Homework 4

1. **p.258 Ex.11 (a)(b)**

   (a) By Theorem 2.23 we can see…

   $A = Q^{-1}(\lambda I)Q = (\lambda I)Q^{-1}Q = (\lambda I)I = \lambda I$

   (b) We let M be the matrix in question, and let λ be the only eigenvalue of the matrix M. By Theorem 5.2 we know that the basis to make M diagonalisable is $(M - \lambda I)v_i = 0$, (where v_i is the vectors of which M consists of). This gives us that $(M - \lambda I)v = 0 \ \forall\ v$, as $\{v_i\}$ forms a basis. Therefore, as $v \neq 0$, matrix must be $M = \lambda I$

2. **p.259 Ex.14**

   By Theorem 4.8 we can see…

   $\det(A - \lambda I) = \det((A - \lambda I)^t) = \det(A^t - \lambda I)$

3. **p.259 Ex.18(a)(b)**

   (a) If B is invertible, we have that B^{-1} exists, and thus $\det(B) \neq 0$.
   We know that $\det(A + cB) = \det(B)\det(B^{-1}A + cI)$, where I is the identity matrix
   Using the properties of determinants and the fact that $\det(I) = 1$, we can simplify that to…
   $\det(B^{-1}A) + \det(cI) = \det(B^{-1}A) + c * \det(I) = \det(B^{-1}A) + c * 1$
   $= \det(B^{-1}A) + c$
   We then have that $\det(B^{-1}A) + c = 0$, we then subtract c from both sides of the equation, we get…
   $\det(B^{-1}A) = -c$
   So for the scalar value of $c = -\det(B^{-1}A)$, then $\det(A + cB) = 0$, therefore there is no inverse

   (b) By part (a) we know that B cannot be invertible, then we have A that is invertible

   $B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ (not invertible) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (invertible)

   We then have…

   $A + cB = \begin{pmatrix} 1 + 0c & 0 + c \\ 1 + 0c & 1 + c \end{pmatrix} = \begin{pmatrix} 1 & c \\ 1 & 1 + c \end{pmatrix}$

4. **p.282 Ex.18(a)(b)**

   (a) According to Theorem 5.1, we know that $[T]_\beta \ and \ [U]_\beta$ are the diagonal matrices where the diagonal entries are the eigenvalues of T and U. We know that diagonal matrices are commutative in nature, therefore $[T]_\beta[U]_\beta = [U]_\beta[T]_\beta$.
   Thus T & U commute

   (b) We know that there exists an invertible matrix $Q \in M_{n\times n}(F)$ where A and $B \in M_{n\times n}(F)$ such that both $Q^{-1}AQ$ and $Q^{-1}BQ$ are both diagonal matrices. Since we know that diagonal matrices are commutative (by properties), we get that…
   $(Q^{-1}AQ)(Q^{-1}BQ) = (Q^{-1}BQ)(Q^{-1}AQ)$.
   $\therefore AB = BA$, that is A and B commute.

5. **p.322 Ex.4**

   Given that $g(t) \in P(R)$, then $g(t) = a_0 + a_1 + \cdots a_{m-1}t^{m-1} + a_m t^m$, (generally), for $a_i \in \mathbb{R}\ \forall i, a_m \neq 0$
   Since W is a T-invariant subspace, we then have…
   $T^k(W) \subseteq T^{k-1}\big(T(W)\big) \subseteq T^{k-1}(W) \subseteq \cdots T(W) \subseteq W,$
   Therefore for any $v \in W$, we know that the sum of some elements $g(T)(v)$ is in W.
   $\therefore W$ is a subspace, we know that $g(T)(v)$ is always an element of W.

6. **p.322 Ex.6(a)**

(a) $z = e_1 = (1,0,0,0)$

$T(z) = (1,0,1,1)$ $\quad [a = 1, b = 0, c = 0, d = 0]$

$T^2(z) = T(T(z)) = (1,-1,2,2)$ $\quad [a = 1, b = 0, c = 1, d = 1]$

$T^3(z) = T\left(T\left(T(z)\right)\right) = (0,-3,3,3)$ $\quad [a = 1, b = -1, c = 2, d = 2]$

We then know that the dimension $= 3$ and, the set $\{z, T(z), T^2(z)\}$ is the basis.

7. **p.323 Ex.18(a)(b)**

(a) From the equation given in the question we can easily conclude, $f(0) = a_0$.
Then by definition we know $f(t) = \det(A - tI)$, and so $f(0) = \det(A)$.
Therefore, A is invertible if and only if $a_0 = \det(A) \neq 0$.

(b) $F(A) = (-1)^n tA^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0In$

$-a_0In = A((-1)^{\mathrm{n}}\mathrm{A}^{\mathrm{n}-1} + a_{n-1}A^{n-2} + \cdots + a_1In)$

$A^{-1}(-a_0In) = -a_0A^{-1}In = -a_0A^{-1} = (-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1In$

$A^{-1} = \frac{-1}{a_0}((-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1In)$

$\therefore A$ is invertible

8. **p.243 Ex.6.19 [HK]**

characteristic polynomial $f(t) = A - tI = 0$

$$\begin{pmatrix} 1-t & 2 & -1 \\ 0 & 5-t & -2 \\ 0 & 6 & -2-t \end{pmatrix} = 0$$

$(1-t)\left((5-t)(-2-t) - (2 \times 6) - 2(0) + \left(-1(0)\right)\right) = (1-t)(t^2 - 3t + 2) = (t-1)^2(t-2) = (t-1)(t-1)(t-2)$

Thus, the eigen values of A are, $\lambda = 1,1, and\ 2$

$\lambda_1 = 1;$

$$A - \lambda_1, I = \begin{pmatrix} 0 & 2 & -1 \\ 0 & 4 & -2 \\ 0 & 6 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ x \end{pmatrix} = 0 \qquad x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^2$$

$$\begin{matrix} 2y - z = 0 \\ 4y - 2z = 0 \\ 6y - 3z \end{matrix} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$\lambda_3 = 2$

$$A - \lambda_3 I = \begin{pmatrix} -1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 6 & -4 \end{pmatrix}$$

$$M\left(L_{B_3}\right) = a\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} : a \in \mathbb{R}$$

The eigenvectors are $V_1 = (1,0,0),\ V_2 = (0,1,2), V_3 = (1,2,3)$

$$Q = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix},\ Q^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & -3 & 2 \\ 0 & -2 & -1 \end{pmatrix} \text{ with } D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\text{And } Q^{-1}x = Q^{-1}\begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$QD^{10}Q^{-1}x = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1^{10} & 0 & 0 \\ 0 & 1^{10} & 0 \\ 0 & 0 & 2^{10} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1024 \\ 0 & 1 & 2048 \\ 0 & 2 & 3072 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1025 \\ 2050 \\ 3076 \end{pmatrix}$$

9. **p.243 Ex.6.21(1) [HK]**

$x_n = 6x_{n-1} - 11x_{n-2} + 6x_{n-3}, \quad n \geq 3$

By lemma 6.10 we know that the characteristic polynomial is…

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$(\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$

$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$

Therefore, $\lambda = 1, 2, 3$, and the geometric sequences $\{1^n\}, \{2^n\}, \{3^n\}$ are linearly independent and by Theorem 6.13 the general solution is a linear combination of them…

$x_n = c_1 1^n + c_2 2^n + c_3 3^n$

10. **p.243 Ex.6.27(1) [HK]**

$A = \begin{pmatrix} -6 & 24 & 8 \\ -1 & 8 & 4 \\ 2 & -12 & -6 \end{pmatrix}$ and $y(1) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -6-t & 24 & 8 \\ -1 & 8-t & 4 \\ 2 & -12 & -6-t \end{pmatrix}$

$= -(t-2)(t+2)(t+4)$

$\therefore$ the eigenvalues are $\lambda = 2, -2, -4$

$\lambda_1 = 2$

$A - \lambda_1 I = \begin{pmatrix} -8 & 24 & 8 \\ -1 & 6 & 4 \\ 2 & -12 & -8 \end{pmatrix}, \ x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^2$

$-8x + 24y + 8z = 0 \quad x = 2$

$-x + 6y + 4z = 0 \quad y = 1$

$2x - 12y - 8z = 0 \quad z = -1$

$\lambda_2 = -2$

$A - \lambda_2 I = \begin{pmatrix} -4 & 24 & 8 \\ -1 & 10 & 4 \\ 2 & -12 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^2$

$-4x + 24y + 8z = 0 \quad x = 2$

$-x + 10y + 4z = 0 \quad y = 1$

$2x - 12y - 4z = 0 \quad z = -2$

$\lambda_3 = -4$

$A - \lambda_3 I = \begin{pmatrix} -2 & 24 & 8 \\ -1 & 12 & 4 \\ 2 & -12 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^2$

$-2x + 24y + 8z = 0 \quad x = 0$

$-x + 12y + 4z = 0 \quad y = 0$

$2x - 12y - 2z = 0 \quad z = 0$

The eigenvectors are: $v_1 = (2,1,-1), v_2 = (2,1,-2), v_3 = (0,0,0)$

The general solution is…

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} + c_3 e^{-4t} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$= c_1e^{2t}v_1 + c_2e^{-2t}v_2 + c_3e^{-4t}v_3$

Where $y(1) = (2,1,0)$

$2c_1e^2 + 2c_2e^{-2} = 2$

$c_1e^2 + c_2e^{-2} = 1$

$-c_1e^2 - 2c_2e^{-2} = 0$

$c_1 = \frac{2}{e^2}, c_2 = -e^2$

Hence the final solution is…

$$y = \frac{2}{e^2}e^{2t}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} - e^2\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} + 0$$