Homework 1

1) EX.7 P.14
Since $f(0) = 2(0) + 1 = 1$ & $g(0) = 1 + 4(0) - 2(0)^2 = 1$, then $f(0) = 1 = g(0)$
And $f(1) = 2(1) + 1 = 3$ & $g(0) = 1 + 4(1) - 2(1)^2 = 3$, then $f(1) = 3 = g(1)$
Therefore, by definition $f = g$
Similarly…
Since $f(0) + g(0) = 1 + 1 = 2$ & $h(0) = 5^0 + 1 = 2$, then $f(0) + g(0) = 2 = h(0)$
And $f(1) + g(1) = 3 + 3 = 6$ & $h(1) = 5^1 + 1 = 6$, then $f(1) + g(1) = 6 = h(1)$
Therefore, by definition $f + g = h$

2) EX. 19 P.21
$W_1 \cup W_2$ *is a subspace of* $V \Leftrightarrow W_1 \subseteq W_2$ *or* $W_2 \subseteq W_1$
$\Longrightarrow$
Let's assume the contrary, that is that $W_1 \nsubseteq W_2$, *and* $W_2 \nsubseteq W_1$. We know $W_1 \cup W_2$ is a subspace of V. Let $x \in W_1 \backslash W_2$, *and* $y \in W_2 \backslash W_1$, then $x + y \in W_1$ *or* $x + y \in W_2$.
If $x + y \in W_1$, then $y = (x + y) - x$, which $\in W_1$ (based on Theorem 1.3 (b))
And similarly, if $x + y \in W_2$, then $x = (y + x) - y$, which $\in W_2$
Since both lead to a contradiction this implies that our original assumption was incorrect and that it follows that $W_1 \subseteq W_2$ *or* $W_2 \subseteq W_1$
$\Longleftarrow$
Since we know that, $W_1 \subseteq W_2$ *or* $W_2 \subseteq W_1$, then $W_1 \cup W_2 = W_1$ *or* $W_1 \cup W_2 = W_2$. In either case, $W_1 \cup W_2$ is a subspace of V, by the fact that W_1 *and* W_2 are subspaces of V.

3) Ex. 28 p.23
As Theorem 1.3 condition hold, we know that W_1 of all nxn matrices (including skew symmetric) are a subspace of $M_{nxn}(F)$.
We also know that the zero matrix is equal to its transpose and thus belongs to W_1.
Based on previous examples, it is easily proven that for any matrices A, B and any scalars a, b, $(aA + bB)^t = aA^t + bB^t$.
If we say $A \in W_1$ *and* $B \in W_1$, then it follows that $A^t = -A$ and $B^t = -B$. Therefore, $(A + B)^t = A^t + B^t = -(A + B)$, which then means $-(A + B) \in W_1$.
Finally, $A \in W_1$, then $A^t = -A$, then for any $a \in F$, we know $(aA)^t = aA^t = -aA$, therefore, $-aA \in W_1$

4) Ex. 29 p.23
Because we know W_1 *and* W_2 are closed under addition and scalar multiplication, it follows that $W_1 \oplus W_2 = M_{nxn}(F)$. We know, $W_1 \cap W_2 = \{0\}$ and similar to Q28, W_1 *and* W_2 are both subspaces of $M_{nxn}(F)$, W_1 contains matrix A, a lower triangular matrix. Therefore, $W_1 \oplus W_2 = M_{nxn}(F)$.

5) EX. 3 P.41
$$a\begin{pmatrix}1 & 1\\0 & 0\\0 & 0\end{pmatrix} + b\begin{pmatrix}0 & 0\\1 & 1\\0 & 0\end{pmatrix} + c\begin{pmatrix}0 & 0\\0 & 0\\1 & 1\end{pmatrix} + d\begin{pmatrix}1 & 0\\1 & 0\\1 & 0\end{pmatrix} + e\begin{pmatrix}0 & 1\\0 & 1\\0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\\0 & 0\end{pmatrix}$$
$a + d = 0, a + e = 0, b + d = 0, b + e = 0, c + d = 0, c + e = 0$
We choose $a = b = c = 1$ and $d = e = -1$
Therefore, it is linearly dependent

6) EX. 9 P.55
Let $(a_1, a_2, a_3, a_4) \coloneqq c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4 \in F$. We can compute,
$c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4 = c_1, (c_1 + c_2), (c_1 + c_2 + c_3), (c_1 + c_2 + c_3 + c_4)$
Then,
$a_1 = c_1, a_2 = c_1 + c_2, a_3 = c_1 + c_2 + c_3, a_4 = c_1 + c_2 + c_3 + c_4$

We then solve for,
$c_1 = a_1, c_2 = a_2 - a_1, c_3 = a_3 - a_2, c_4 = a_4 - a_3$
Therefore, we conclude,
$(a_1, a_2, a_3, a_4) = a_1u_1 + (a_2 - a_1)u_2 + (a_3 - a_2)u_3 + (a_4 - a_3)u_4$

7) EX. 17 P.56
By definition, a skew-symmetric matrix is $A^T = -A$, in terms of entries in the matrix this means $a_{ji} = a_{-ij}$.
The basis of a matrix M_{mxn} is $\{E^{ij}: 1 \le i \le m, 1 \le j \le n\}$, therefore, the basis for skew-symmetric matrix M_{nxn} is $\{E^{ij} - E^{ji}: 1 \le i \le m, 1 \le j \le n\}$.
By definition we know, the vector space $M_{mxn}(F)$ has dimension mn, thus we know that a vector space $M_{nxn}(F)$ has dimension $\frac{n^2+n}{2}$ (n^2 – both identical sides of the diagonal, n the diagonal itself).
It then follows that a skew-symmetric vector space has the dimension $\frac{n^2-n}{2}$, ($-n$ to remove the diagonal, which is all zeroes).

8) EX. 10 P.75
$T(2,3) = 3\big(T(1,1)\big) - T(1,0) = (6,15) - (1,4) = (5,11)$
From Example 12, we know that $T: R^2 \to R^2$ can be defined by
$T(a_1, a_2) = (a_1 + a_2, a_1)$
Therefore, we can see that $N(T) = \{0\}$; so T is one-to-one.

9) EX. 3 P.84
$x_1 + 0 + 2x_3 = 1 \longrightarrow x_1 = -\frac{1}{3}$
$x_1 + x_2 + 2x_3 = 1 \longrightarrow x_2 = 0$
$0 + x_2 + 3x_3 = 2 \longrightarrow x_3 = \frac{2}{3}$
$T(1,0) = (1,1,2) = -\frac{1}{3}(1,1,0) + 0(0,1,1) + \frac{2}{3}(2,2,3)$
$x_1 + 0 + 2x_3 = -1 \longrightarrow x_1 = -1$
$x_1 + x_2 + 2x_3 = 0 \longrightarrow x_2 = 1$
$0 + x_2 + 3x_3 = 1 \longrightarrow x_3 = 0$
$T(0,1) = (-1,0,1) = -1(1,1,0) + 1(0,1,1) + 0(2,2,3)$
$x_1 + 0 + 2x_3 = -1 \longrightarrow x_1 = -\frac{7}{3}$
$x_1 + x_2 + 2x_3 = 1 \longrightarrow x_2 = 2$
$0 + x_2 + 3x_3 = 4 \longrightarrow x_3 = \frac{2}{3}$
$T(1,2) = (-1,1,4) = -\frac{7}{3}(1,1,0) + 2(0,1,1) + \frac{2}{3}(2,2,3)$
$x_1 + 0 + 2x_3 = -1 \longrightarrow x_1 = -\frac{11}{3}$
$x_1 + x_2 + 2x_3 = 2 \longrightarrow x_2 = 3$
$0 + x_2 + 3x_3 = 7 \longrightarrow x_3 = \frac{4}{3}$
$T(2,3) = (-1,2,7) = -\frac{11}{3}(1,1,0) + 3(0,1,1) + \frac{4}{3}(2,2,3)$

$$[T]_\beta^\gamma = \begin{pmatrix} -\frac{1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{pmatrix}$$

$$[T]_\alpha^\gamma = \begin{pmatrix} -\frac{7}{3} & -\frac{11}{3} \\ 2 & 3 \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix}$$

10) <u>EX.4 P.84</u>

$$T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} = (1+0) + 2(0)x + 0x^2 = 1 + 0x + 0x^2$$
$$T\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} = (0+1) + 2(0)x + 1x^2 = 1 + 0x + 1x^2$$
$$T\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} = (0+0) + 2(0)x + 0x^2 = 0 + 0x + 0x^2$$
$$T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = (0+0) + 2(1)x + 0x^2 = 0 + 2x + 0x^2$$

$$[T]_\alpha^\gamma = \begin{pmatrix}1 & 1 & 0 & 0\\0 & 0 & 0 & 2\\0 & 1 & 0 & 0\end{pmatrix}$$