Expectation of Sum Proof

Now that we have learned about joint probabilities, we have all the tools we need to prove one of the most useful properties of Expectation: the fact that the expectation of a sum of random variables is equal to the sum of expectation (even if the variables are not independent). In other words:

For any two random variables X and Y,

$$\mathrm{E}[X+Y] = \mathrm{E}[X] + \mathrm{E}[Y]$$

The proof is going to use the Law of Unconcious statistician (LOTUS) where the function is addition!

Proof: Expectation of Sum

Let X and Y be any two random variables:

$$\begin{aligned}
&\mathrm{E}[X+Y] \\
&\quad = \sum_x \sum_y (x+y) \cdot P(X=x, Y=y) && \mathit{LOTUS} \\
&\quad = \sum_x \sum_y x \cdot P(X=x, Y=y) + y \cdot P(X=x, Y=y) && \text{Distribute} \\
&\quad = \sum_x \sum_y x \cdot P(X=x, Y=y) + \sum_y \sum_x y \cdot P(X=x, Y=y) && \text{Rearrange Sums} \\
&\quad = \sum_x x \sum_y P(X=x, Y=y) + \sum_y y \sum_x P(X=x, Y=y) && \text{Factor Out} \\
&\quad = \sum_x x \cdot P(X=x) + \sum_y y \cdot P(Y=y) && \text{Def of Marginal} \\
&\quad = \mathrm{E}[X] + \mathrm{E}[Y] && \text{Def of Expectation}
\end{aligned}$$

At no point in the proof do we need to assume that X and Y are independent. In the second step the joint probability ends up in each sum, and in both cases, one of the sums ends up marginalizing over the joint probability!