Approximate Counting

What if you wanted a counter that could count up to the number of atoms in the universe, but you wanted to store the counter in 8 bits? You could use the amazing probabilistic algorithm described below! In this example we are going to show that the expected return value of `stochastic_counter(4)`, where `count` is called four times, is in fact equal to four.

```
def stochastic_counter(true_count):
   n = -1
   for i in range(true_count):
       n += count(n)
   return 2 ** n  # 2^n, aka 2 to the power of n

def count(n):
   # To return 1 you need n heads. Always returns 1 if n is <= 0
   for i in range(n):
       if not coin_flip():
           return 0
   return 1

def coin_flip():
   # returns true 50% of the time
   return random.random() < 0.5
```

Let X be a random variable for the value of n at the end of \texttt{stochastic_counter(4)}. Note that X is not a binomial because the probabilities of each outcome change.

Let R be the return value of the function. $R = 2^X$ which is a function of X. Use the law of unconscious statistician

$$E[R] = \sum_x 2^x \cdot P(X = x)$$

We can compute each of the probabilities $P(X = x)$ separately. Note that the first two calls to count will always return 1. Let H_i be the event that the ith call returns 1. Let T_i be the event that the ith call returns 0. X can't be less than 1 because the first two calls to count always return 1. $P(X = 1) = P(T_3, T_4)$ \ $P(X = 2) = P(H_3, T_4) + P(T_3, H_4)$ \ $P(X = 3) = P(H_3, H_4)$

At the point of the third call to count, $n = 1$. If H_3 then $n = 2$ for the fourth call and the loop runs twice.

$$\begin{aligned} P(H_3, T_4) &= P(H_3) \cdot P(T_4 | H_3) \\ &= \frac{1}{2} \cdot (\frac{1}{2} + \frac{1}{4}) \\ P(H_3, H_4) &= P(H_3) \cdot P(H_4 | H_3) \\ &= \frac{1}{2} \cdot \frac{1}{2} \end{aligned}$$

If T_3 then $n = 1$ for the fourth call.

$$\begin{aligned} P(T_3, H_4) &= P(T_3) \cdot P(H_4 | T_3) \\ &= \frac{1}{2} \cdot \frac{1}{2} \\ P(T_3, T_4) &= P(T_3) \cdot P(T_4 | T_3) \\ &= \frac{1}{2} \cdot \frac{1}{2} \end{aligned}$$

Plug everything in:

$$
\begin{aligned}
E[R] &= \sum_{x=1}^{3} 2^x \cdot P(X = x) \\
&= 2 \cdot \frac{1}{4} + 4 \cdot \frac{5}{8} + 8 \cdot \frac{1}{8} \\
&= 4
\end{aligned}
$$