

# Approximate Counting

What if you wanted a counter that could count up to the number of atoms in the universe, but you wanted to store the counter in 8 bits? You could use the amazing probabilistic algorithm described below! In this example we are going to show that the expected return value of `stochastic_counter(4)`, where `count` is called four times, is in fact equal to four.

```
def stochastic_counter(true_count):
    n = -1
    for i in range(true_count):
        n += count(n)
    return 2 ** n # 2^n, aka 2 to the power of n

def count(n):
    # To return 1 you need n heads. Always returns 1 if n is <= 0
    for i in range(n):
        if not coin_flip():
            return 0
    return 1

def coin_flip():
    # returns true 50% of the time
    return random.random() < 0.5
```

Let  $X$  be a random variable for the value of  $n$  at the end of `stochastic_counter(4)`. Note that  $X$  is not a binomial because the probabilities of each outcome change.

Let  $R$  be the return value of the function.  $R = 2^X$  which is a function of  $X$ . Use the law of unconscious statistician

$$E[R] = \sum_x 2^x \cdot P(X = x)$$

We can compute each of the probabilities  $P(X = x)$  separately. Note that the first two calls to count will always return 1. Let  $H_i$  be the event that the  $i$ th call returns 1. Let  $T_i$  be the event that the  $i$ th call returns 0.  $X$  can't be less than 1 because the first two calls to count always return 1.  $P(X = 1) = P(T_3, T_4) \setminus P(X = 2) = P(H_3, T_4) + P(T_3, H_4) \setminus P(X = 3) = P(H_3, H_4)$

At the point of the third call to count,  $n = 1$ . If  $H_3$  then  $n = 2$  for the fourth call and the loop runs twice.

$$\begin{aligned} P(H_3, T_4) &= P(H_3) \cdot P(T_4|H_3) \\ &= \frac{1}{2} \cdot \left(\frac{1}{2} + \frac{1}{4}\right) \end{aligned}$$

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If  $T_3$  then  $n = 1$  for the fourth call.

$$\begin{aligned} P(T_3, H_4) &= P(T_3) \cdot P(H_4|T_3) \\ &= \frac{1}{2} \cdot \frac{1}{2} \end{aligned}$$

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Plug everything in:

$$\begin{aligned}
 E[R] &= \sum_{x=1}^3 2^x \cdot P(X=x) \\
 &= 2 \cdot \frac{1}{4} + 4 \cdot \frac{5}{8} + 8 \cdot \frac{1}{8} \\
 &= 4
 \end{aligned}$$