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CS109

Lecture #12
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Continuous Joints

Continuous Joint Distributions

Random variables X and Y are Jointly Continuous if there exists a Probability Density Function (PDF) $f_{X,Y}$ such that:

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = \int_{a_1}^{a_2} \int_{b_1}^{b_2} f_{X,Y}(x,y)dy\,dx$$

Using the PDF we can compute marginal probability densities:

$$f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a,y)dy$$
$$f_Y(b) = \int_{-\infty}^{\infty} f_{X,Y}(x,b)dx$$

Lemmas

Here are two useful lemmas. Let $F(a,b)$ be the Cumulative Density Function (CDF):

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F(a_2,b_2) - F(a_1,b_2) + F(a_1,b_1) - F(a_2,b_1)$$

And did you know that if Y is a non-negative random variable the following hold (for discrete and continuous random variables respectively):

$$E[Y] = \sum_{i=1}^{n} P(Y \geq i)$$
$$E[Y] = \int_{0}^{\infty} P(Y \geq i)di$$

Example 3

A disk surface is a circle of radius R. A single point imperfection is uniformly distributed on the disk with joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi R^2} & \text{if } x^2+y^2 \leq R^2 \\ 0 & \text{else} \end{cases}$$

Let D be the distance from the origin: $D = \sqrt{X^2+Y^2}$. What is $E[D]$? Hint: use the lemmas

Example 4

Lets make a weight matrix used for Gaussian blur. In the weight matrix, each location in the weight matrix will be given a weight based on the probability density of the area covered by that grid square in a 2D Gaussian with variance σ^2. For this example lets blur using $\sigma = 3$.

In image processing, a Gaussian blur is the result of blurring an image by a Gaussian function. It is a widely used effect in graphics software, typically to reduce image noise.

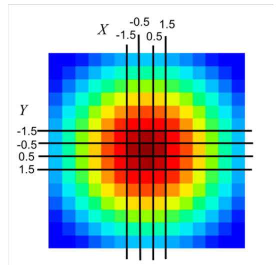
Gaussian blurring with StDev = 3, is based on a joint probability distribution:

Joint PDF

$$f_{X,Y}(x,y) = \frac{1}{2\pi \cdot 3^2} e^{-\frac{x^2+y^2}{2\cdot 3^2}}$$

Joint CDF

$$F_{X,Y}(x,y) = \Phi\left(\frac{x}{3}\right) \cdot \Phi\left(\frac{y}{3}\right)$$



Used to generate this weight matrix

Each pixel is given a weight equal to the probability that X and Y are both within the pixel bounds. The center pixel covers the area where $-0.5 \leq x \leq 0.5$ and $-0.5 \leq y \leq 0.5$ What is the weight of the center pixel?

$$\begin{aligned}
&P(-0.5 < X < 0.5, -0.5 < Y < 0.5) \\
=&P(X < 0.5, Y < 0.5) - P(X < 0.5, Y < -0.5) \\
&- P(X < -0.5, Y < 0.5) + P(X < -0.5, Y < -0.5) \\
=&\phi\left(\frac{0.5}{3}\right) \cdot \phi\left(\frac{0.5}{3}\right) - 2\phi\left(\frac{0.5}{3}\right) \cdot \phi\left(\frac{-0.5}{3}\right) \\
&+ \phi\left(\frac{-0.5}{3}\right) \cdot \phi\left(\frac{-0.5}{3}\right) \\
=&0.5662^2 - 2 \cdot 0.5662 \cdot 0.4338 + 0.4338^2 = 0.206
\end{aligned}$$