Uniform Distribution

The most basic of all the continuous random variables is the uniform random variable, which is equally likely to take on any value in its range (α, β). X is a *uniform random variable* ($X \sim \text{Uni}(\alpha, \beta)$) if it has PDF:

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{when } \alpha \le x \le \beta \\ 0 & \text{otherwise} \end{cases}$$

Notice how the density $1/(\beta - \alpha)$ is exactly the same regardless of the value for x. That makes the density uniform. So why is the PDF $1/(\beta - \alpha)$ and not 1? That is the constant that makes it such that the integral over all possible inputs evaluates to 1.

Uniform Random Variable

Notation: $X \sim \text{Uni}(\alpha, \beta)$

Description: A continuous random variable that takes on values, with equal likelihood, between α and β

Parameters: $\alpha \in \mathbb{R}$, the minimum value of the variable.
$\beta \in \mathbb{R}$, $\beta > \alpha$, the maximum value of the variable.

Support: $x \in [\alpha, \beta]$

PDF equation: $f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{for } x \in [\alpha, \beta] \\ 0 & \text{else} \end{cases}$

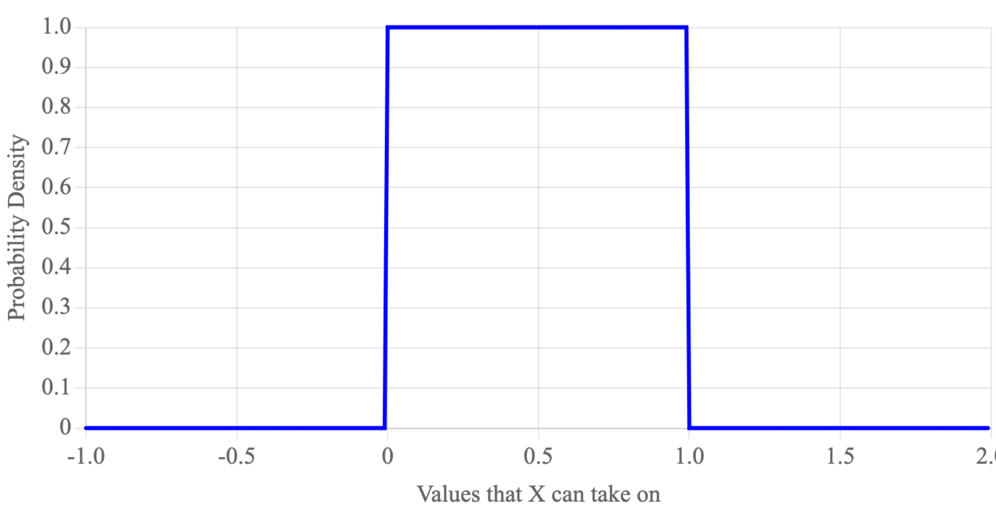
CDF equation: $F(x) = \begin{cases} \frac{x-\alpha}{\beta-\alpha} & \text{for } x \in [\alpha, \beta] \\ 0 & \text{for } x < \alpha \\ 1 & \text{for } x > \beta \end{cases}$

Expectation: $\text{E}[X] = \frac{1}{2}(\alpha + \beta)$

Variance: $\text{Var}(X) = \frac{1}{12}(\beta - \alpha)^2$

PDF graph:

Parameter α: 0 Parameter β: 1



Example: You are running to the bus stop. You don't know exactly when the bus arrives. You believe all times between 2 and 2:30 are equally likely. You show up at 2:15pm. What is P(wait < 5 minutes)?

Let T be the time, in minutes after 2pm that the bus arrives. Because we think that all times are equally likely in this range, $T \sim \text{Uni}(\alpha = 0, \beta = 30)$. The probability that you wait 5 minutes is equal to the probability that the bus shows up between 2:15 and 2:20. In other words $\text{P}(15 < T < 20)$:

$$
\begin{aligned}
\text{P(Wait under 5 mins)} &= \text{P}(15 < T < 20) \\
&= \int_{15}^{20} f_T(x)\partial x \\
&= \int_{15}^{20} \frac{1}{\beta - \alpha}\partial x \\
&= \frac{1}{30}\partial x \\
&= \frac{x}{30}\Big|_{15}^{20} \\
&= \frac{20}{30} - \frac{15}{30} = \frac{5}{30}
\end{aligned}
$$

We can come up with a closed form for the probability that a uniform random variable X is in the range a to b, assuming that $\alpha \le a \le b \le \beta$:

$$
\begin{aligned}
\text{P}(a \le X \le b) &= \int_a^b f(x)\,dx \\
&= \int_a^b \frac{1}{\beta - \alpha}\,dx \\
&= \frac{b - a}{\beta - \alpha}
\end{aligned}
$$