Homework 3

1. <u>p.168 Ex.17</u>

As given in the question, $B = 3 \times 1$ matrix, $C = 1 \times 3$ matrix, thus giving us…

$BC = 3 \times 3$ matrix, with rank at most 1

So as an example we let…

$B = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}, C = \begin{pmatrix} 3 & 5 & 7 \end{pmatrix}$, then $BC = \begin{pmatrix} 9 & 15 & 21 \\ 15 & 25 & 35 \\ 21 & 35 & 49 \end{pmatrix}$

Based on our example we can clearly see that the rank of $BC = 1$, as BC has at most one independent row or column.

<u>Converse: $A = 3 \times 3$ matrix, with rank 1</u>

If $A_{3\times 3}$ has rank 1, this means according to theorem 3.5 that the j_{th} column of A is equal to the maximum linear set. Hence, the remaining two columns can be found by multiplying some scalar value (which could be 0).

Therefore, we can assume since $B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, C = \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix}$,

then $A = BC = \begin{pmatrix} b_1c_1 & b_1c_2 & b_1c_3 \\ b_2c_1 & b_2c_2 & b_2c_3 \\ b_3c_1 & b_3c_2 & b_3c_3 \end{pmatrix}$

2. <u>p.168 Ex.21</u>

We know that $rank(A) = m$, and by theorem 3.4, then $rank(I_m) = m$.

$R(L_{AB}) = R(L_A L_B) = L_A L_B(F^n) = L_A\big(L_B(F^n)\big) = L_A(F^n) = R(L_A)$, since L_B is onto.

$\therefore rank(AB) = \dim\big(R(L_{AB})\big) = \dim\big(R(L_A)\big) = rank(A) = m$

Finally by theorem 3.6, we know that rank $r \leq m, r \leq n$, and where $r = m$, we have $O = \begin{pmatrix} I_m & O_1 \\ O_2 & O_3 \end{pmatrix}$, thus $AB = I_m$.

3. <u>p.180 Ex.2 (g)</u>

$x_1 + 2x_2 + x_3 + x_4 = 0$

$x_2 - x_3 + x_4 = 0$

$x_2 = x_3 - x_4$

$x_1 = -2(x_3 - x_4) - x_3 - x_4 = -3x_3 + x_4$

We can set $x_3 = s, x_4 = t$, giving us…

$x_1 = -3s + t$

$x_2 = s - t$

So we have…

$\{(-3s + t, s - t, s, t) \mid s, t \in \mathbb{R}\}$

$\{s(-3,1,1,0) + t(1,-1,0,1) \mid s, t \in \mathbb{R}\}$

So the basis for the solution space is…

$\begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

And the dimension is…

$\dim(k) = 4 - 2 = 2$

4. p.180 Ex.3 (g)
$x_1 + 2x_2 + x_3 + x_4 = 1$
$x_2 - x_3 + x_4 = 1$
Subtracting equation 2 from equation 1, we get…
$x_1 + x_2 + 2x_3 = 0$
Giving us…
$x_1 = -x_2 - 2x_3$
Then we put equation 2 in terms of x_2, x_3 (to match equation 1), and we get…
$x_4 = 1 - x_2 + x_3$
Which then gives us…
$\begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \\ 1 - x_2 + x_3 \end{pmatrix}$, which then gives us our solution set $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

Our span is $\begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$.

5. p.180 Ex.6
we set…
$a + b = 1, 2a - c = 11$
We set $a = t$, giving us….
$a = t$
$b = 1 - t$
$c = 2t - 11$
$T^{-1}\{(1,11)\} = \left\{ \begin{pmatrix} 0 \\ 1 \\ -11 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} t \in \mathbb{R} \right\}$

6. p.197 Ex.12(a)(b)
(a) Using given equations…
$$x_1 - x_2 + 2x_4 + 3x_5 + x_6 = 0$$
$$2x_1 - x_2 - x_3 + 3x_4 - 4x_5 + 4x_6 = 0$$
We can make the augmented matrix…
$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then changing it to reduced row echolen form, we get
$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
Therefore, because the first and second columns of B (our reduced row echelon form matrix), are e_1 & e_2, from theorem 3.16, we can conclude that S, (the first two columns of matrix A), are a linearly independent subset of V.

(b) We first obtain basis β for V, which gives us…
$\beta = \{(1,1,1,0,0,0),(-1,1,0,1,0,0),(1,-2,0,0,1,0),(-3,-2,0,0,0,1)\}$
Which we can then (combine with matrix A above) and use to obtain the augmented matrix…

$\begin{pmatrix} 0 & 1 & 1 & -1 & 1 & -3 \\ -1 & 0 & 1 & 1 & -2 & -2 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, which in reduced row echelon form gives us…

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ the basis for V containing S are…
$\{(0,-1,0,0,1,1,0),(1,0,1,1,1,0),(-1,1,0,1,0,0),(-3,-2,0,0,0,1)\}$

7. p.222 Ex.19

$$\begin{pmatrix} i & 2 & -1 \\ 3 & 1+i & 2 \\ -2i & 1 & 4-i \end{pmatrix} \xrightarrow{R1=\frac{R1}{i}} \begin{pmatrix} 1 & \frac{2}{i} & -\frac{1}{i} \\ 3 & 1+i & 2 \\ -2i & 1 & 4-i \end{pmatrix} \xrightarrow{R2=R2-3R1} \begin{pmatrix} 1 & \frac{2}{i} & -\frac{1}{i} \\ 0 & \frac{-7+i}{i} & \frac{3+2i}{i} \\ -2i & 1 & 4-i \end{pmatrix}$$

$$\xrightarrow{R3=R3-(-2i(R1))} \begin{pmatrix} 1 & \frac{2}{i} & -\frac{1}{i} \\ 0 & \frac{-7+i}{i} & \frac{3+2i}{i} \\ 0 & 5 & 2-i \end{pmatrix} \xrightarrow{R1=R1\times i} \begin{pmatrix} i & 2 & -1 \\ 0 & \frac{-7+i}{i} & \frac{3+2i}{i} \\ 0 & 5 & 2-i \end{pmatrix} \xrightarrow{R2=\frac{R2}{\frac{-7+i}{i}}} \begin{pmatrix} i & 2 & -1 \\ 0 & 1 & \frac{2-3i}{1+7i} \\ 0 & 5 & 2-i \end{pmatrix}$$

$$\xrightarrow{R3=R3-5R2} \begin{pmatrix} i & 2 & -1 \\ 0 & 1 & \frac{2-3i}{1+7i} \\ 0 & 0 & \frac{-1+28i}{1+7i} \end{pmatrix} \xrightarrow{R2=R2\times\frac{-7+i}{i}} \begin{pmatrix} i & 2 & -1 \\ 0 & \frac{-7+i}{i} & \frac{3-2i}{i} \\ 0 & 0 & \frac{-1+28i}{1+7i} \end{pmatrix}$$

The matrix has now been converted into an upper triangular matrix, so we can calculate the determinant by getting the product of the diagonal entries…
$i \times \frac{-7+i}{i} \times \frac{-1+28i}{1+7i} = -28-i$

8. p.222 Ex.20

$$\begin{pmatrix} -1 & 2+i & 3 \\ 1-i & i & 1 \\ 3i & 2 & -1+i \end{pmatrix} \xrightarrow{R1=-R1} \begin{pmatrix} 1 & -2-i & -3 \\ 1-i & i & 1 \\ 3i & 2 & -1+i \end{pmatrix}$$

$$\xrightarrow{R2=R2-(1-i(R1))} \begin{pmatrix} 1 & -2-i & -3 \\ 0 & 3 & 4-3i \\ 3i & 2 & -1+i \end{pmatrix} \xrightarrow{R3=R3-3i(R1)} \begin{pmatrix} 1 & -2-i & -3 \\ 0 & 3 & 4-3i \\ 0 & -1+6i & -1+10i \end{pmatrix}$$

$$\xrightarrow{R1=-R1}\begin{pmatrix}-1 & 2+i & 3\\ 0 & 3 & 4-3i\\ 0 & -1+6i & -1+10i\end{pmatrix}\xrightarrow{R2=\frac{R2}{3}}\begin{pmatrix}-1 & 2+i & 3\\ 0 & 1 & \frac{4-3i}{3}\\ 0 & -1+6i & -1+10i\end{pmatrix}$$

$$\xrightarrow{R3=R3-(-1+6i(R2))}\begin{pmatrix}-1 & 2+i & 3\\ 0 & 1 & \frac{4-3i}{3}\\ 0 & 0 & \frac{-17+3i}{3}\end{pmatrix}\xrightarrow{R2=R2\times 3}\begin{pmatrix}-1 & 2+i & 3\\ 0 & 3 & 4-3i\\ 0 & 0 & \frac{-17+3i}{3}\end{pmatrix}$$

The matrix has now been converted into an upper triangular matrix, so we can calculate the determinant by getting the product of the diagonal entries…

$-1\times 3\times\frac{-17+3i}{3}=17-3i$

9. <u>p.228 Ex7</u>

Using Cramer's rule, the following equations in the form, $Ax=b$, gives us…

$3x_1+x_2+x_3=4$

$-2x_1-x_2=12$

$x_1+2x_2+x_3=-8$

$A=\begin{pmatrix}3 & 1 & 1\\ -2 & -1 & 0\\ 1 & 2 & 1\end{pmatrix}$ & $b=\begin{pmatrix}4\\ 12\\ -8\end{pmatrix}$

We calculate $\det(A)$ first…

$\det(A)=\begin{vmatrix}1 & 1\\ -3 & 1\end{vmatrix}=(1\times 1)-(1\times -3)=4$, then $4\times -1=-4$

$$x_1=\frac{\det(M_1)}{\det(A)}=\frac{\det\begin{pmatrix}4 & 1 & 1\\ 12 & -1 & 0\\ -8 & 2 & 1\end{pmatrix}}{\det(A)}=\frac{0}{-4}=0$$

$$x_2=\frac{\det(M_2)}{\det(A)}=\frac{\det\begin{pmatrix}3 & 4 & 1\\ -2 & 12 & 0\\ 1 & -8 & 1\end{pmatrix}}{\det(A)}=\frac{48}{-4}=-12$$

$$x_3=\frac{\det(M_3)}{\det(A)}=\frac{\det\begin{pmatrix}3 & 1 & 4\\ -2 & -1 & 12\\ 1 & 2 & -8\end{pmatrix}}{\det(A)}=\frac{-64}{-4}=16$$

Thus giving us the unique solutions to the given system of equations…

$x_1=0, x_2=-12, x_3=16$

10. <u>p.237 Ex.4(g)</u>

let matrix $A=\begin{pmatrix}1 & 0 & -2 & 3\\ -3 & 1 & 1 & 2\\ 0 & 4 & -1 & 1\\ 2 & 3 & 0 & 1\end{pmatrix}$

We first convert the matrix $A$to an upper triangular matrix form…

$$\begin{pmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix} \xrightarrow{R2=R2-(-3R1)} \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix} \xrightarrow{R4=R4-2R1} \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 4 & -1 & 1 \\ 0 & 3 & 4 & -5 \end{pmatrix}$$

$$\xrightarrow{R3=R3-4R2} \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 0 & 19 & -43 \\ 0 & 3 & 4 & -5 \end{pmatrix} \xrightarrow{R4=R4-3R2} \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 0 & 19 & -43 \\ 0 & 0 & 19 & -38 \end{pmatrix}$$

$$\xrightarrow{R4=R4-R3} \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 0 & 19 & -43 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

Then we can multiply the diagonal to calculate the determinant

$\det(A) = 1 \times 1 \times 19 \times 5 = 95$