $-\cos x + g_1(y,z) = g_3(x,y)$, so $g_1(y,z)$ cannot be a function of z.
So $g_1(y,z) = c$ where $c \in \mathbb{R}$ constant. Hence $f(x,y,z) = x^2yz - \cos x + c$.

$\boxed{4}\boxed{(a)}$ Parametrize the unit circle by $\vec{c}: [0,2\pi] \to \mathbb{R}^2$, $t \mapsto (\cos t, \sin t)$. Then

$$\int_C \frac{x\,dy - y\,dx}{x^2+y^2} = \int_0^{2\pi} \frac{\cos t \cdot \cos t - \sin t(-\sin t)}{\cos^2 t + \sin^2 t}\,dt = \int_0^{2\pi} dt = 2\pi.$$

Since C is a simple closed curve and $\int_C \vec{F}\cdot d\vec{s} \neq 0$ where $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$, $\vec{F}$ is not conservative.

$\boxed{(b)}$ $P = \frac{-y}{x^2+y^2} \Rightarrow \frac{\partial P}{\partial y} = \frac{(x^2+y^2)(-1)-(-y)(2y)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$

$Q = \frac{x}{x^2+y^2} \Rightarrow \frac{\partial Q}{\partial x} = \frac{(x^2+y^2)(1)-(x)(2x)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$

Although $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, this vector field is not conservative. This does not violate the proposition about conservative vector fields we covered in class since that proposition requires that $\vec{F}$ be defined on all of $\mathbb{R}^2$, but $\vec{F}$ here is not defined at the origin $(0,0)$.