CO759 Term Project

Cutting Plane Method for Matching and Its Application in Solving Uncapacitied b-Matching via Data Reduction

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Uncapacitied b-matching

First, what is a matching?

In graph G=(V,E), a matching M is a subset of edges in E that each vertex in V only meet M at most once.

If each vertex in V meets M exactly once, the it is called a perfect matching

The indicator for matching:

$$x_e = \begin{cases} 1 & \text{,if e is in M} \\ 0 & \text{,otherwise} \end{cases}$$

The Integer Programming for Perfect Matching

$$min\ c_e^T x_e$$
$$x(\delta(v)) = 1, \forall v \in V$$
$$x = \{0, 1\};$$

The IP is generally hard to solve, so we take the relaxation of it

The LP Relaxation

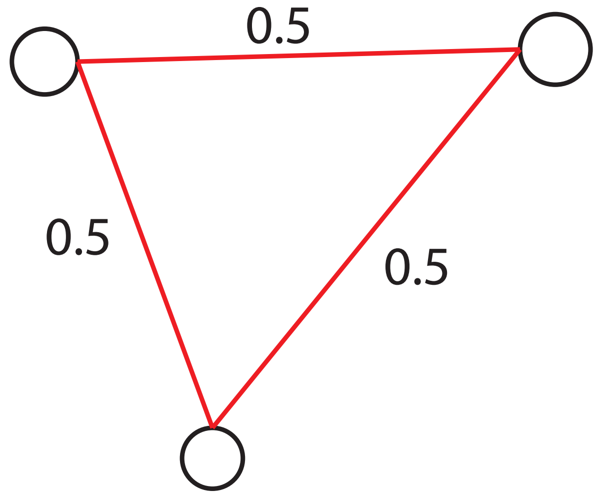
(P) $\quad min\ c_e^T x_e$

$$x(\delta(v)) = 1, \forall v \in V$$

$$x \geq 0;$$

However, solving (P) will not give us what we want

An counter example:



In order to avoid this, we need cutting planes:

Blossom Inequality

$$x(\gamma(S)) \leq \frac{|S|-1}{2}$$

$$\forall S \subset V, S\ is\ odd$$

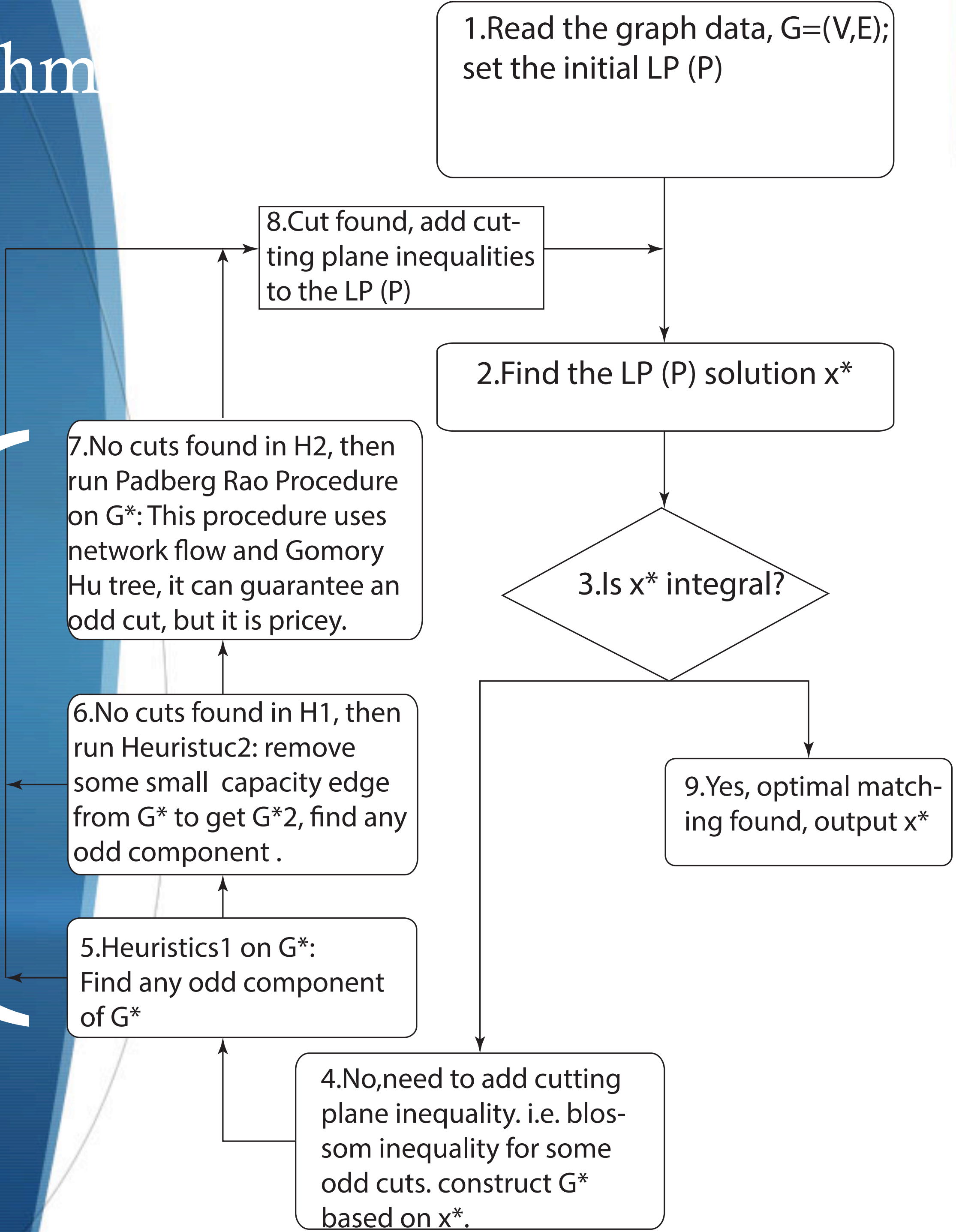
Or equivalently

$$x(\delta(S)) \geq 1$$

$$\forall S \subset V, S\ is\ odd$$

Cutting Plane Algorithm for Perfect Matching

This is for finding odd cuts that violate the blossom inequality, add them as cutting plane.



Heuristics

The LP graph G* is a graph with same vertices as G, and non-zero LP solution edges of G. The capacity of edges of G* is the LP solution .

Heuristics1 use DFS find odd components of G*. These components have cut capacity zero , and the odd components will definitely violate the blossom inequalities.

Heuristics2 is based on Heuristic1. But remove some weak edge i.e. edges with capacity less than 0.3. Heuristics2 will give some odd cut with very small cut capacity.

Death of Heuristics

Heuristics are cheap. Only O(n) complexity.
However, they could **fail**!
When there are lots of Combs.
When there are no odd componenets.

What should we do it they **fail**?
We have an ultimate weapon of massive destruction :

Padberg and Rao Procedure

But before it, we need another definition

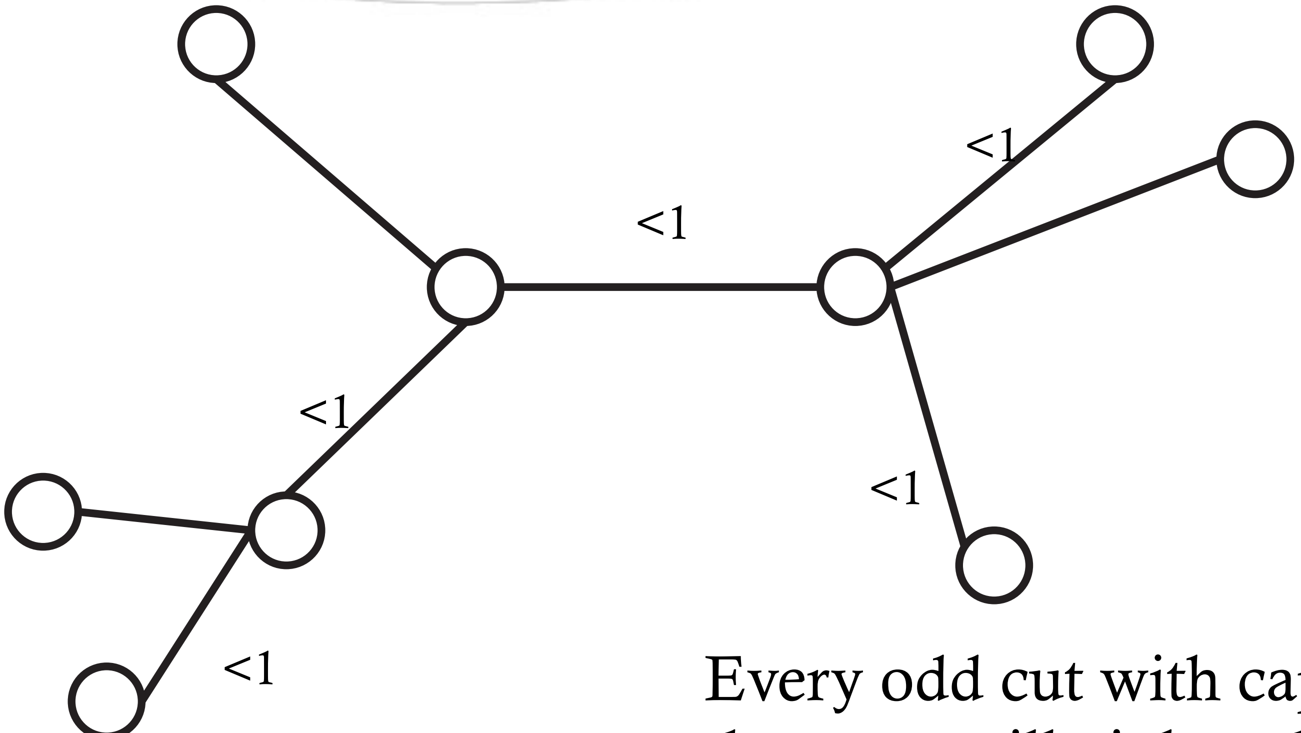
Gomory-Hu Tree

Let MAXFLOW(u,v, G*) be the maximum flow between u, v in G*
Let PATH(u,v,T) be the unique path between u,v in tree T
Let GH(G*) be the Gmoory-Hu tree of G* which is defined as:

$$V(GH(G^*)) = V(G^*), \forall u, v \in V(G^*),$$
$$if\ uv \in GH(G^*), then\ w_{uv} = MAXFLOW(u, v, G^*);$$
$$if\ uv \notin GH(G^*),$$
$$then\ min\{w_{ij} : ij \in PATH(u, v, GH(G^*)\} = MAXFLOW(u, v, G^*);$$
$$let\ mn = \{mn \in GH(G^*) : w_{mn} = min\{w_{ij} : ij \in PATH(u, v, GH(G^*))\}\},$$
$$then\ disconnect\ mn\ inGH(G^*)\ gives\ two\ components\ and\ the\ components$$
$$gives\ a\ minimum\ capacity\ u - v\ cut\ in\ G^*$$

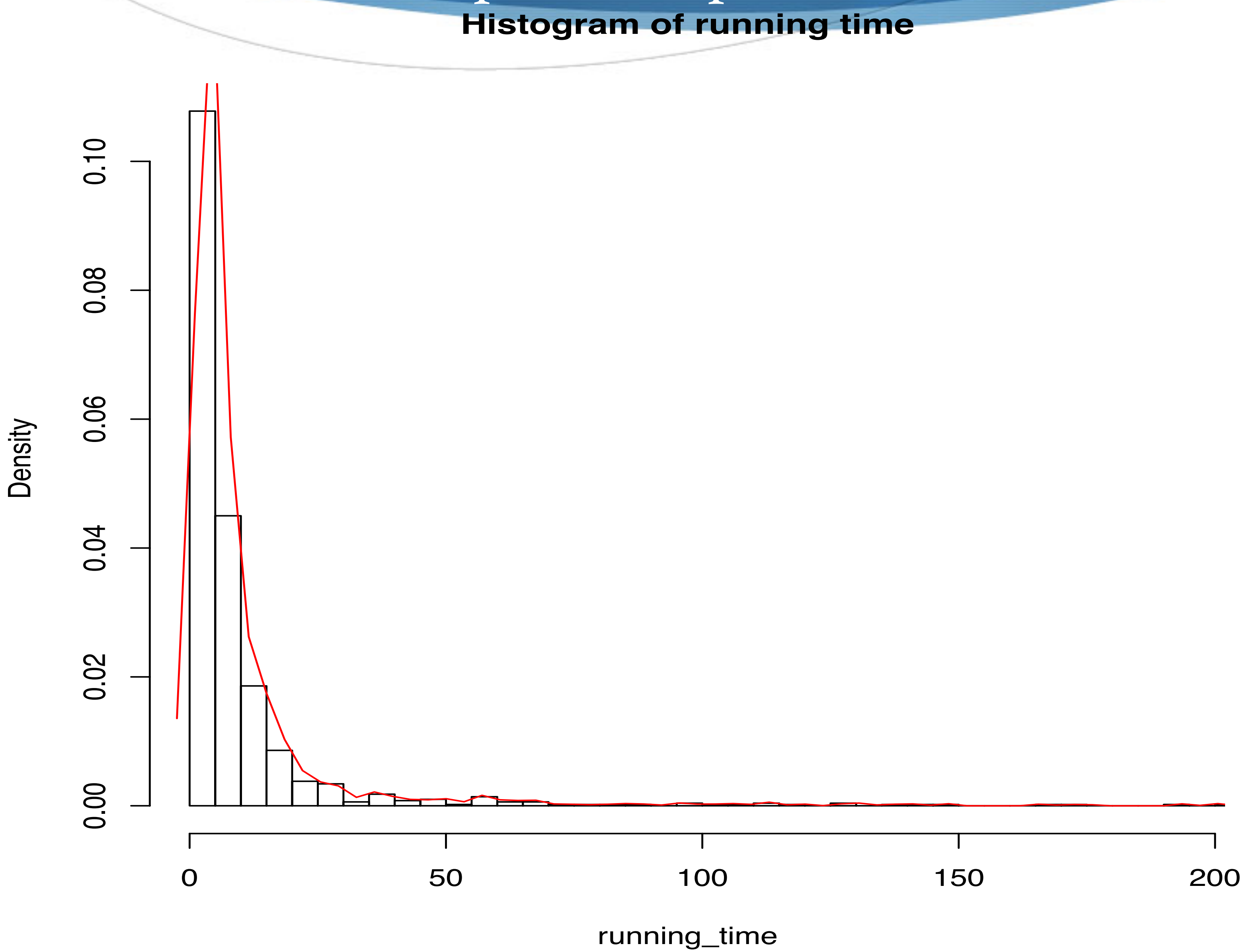
Padberg and Rao $O(V^3E^2)$

GH(G*):

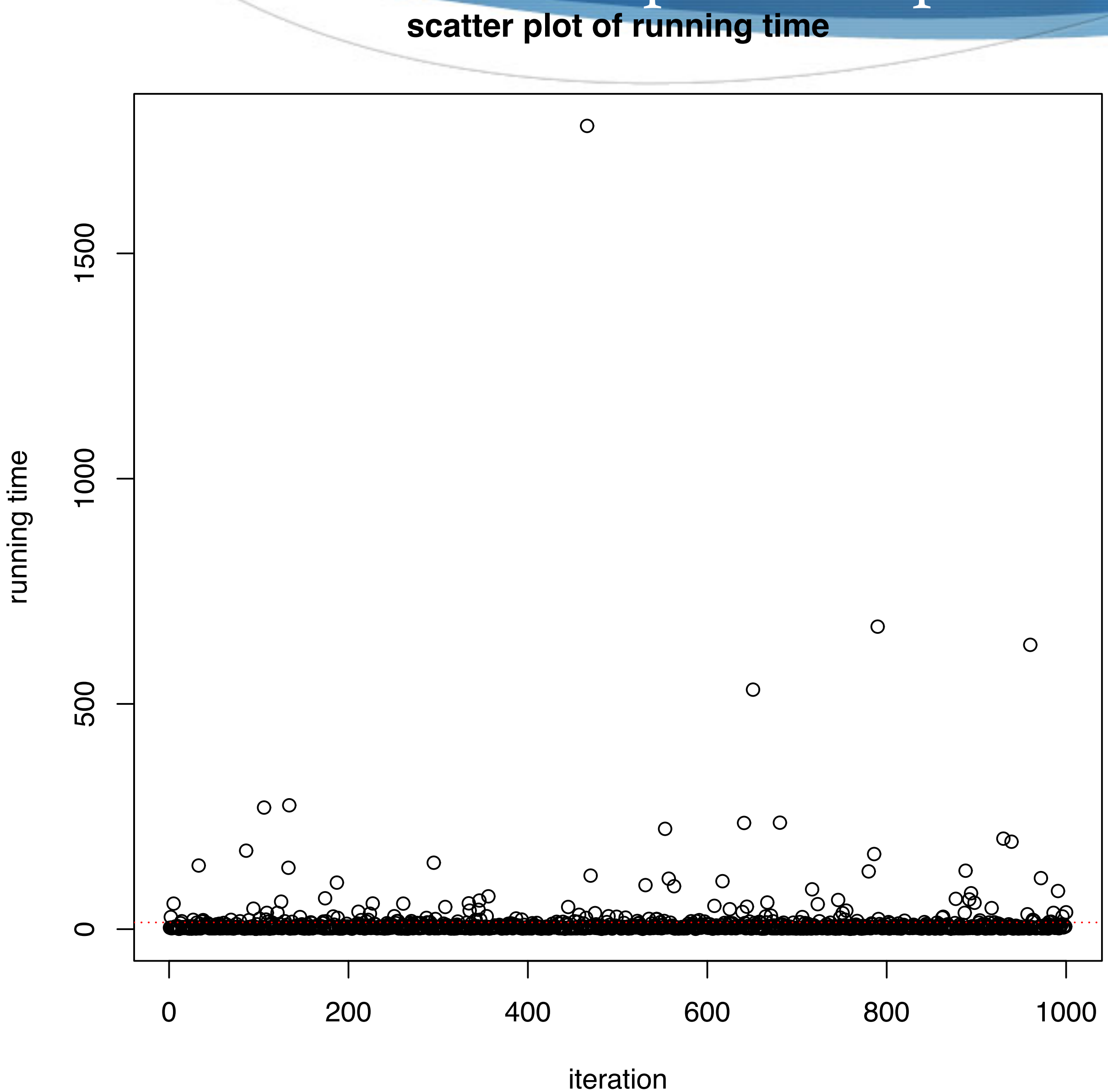


Every odd cut with capacity less than one will violate the blossom inequality.

Computational Test on 1000 Random Complete Graph K1000



Computational Test on 1000 Random Complete Graph K1000



Median : 4.68

Mean: 14.92s

3rd Quantile: 9.344s

Peak:1783s

Uncapacitied b-matching

Just like the perfect matching, the b-matching is a matching Mb such that

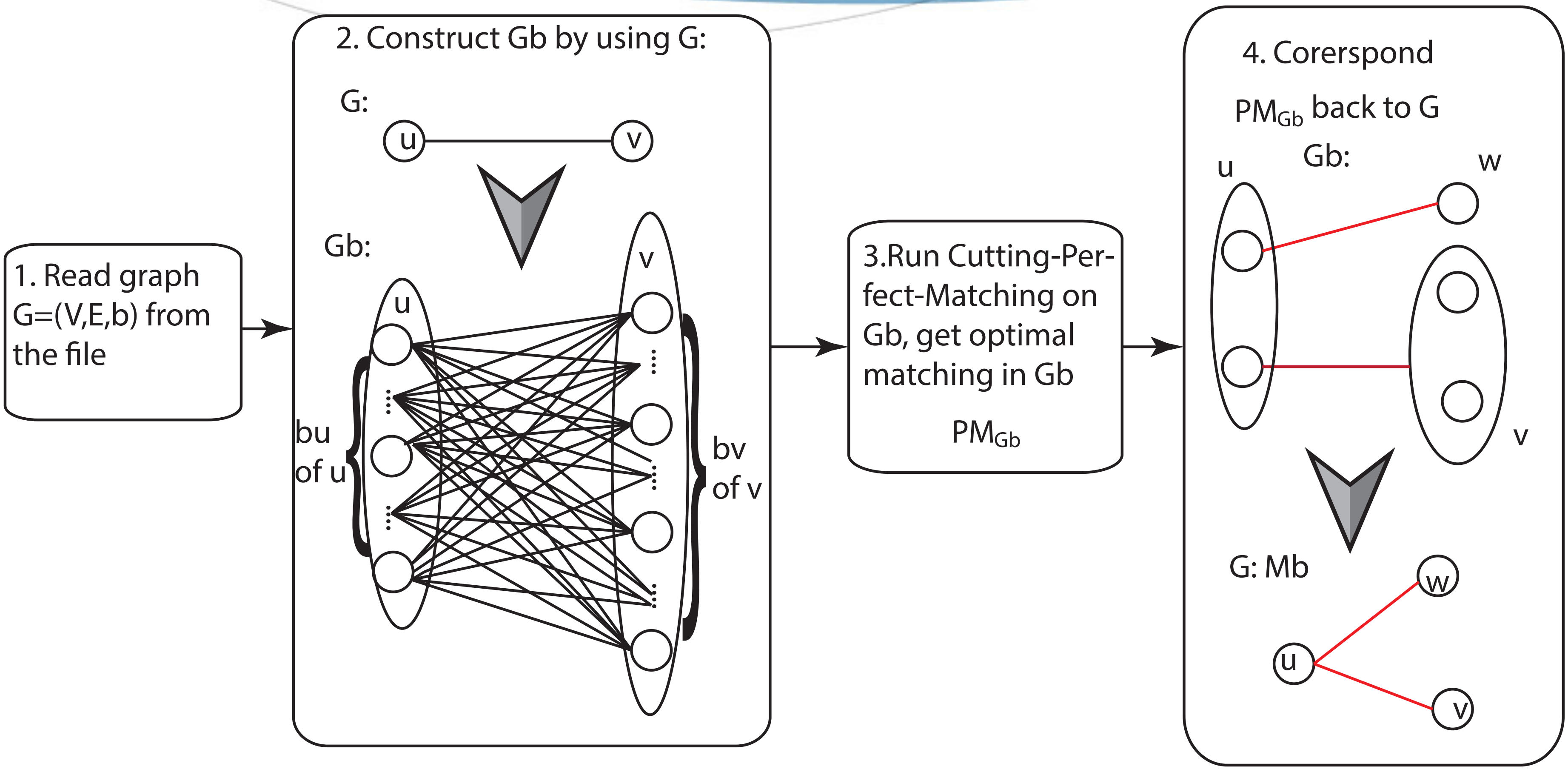
$$|M_b \cap v| = b_v, \forall v \in V$$

Where b is a vector with dim(b)=|V|

If each edge in Mb can take arbitrary value, then this is an uncapacitied b-matching.

One can reduce b-matching to perfect-matching.

Graph Reduction



Cons: Increase the usage of storage.

Pros: Strong polynomial time

Looking forword

- Use blossom algorithm instead of cutting plane. More stable.
- Modify the graph further to solve capacitied b-marching

Thank you!