Expectation of Sum Proof

Now that we have learned about joint probabilities, we have all the tools we need to prove one of the most useful properties of Expectation: the fact that the expectation of a sum of random variables is equal to the sum of expectation (even if the variables are not independent). In other words:

For any two random variables X and Y,

$$\text{E}[X+Y] = \text{E}[X] + \text{E}[Y]$$

The proof is going to use the Law of Unconcious statistician (LOTUS) where the function is addition!

Proof: Expectation of Sum

Let X and Y be any two random variables:

$$\begin{aligned}
&\text{E}[X+Y] \\
&\quad= \sum_x \sum_y (x+y) \cdot P(X=x, Y=y) && \textit{LOTUS} \\
&\quad= \sum_x \sum_y x \cdot P(X=x, Y=y) + y \cdot P(X=x, Y=y) && \text{Distribute} \\
&\quad= \sum_x \sum_y x \cdot P(X=x, Y=y) + \sum_y \sum_x y \cdot P(X=x, Y=y) && \text{Rearrange Sums} \\
&\quad= \sum_x x \sum_y P(X=x, Y=y) + \sum_y y \sum_x P(X=x, Y=y) && \text{Factor Out} \\
&\quad= \sum_x x \cdot P(X=x) + \sum_y y \cdot P(Y=y) && \text{Def of Marginal} \\
&\quad= \text{E}[X] + \text{E}[Y] && \text{Def of Expectation}
\end{aligned}$$

At no point in the proof do we need to assume that X and Y are independent. In the second step the joint probability ends up in each sum, and in both cases, one of the sums ends up marginalizing over the joint probability!

Visualization of the Proof

Here is a visualization to show the idea behind the proof. This table shows the joint probabilities $P(X, Y)$ for two random variables X and Y that are not independent.

	Y	
X	**4**	**5**
1	0.1	0.3
2	0.2	0.4

Computing $E[X]$ using joint probabilities:

A key insight from the proof is that we can compute $E[X]$ using values from the joint:

$$\begin{aligned}
E[X] &= \sum_x x P(X=x) \\
&= \sum_x x \sum_y \cdot P(X=x, Y=y) \\
&= \sum_x \sum_y x \cdot P(X=x, Y=y)
\end{aligned}$$

X	Y	$P(X,Y)$	$x \cdot P(X,Y)$
1	4	0.1	$1 \times 0.1 = 0.1$
1	5	0.3	$1 \times 0.3 = 0.3$
2	4	0.2	$2 \times 0.2 = 0.4$
2	5	0.4	$2 \times 0.4 = 0.8$

Summing up the contributions:

$E[X] = 0.1 + 0.3 + 0.4 + 0.8 = 1.6$

Computing $E[Y]$ using joint probabilities:

We compute $E[Y] = \sum_x \sum_y y \cdot P(X = x, Y = y)$.

X	Y	$P(X,Y)$	$y \cdot P(X,Y)$
1	4	0.1	$4 \times 0.1 = 0.4$
1	5	0.3	$5 \times 0.3 = 1.5$
2	4	0.2	$4 \times 0.2 = 0.8$
2	5	0.4	$5 \times 0.4 = 2.0$

Summing up the contributions:

$E[Y] = 0.4 + 1.5 + 0.8 + 2.0 = 4.7$

Computing $E[X + Y]$ using joint probabilities:

We compute $E[X + Y] = \sum_x \sum_y (x + y) \cdot P(X = x, Y = y)$.

We expand $(x + y) \cdot P(X,Y)$ into $x \cdot P(X,Y) + y \cdot P(X,Y)$ and highlight the contributions:

X	Y	$P(X,Y)$	$x \cdot P(X,Y)$	$y \cdot P(X,Y)$	$(x + y) \cdot P(X,Y)$
1	4	0.1	0.1	0.4	$0.1 + 0.4 = 0.5$
1	5	0.3	0.3	1.5	$0.3 + 1.5 = 1.8$
2	4	0.2	0.4	0.8	$0.4 + 0.8 = 1.2$
2	5	0.4	0.8	2.0	$0.8 + 2.0 = 2.8$

Summing up the contributions:

- Sum of $x \cdot P(X,Y)$: $0.1 + 0.3 + 0.4 + 0.8 = 1.6$
- Sum of $y \cdot P(X,Y)$: $0.4 + 1.5 + 0.8 + 2.0 = 4.7$
- Sum of $(x + y) \cdot P(X,Y)$: $0.5 + 1.8 + 1.2 + 2.8 = 6.3$ (This is $E[X + Y]$)

Note that for each term:

$(x + y) \cdot P(X,Y) = x \cdot P(X,Y) + y \cdot P(X,Y)$

Conclusion:

We can see that:

$E[X] = 1.6$, $E[Y] = 4.7$, and $E[X + Y] = E[X] + E[Y] = 1.6 + 4.7 = 6.3$.

By coloring the values with shades of light blue for $x \cdot P(X, Y)$ and shades of pink for $y \cdot P(X, Y)$, and varying their brightness, it becomes clear how each term contributes to the expectations.