Regression and Simulation Methods

Week 5: Inference with Likelihood

Chris Oldnall, 7th November 2023

What do we need to consider with likelihood?

- How do we perform some sort of confidence procedure for any likelihood function?
- What are some general properties of the likelihood?

Confidence Intervals

The confidence interval in general: 1D

- The Wilks Interval

$$2\left[l(\hat{\theta}) - l\left(\theta_0\right)\right] \sim \chi_1^2$$

$$\mathbb{P}\left\{2\left[l(\hat{\theta}) - l\left(\theta_0\right)\right] \leq \chi_1^2(c)\right\} \approx c$$

$$\left\{\theta : r(\theta) \geq -\frac{1}{2}\chi_1^2(c)\right\}$$

The confidence interval in general: 1D

- The Wald Interval

$$\hat{\theta} \sim N\left(\theta_0, 1/k(\underline{x})\right)$$

$$(\hat{\theta} - z\sqrt{1/k(\underline{x})}, \hat{\theta} + z\sqrt{1/k(\underline{x})})\,.$$

The confidence interval in general: K-D

- The Wilks Interval

$$2\left[l(\hat{\theta}) - l\left(\theta_0\right)\right] \sim \chi_k^2$$

$$\mathbb{P}\left\{2\left[l(\hat{\theta}) - l\left(\theta_0\right)\right] \leq \chi_k^2(c)\right\} \approx c$$

$$\left\{\theta : r(\theta) \geq -\frac{1}{2}\chi_k^2(c)\right\}$$

The confidence interval in general: K-D

- The Wald Interval

$$l(\boldsymbol{\theta}) \simeq l(\hat{\boldsymbol{\theta}}) + (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^\top \mathbf{g} + \frac{1}{2}(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^\top \mathbf{H}(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})$$

$$l(\boldsymbol{\theta}) \simeq l(\hat{\boldsymbol{\theta}}) + \frac{1}{2}(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^\top \mathbf{H}(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) \quad \text{or} \quad r(\boldsymbol{\theta}) \simeq \frac{1}{2}(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^\top \mathbf{H}(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})$$

$$\left\{\boldsymbol{\theta} : \frac{1}{2}(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^\top \mathbf{H}(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) \geq -\frac{1}{2}\chi_k^2(c)\right\} \quad \text{or} \quad \left\{\boldsymbol{\theta} : (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^\top \mathbf{K}(\underline{\mathbf{x}})(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) \leq \chi_k^2(c)\right\}$$

Other Properties

Invariance

$$\beta = g(\theta) \iff \hat{\beta} = g(\hat{\theta})$$

Property 1 of Expected Likelihood: Expectation

$$E_0\left(\left.\frac{\partial l}{\partial \theta}\right|_{\theta_0}\right) = 0$$

Property 2 of Expected Likelihood: Variance

$$\operatorname{var}\left(\left.\frac{\partial l}{\partial \theta}\right|_{\theta_0}\right) = E_0\left[\left(\left.\frac{\partial l}{\partial \theta}\right|_{\theta_0}\right)^2\right]$$

Property 3 of Expected Likelihood: Fisher Information

$$\mathscr{I} \equiv E_0 \left[\left(\left. \frac{\partial l}{\partial \theta} \right|_{\theta_0} \right)^2 \right] = -E_0 \left[\left. \frac{\partial^2 l}{\partial \theta^2} \right|_{\theta_0} \right]$$

Property 4 of Expected Likelihood: Global Maximum

$$E_0\left[l\left(\theta_0\right)\right] \geq E_0[l(\theta)] \quad \forall \theta$$

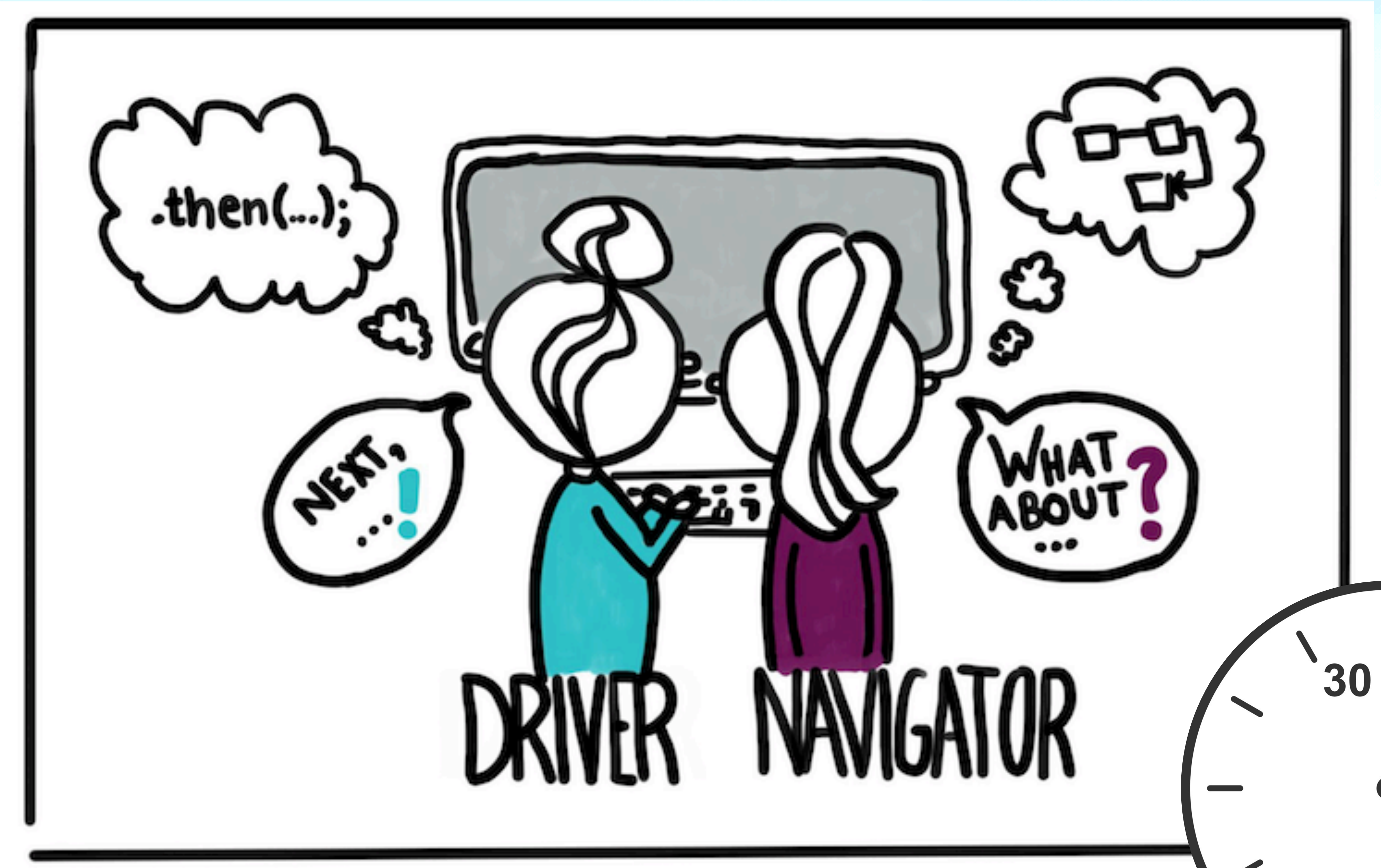
Consistency

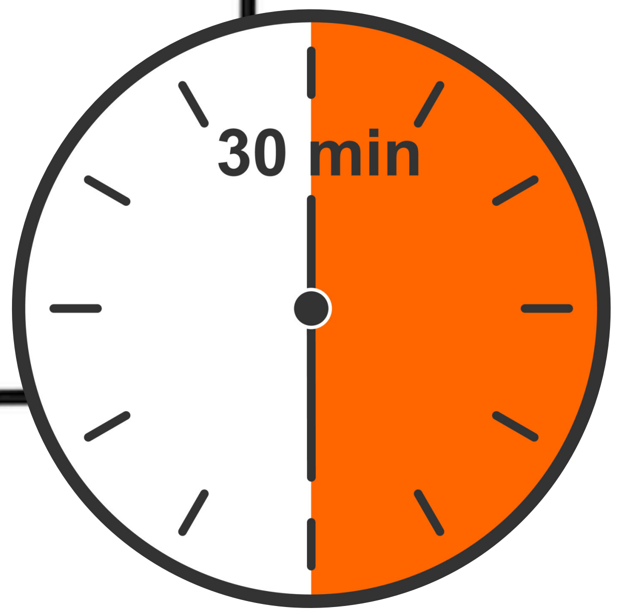
$$\mathbb{P}\left[\left|\hat{\theta}_n - \theta_0\right| < \epsilon\right] \to 1$$

As n tends to infinity.

Rest of the tutorial...

- In pairs work on the fifth notebook.
  Paired programming will continue!





Rest of the week…

The notebook is quite long and will require lots of thinking this week. If you finish it consider if there's any of the properties you might want to test from the scenario.

Otherwise consider completing exercise 5-1 from the SMSTC exercise sheet.