Eulerian trails and circuits.
Hamiltonian paths and cycles

Trails, circuits, paths and cycles

Remember that ...

If $G = (V, E)$ is a simple graph, then

- A walk or path in G is a sequence of (not necessarily distinct) nodes $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $i = 1, 2, \ldots, k-1$. Such a walk is sometimes called a $v_1 - v_k$ walk.
  - v_1 and v_k are the end vertices of the walk.
  - If the vertices in a walk are distinct, then the walk is called a simple path.
  - If the edges in a walk are distinct, then the walk is called a trail.
- A cycle is a simple path $v_1, \ldots, v_k$ (where $k \geq 3$) together with the edge (v_k, v_1).
- A circuit or closed trail is a trail that begins and ends at the same node.
- The length of a walk (or simple path, trail, cycle, circuit) is its number of edges, counting repetitions.

Eulerian trails and circuits

- An Eulerian trail in a simple graph $G = (V, E)$ is a trail which includes every edge of G.
- An Eulerian circuit in a simple graph $G = (V, E)$ is a circuit which includes every edge of G.
- An Eulerian graph is a simple graph which contains an Eulerian circuit.

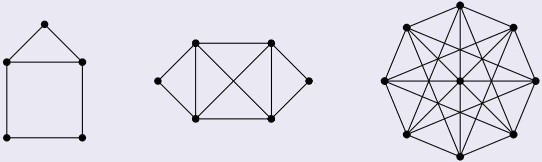
Note that

- Cycles C_n are Eulerian graphs.
- Paths P_n have no circuits at all $\Rightarrow$ P_n are not Eulerian graphs.

Eulerian trails and circuits

Quiz

Which of the following graphs is Eulerian?



Q: How can we recognize Eulerian graphs?

A: Two well-known characterizations:

1. based on node degrees
2. based on the existence of a special collection of cycles.

Theorem

For a connected graph G, the following statements are equivalent.

1. G is Eulerian.
2. Every vertex of G has even degree.
3. The edges of G can be partitioned into (edge-disjoint) cycles.

PROOF OF $1 \Rightarrow 2$. Let $G = (V, E)$ be an Eulerian graph, and $v \in V$. Every time a circuit enters v on an edge, it must leave on a different edge. Since the circuit never repeats an edge, there number of edges incident with v is even $\Rightarrow \deg(v)$ is even.

PROOF OF $2 \Rightarrow 3$. Suppose every node of G has even degree. We use induction on the number of cycles in G. G is connected and without nodes of degree 1 $\Rightarrow G$ is not a tree $\Rightarrow G$ has at least one cycle C_n. Let G' be the graph obtained by removing C_n from $G \Rightarrow$ all edges of G' have even degree and we can proceed recursively to prove that G' is Eulerian. This means that $G = C$ where C is an Eulerian circuit which shares no edges with C_n. But C and C_n must share a node (because G is connected) $\Rightarrow C_n$ and C' can be combined into an Eulerian circuit of G.

PROOF OF $3 \Rightarrow 1$. Suppose that the edges of G can be partitioned into k edge-disjoint cycles $S_1, \ldots, S_k$. Because G is connected, every such cycle is an Eulerian circuit which must share a node with another cycle $\Rightarrow$ these circuits can be combined until we obtain one Eulerian circuit which i the whole graph G. □

Q: How can we recognize graphs which contain an Eulerian trail?

A: Note that:

- If the graph is Eulerian, then it contains an Eulerian trail too, because every Eulerian circuit is also a trail.
- **There are non-Eulerian graphs with Eulerian trails.**

Corollary

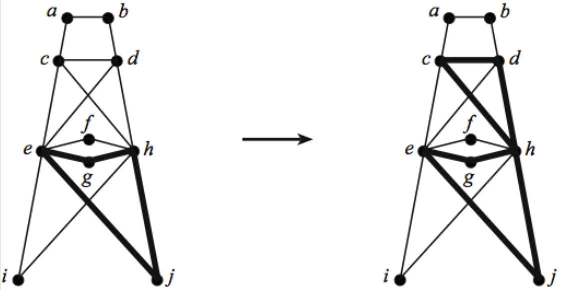
A connected graph G contains an Eulerian trail if and only if there are at most two vertices of odd degree.

Given: an Eulerian graph G

Find an Eulerian circuit of G.

1. Identify a circuit in G and call it R_1. Mark the edges of R_1. Let $i = 1$.
2. If R_i contains all edges of G, then stop (since R_i is an Eulerian circuit).
3. If R_i does not contain all edges of G, then let v_i be a node on R_i that is incident with an unmarked edge, e_i.
4. Build a circuit, Q_i, starting at node v_i and using edge e_i. Mark the edges of Q_i.
5. Create a new circuit, R_{i+1}, by patching the circuit Q_i into R_i at v_i.
6. Increment i by 1, and go to step (2).

Example



R_1: e, g, h, j, e

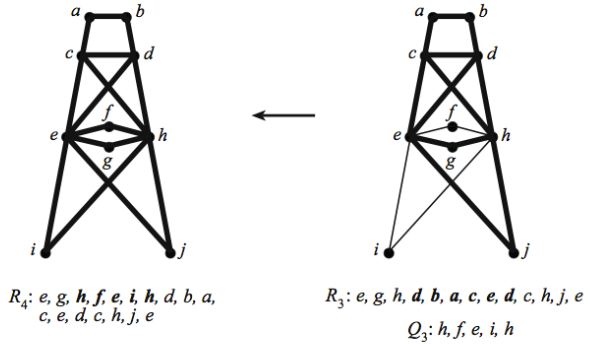
Q_1: h, d, c, h

R_2: $e, g, \mathbf{h}, \mathbf{d}, \mathbf{c}, \mathbf{h}, j, e$

Q_2: d, b, a, c, e, d

Finding Eulerian circuits
Hierholzer's Algorithm

Example (continued)

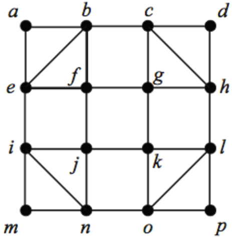


Finding Eulerian circuits
Fleury's algorithm

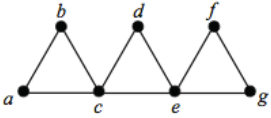
GIVEN: an Eulerian graph G with all edges unmarked.

1. Choose a node v and call it the lead node.
2. If all edges of G have been marked, then stop. Otherwise continue with next step.
3. Among all edges incident with the lead node, choose, if possible, one that is not a bridge of the subgraph formed by the unmarked edges. If this is not possible, choose any edge incident with the lead node. Mark this edge and let its other end node be the new lead node.
4. Go to step (2).

1. For each of the following, draw an Eulerian graph that satisfies the conditions, or prove that no such graph exists.
   (a) An even number of vertices, an even number of edges.
   (b) An even number of vertices, an odd number of edges.
   (c) An odd number of vertices, an even number of edges.
   (d) An odd number of vertices, an odd number of edges.
2. Use Hierholzer's algorithm to find an Eulerian circuit in the following graph. Use $R_1 = a, b, c, g, f, j, i, e, a$ as your initial circuit.



1. Use Fleury's algorithm to find an Eulerian circuit for the graph depicted below. Let a be your initial node.



2. Prove that if every edge of a graph G lies on an odd number of cycles, then G is Eulerian.
3. Let $G = K_{m,n}$.
    1. Find conditions on m and n that characterize when G will have an Eulerian trail.
    2. Find conditions that characterize when G will be Eulerian.

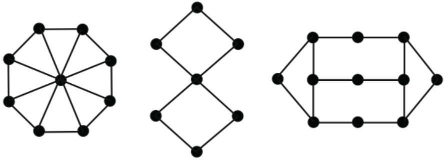
Hamiltonian paths and cycles

- A Hamiltonian path P of a simple graph G is a simple path that contains all nodes of G.
- A traceable graph is a simple graph containing a Hamiltonian path.
- A Hamiltonian cycle of a graph is a cycle that contains all nodes of the graph.
- A Hamiltonian graph is a graph containing a Hamiltonian cycle.

REMARKS

1. All Hamiltonian graphs are traceable.
2. There are traceable graphs which are not Hamiltonian.

Look at the following graphs and try to determine which ones are traceable, Hamiltonian, or neither.



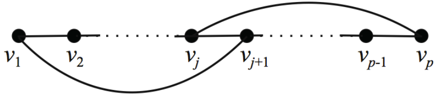
G_1 G_2 G_3

REMARKS. Hamiltonian graphs can have all even degrees (C_{10}), all odd degrees (K_{10}), or a mixture (G_1 in the previous figure).

Dirac's Theorem

Let G be a graph of order $n \geq 3$. If $\delta(G) \geq n/2$, then G is Hamiltonian.

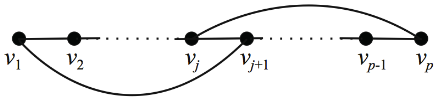
PROOF. Let G satisfy the given conditions. Assume G his not Hamiltonian, and let $P = v_1, \ldots, v_p$ be a simple path in G with maximum length. Since P is maximal, all neighbors of v_1 and of v_p are on P. Also, since $\delta(G) \geq n/2$, each of v_1 and v_p has at least $n/2$ neighbors on P. We claim that $\exists j \in \{1, \ldots, p-1\}$ such that $v_j \in N(v_p)$ and $v_{j+1} \in N(v_1)$. If this was not the case, then for every neighborr v_i of v_p on P (and there are at least $n/2$ of them), v_{i+1} is **not** a neighbor of v_1. This means that $\deg(v_1) \leq p - 1 - \frac{n}{2} < n - \frac{n}{2} = \frac{n}{2}$ contradicting the fact that $\delta(G) \geq n/2$. Thus, such a j exists, as shown in the following figure:



Dirac's Theorem

Let G be a graph of order $n \geq 3$. If $\delta(G) \geq n/2$, then G is Hamiltonian.

PROOF. (CONTINUED)



Let C be the cycle $v_1, v_2, \ldots, v_j, v_p, v_{p-1}, \ldots, v_{j+1}, v_1$. Since we assume G is not hamiltonian, there there is a node of G not on P. $\delta(G) \geq n/2$ implies G is connected $\Rightarrow$ G has a node w not on G that is adjacent to some node v_i on P. But then the path starting with w, v_i and then continuing around the cycle C is longer than the maximal path P, contradiction.
We conclude that G is Hamiltonian. $\square$

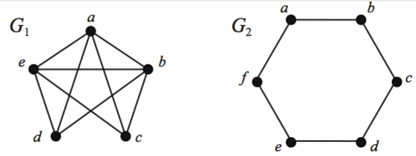
Theorem (A generalization Dirac's Theorem)

Let G be a graph of order $n \geq 3$. If $\deg(x) + deg(y) \geq n$ for all pairs of nonadjacent nodes x, y, then G is Hamiltonian.

A set of nodes in a graph G is independent if they are pairwise nonadjacent. The independence number of a graph G, denoted by $\alpha(G)$, is the largest size of an independent set of nodes from G.

Example

Consider the following two graphs



The only independent set of size 2 in G_1 is $\{c, d\}$, so $\alpha(G_1) = 2$. There are two independent sets of size 3 in G_2 : $\{a, c, e\}$ and $\{b, d, f\}$, and none of size 4, so $\alpha(G_2) = 3$.

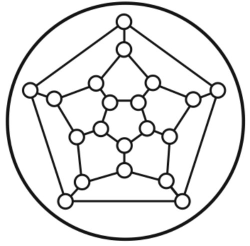
We recall that the vertex connectivity $\kappa(G)$ of a graph G is the minimum size of a node cut set of G.

Theorem (Chvátal and Erdös, 1972)

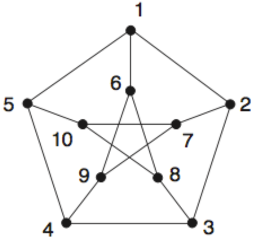
Let G be a connected graph of order $n \geq 3$ with vertex connectivity $\kappa(G)$ and independence number $\alpha(G)$. If $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian.

Exercise (The Icosian game of Hamilton)

Show that the graph depicted in the circle below is Hamiltonian.

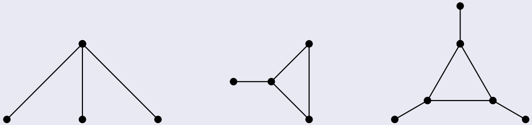


1. Prove that if G is Hamiltonian, then G is 2-connected.
2. Give the connectivity and independence number of the Petersen graph depicted below.



- Given two graphs G and H, we say that G is H-free if G does not contain a copy of H as an induced graph.
- If S is a collection of graphs, we say that G is S-free if G does not contain any of the graphs of S as induced subgraph.

Three special graphs



$K_{1,3}$ $\quad Z_1$ $\quad N$

Theorem (Goodman and Hedetniemi, 1974)

If G is a 2-connected, $\{K_{1,3}, Z_1\}$-free graph, then G is Hamiltonian.

PROOF. Let G be 2-connected and $\{K_{1,3}, Z_1\}$-free, and let C be a longest cycle in G. Since G is 2-connected, the cycle C exists. We show that C must be Hamiltonian.
If C is not Hamiltonian, there must be a node v not on C that is adjacent to a node w in C. Let a and b be the immediate predecessor and successor of w on C.

- A longer cycle would exist if $\{a, b\} \cap N(v) \neq \emptyset$, thus $\{a, b\} \cap N(v) = \emptyset$.
- If a is not adjacent to b then the subgraph induced by $\{w, v, a, b\}$ is $K_{1,3}$, contradiction with the assumption that G is $K_{1,3}$-free $\Rightarrow$ ab is an edge in G. But in this case the subgraph induced by $\{w, v, a, b\}$ is Z_1, a contradiction with the assumption that G is Z_1-free.

$\Rightarrow$ C is a Hamiltonian cycle. $\square$

Theorem (Duffus, Gould, and Jacobson, 1981)

Let G be a $\{K_{1,3}, N\}$-free graph.

1. *If G is connected, then G is traceable.*
2. *If G is 2-connected, then G is Hamiltonian.*

REMARK.

- The graph $K_{1,3}$ is forbidden to appear as a subgraph by both last two theorems. The graph $K_{1,3}$ is usually called the "claw", and appears as forbidden subgraph in many theorems from graph theory.

References

- J. M. Harris, J. L. Hirst, M. J. Mossinghoff. *Combinatorics and Graph Theory. Second Edition.* Springer 2008.

  Section 1.4. Trails, Circuits, Paths, and Cycles.