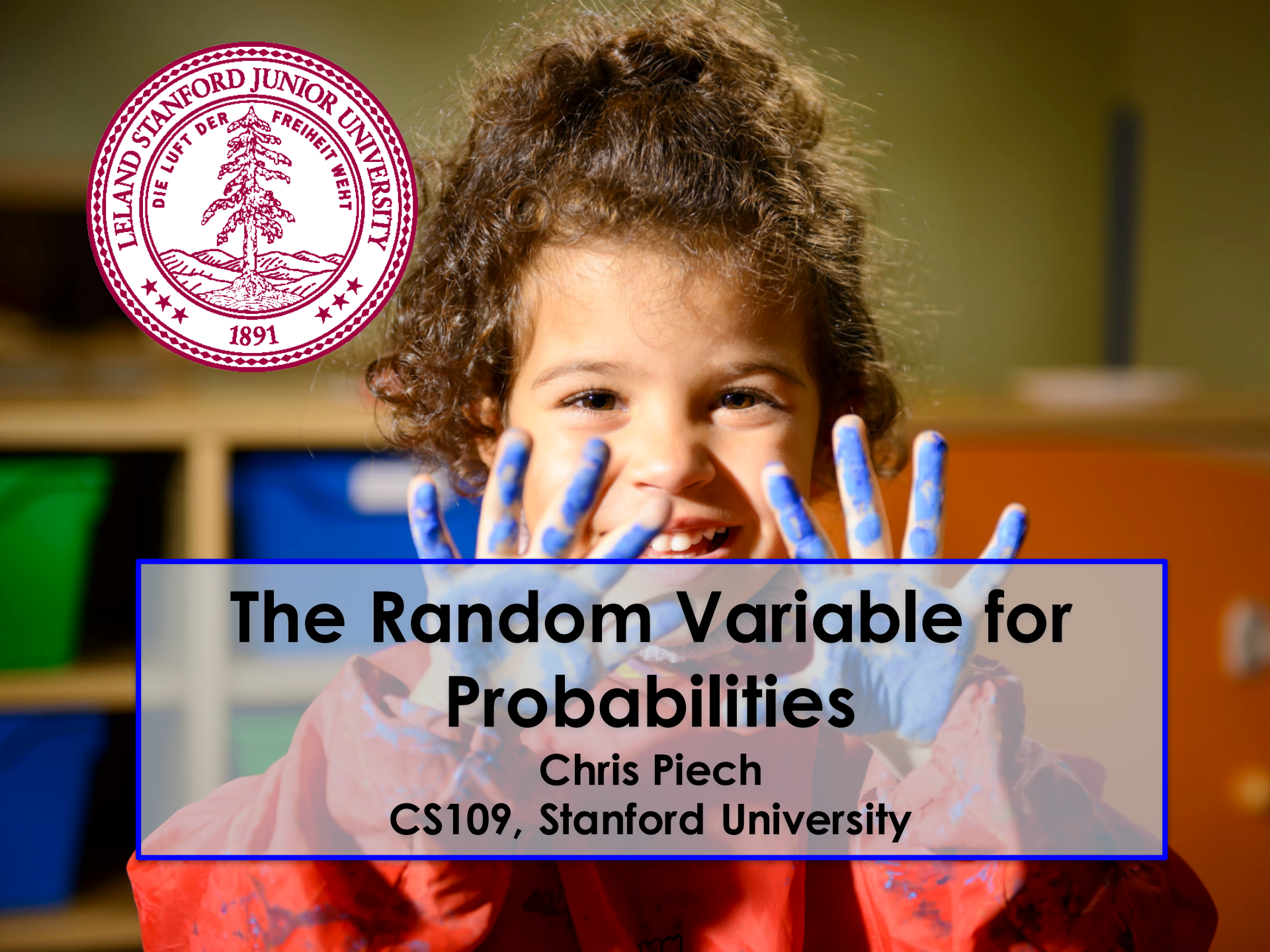
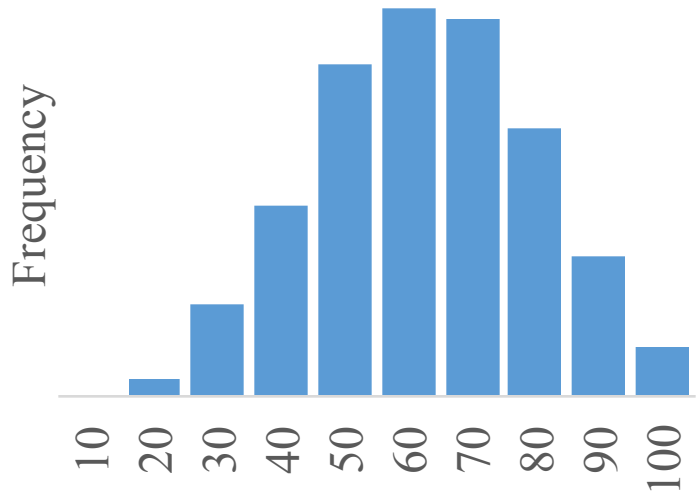
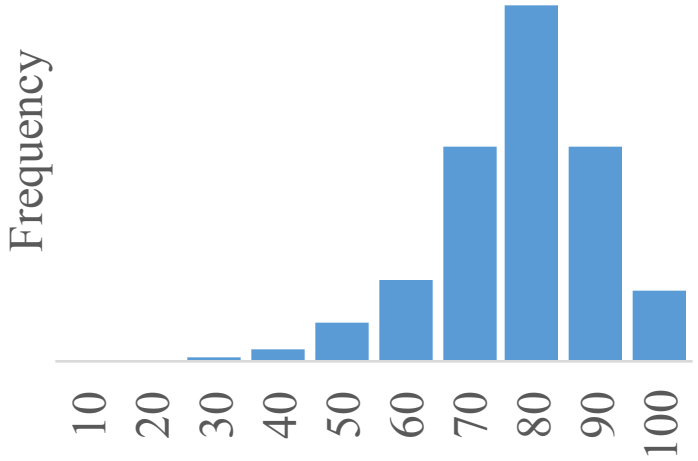


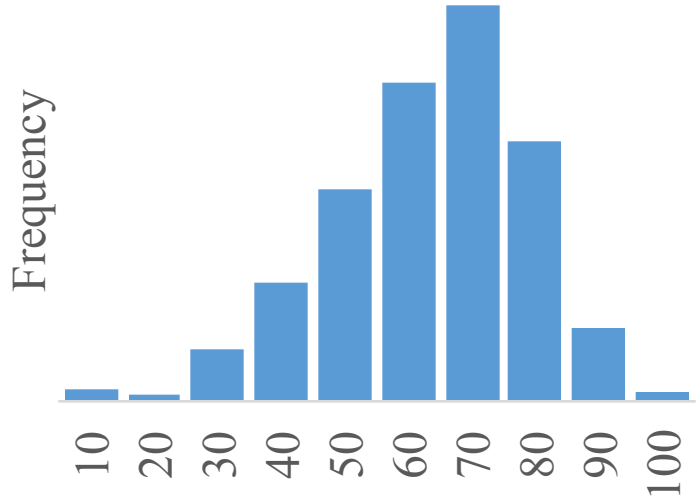
# The Random Variable for Probabilities

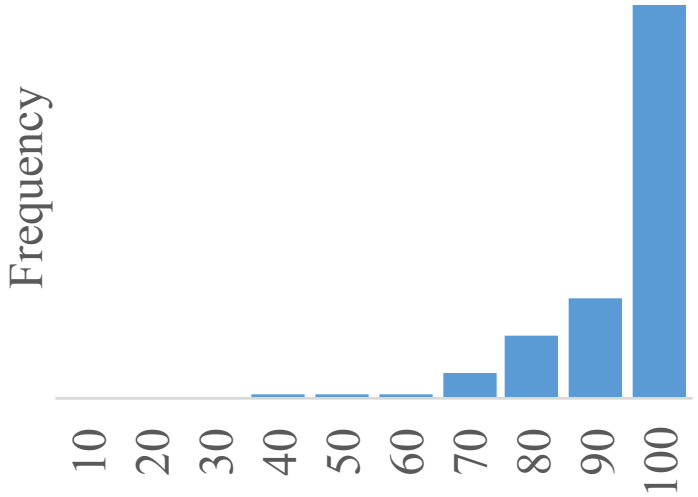
Chris Piech
CS109, Stanford University

# Assignment Grades




We have 2055 assignment distributions from grade scope

# Flip a Coin With Unknown Probability

Demo

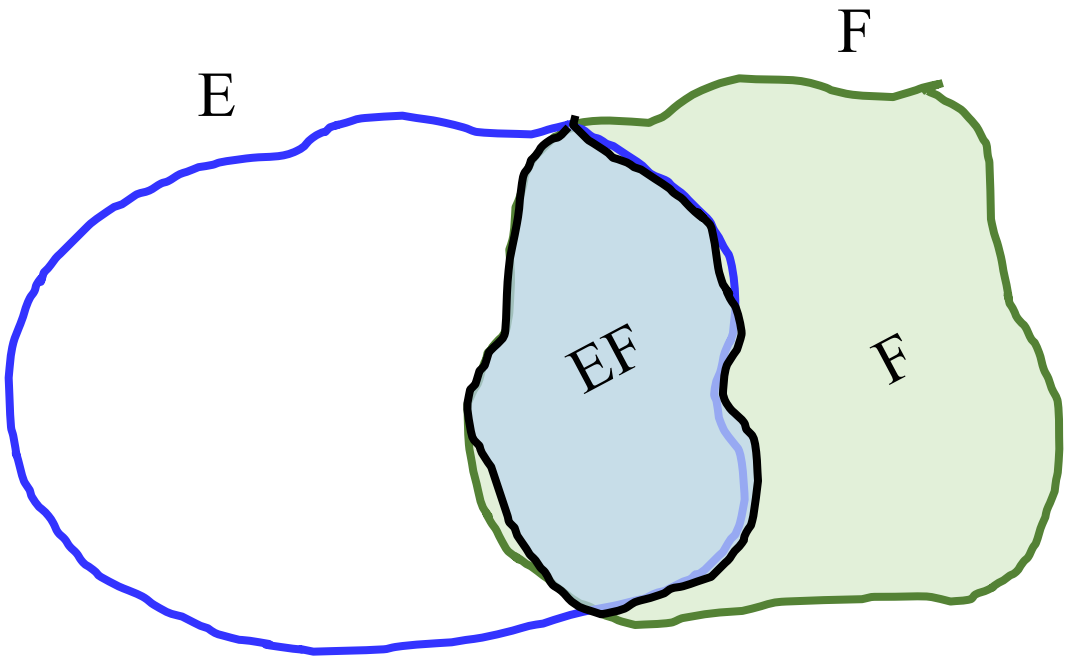
Today we are going to learn something unintuitive, beautiful and useful

# Review

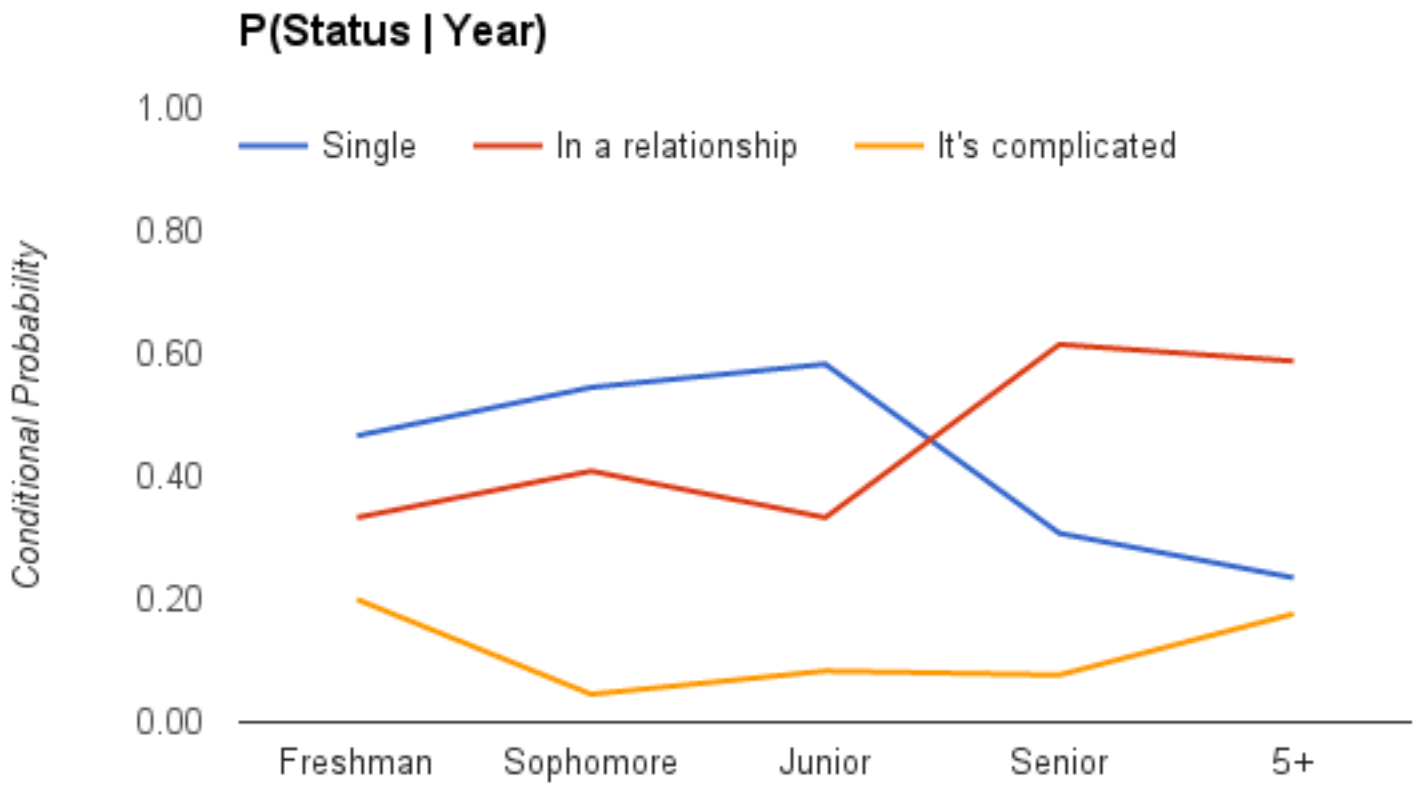
# Conditional Events

- Recall that for *events* E and F:

$$P(E \mid F) = \frac{P(EF)}{P(F)} \quad \text{where} \quad P(F) > 0$$



# Discrete Conditional Distributions



# Continuous Conditional Distributions

- Let X and Y be continuous random variables
  - Conditional PDF of X given Y (where $f_Y(y) > 0$):

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

$$f_{X|Y}(x, y)\partial x = \frac{f_{X,Y}(x, y)\partial x \partial y}{f_{X|Y}(y)\partial y}$$

$$f_{X|Y}(x, y)\partial x = \frac{f_{X,Y}(x, y)\partial x}{f_{X|Y}(y)}$$

$$f_{X|Y}(x, y) = \frac{f_{X,Y}(x, y)}{f_{X|Y}(y)}$$

Conditioning with a continuous random variable feels weird at first. But then it gets good.

Its like biking with a helmet…

# Continuous Conditional Distributions

- Let X be continuous random variable
- Let E be an event:

$$
\begin{aligned}
P(E|X=x) &= \frac{P(X=x,E)}{P(X=x)} \\
&= \frac{P(X=x|E)P(E)}{P(X=x)} \\
&= \frac{f_X(x|E)P(E)\partial x}{f_X(x)\partial x} \\
&= \frac{f_X(x|E)P(E)}{f_X(x)}
\end{aligned}
$$

# Anomaly Detection

- Let X be a measure of time to answer a question
- Let E be the event that the user is a human:

$$
\begin{aligned}
P(E|X=x) &= \frac{P(X=x,E)}{P(X=x)} \\
&= \frac{P(X=x|E)P(E)}{P(X=x)} \\
&= \frac{f_X(x|E)P(E)\partial x}{f_X(x)\partial x} \\
&= \frac{f_X(x|E)P(E)}{f_X(x)}
\end{aligned}
$$

# Anomaly Detection

- Let X be a measure of time to answer a question
- Let E be the event that the user is a human
- What if you don't know unconditional?:

Normal pdf

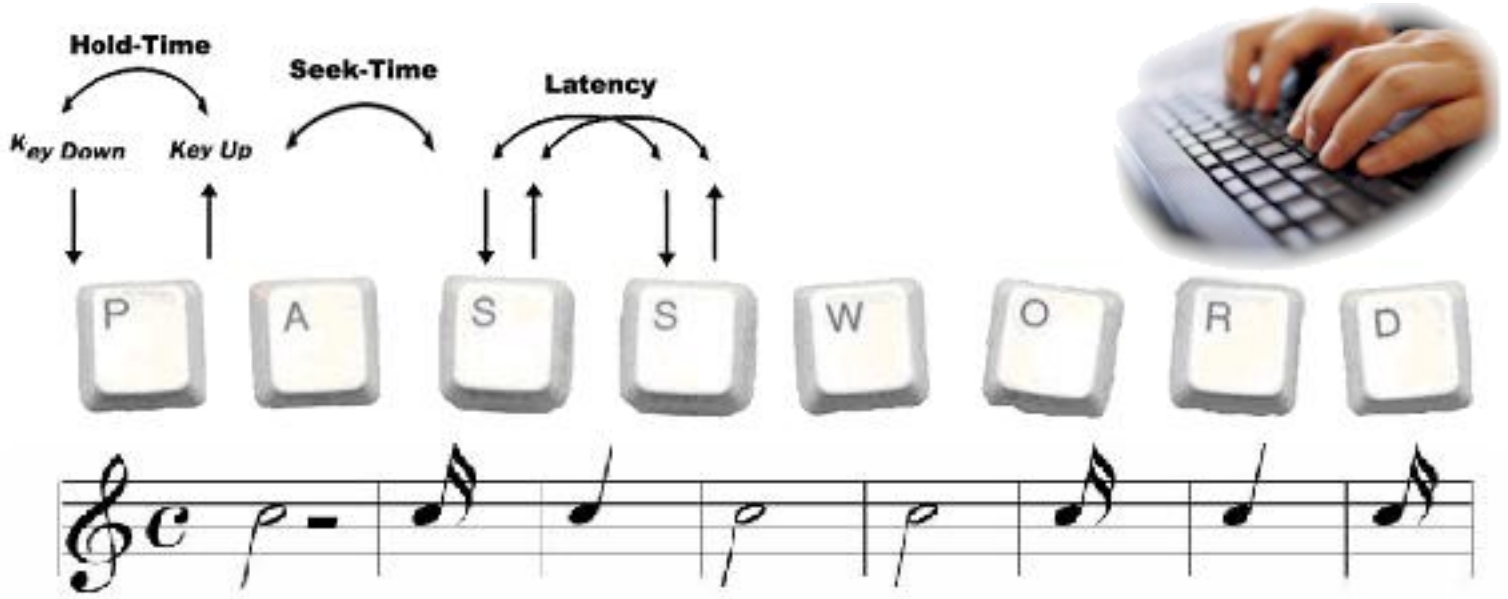
Prior

$$P(E|X=x) = \frac{f_X(x|E)P(E)}{f_X(x)}$$

???

$$\frac{P(E|X=x)}{P(E^C|X=x)}$$

# Biometric Keystroke



# Mixing Discrete and Continuous

- Let X be a continuous random variable
- Let N be a discrete random variable
  - Conditional PDF of X given N:

$$f_{X|N}(x \mid n) = \frac{p_{N|X}(n \mid x) f_X(x)}{p_N(n)}$$

  - Conditional PMF of N given X:

$$p_{N|X}(n \mid x) = \frac{f_{X|N}(x \mid n) p_N(n)}{f_X(x)}$$

  - If X and N are independent, then:

$$f_{X|N}(x \mid n) = f_X(x) \qquad p_{N|X}(n \mid x) = p_N(n)$$

End Review

We are going to think of probabilities as random variables!!!

# Flip a Coin With Unknown Probability

- Flip a coin (n + m) times, comes up with n heads
  - We don't know probability X that coin comes up heads

Frequentist

$$X = \lim_{n+m\to\infty} \frac{n}{n+m} \approx \frac{n}{n+m}$$

X is a single value

Bayesian

$$f_{X|N}(x|n) = \frac{P(N=n|X=x)f_X(x)}{P(N=n)}$$

X is a random variable

# Flip a Coin With Unknown Probability

- Flip a coin (n + m) times, comes up with n heads
  - We don't know probability X that coin comes up heads
  - Our belief before flipping coins is that: X ~ Uni(0, 1)
  - Let N = number of heads
  - Given X = x, coin flips independent: (N | X) ~ Bin(n + m, x)

$$f_{X|N}(x|n) = \frac{P(N = n|X = x) f_X(x)}{P(N = n)}$$

Bayesian "posterior" probability distribution

Bayesian "prior" probability distribution

# Flip a Coin With Unknown Probability

- Flip a coin (n + m) times, comes up with n heads
  - We don't know probability X that coin comes up heads
  - Our belief before flipping coins is that: X ~ Uni(0, 1)
  - Let N = number of heads
  - Given X = x, coin flips independent: (N | X) ~ Bin(n + m, x)

$$f_{X|N}(x|n) = \frac{P(N=n|X=x)f_X(x)}{P(N=n)}$$

1

Binomial

$$= \frac{\binom{n+m}{n}x^n(1-x)^m}{P(N=n)}$$

Move terms around

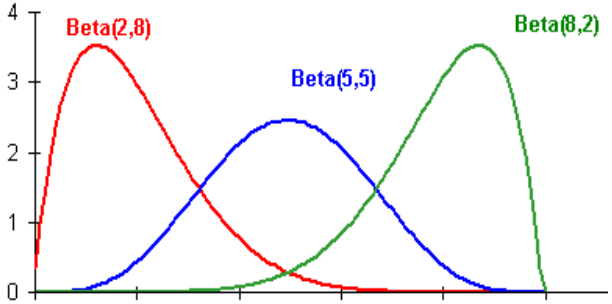
$$= \frac{\binom{n+m}{n}}{P(N=n)}x^n(1-x)^m$$

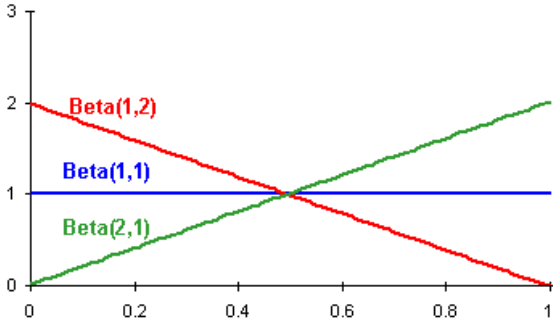
$$= \frac{1}{c}\cdot x^n(1-x)^m \qquad \text{where } c = \int_0^1 x^n(1-x)^m dx$$

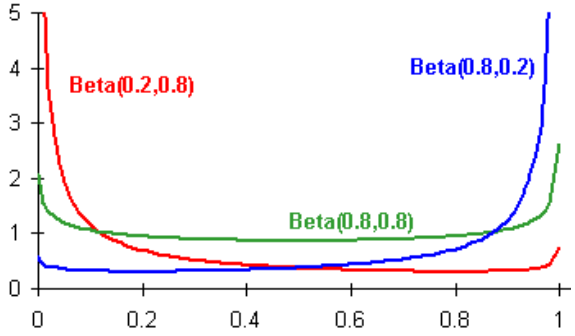
# Beta Random Variable

- X is a **Beta Random Variable**: X ~ Beta($a$, $b$)
  - Probability Density Function (PDF): (where $a$, $b > 0$)

$$f(x) = \begin{cases} \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{where} \quad B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$

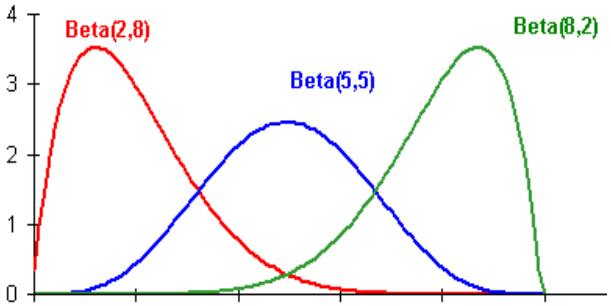


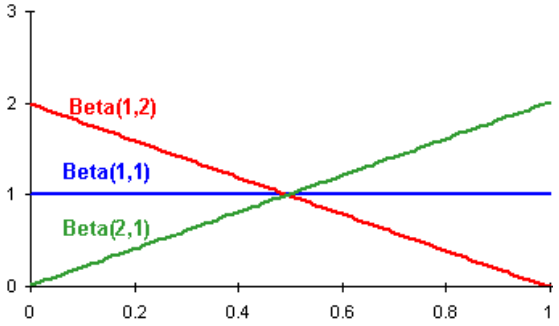



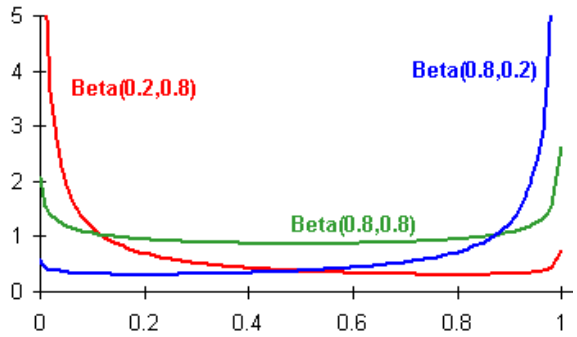




  - Symmetric when $a = b$
  - $E[X] = \frac{a}{a+b}$ $\quad Var(X) = \frac{ab}{(a+b)^2(a+b+1)}$

# Meta Beta




Used to represent a distributed belief of a probability

# Beta is a distribution for probabilities

# Back to flipping coins

- Flip a coin (n + m) times, comes up with n heads
  - We don't know probability X that coin comes up heads
  - Our belief before flipping coins is that: X ~ Uni(0, 1)
  - Let N = number of heads
  - Given X = x, coin flips independent: (N | X) ~ Bin(n + m, x)

$$f_{X|N}(x|n) = \frac{P(N=n|X=x)f_X(x)}{P(N=n)}$$

$$= \frac{\binom{n+m}{n}x^n(1-x)^m}{P(N=n)}$$

$$= \frac{\binom{n+m}{n}}{P(N=n)}x^n(1-x)^m$$

$$= \frac{1}{c}\cdot x^n(1-x)^m \qquad \text{where } c = \int_0^1 x^n(1-x)^m dx$$

# Dude, Where's My Beta?

- Flip a coin (n + m) times, comes up with n heads
  - Conditional density of X given N = n

$$f_{X|N}(x \mid n) = \frac{1}{c} \cdot x^n (1-x)^m \quad \text{where} \quad c = \int_0^1 x^n (1-x)^m \, dx$$

  - Note: $0 < x < 1$, so $f_{X|N}(x \mid n) = 0$ otherwise
  - Recall Beta distribution:

$$f(x) = \begin{cases} \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \qquad B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} \, dx$$

  - Hey, that looks more familiar now...
  - X | (N = n, n + m trials) ~ Beta(n + 1, m + 1)

# Understanding Beta

- X | (N = n, m + n trials) ~ Beta(n + 1, m + 1)
  - X ~ Uni(0, 1)
  - Check this out, boss:
    - Beta(1, 1) =?

$$f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} = \frac{1}{B(a,b)} x^{0}(1-x)^{0}$$

$$= \frac{1}{\int_0^1 1\,dx} 1 = 1 \quad \text{where} \quad 0 < x < 1$$

    - Beta(1, 1) = Uni(0, 1)
  - So, X ~ Beta(1, 1)

# If the Prior was a Beta…

If our belief about X (that random variable for probability) was beta

$$f_X(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}$$

What is our belief about X after observing N heads?

$$f_{X|N}(x,n) = ???$$

# If the Prior was a Beta…

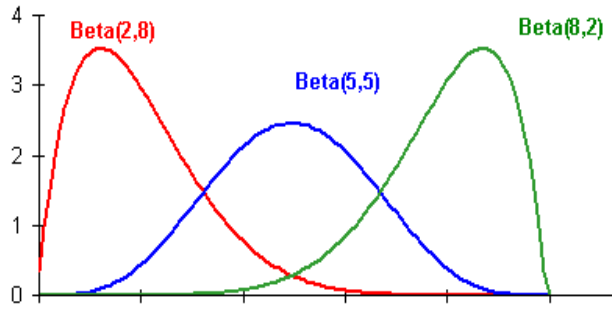
$$
\begin{aligned}
f_{X|N}(x,n) &= \frac{P(N=n|X=x)f_X(x)}{P(N=n)} \\
&= \frac{\binom{n+m}{n}x^n(1-x)^m f_X(x)}{P(N=n)} \\
&= \frac{\binom{n+m}{n}x^n(1-x)^m \frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}}{P(N=n)} \\
&= K_1 \cdot \binom{n+m}{n}x^n(1-x)^m \frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1} \\
&= K_2 \cdot x^n(1-x)^m \frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1} \\
&= K_2 \cdot x^n(1-x)^m x^{a-1}(1-x)^{b-1} \\
&= K_2 \cdot x^{n+a-1}(1-x)^{m+b-1} \\
X|N &\sim \text{Beta}(n+a, m+b)
\end{aligned}
$$

# Understanding Beta

- If “Prior” distribution of X (before seeing flips) is Beta
- Then “Posterior” distribution of X (after flips) is Beta
- Beta is a **conjugate** distribution for Beta
  - Prior and posterior parametric forms are the same!
  - Practically, conjugate means easy update:
    - Add number of “heads” and “tails” seen to Beta parameters

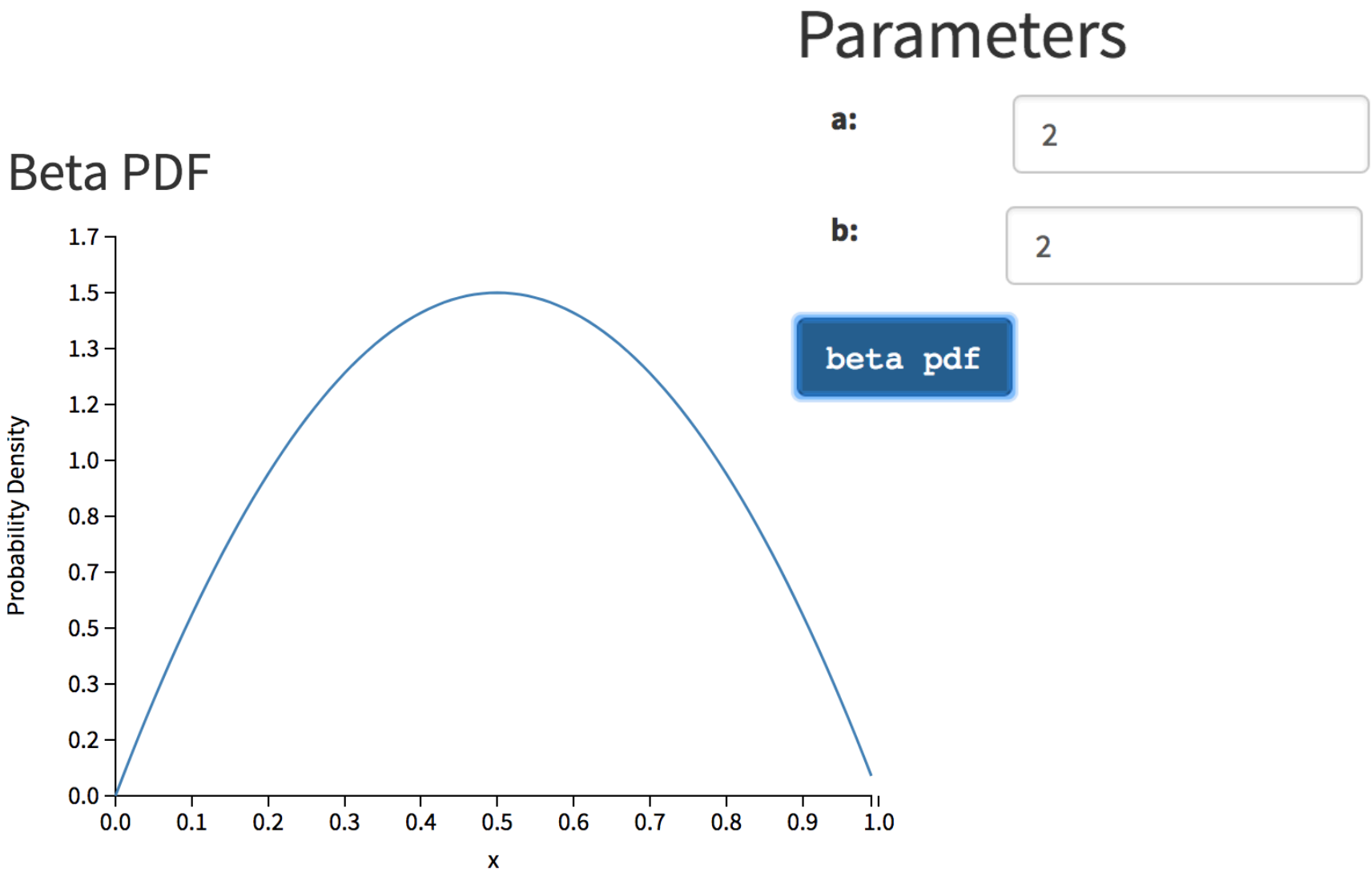
# Further Understanding Beta

- Can set X ~ Beta($a$, $b$) as prior to reflect how biased you think coin is apriori
  - This is a subjective probability!
  - Then observe n + m trials, where n of trials are heads




- Update to get posterior probability
  - X | (n heads in n + m trials) ~ Beta(a + n, b + m)
  - Sometimes call $a$ and $b$ the "equivalent sample size"
  - Prior probability for X based on seeing ($a$ + $b$ – 2) "imaginary" trials, where ($a$ – 1) of them were heads.
  - Beta(1, 1) ~ Uni(0, 1) → we haven't seen any "imaginary trials", so apriori know nothing about coin

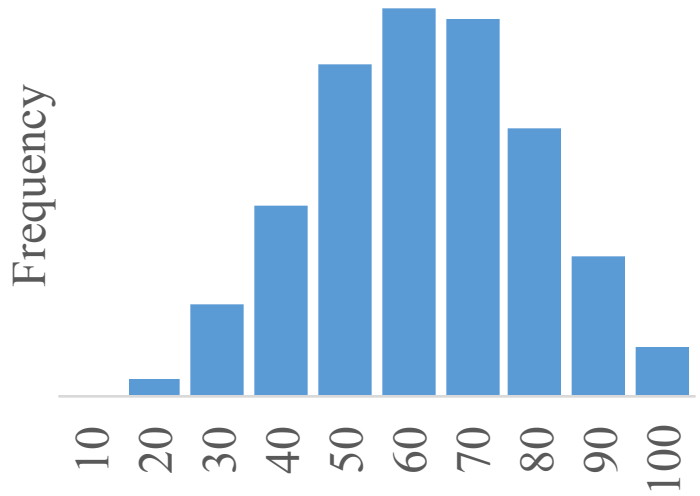
# Check out Demo!

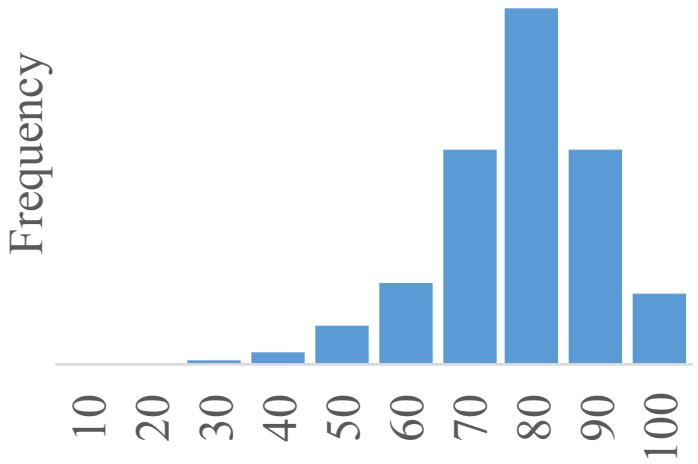


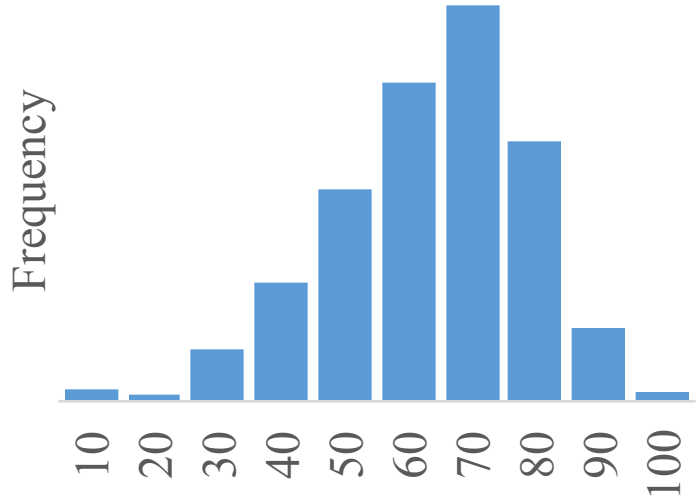
Damn

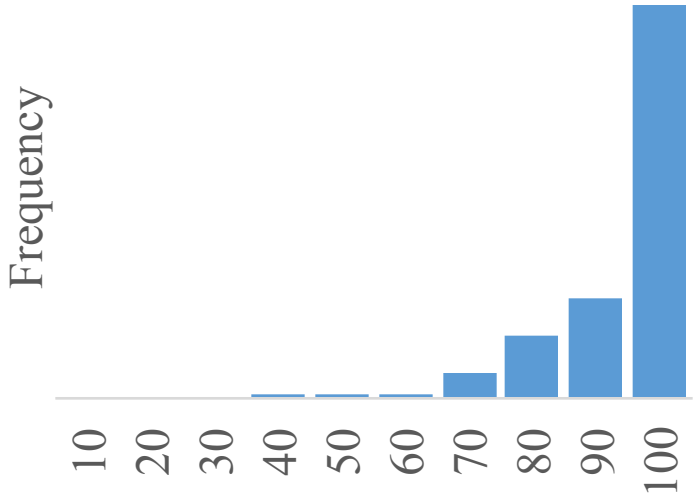
Next level?

# Assignment Grades



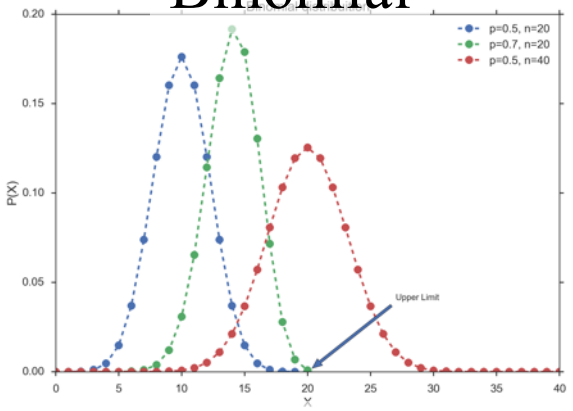



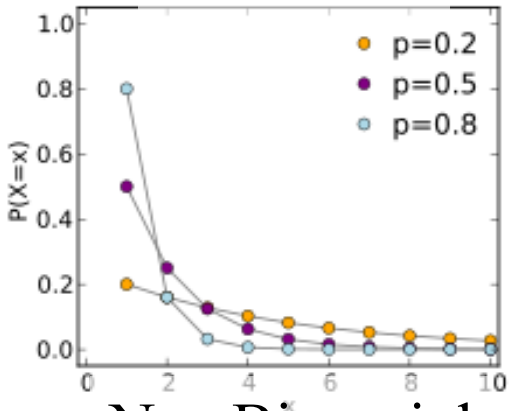

We have 2055 assignment distributions from gradescope

# Distributions

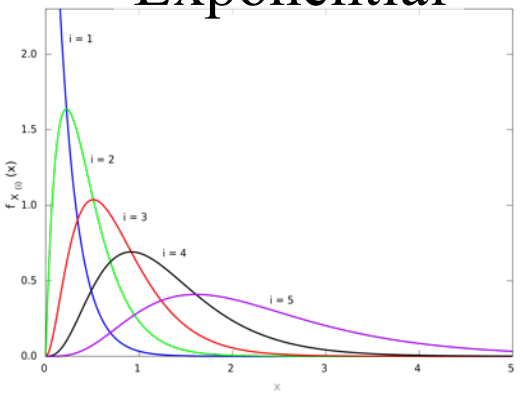
## Binomial



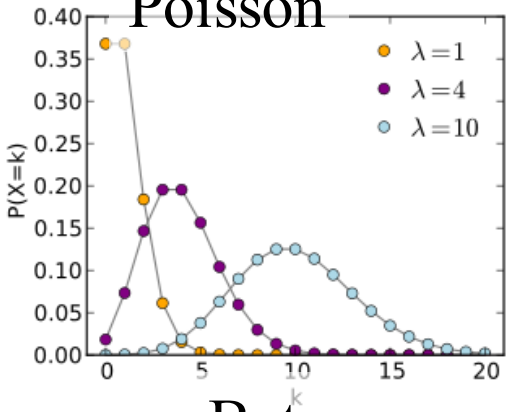

## Geometric



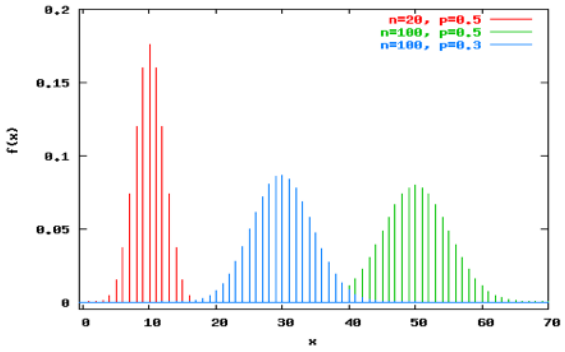

## Exponential



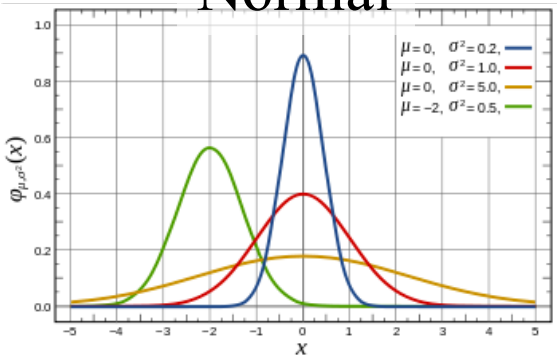

## Poisson



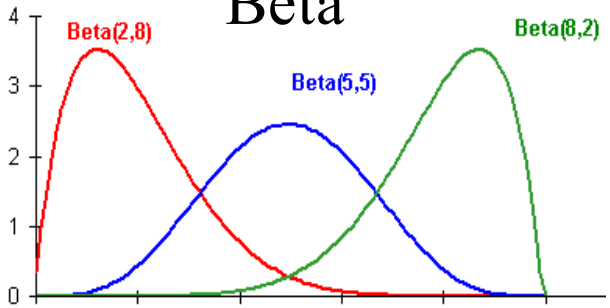

## Neg Binomial



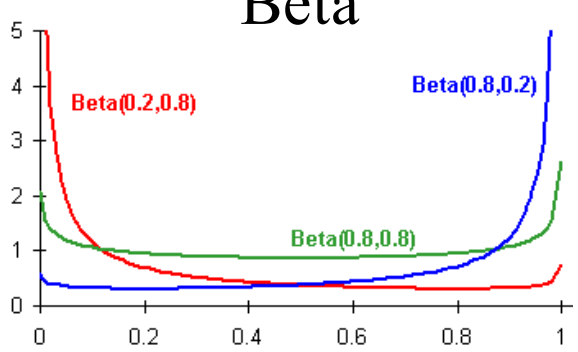

## Normal



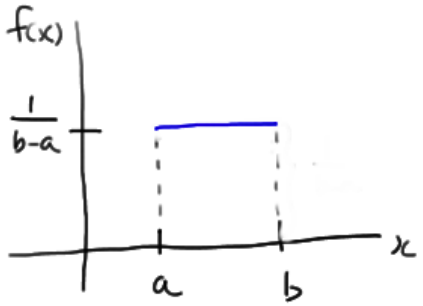

## Beta




## Beta




## Uniform



Grades must be bounded

Normal: No

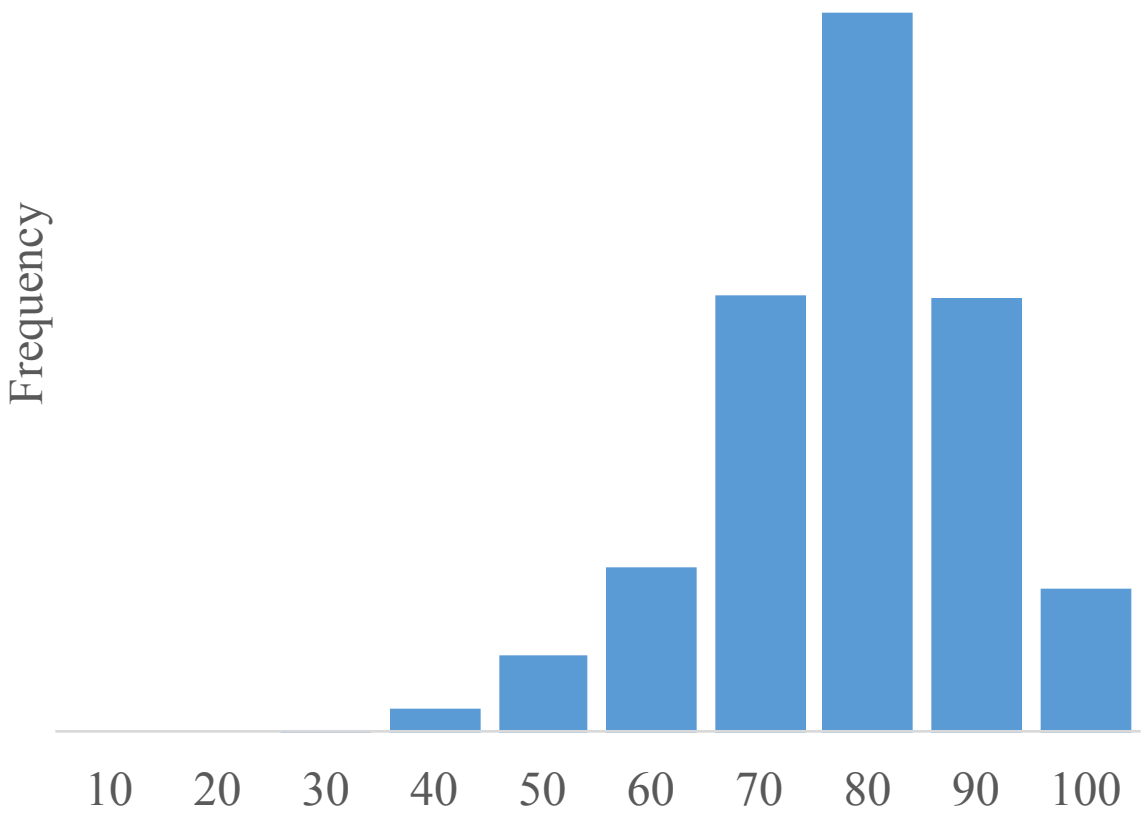
Poisson: No

Exponential: No

Beta: Yes

# Assignment Grades Demo

```
Assignment id = '1613'
```



# Assignment Grades Demo

```
Assignment id = '1613'
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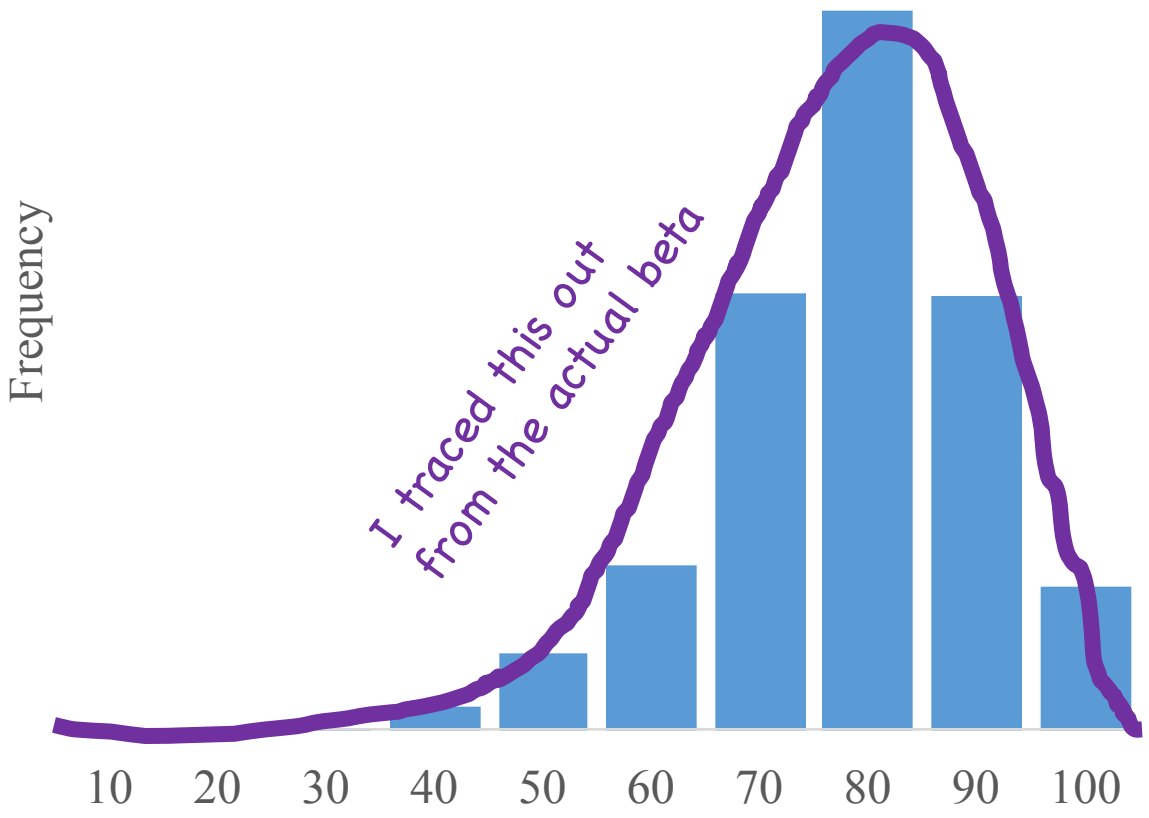
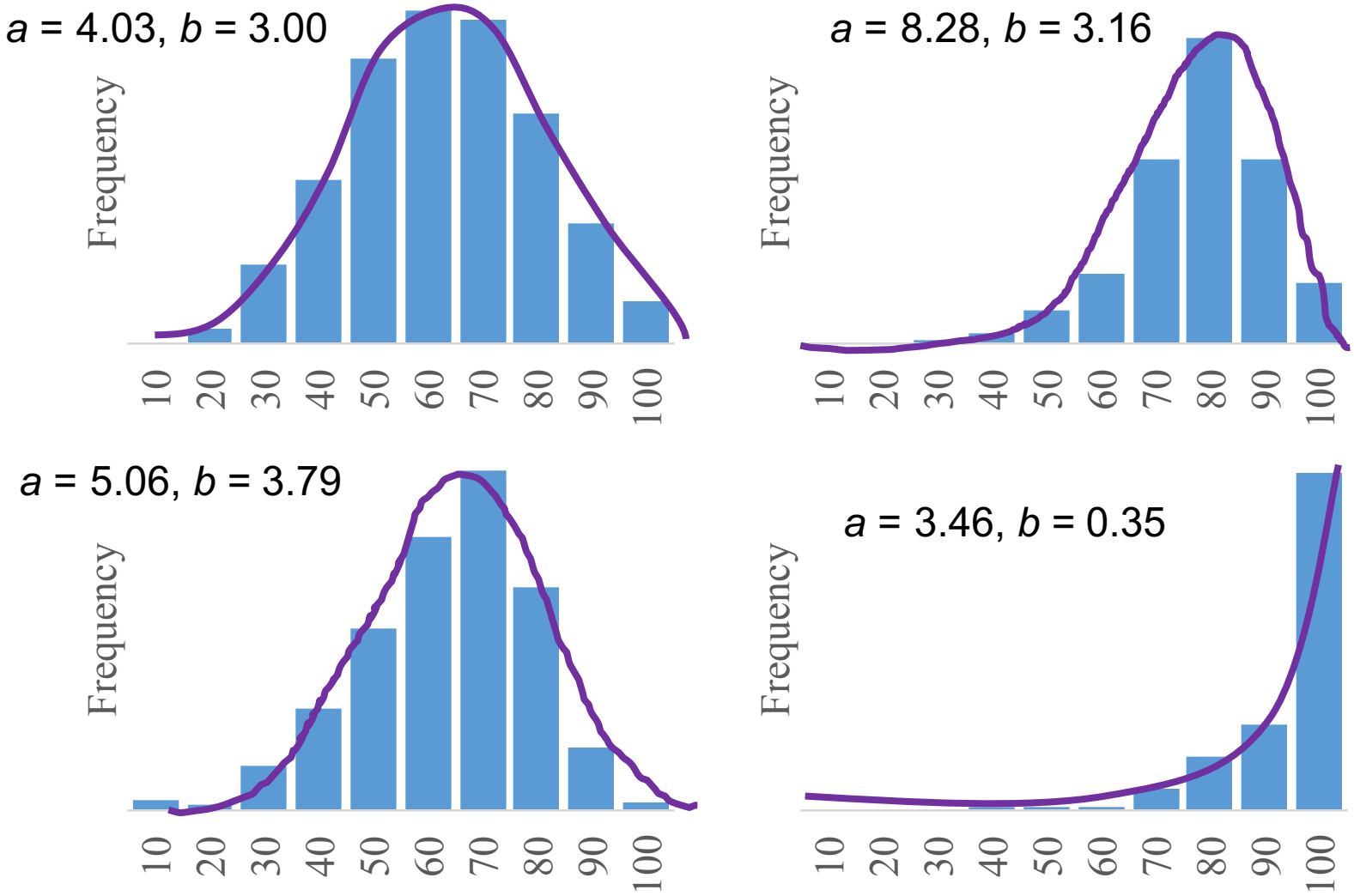


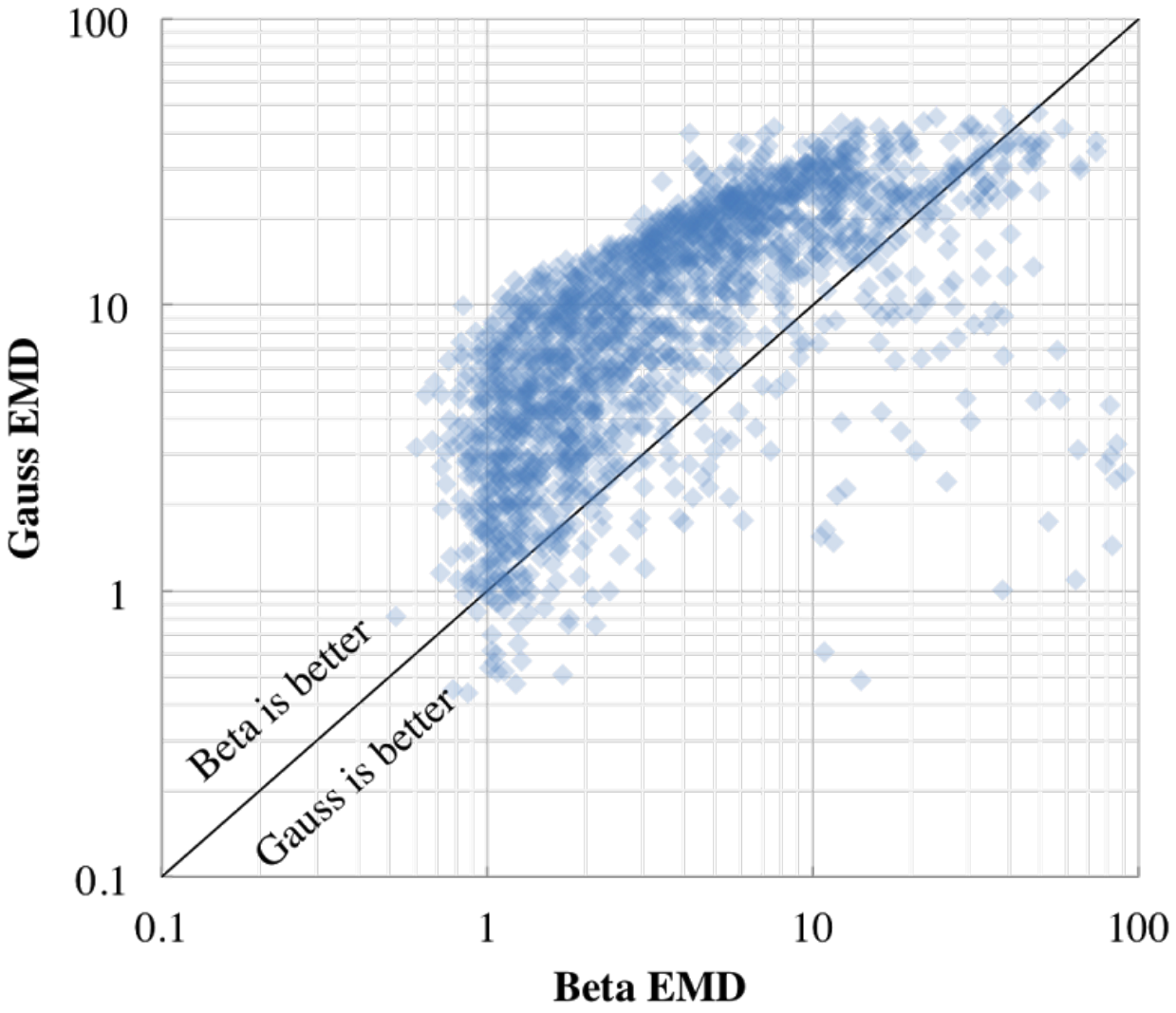

$$X \sim Beta(a = 8.28, b = 3.16)$$

# Assignment Grades



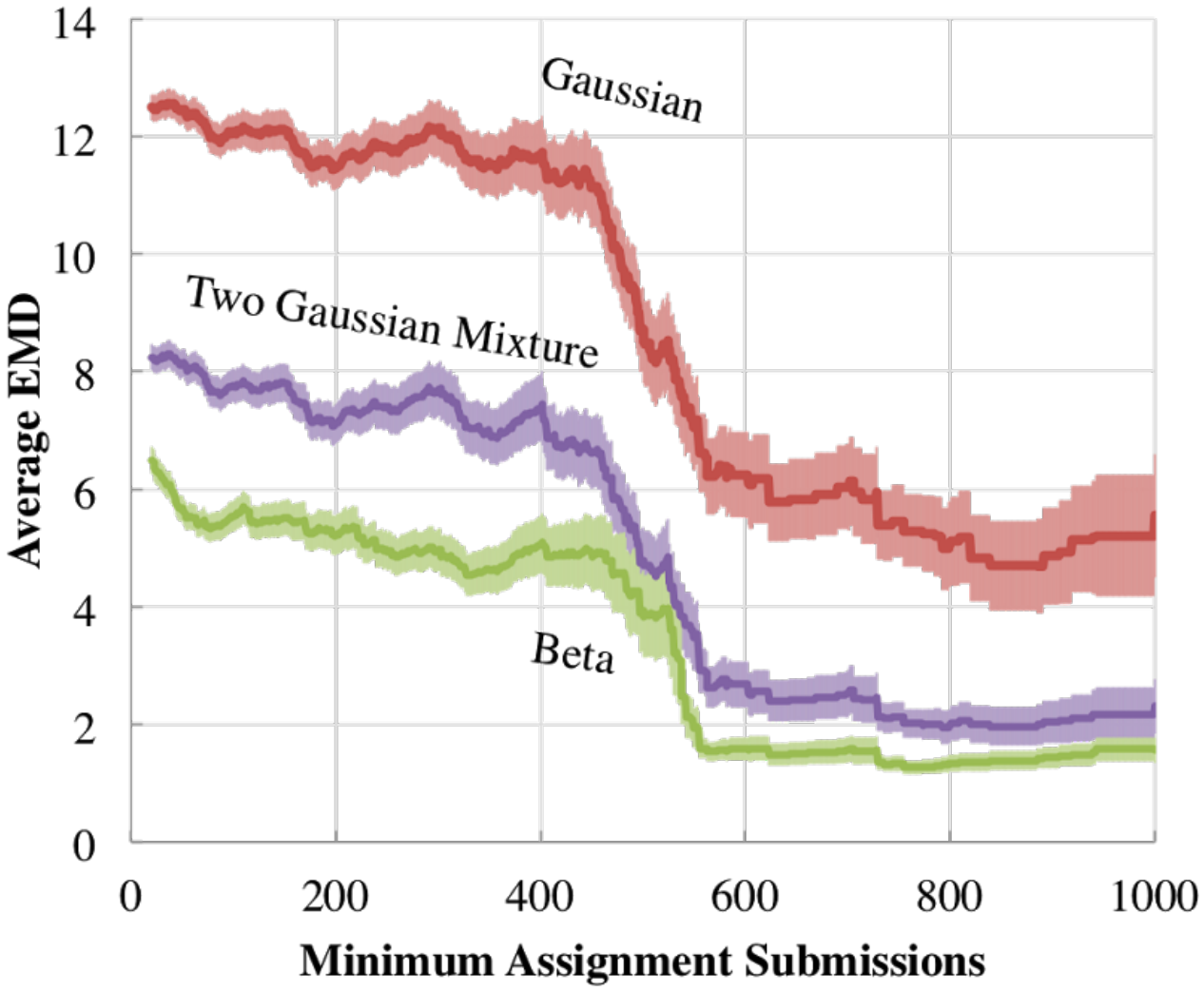

We have 2055 assignment distributions from grade scope

# Beta is a Better Fit




Unpublished results. Based on Gradescope data

# Beta is a Better Fit For All Class Sizes




Unpublished results. Based on Gradescope data

# Binomial Interpretation

Each student has **the same** probability of getting each point. Generate grades by flipping a coin 100 times for each student. The resulting distribution is binomial.

- Binomial

# Normal Interpretation

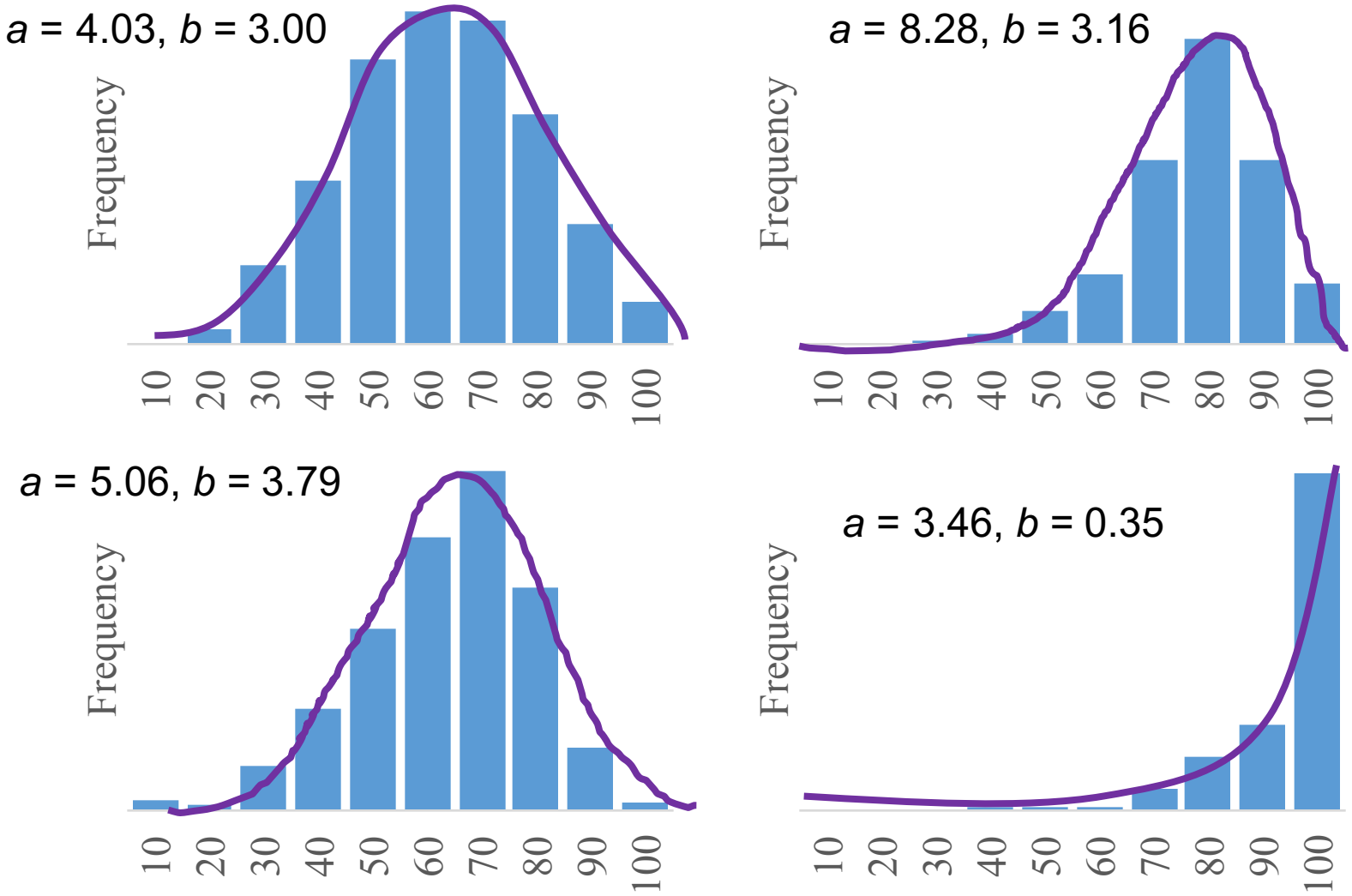
> What the Binomial said, but approximated.
>
> - Normal

# Beta Interpretation

> Each student's ability is represented as a probability. The distribution of probabilities is a Beta distribution. Each student has **a different** probability of getting points, and that probability is sampled from a Beta distribution.
>
> - Beta

- This is Chris Piech's opinion. It is open for debate

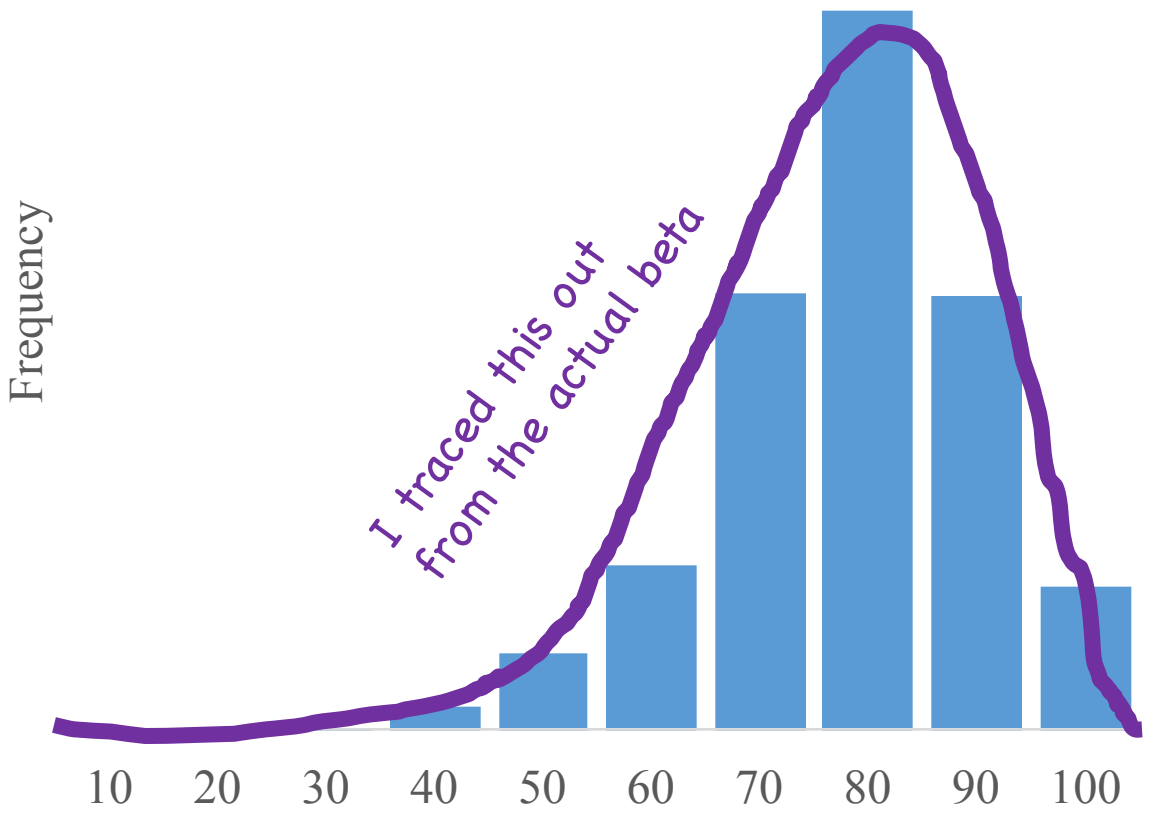
# Assignment Grades




These are the distribution of student *point probabilitities*

# Assignment Grades Demo

What is the semantics of E[X]?




$$X \sim Beta(a = 8.28, b = 3.16)$$

# Assignment Grades

What is the probability that a student is bellow the mean?

$$X \sim Beta(a = 8.28, b = 3.16)$$

---

$$E[X] = \frac{a}{a+b} = \frac{8.28}{8.28 + 3.16} \approx 0.7238$$

$$P(X < 0.7238) = F_X(0.7238)$$

Wait what? Chris are you holding out on me?

```
stats.beta.cdf(x, alpha, beta)
```

$$P(X < E[X]) = 0.46$$

As far as I know, this is an unpublished result

# Implications

- Will be combined with Item Response Theory which models how assignment difficulty and student ability combine to give *point probabilities*.
- Suggests a way to calculate final grades as a probabilistic most likely estimate of “ability”.
- Machine learning on education data will be more accurate.
- Analysis of “mixture” distributions can be fixed.

Will you use this on us?

Not yet ☺

# Beta:
# The probability density for probabilities

Any parameter for a “parameterized” random variable can be thought of as a random variable.

# Course Mean

$$E[CS109]$$

*This is actual midpoint of course*
*(Just wanted you to know)*