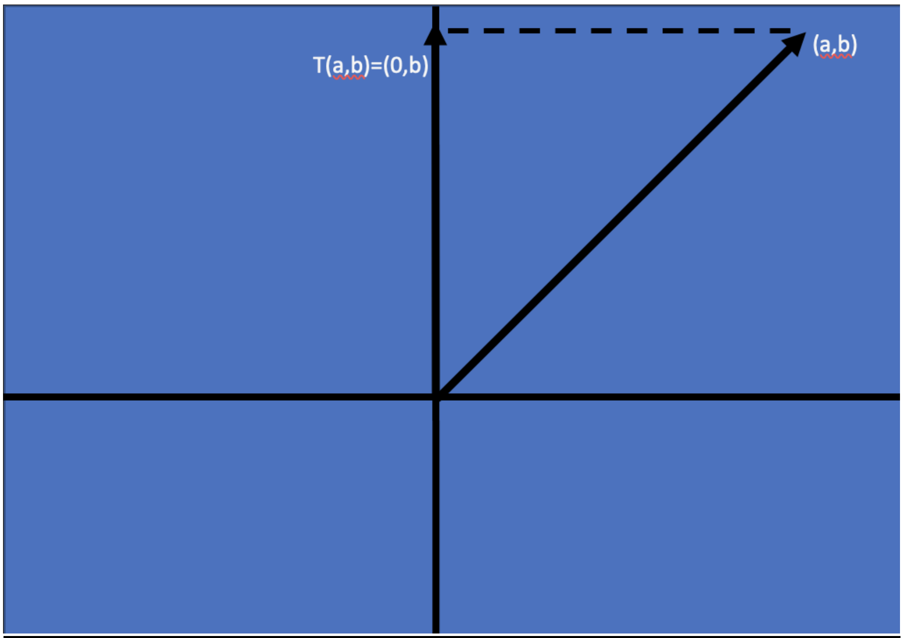
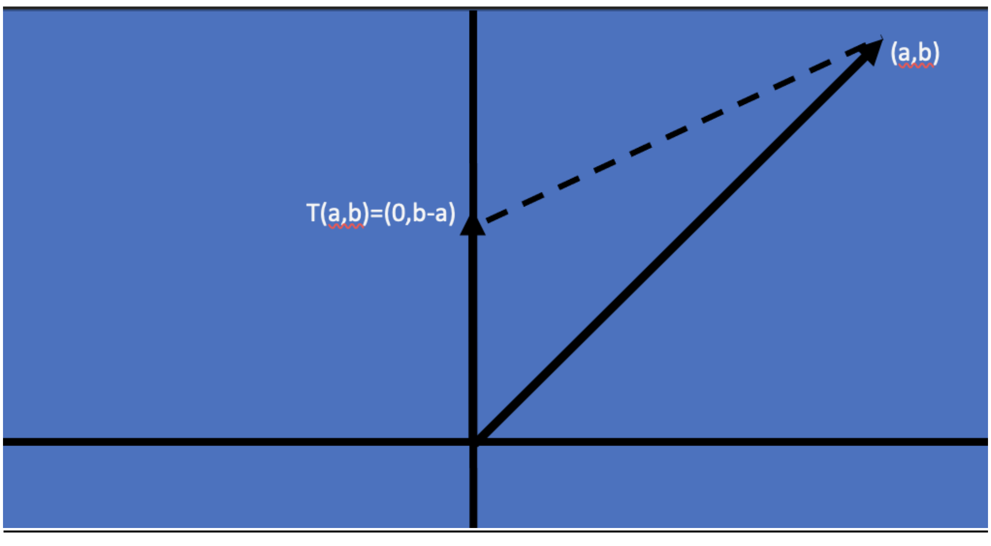
Homework 2

1. p.76-77 Ex.24(a)(b)
   a. V is the vector space, and $W_1, and\ \ W_2$ are subspaces of V, such that $V = W_1 \oplus W_2$
   For $T: V \to V$, we have $T(a, b)$,
   meaning that [y-axis] $W_1 = \{(a, 0) : a\ \in \mathbb{R}\}$ and [x-axis] $W_2 = \{(0, b) : b\ \in \mathbb{R}\}$,
   as $= W_1 \oplus W_2$ and $(\mathrm{a}, \mathrm{b}) = (\mathrm{a}, 0) + (0, \mathrm{b})$.
   $\therefore T(a, b) = T(0, b)$ [projection on W_1, along W_2].



   b. V is the vector space, and $W_1, and\ \ W_2$ are subspaces of V, such that $V = W_1 \oplus W_2$
   For $T: V \to V$, we have $T(a, b)$.
   We are given that, $W_2 = \{(s, s) : s\ \in \mathbb{R}\}$ which is $W_2 = \{(a, a) : a\ \in \mathbb{R}\}$,
   meaning that [y-axis] $W_1 = \{(0, b - a) : a, b\ \in \mathbb{R}\}$ and $[L]$ $W_2 = \{(a, a) : a\ \in \mathbb{R}\}$,
   as $= W_1 \oplus W_2$ and $(\mathrm{a}, \mathrm{b}) = (0, \mathrm{b} - \mathrm{a}) + (\mathrm{a}, \mathrm{a})$.
   $\therefore T(a, b) = T(0, b - a)$ [projection on W_1, along W_2].



2. p.77 Ex.28

subspace W of V is said to be T-invariant if $T(x) \in W$ for every $x \in W$, that is $T(W) \subseteq W$.

   1. $T(0) = 0 \in \{0\}$ $\therefore$ $\{0\}$ is T-invariant
   2. $T: V \to V$ $\therefore$ V is T-invariant
   3. If, $x \in R(T) \subseteq V$, then $x \in V$. Therefore, we have that, $T(x) \in R(T)$, hence it follows that $T\big(R(T)\big) \subseteq R(T)$ $\therefore$ $R(T)$ is T-invariant
   4. If, $T: V \to V$, then $N(T) = \{0\}$. Then if, $x \in N(T)$, then $T(x) = 0$.
      Also, $T(0) = 0$, and hence, $0 \in N(T)$
      Therefore, $T(x) = 0 \in N(T), \forall x \in N(T) \to T\big(N(T)\big) \subseteq N(T)$, $\therefore$ $N(T)$ is T-invariant

3. p.85 Ex.11

dim$(W) = k$, dim$(V) = n$, and W is T-invariant (i.e., $T(x) \in W$ for every $x \in W$, that is $T(W) \subseteq W$).

We have vector space V, and its T-invariant subspace W, we take $\{v_1, v_2 \dots v_k\}$ to be the basis of W, we know we can extend it to be $\beta = \{v_1, v_2, \dots v_n\}$, the basis of V.

Since $T(W) \subseteq W$, $T\big(v_j\big) \in W, \forall\, 1 \le j \le k$, that is $T\big(v_j\big) = \sum_{i=1}^{k} a_{ij} v_i$ for unique scalars a_{ij}.

Let $A \coloneqq [T]_\gamma,\ k \times k\ matrix$ and O is the $(n-k) \times k\ zero\ matrix$.

Then, $[T]_\beta = \begin{pmatrix} A & B \\ O & C \end{pmatrix}$.

4. p.97 Ex.9

We take $[U]_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $[T]_\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, we then define:

$U: F^2 \to F^2$ by $(a,b) \to (b,b)$ and $T: F^2 \to F^2$ by $(a,b) \to (a+b, 0)$

If we say that, $x \in F^2$, then $UT(x) = U(a,b) = U(a+b,0) = (0,0), \forall a,b\ \in F$, which then gives us that $UT = T_0$.

We then take, $TU(x) = TU(a,b) = T(b,b) \neq T_o(a,b)$

$\therefore TU \neq T_0$

We then use the above example…

$A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$,

then we have $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = T_0$ and $BA = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \neq T_0$.

5. p.97 Ex.11

Let $x \in V$ such that $T(x) = 0$. We can assume thar $T^2 = T$. We then know that as $T(x) = 0$, then $T\big(T(x)\big) = 0$. Hence, $x \in N(T)$.

Since we know that $V \to V$ is linear, then we know $x \in R(T)$. Finally, since $T(x)$ consists of $R(T)\ and\ N(T)$.

$\therefore R(T) \subseteq N(T)$, when $T^2 = 0 = T_0$

6. p.97 Ex.13

We have, $tr(AB)_{ii} = \sum_{k=1}^{n} A_{ik}B_{ki} = \sum_{k=1}^{n} B_{ki}A_{ik} = \sum_{k=1}^{n} (BA)_{kk} = tr(BA)_{kk} = tr(BA)$

Let: $A = \begin{pmatrix} 3 & 7 & 9 \\ 5 & 2 & 4 \\ 1 & 6 & 8 \end{pmatrix}, B = \begin{pmatrix} 9 & 1 & 8 \\ 2 & 7 & 3 \\ 6 & 4 & 5 \end{pmatrix}$

$$AB = \begin{pmatrix} 3 & 7 & 9 \\ 5 & 2 & 4 \\ 1 & 6 & 8 \end{pmatrix} \times \begin{pmatrix} 9 & 1 & 8 \\ 2 & 7 & 3 \\ 6 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 95 & 88 & 90 \\ 73 & 35 & 66 \\ 69 & 75 & 66 \end{pmatrix}$$

$Tr(AB) = 95 + 35 + 66 = 196$

$BA = \begin{pmatrix} 9 & 1 & 8 \\ 2 & 7 & 3 \\ 6 & 4 & 5 \end{pmatrix} \times \begin{pmatrix} 3 & 7 & 9 \\ 5 & 2 & 4 \\ 1 & 6 & 8 \end{pmatrix} = \begin{pmatrix} 40 & 113 & 149 \\ 44 & 46 & 70 \\ 43 & 80 & 110 \end{pmatrix}$

$Tr(BA) = 40 + 46 + 110$

$\therefore tr(AB) = tr(BA)$

Since we know that $A_{ij}^T = A_{ji}$, it can be easily seen in the examples below…

$A = \begin{pmatrix} 2 & 5 & 7 \\ 3 & 2 & 4 \\ 9 & 7 & 5 \end{pmatrix}, A^T = \begin{pmatrix} 2 & 3 & 9 \\ 5 & 2 & 7 \\ 7 & 4 & 5 \end{pmatrix}$

$Tr(A) = 2 + 2 + 5 = 9$

$Tr(A^T) = 2 + 2 + 5 = 9$

$\therefore tr(A) = tr(A^T)$

7. <u>p.116 Ex.2(d)</u>

   $\beta = \{(-4,3),(2,-1)\}\ and\ \beta' = \{(2,1),(-4,1)\}$

   $(2,1) = 2(-4,3) + 5(2,-1)\ \ and\ (-4,1) = -1(-4,3) - 4(2,-1)$

   $\therefore [I]_{\beta'}^{\beta} \begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}$

8. <u>p.116 Ex.4</u>

   $[T]_{\beta'} = [T]_{\beta}^{\beta'}[T]_{\beta}[T]_{\beta'}^{\beta} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ -2 & -5 \end{pmatrix} =$

   $\begin{pmatrix} 8 & 13 \\ -5 & -9 \end{pmatrix} = [T]_{\beta'}$

9. p.141 Ex.3(a)

   $y'' + 2y' + y = 0$ gives $t^2 + 2t + 1 = (t+1)(t+1)\ which\ gives\ us \to \{e^{-t}, te^{-t}\}$

   $y(0) = 3, y'(0) = 2$

10. p.141 Ex.3(b)

    $y''' - y' = 0$ gives $t^3 - t = t(t-1)(t+1)\ which\ give\ us \to \{1, e^t, e^{-t}\}$

    $y(0) = 9, y'(0) = 1, y''(0) = 5$