Problem Set 10 — Solutions

1. Parametrize ∂B by $\vec{\Phi}: [0,\pi]\times[0,2\pi] \to \mathbb{R}^3$, $(\varphi,\theta) \mapsto (\cos\theta \sin\varphi,\ \sin\theta \sin\varphi,\ \cos\varphi)$.

Then $\vec{T}_\varphi \times \vec{T}_\theta = (\cos\theta \sin^2\varphi,\ \sin\theta\sin^2\varphi,\ \sin\varphi\cos\varphi)$.

$$\iint_{\partial B} \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi} (-\sin\theta\sin\varphi,\ \cos\theta\sin\varphi,\ \cos\varphi)\cdot(\cos\theta\sin^2\varphi,\ \sin\theta\sin^2\varphi,\ \sin\varphi\cos\varphi)\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(-\sin\theta\cos\theta\sin^3\varphi + \sin\theta\cos\theta\sin^3\varphi + \sin\varphi\cos^2\varphi\right) d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin\varphi\cos^2\varphi\, d\varphi\, d\theta$$

$$= 2\pi\cdot\frac{-1}{3}\cos^3\varphi\Big|_0^{\pi} = \frac{4\pi}{3}$$

$$\iiint_B (\nabla\cdot\vec{F})\, dV = \iiint_B (0+0+1)\, dV = \iiint_B dV = \frac{4\pi}{3}$$

2.
$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_B (\nabla\cdot\vec{F})\, dV = 3\iiint_B (x^2+y^2+z^2)\, dx\, dy\, dz \overset{\text{switch to spherical coordinates}}{=} 3\int_0^1\int_0^{2\pi}\int_0^{\pi} r^2(r^2\sin\varphi)\, d\varphi\, d\theta\, dr$$

$$= 6\pi\int_0^1 r^4\, dr\int_0^{\pi}\sin\varphi\, d\varphi = 6\pi\left(\frac{1}{5}\right)\left(-\cos\varphi\Big|_0^{\pi}\right) = \frac{12\pi}{5}$$

3. $\nabla\cdot\vec{F} = 3+(-2)+8y = 1+8y$

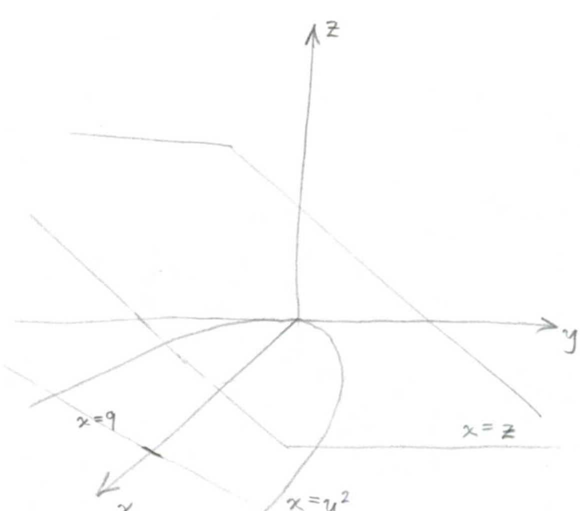
$$\iint_{\partial W} \vec{F}\cdot d\vec{S} = \iiint_W (\nabla\cdot\vec{F})\, dV$$

$$= \int_{x=0}^{x=9}\int_{z=0}^{z=x}\int_{y=-\sqrt{x}}^{y=\sqrt{x}} (1+8y)\, dy\, dz\, dx$$

$$= \int_0^9\int_0^x \left[y+4y^2\right]_{-\sqrt{x}}^{\sqrt{x}}\, dz\, dx$$

$$= \int_0^9\int_0^x (2\sqrt{x})\, dz\, dx$$

$$= \int_0^9 (2x^{3/2})\, dx = \frac{2}{5}\cdot 2\cdot 9^{5/2} = \frac{972}{5}$$



$\boxed{4}$ $\nabla \cdot \vec{F} = 0 + 0 + (x^2+y^2)^2$

$$\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iiint_W (\nabla \cdot \vec{F})\, dV = \iiint_W (x^2+y^2)^2\, dx\, dy\, dz = \int_0^1 \int_0^{2\pi} \int_0^1 r^{2^2}\, r\, dr\, d\theta\, dz = 2\pi \int_0^1 r^5\, dr = \frac{\pi}{3}$$