Grassmann Manifolds

February 24, 2025

Broad Outline for today

- Real Grassmann Manifolds
- First Applications

Broad goal for the course:

The interplay between geometry, dimension, volume, and randomness.

How will we explore this interplay?

Through experimentation, computation, and applications.

First things first

Real Projective space: $\mathbb{RP}^n$ equals S^n modulo antipodal points.

Grassmann manifolds are a generalization of Projective space.

Real Grassmann manifolds have models as homogeneous spaces via "quotients" of the matrix group $O(n)$.

We will focus on such a quotient model today.

In order to get to applications quickly, today will be an overview.

Additional reading on Grassmannians is left to the student.

The Orthogonal group $O(n)$

Let $O(n) = \{A \in \mathbb{R}^{n\times n} \mid A^T A = I\}$

Let $SO(n) = \{A \in \mathbb{R}^{n\times n} \mid A^T A = I \text{ and } det(A) = 1\}$

$O(n)$ and $SO(n)$ are compact real Lie groups of dimension $\frac{n(n-1)}{2}$.

Grassmann manifolds

Let V be a vector space. The Grassmannian, $Gr(k, V)$, denotes the collection of k-dimensional subspaces of V.

If V is n-dimensional then we write $Gr(k, n)$ for $Gr(k, V)$.

There are many different constructions for $Gr(k, n)$. Read the wikipedia page on **Grassmannians** for several examples.

Today, we assume that $V = \mathbb{R}^n$.

We can express $Gr(k, n)$ as a homogeneous space:

$$Gr(k, n) = O(n)/O(k) \times O(n-k)$$

Grassmannian revisited

Let $Q \in O(n)$. When we write $Gr(k,n) = O(n)/O(k) \times O(n-k)$, we identify points in $Gr(k,n)$ with equivalence classes $[Q]$ where

$$[Q] = \left\{ Q \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \mid M_1 \in O(k),\ M_2 \in O(n-k) \right\}.$$

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We can also express $Gr(k,n)$ in terms of $SO(n)$:

$Gr(k,n) = SO(n)/S(O(k) \times O(n-k))$.

Grassmannian revisited

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$$[Q] = \left\{ Q \left[\begin{array}{cc} M_1 & 0 \\ 0 & M_2 \end{array} \right] \mid M_1 \in O(k),\ M_2 \in O(n-k) \right\}.$$

We can also express $Gr(k,n)$ in terms of $SO(n)$:

$Gr(k,n) = SO(n)/S(O(k) \times O(n-k))$.

Note that $S(O(k) \times O(n-k))$ means that $\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \in SO(n)$ but $M_1 \in O(k),\ M_2 \in O(n-k)$.

Oriented Grassmannian

The Grassmannian is

$$Gr(k,n) = O(n)/O(k) \times O(n-k)$$
$$= SO(n)/S(O(k) \times O(n-k))$$

The oriented Grassmannian is

$$Gr(k,n)^\circ = SO(n)/SO(k) \times SO(n-k).$$

$Gr(k,n)^\circ$ is a 2 : 1 cover of $Gr(k,n)$.

$Gr(k,n) = SO(n)/S(O(k) \times O(n-k))$

$Gr(k,n)^{or} = SO(n)/SO(k) \times SO(n-k)$

$Gr(k,n)^{or}$ is a $2:1$ cover of $Gr(k,n)$.

$\mathbb{RP}^{n-1} = Gr(1,n) = SO(n)/S(O(1) \times O(n-1))$

$S^{n-1} = Gr(1,n)^{or} = SO(n)/SO(1) \times SO(n-1)$

The sphere is a $2:1$ cover of projective space.

Dimension of a Grassmann manifold

The dimension of $O(n)$ is

$$\dim(O(n)) = (n-1) + (n-2) + \cdots + 1 = \frac{n(n-1)}{2}$$

The dimension of $O(n)/O(k) \times O(n-k)$ is

$$\dim(O(n)) - \dim(O(k)) - \dim(O(n-k))$$

Example: $\dim(Gr(2,5)) = 10 - 3 - 1 = 6$

Example: $\dim(Gr(k,n)) = k(n-k)$

$\dim(Gr(k,n))^\circ = \dim(Gr(k,n))$

Volume of Spheres and Balls

Let B^n denote the unit n-ball and let S^n denote the unit n-sphere.

$$Vol(B^n, r) = \frac{2\pi^{n/2}}{n\Gamma(n/2)} r^n$$

$$Vol(S^{n-1}, r) = \frac{2\pi^{n/2}}{\Gamma(n/2)} r^{n-1}$$

$\Gamma(1/2) = \sqrt{\pi} \quad \Gamma(1) = 1 \quad \Gamma(n+1) = n\Gamma(n)$

If n is an integer then $\Gamma(n+1) = n!$

Volume of Spheres

$$\text{Volume of } S^n: 2,\ \ 2\pi,\ 4\pi,\ \ 2\pi^2,\ \frac{8}{3}\pi^2,\ \ \pi^3,\ \frac{16}{15}\pi^3, \ldots$$

$$Vol(S^{2n-1}) = \frac{2\pi^n}{(n-1)!}$$

$$Vol(S^{99}) \approx 2.3 \times 10^{-38}$$

Probability of a point in a box landing inside the inscribed sphere is 1.8×10^{-68}

Volume of O(n) and Gr(k,n)

$$Vol(O(n)) = \prod_{k=0}^{n-1} Vol(S^k)$$

$$Vol(Gr(k,n)) = \frac{Vol(O(n))}{Vol(O(k)) \cdot Vol(O(n-k))}$$

Volumes of S^n: $2,\ \ 2\pi,\ 4\pi,\ \ 2\pi^2,\ \frac{8}{3}\pi^2, \ldots$

Volumes of $O(n)$: $2,\ \ 4\pi,\ \ 16\pi^2,\ \ 32\pi^4,\ \ \frac{256}{3}\pi^6, \ldots$

$Gr(2,5) = O(5)/O(2) \times O(3)$

$Vol(Gr(2,5)) = \frac{4}{3}\pi^3 \qquad Vol(Gr(2,5))^{\circ} = \frac{8}{3}\pi^3$

Principal angles between subspaces

Let U and V be subspaces of $\mathbb{R}^n$.

Let $k = \min\{\dim(U), \dim(V)\}$.

Between U and V there exist:

- Angles, $\theta_1, \ldots, \theta_k$, between U, V with $|\theta_1| \leq \cdots \leq |\theta_k| \leq \frac{\pi}{2}$
- Orthonormal vectors $u_1, \ldots, u_k$ with $u_i \in U$
- Orthonormal vector $v_1, \ldots, v_k$ with $v_k \in V$

with the relationship that $u_i \cdot v_i = \cos(\theta_i)$.

θ_i is called the i^{th} principal angle between U and V.

u_i, v_i are the i^{th} principal vectors.

Recursive definition of principal angles and principal vectors

$$\theta_1 = \min_{\substack{u \in U, v \in V \\ ||u||=||v||=1}} \{\cos^{-1}(u \cdot v)\}$$

$$u_1, v_1 = \operatorname*{arg\,min}_{\substack{u \in U, v \in V \\ ||u||=||v||=1}} \{\cos^{-1}(u \cdot v)\}$$

Now define θ_i recursively by

$$\theta_i = \min_{\substack{u \in U, v \in V \\ ||u||=||v||=1}} \{\cos^{-1}(u \cdot v) \mid u \perp u_j, v \perp v_j \text{ for } j < i\}$$

$$u_i, v_i = \operatorname*{arg\,min}_{\substack{u \in U, v \in V \\ ||u||=||v||=1}} \{\cos^{-1}(u \cdot v) \mid u \perp u_j, v \perp v_j \text{ for } j < i\}$$

SVD method for principal angles between subspaces

Let A and B be $n \times k$ matrices with orthonormal columns. We will denote the column space of A by $[A]$.

$[A]$ and $[B]$ correspond to points on $Gr(k, n)$.

Let θ_i denote the i^{th} principal angle between $[A]$ and $[B]$.

Consider the singular value decomposition $A^T B = U \Sigma V^T$.

We have $[AU] = [A], \;\; [BV] = [B], \;\; (AU)^T (BV) = \Sigma$.

Theorem: $\Sigma_{i,i} = \cos(\theta_i) \quad$ and $\quad u_i, v_i = i^{th}$ column of AU, BV

Orthogonally invariant distances on Grassmann manifolds

An orthogonally invariant distance measure between subspaces can be expressed in terms of principal angles between the subspaces.

Let $\Theta(U, V) = (\theta_1, \ldots, \theta_k)$.

- $d(U, V) = ||\Theta(U, V)||_2$ is the geodesic distance
- $d(U, V) = 2||\sin(\Theta(U, V))||_2$ is the projection distance
- $d(U, V) = \cos^{-1}(\prod_i \cos(\theta_i))$ is the Fubini-Study distance

We will focus today on the geodesic distance.

Geodesics on Grassmann manifolds

Two perspectives that will now consider are:

1) Geodesics via the CS decomposition.

2) Geodesics via exponentiation of a skew symmetric matrix.

The CS decomposition

Let $A, B \in O(n)$. Denote the first k columns of A, B by $\underline{A}, \underline{B}$ and the last $n-k$ columns by $\overline{A}, \overline{B}$ (assume $n-2k \geq 0$).

From the SVD: $\underline{A}^T\underline{B} = \underline{U}\ \underline{\Sigma}\ \underline{V}^T$ and $\overline{A}^T\overline{B} = \overline{U}\ \overline{\Sigma}\ \overline{V}^T$.

Let $\mathbf{A} = \underline{A}\ \underline{U}|\overline{A}\ \overline{U}$ and $\mathbf{B} = \underline{B}\ \underline{V}|\overline{B}\ \overline{V}$ then

$$\mathbf{A}^\mathbf{T}\mathbf{B} = \begin{bmatrix} C & 0 & -S \\ 0 & I & 0 \\ S & 0 & C \end{bmatrix}$$

where

$$C = Diag(Cos(\Theta)) \qquad S = Diag(Sin(\Theta)) \qquad I = I_{n-2k}$$

Parameterizing the CS decomposition

$$\mathbf{A}^{\mathbf{T}}\mathbf{B} = \begin{bmatrix} C & 0 & -S \\ 0 & I & 0 \\ S & 0 & C \end{bmatrix}. \text{ Let } Q_t = \begin{bmatrix} C_t & 0 & -S_t \\ 0 & I & 0 \\ S_t & 0 & C_t \end{bmatrix}$$

$$\text{where } \; C_t = Diag(Cos(\Theta t)) \;\; S_t = Diag(Sin(\Theta t))$$

then $\mathbf{A}Q_t$ parametrizes a family of orthogonal matrices that start at $\mathbf{A}$ when $t = 0$ and end at $\mathbf{B}$ when $t = 1$.

This is a geodesic path on $SO(n)$. The first k columns of $\mathbf{A}Q_t$ gives a geodesic path between points on $Gr(k, n)$.

Geodesics on $O(n)$ through skew symmetric matrices

Let A be a square matrix with $A^T = -A$ (A is a skew-symmetric).

An exercise shows that $\exp(A) \in O(n)$.

$\exp(At)$ gives a geodesic on $O(n)$ from $\exp(A \cdot 0)$ to $\exp(A \cdot 1)$.

Since this path is connected, we see that $\exp(A) \in SO(n)$.

What is the relation between the geodesics via the CS decomposition and geodesics via exponentiation?

CS decomposition recap

From an earlier slide, we started with $A, B \in O(n)$ and we choose a k satisfying $n - 2k \geq 0$.

We constructed $\mathbf{A}, \mathbf{B}$ with the property that

$$\mathbf{A}^{\mathbf{T}}\mathbf{B} = \begin{bmatrix} C & 0 & -S \\ 0 & I & 0 \\ S & 0 & C \end{bmatrix}$$

$$C = Diag(Cos(\Theta)) \qquad S = Diag(Sin(\Theta)) \qquad I = I_{n-2k}$$

Θ is the vector of (signed) principal angles between the first k columns of A and the first k columns of B.

CS decomposition and Skew symmetric matrices

We have

$$\exp\left(\begin{bmatrix} 0 & 0 & -\Theta \\ 0 & 0 & 0 \\ \Theta & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} C & 0 & -S \\ 0 & I & 0 \\ S & 0 & C \end{bmatrix} = \mathbf{A}^{\mathbf{T}}\mathbf{B}$$

where

$$C = Diag(Cos(\Theta)) \qquad S = Diag(Sin(\Theta)) \qquad I = I_{n-2k}$$

and $\Theta = \arcsin(Sin(\Theta))$ is the vector of signed principal angles between the first k columns of A and the first k columns of B.

Skew symmetric matrices and Geodesic distance

$$\mathbf{A^T B} = \begin{bmatrix} C & 0 & -S \\ 0 & I & 0 \\ S & 0 & C \end{bmatrix} \qquad \Theta = \arcsin(Sin(\Theta))$$

$$\exp(\begin{bmatrix} 0 & 0 & -\Theta \\ 0 & 0 & 0 \\ \Theta & 0 & 0 \end{bmatrix}) = \begin{bmatrix} C & 0 & -S \\ 0 & I & 0 \\ S & 0 & C \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 0 & -\Theta \\ 0 & 0 & 0 \\ \Theta & 0 & 0 \end{bmatrix} \implies ||\Theta||_2 = \sqrt{\frac{1}{2} Trace(M^T M)}.$$

Now a look at some applications

The next part will focus on applications to digital images.

How do ordered bases show up in data analysis?

Four common ways are:

- Singular Value Decomposition.
- Wavelets and related decompositions.
- Osculating spaces to manifolds.
- Approximations via an ordered basis (e.g. Spherical Harmonics).

Ordered bases lead to subspaces (and flags).

Geometric/Topological Data Analysis?

Geometric/Topological Data analysis is the application of tools from geometry/topology to elucidate structure in large data sets.

The following slides describe some examples in the context of digital photographs.

Black and white images as rectangular arrays

A black and white digital picture is stored as a matrix of numbers.

Each number in the matrix corresponds to a pixel and is a measure of the energy arriving from a corresponding portion of the image.

Frequently, the numbers lie between 0 and 255.

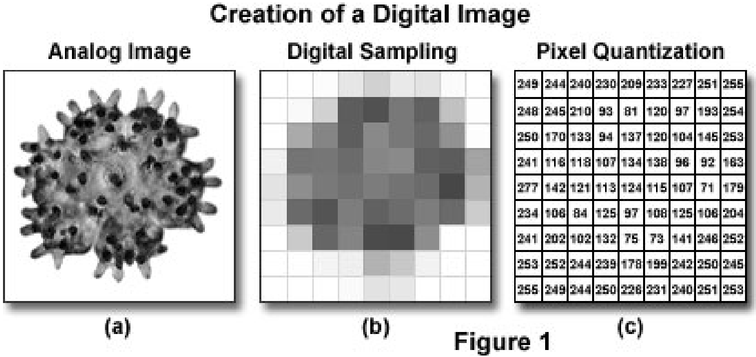


Figure 1

http://micro.magnet.fsu.edu/primer/digitalimaging/digitalimagebasics.html

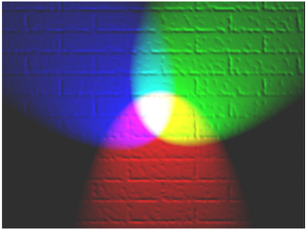
Color images as rectangular hyper-arrays

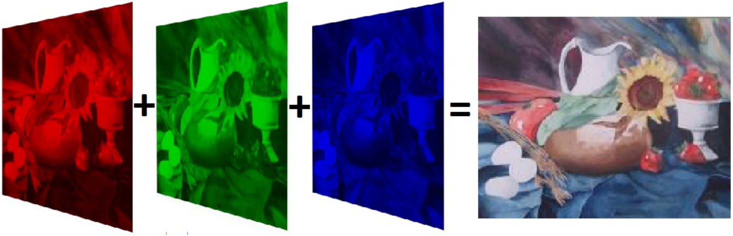
In a digital camera, a color image is typically stored as a three sheeted rectangular array of numbers.

Each sheet in the array corresponds to energy arriving in a particular color band. In many cameras, these sheets are the red, green, and blue contributions.

We can visualize each sheet as a black and white photograph or as a monochromatic image.

By combining basic colors, we can produce a wider range of colors.





Color is really a function of wavelength

We think of color as being composed of three basic colors. We think this way because this is how (most) humans are built. But colors are much richer than what we can see.

Humans are typically tri-chromats. We see three dimensions worth of color.

Some people are bi-chromats (color blind) and see two dimensions worth of color. There are even mono-chromats (who only see in "black and white").

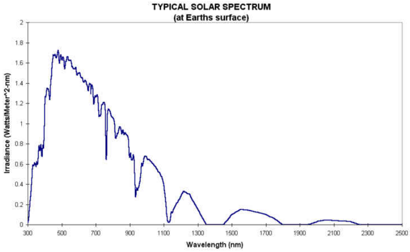
In the other direction, there are some rare people who are tetra-chromats. They see a richer collection of colors.

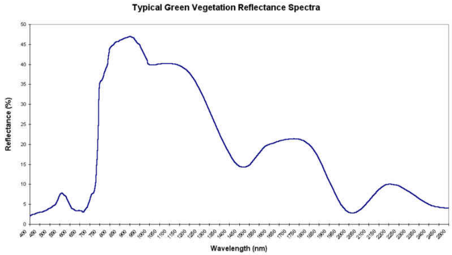
The color you see depends on what wavelengths are being reflected by an object.

These reflected wavelengths, in turn, depend on what wavelengths are illuminating an object.

Colors form an infinite dimensional cone; the cone of non-negative functions on an interval. Humans pick a particular three dimensional projection.

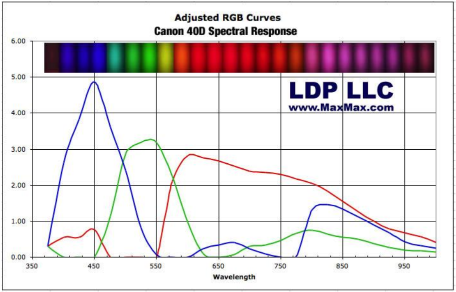
http://www.cropscan.com/response.html



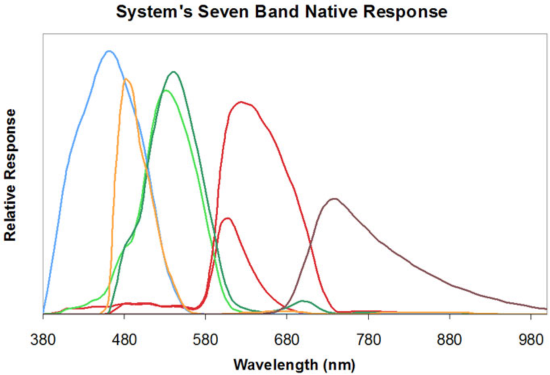


Intensity is a function of wavelength, $I(\lambda)$. A pixel value (roughly) corresponds to an integral of $I(\lambda)$ multiplied against a frequency response function, $F(\lambda)$.

In other words: Pixel Value $= \int_\lambda I(\lambda) \cdot F(\lambda)\, \mathrm{d}\lambda$.



Some cameras can collect more samples of the spectral curve by integrating against more frequency response curves. Below are some curves from a higher end multi-spectral camera.



http://www.fluxdata.com/products/high-resolution-3-ccd-multispectral-camera

Most digital cameras have the ability to collect data from outside the visible spectrum (but dampen this ability with filters).

It can be surprising when you “shift” data outside the visible spectrum to lie in the visible spectrum.

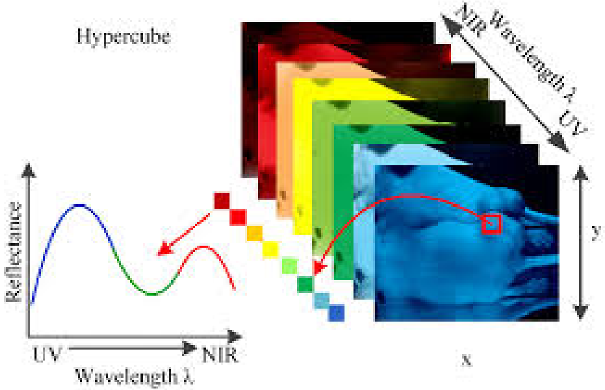
Below is a flower imaged in the visible spectrum and the same flower imaged in the UV-spectrum then ”shifted” so that we may view the UV information.

http://www.naturfotograf.com

You are already familiar with this idea of “shifting” non-visible colors into the realm of visible colors if you are a fan of astronomy.

Orion Nebula as imaged by the Hubble space telescope

https://hubblesite.org/contents/media/images/2006/01/1826-Image.html



https://link.springer.com/chapter/10.1007/978-3-030-03000-1_16/figures/1

Variability and parameter spaces

Variations of state

Geometric variations of state: $SO(n), O(n), \mathbb{R}^n, E(n), SE(n)$

Other variations of state: identity, illumination, etc

Parameter spaces

Parameter spaces useful for data collected under variations:

Grassmann, Veronese, and Segre Manifolds

Secant Varieties, Flag and Siefel Manifolds

How to involve data analysis with these parameter spaces?

One way to get data to lie on one of these parameter spaces is through an application of the Singular Value Decomposition.

This is a method of capturing as much "Energy" as possible in a data set in an efficient manner.

By considering a photograph as a matrix, you can apply linear algebra techniques such as the **Singular Value Decomposition:**

$$M = U\ \Sigma\ V^T$$

where:

U is an orthogonal matrix,

Σ is a monotone decreasing non-negative "diagonal" matrix,

V^* is an orthogonal matrix.

Columns of $U \quad \leftrightarrow \quad$ eigenvectors of $M\ M^T$.

Columns of $V \quad \leftrightarrow \quad$ eigenvectors of $M^T\ M$.

Diagonal entries in $\Sigma \quad \leftrightarrow \quad$ square roots of eigenvalues of $M\ M^T$

There is a great deal of geometric information that can be extracted from the SVD.

For instance, finding the closest rank r matrix to M.

It lets you find a rank r matrix M_r that minimizes $||M - M_r||$.

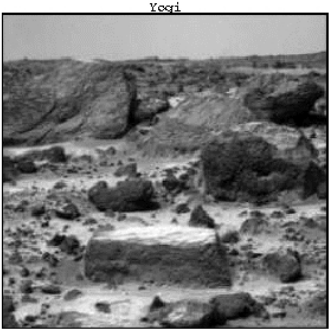
In particular,

$$M_r = U\ \Sigma_r\ V^T.$$

If M is square, the closest orthogonal matrix is UV^T

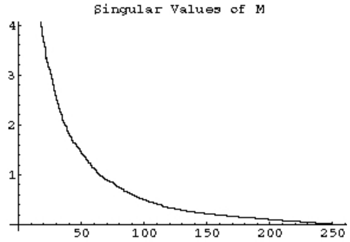
Low Rank Image: Image Compression

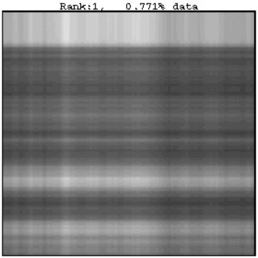
Picture sent back from Mars: Resolution is 256 x 264



http://www.uwlax.edu/faculty/will/svd/compression/index.html

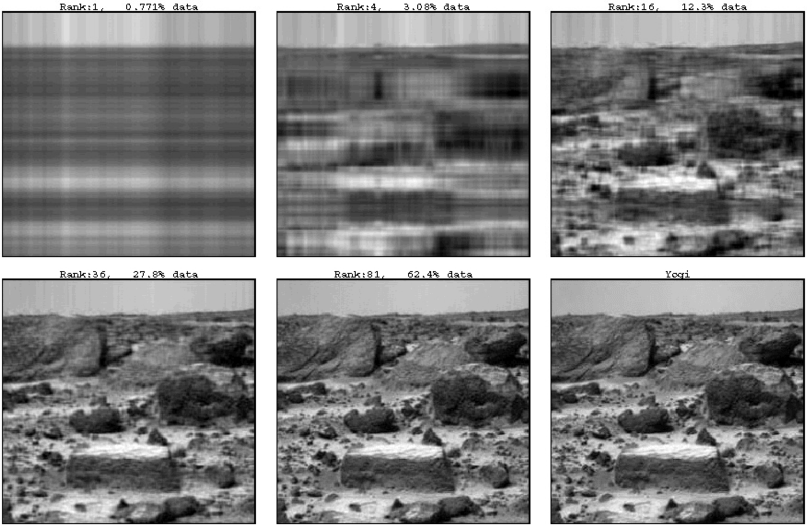
Matrix has rank 256, plot of singular values of normalized matrix





$$= \sigma_1 U_1 V_1^*$$

Original picture $= \sigma_1 U_1 V_1^* + \sigma_2 U_2 V_2^* + \cdots + \sigma_k U_k V_k^* + \dots$


Rank:1, 0.771% data
Rank:4, 3.08% data
Rank:16, 12.3% data
Rank:36, 27.8% data
Rank:81, 62.4% data
Yogi

SVD, Segre Varieties, Secant Varieties

The Segre variety $S(a, b)$ is (essentially) the locus of outer products of unit length vectors with a entries and b entries. Its points parameterize rank one $a \times b$ matrices modulo scalars.

The singular value decomposition expresses a picture as a linear combination of points on a Segre Variety.

Original picture $= \sigma_1 U_1 V_1^T + \sigma_2 U_2 V_2^T + \cdots + \sigma_k U_k V_k^T + \ldots$

A rank r approximation of a picture represents the picture as a point on an r-secant variety to a Segre variety.

The matrices $U_1 V_1^T, U_2 V_2^T, U_3 V_3^T, \ldots$ are mutually orthogonal.

Digital images as vectors

A black and white digital picture determines a point in $\mathbb{R}^{r\times c}$.

By "flattening" the matrix, we obtain a point in $\mathbb{R}^{rc\times 1}$.

Here a 2×3 matrix is flattened to a 6×1 matrix.

$$\begin{bmatrix} 1 & 3 & 17 \\ 2 & 6 & 19 \end{bmatrix} \to \begin{bmatrix} 1 \\ 2 \\ 3 \\ 6 \\ 17 \\ 19 \end{bmatrix}. \text{ Another flattening is } \begin{bmatrix} 1 \\ 3 \\ 17 \\ 2 \\ 6 \\ 19 \end{bmatrix}.$$

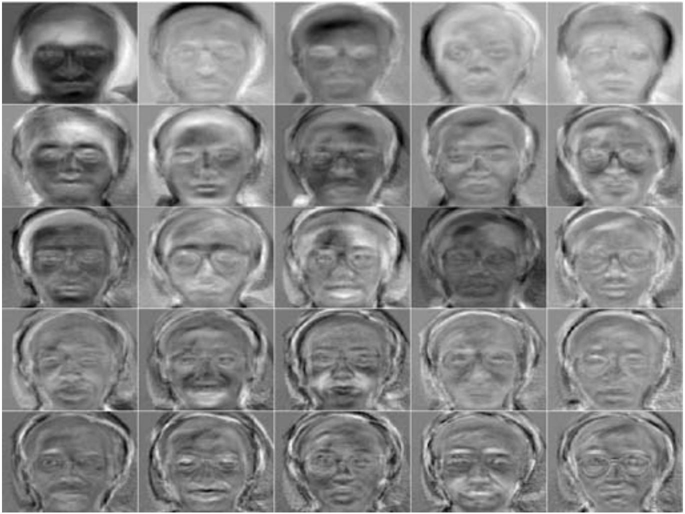
Eigenfaces

Consider a matrix, M, whose columns are flattened photographs.

From the SVD: $M = U \Sigma V^T$, the columns of U give an ordered orthogonal basis for the column space of M.

If the columns of M have some correlated property then this property is reflected in the columns of U.

For instance, if the columns of M are faces, we get "Eigenfaces".



These pictures are an ordered orthonormal basis spanning an approximation of "facespace".

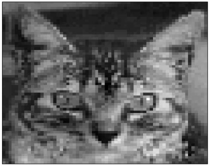
Any photograph can be written as a linear combination of these pictures plus a "residual" in the orthogonal complement of the span of the eigenfaces.

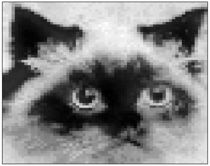
If the photograph is a face, one expects the residual to be small.

To a face we can attach a coordinate tag and measure the "novelty" or length of the residual.

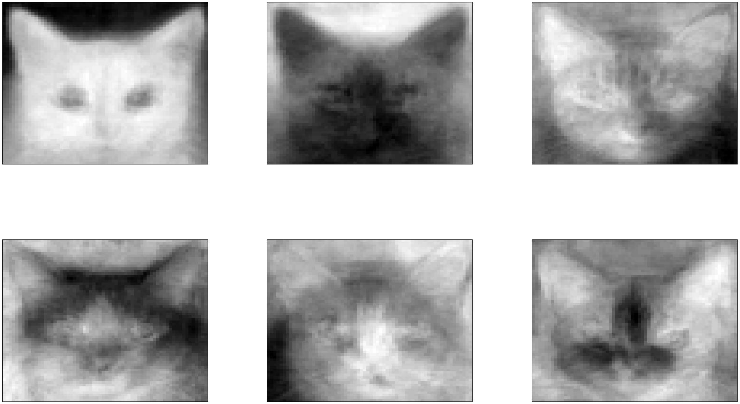
Of course the same process can be carried out with other data sets.

Collection of cats

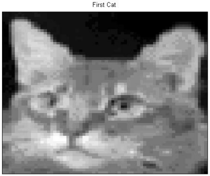


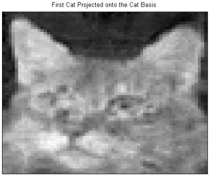


Eigencats



Eigencats and Projection

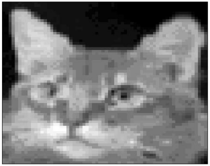




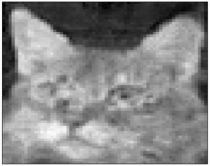
Novelty$= 5.83 \times 10^{-12}$

Eigencats and Projection

First Cat



First Cat Projected onto the Cat Basis



Novelty$= 5.83 \times 10^{-12}$

Adrienne Projected onto the Cat Basis

Residual of Adrienne

Novelty$= 1.8781 \times 10^{3}$

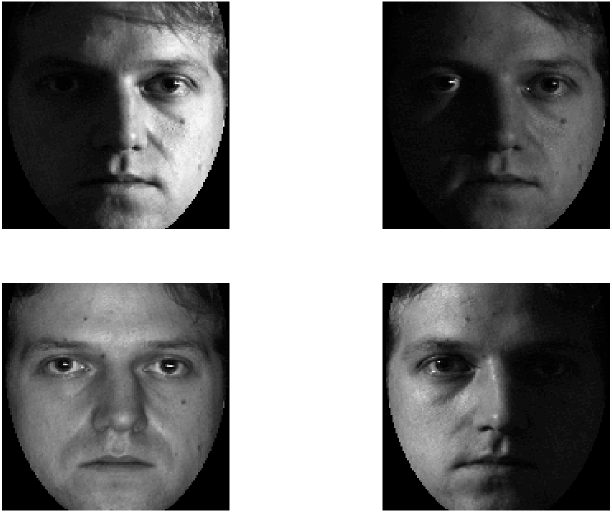
A Grassmann example: Illumination space

Consider the "illumination space" of an object.

This means the span of a collection of images of a fixed object under varying illumination.

As a specific example, consider the illumination space of a person.

Images of a person under different illumination conditions



Illumination images form a convex set

If $\mathbf{A}, \mathbf{B}$ are vectors representing two images of the same object under two different illumination conditions, then any convex combination of $\mathbf{A}$ and $\mathbf{B}$ represents the object under a convex combination of these lighting conditions.

Consequence: the set of images of a fixed object collected under varying illumination conditions form a convex set.

I.e. any weighted "average" of two such images is a valid image.

An average illumination image.

The picture in the middle is artificial. It was created by adding the matrices and dividing by 8.

Singular values of an illumination data matrix

Basri and Jacobs (2000) showed that a collection of digital photos of a convex Lambertian object under distant lighting lies close to a 9D linear subspace.

An implication is that the illumination space of many objects is well approximated by a low-dimensional linear space.

If the columns of a matrix consist of images of a fixed object taken under a wide variety of illumination conditions then the singular values of the matrix decay rapidly.

Objects which do not reflect light well are Lambertian, they have the effect of smoothing (rugs and fur are very Lambertian, human faces are somewhat Lambertian).

Objects which reflect light very well are non-Lambertian (mirror disco balls and water are non-Lambertian).

Non-Lambertian images - slower decay in singular values

Illumination data matrices of non-Lambertian objects have a slower decay of singular values.

They are less well approximated by a 9-dimensional space; it takes more dimensions to capture their illumination variance.

The Grassmann manifold $Gr(k, n)$ parameterizes the k-dimensional subspaces of a fixed n-dimensional vector space.

So, a point on Gr(k,n) corresponds to a k-dimensional subspace.

The singular values of an illumination data matrix decay rapidly.

Thus there is a low dimensional linear space that captures the majority of the energy of an illumination data matrix.

Using this geometry of the Grassmannian, we can tell illumination spaces apart by their low dimensional illumination space.

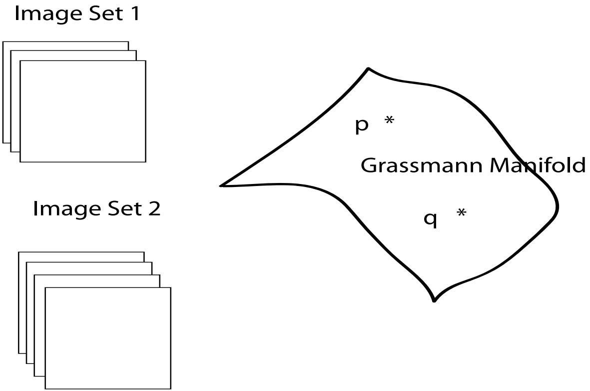


Image Set 1
Image Set 2
p *
Grassmann Manifold
q *

The CMU-PIE data base is a database (developed at Carnegie-Mellon University) containing images of people with variations in **I**llumination, **P**ose and **E**xpression.

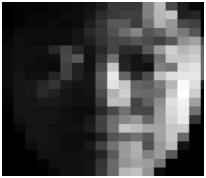
An experiment can be run using images in the CMU-PIE database with a variation only in Illumination:

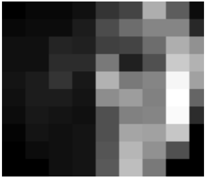
There are 67 subjects in the CMU-PIE database with lots of illumination pictures of each.

Selecting 9 images at random, their spans produce 9-dimensional estimates of the illumination space for the subject.

One can compare these proxy spaces with other proxy spaces.

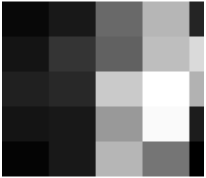
Harr Wavelet

Application of the 2D Haar wavelet transform.

The resulting LL image is displayed at each level.

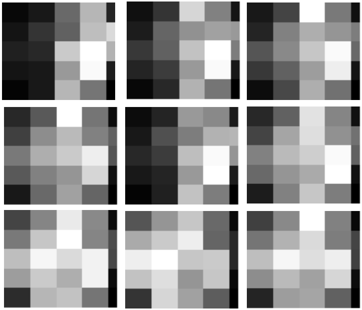
25 pixels

This transform mimics a low resolution camera.

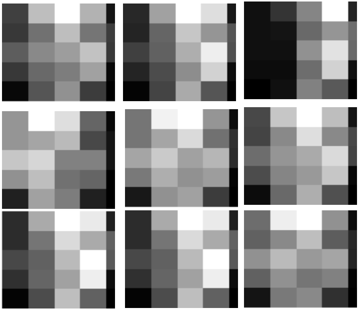
Under drastic resolution reduction, the CMU-PIE data base is still Grassmann separable.

9 dimensional subspaces in 25 dimensions is the smallest case that still worked

Low resolution illumination spaces

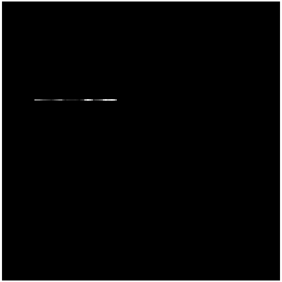


Subject A

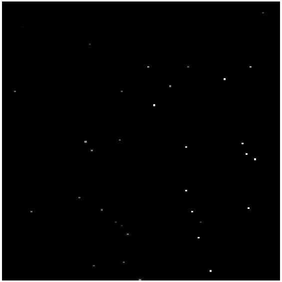


Subject B

Sampling of a strip of pixels



Sampling of a random set of pixels



Some additional applications of Grassmann manifolds

- Random polygons
- Projections of hypercubes, cross polytopes, and simplexes
- Euclidean Distance Matrices
- Morphing between Discrete Daubechies Wavelets

Random Polygons in $\mathbb{R}^2$

Let a, b be a pair of orthonormal vectors in $\mathbb{R}^n$.

Consider $c = a + bi \in \mathbb{C}^n$ and let $d = c \odot c$.

Write $d = e + fi$ with $e, f \in \mathbb{R}^n$ then form the matrix U whose columns are e and f.

The rows of U are a collection of vectors in $\mathbb{R}^2$ whose total length is 2 and whose sum is the zero vector.

This ordered set of vectors can be used to build a polygon in $\mathbb{R}^2$.

Random Polygons in $\mathbb{R}^3$

Let a, b be a pair of orthonormal vectors in $\mathbb{C}^n$.

Consider $c = a + bj \in \mathbb{Q}^n$ and let $d = \bar{c}i \odot c$.

d is a quaternionic vector with no real component.

Write $d = ei + fj + gk$ with $e, f, g \in \mathbb{R}^n$ then form the matrix U whose columns are e, f and g.

The rows of U are a collection of vectors in $\mathbb{R}^3$ whose total length is 2 and whose sum is the zero vector.

This ordered set of vectors can be used to build a polygon in $\mathbb{R}^3$.

Zonotopes

A zonotope is the projection of a hypercube.

Up to translation and scaling, a zonotope is the projection of a hypercube whose vertices consist of all ± 1 n-tuples

What does a "typical" zonotope in $\mathbb{R}^2$ and $\mathbb{R}^3$ look like?

How many vertices and faces does one expect?

What is the expected volume?

What about the projection of a high dimensional hypercube?

What about the projection of the dual of a hypercube (cross polytope)?

Polytopes

A polytope is the convex hull of a set of points in $\mathbb{R}^n$.

Non degenerate polytopes are projections of a regular simplex.

Up to translation and scaling, a polytope is the projection of the standard regular simplex.

The standard regular simplex can be made by taking the convex hull of the standard basis vectors then shifting by their average.

What does a "typical" projection to $\mathbb{R}^2$ and $\mathbb{R}^3$ look like?

How many vertices and faces does one expect? What is the expected volume?

What about the projection of a high dimensional standard simplex?