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CS109

Section #1
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Section #1 Solutions

1. a. This problem is much easier to solve if you try and calculate the probability that there are zero matches from the n people in section. We can solve this problem by setting up a sample space with equally likely outcomes. If the sample space is, how many ways can we assign birthdays to the n students, the outcomes are equally likely, and the number of outcomes can be calculated using the product rule. Note that in this setting I am thinking of the students as being distinct and ordered:

$$|S| = (365)^n$$

How many outcomes from that sample space (assignments of birthdays to students) have no birthday matches? Again we can use the product rule. There are 365 choices of birthdays for the first student, 364 for the second (since it can't be the same birthday as the first student) and so on.

$$|E| = (365) \cdot (364) \dots (365 - n + 1)$$

$$\begin{aligned} P(\text{birthday match}) &= 1 - P(\text{no matches}) \\ &= 1 - \frac{|E|}{|S|} \\ &= 1 - \frac{(365) \cdot (364) \dots (365 - n + 1)}{(365)^n} \end{aligned}$$

Interesting values. ($n = 13 : p \approx 0.19$), ($n = 23 : p \approx 0.5$), ($n = 70 : p \geq 0.99$)

b. Again, let's solve this problem by working out the probability of the complement of the event. We can use the exact same sample space as in the above problem, but now for each student there are (365 - 8) = 357 possible days:

$$|E| = (357)^n$$

$$\begin{aligned} P(\text{birthday during section day}) &= 1 - P(\text{no birthdays during section}) \\ &= 1 - \frac{|E|}{|S|} \\ &= 1 - \frac{(357)^n}{(365)^n} \end{aligned}$$

For a section of size 13, $p \approx 0.25$. For a group of size 200, $p \approx 0.99$.

2. This problem is requires an application of Bayes' theorem.

$$P(X_1|H) = \frac{P(H|X_1)P(X_1)}{P(H)}$$

The hardest part is to calculate $P(H)$. You can do this using the law of total probability. If it helps: first solve the easier question where song 3 doesn't exist (you can directly apply the expanded Bayes' formula). Given that there are three songs:

$$\begin{aligned}
P(X_1|H) &= \frac{P(H|X_1)P(X_1)}{P(H|X_1)P(X_1) + P(H|X_2)P(X_2) + P(H|X_3)P(X_3)} \\
&= \frac{0.50 \cdot 0.80}{0.50 \cdot 0.80 + 0.90 \cdot 0.15 + 0.30 \cdot 0.05} \\
&\approx 0.72
\end{aligned}$$

3. a. We can directly leverage that these are independent events:

$$\begin{aligned}
P(M_1 \cap M_2 \cap M_3) &= P(M_1) \cdot P(M_2) \cdot P(M_3) && \text{Independence} \\
&= p_1 \cdot p_2 \cdot p_3
\end{aligned}$$

b. You have two options. You can either use the inclusion exclusions principle, or, you could use De Morgan's rule to turn your or probability into an and probability.

$$\begin{aligned}
P(M_1 \cup M_2 \cup M_3) &= 1 - P((M_1 \cup M_2 \cup M_3)^C) && P(E) = 1 - P(E^C) \\
&= 1 - P(M_1^C \cap M_2^C \cap M_3^C) && \text{DeMorgan's} \\
&= 1 - P(M_1^C) \cdot P(M_2^C) \cdot P(M_3^C) && \text{Independence} \\
&= 1 - (1 - p_1) \cdot (1 - p_2) \cdot (1 - p_3)
\end{aligned}$$

Alternatively:

$$\begin{aligned}
P(M_1 \cup M_2 \cup M_3) = &\; P(M_1) + P(M_2) + P(M_3) && \text{Inclusion Exclusion} \\
&- P(M_1 \cap M_2) - P(M_1 \cap M_3) - P(M_1 \cap M_2) \\
&+ P(M_1 \cap M_2 \cap M_3) \\
= &\; p_1 + p_2 + p_3 - p_1p_2 - p_1p_3 - p_2p_3 + p_1p_2p_3 && \text{Independence}
\end{aligned}$$

4. Counting (especially the product rule) quickly leads to exponential amount of work, which can be sad times for Computer Scientists. Some times, we can develop algorithms that have logrithmic amount of work, and that is amazing.

In this case, call the number of binary decisions n. We can use the product rule to know the total number of breakouts defined are 2^n:

$$\begin{aligned}
2^n &\geq 10{,}000{,}000 \\
n &= \lceil \log_2(10{,}000) \rceil = 24
\end{aligned}$$