Problem Set 6 — Solutions

[1] Let H denote the top hemisphere of S and let D denote the unit disk in the xy-plane. Then $S = D \cup H$ and $\iint_S \vec{F}\cdot d\vec{S} = \iint_H \vec{F}\cdot d\vec{S} + \iint_D \vec{F}\cdot d\vec{S}$.

Parametrize H by $\vec{\Phi}_H : [0,\frac{\pi}{2}]\times[0,2\pi] \longrightarrow \mathbb{R}^3$, $(\varphi,\theta)\mapsto(\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$.

$$\left.\begin{aligned}\vec{T}_\varphi &= (\cos\theta\cos\varphi, \sin\theta\cos\varphi, -\sin\varphi)\\ \vec{T}_\theta &= (-\sin\theta\sin\varphi, \cos\theta\sin\varphi, 0)\end{aligned}\right\} \Rightarrow \vec{T}_\varphi\times\vec{T}_\theta = (\cos\theta\sin^2\varphi, \sin\theta\sin^2\varphi, \cos\varphi\sin\varphi).$$

$$\vec{F}(\vec{\Phi}_H(\varphi,\theta)) = (\cos\theta\sin\varphi + 3\sin^5\theta\sin^5\varphi,\ \sin\theta\sin\varphi + 10\cos\theta\sin\varphi\cos\varphi,\ \cos\varphi - \cos\theta\sin\theta\sin^2\varphi)$$

$$\vec{F}(\vec{\Phi}_H(\varphi,\theta))\cdot(\vec{T}_\varphi\times\vec{T}_\theta) = \cos^2\theta\sin^3\varphi + 3\cos\theta\sin^5\theta\sin^7\varphi + \sin^2\theta\sin^3\varphi + 10\cos\theta\sin\theta\cos\varphi\sin^3\varphi$$
$$+ \cos^2\varphi\sin\varphi - \cos\theta\sin\theta\cos\varphi\sin^3\varphi$$

$$\iint_H \vec{F}\cdot d\vec{S} = \underbrace{\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\cos^2\theta\sin^3\varphi\, d\theta\, d\varphi}_{I_1} + \underbrace{\int_0^{\frac{\pi}{2}}\int_0^{2\pi} 3\cos\theta\sin^5\theta\sin^7\varphi\, d\theta\, d\varphi}_{I_2} + \underbrace{\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\sin^2\theta\sin^3\varphi\, d\theta\, d\varphi}_{I_3}$$
$$+ \underbrace{\int_0^{\frac{\pi}{2}}\int_0^{2\pi} 10\cos\theta\sin\theta\cos\varphi\sin^3\varphi\, d\theta\, d\varphi}_{I_4} + \underbrace{\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\cos^2\varphi\sin\varphi\, d\theta\, d\varphi}_{I_5} - \underbrace{\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\cos\theta\sin\theta\cos\varphi\sin^3\varphi\, d\theta\, d\varphi}_{I_6}$$

$$I_1 = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\cos^2\theta\sin^3\varphi\, d\theta\, d\varphi = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\left(\tfrac{1}{2}\cos 2\theta + \tfrac{1}{2}\right)\sin^3\varphi\, d\theta\, d\varphi = \int_0^{\frac{\pi}{2}} \tfrac{1}{2}\cdot 2\pi\cdot\sin^3\varphi\, d\varphi$$
$$= \pi\int_0^{\frac{\pi}{2}}(1-\cos^2\varphi)\sin\varphi\, d\varphi = \pi\left[-\cos\varphi\right]_0^{\frac{\pi}{2}} - \pi\left[-\tfrac{1}{3}\cos^3\varphi\right]_0^{\frac{\pi}{2}} = -\pi(0-1) - \tfrac{\pi}{3}(0-1) = \tfrac{2\pi}{3}$$

$$I_2 = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} 3\cos\theta\sin^5\theta\sin^7\varphi\, d\theta\, d\varphi = 3\left[\tfrac{1}{6}\sin^6\theta\right]_0^{2\pi}\int_0^{\frac{\pi}{2}}\sin^7\varphi\, d\varphi = 3\cdot 0\cdot\int_0^{\frac{\pi}{2}}\sin^7\varphi\, d\varphi = 0$$

$$I_3 = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\sin^2\theta\sin^3\varphi\, d\theta\, d\varphi = \int_0^{2\pi}\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 2\theta\right)d\theta\int_0^{\frac{\pi}{2}}(1-\cos^2\varphi)\sin\varphi\, d\varphi = \tfrac{1}{2}(2\pi-0)\int_0^{\frac{\pi}{2}}(1-\cos^2\varphi)\sin\varphi\, d\varphi$$
$$= \pi\left[-\cos\varphi\right]_0^{\frac{\pi}{2}} - \pi\left[-\tfrac{1}{3}\cos^3\varphi\right]_0^{\frac{\pi}{2}} = -\pi(0-1) + \tfrac{\pi}{3}(0-1) = \tfrac{2\pi}{3}$$

$$I_4 = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} 10\cos\theta\sin\theta\cos\varphi\sin^3\varphi\, d\theta\, d\varphi = 10\int_0^{2\pi}\tfrac{1}{2}\sin 2\theta\, d\theta\int_0^{\frac{\pi}{2}}\cos\varphi\sin^3\varphi\, d\varphi = 10\cdot 0\cdot\int_0^{\frac{\pi}{2}}\cos\varphi\sin^3\varphi\, d\varphi = 0$$

$$I_5 = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \cos^2\varphi \sin\varphi \, d\varphi = 2\pi\left[-\frac{1}{3}\cos^3\varphi\right]_0^{\frac{\pi}{2}} = -\frac{2\pi}{3}(0-1) = \frac{2\pi}{3}$$

$$I_6 = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \cos\theta\sin\theta\cos\varphi\sin^3\varphi \, d\theta \, d\varphi = \left[\frac{1}{2}\sin^2\theta\right]_0^{2\pi}\int_0^{\frac{\pi}{2}}\cos\varphi\sin^3\varphi \, d\varphi = 0$$

$$\Rightarrow \iint_H \vec{F}\cdot d\vec{S} = \sum_{j=1}^{6} I_j = 3\cdot\frac{2\pi}{3} = 2\pi$$

Parametrize D by $\vec{\Phi}_D : [0,2\pi]\times[0,1] \to \mathbb{R}^3$, $(\theta, r) \mapsto (r\cos\theta, r\sin\theta, 0)$

$$\left.\begin{aligned} \vec{T}_\theta &= (-r\sin\theta, r\cos\theta, 0) \\ \vec{T}_r &= (\cos\theta, \sin\theta, 0) \end{aligned}\right\} \Rightarrow \vec{T}_\theta \times \vec{T}_r = (0,0,-r).$$

Note that D should be oriented such that the normal vector points downward, which is given by $\vec{T}_\theta\times\vec{T}_r$, not $\vec{T}_r\times\vec{T}_\theta$.

$$\vec{F}(\vec{\Phi}_D(\theta,r)) = (r\cos\theta + 3r^3\sin^3\theta, \; r\sin\theta, \; -r^2\cos\theta\sin\theta)$$

$$\vec{F}(\vec{\Phi}_D(\theta,r))\cdot(\vec{T}_\theta\times\vec{T}_r) = 0 + 0 + r^3\cos\theta\sin\theta$$

$$\iint_D \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^1 r^3\cos\theta\sin\theta \, dr \, d\theta = \left[\frac{1}{2}\sin^2\theta\right]_0^{2\pi}\int_0^1 r^3 dr = 0\cdot\int_0^1 r^3 dr = 0$$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_H \vec{F}\cdot d\vec{S} + \iint_D \vec{F}\cdot d\vec{S} = 2\pi + 0 = 2\pi.$$

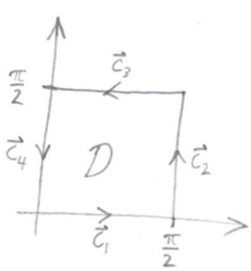
$\boxed{2}$ Parametrize ∂D by four curves

$\vec{c}_1 : [0, \frac{\pi}{2}] \to \mathbb{R}^2$, $t \mapsto (t, 0)$

$\vec{c}_2 : [0, \frac{\pi}{2}] \to \mathbb{R}^2$, $t \mapsto (\frac{\pi}{2}, t)$

$\vec{c}_3 : [0, \frac{\pi}{2}] \to \mathbb{R}^2$, $t \mapsto (\frac{\pi}{2} - t, \frac{\pi}{2})$

$\vec{c}_4 : [0, \frac{\pi}{2}] \to \mathbb{R}^2$, $t \mapsto (0, \frac{\pi}{2} - t)$



$$\int_{\partial D} P\,dx + Q\,dy = \int_{\vec{c}_1} P\,dx + Q\,dy + \int_{\vec{c}_2} P\,dx + Q\,dy + \int_{\vec{c}_3} P\,dx + Q\,dy + \int_{\vec{c}_4} P\,dx + Q\,dy$$

$$\int_{\vec{c}_1} P\,dx + Q\,dy = \int_0^{\frac{\pi}{2}} (\sin t \cdot 1 + \cos 0 \cdot 0)\,dt = -\cos t\Big|_0^{\frac{\pi}{2}} = -(0-1) = 1$$

$$\int_{\vec{c}_2} P\,dx + Q\,dy = \int_0^{\frac{\pi}{2}} (\sin\frac{\pi}{2}\cdot 0 + \cos t \cdot 1)\,dt = \sin t\Big|_0^{\frac{\pi}{2}} = 1 - 0 = 1$$

$$\int_{\vec{c}_3} P\,dx + Q\,dy = \int_0^{\frac{\pi}{2}} (\sin(\frac{\pi}{2}-t)\cdot(-1) + \cos\frac{\pi}{2}\cdot 0)\,dt = -\int_0^{\frac{\pi}{2}}\cos t\,dt = -\sin t\Big|_0^{\frac{\pi}{2}} = -(1-0) = -1$$

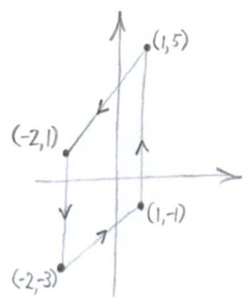
$$\int_{\vec{C}_4} P\,dx + Q\,dy = \int_0^{\frac{\pi}{2}} \left(\sin 0 \cdot 0 + \cos\left(\tfrac{\pi}{2} - t\right)(-1)\right) dt = \int_0^{\frac{\pi}{2}} -\sin t\,dt = \cos t \Big|_0^{\frac{\pi}{2}} = -1$$

$$\Rightarrow \int_{\partial D} P\,dx + Q\,dy = 1 + 1 - 1 - 1 = 0$$

Alternatively, let $f(x) = -\cos x + \sin y$. Then $\nabla f = P\vec{i} + Q\vec{j}$. Since ∂D is a simple closed curve, $\int_{\partial D} P\,dx + Q\,dy = 0$.

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_D (0 - 0)\,dx\,dy = 0.$$

$\boxed{3}$



Let D be the region bound by the quadrilateral.
The top edge of the quadrilateral is part of the line $y - 1 = \frac{4}{3}(x+2)$, i.e., $y = \frac{4}{3}x + \frac{11}{3}$. The bottom edge is part of the line $y + 1 = \frac{2}{3}(x-1)$, i.e., $y = \frac{2}{3}x - \frac{5}{3}$.

Let $P = 2xy$ and let $Q = xy^2$. Then by Green's Theorem,

$$\int_{C^+} 2xy\,dx + xy^2\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{-2}^{1} \int_{\frac{2}{3}x - \frac{5}{3}}^{\frac{4}{3}x + 11} (y^2 - 2x)\,dy\,dx$$

$$= \int_{-2}^{1} \left[\frac{1}{3}y^3 - 2xy\right]_{\frac{2}{3}x - \frac{5}{3}}^{\frac{4}{3}x + \frac{11}{3}} dx = \int_{-2}^{1} \left(\frac{1}{3}\left(\frac{4}{3}x + \frac{11}{3}\right)^3 - \frac{1}{3}\left(\frac{2}{3}x - \frac{5}{3}\right)^3 - 2x\left(\frac{4}{3}x + \frac{11}{3}\right) + 2x\left(\frac{2}{3}x - \frac{5}{3}\right)\right) dx$$

$$= 61$$