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Homework 6

Section 7.1

6. Since there are 4 aces in the deck and 13 hearts, the probability of selecting an ace or a heart but not the ace of hearts is $\frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{4}{13}$.

12. Since there are $C(52, 5)$ ways to choose a 5 card hand and 4 aces in the deck, the probability of a 5 card hand containing an ace is $\frac{C(4,1) \cdot C(48,4)}{C(52,5)}$.

30. There are $C(40, 6)$ ways to choose the 6 numbers. The number of ways to choose 5 correct numbers and 1 incorrect number is $C(6, 5) \cdot C(40 - 6, 1)$. $\therefore$ the probability of choosing 5 correct numbers and 1 incorrect number is $\frac{C(6,5) \cdot C(34,1)}{C(40,6)}$.

32. The probability of all three getting a prize is $\frac{3}{100} \cdot \frac{2}{99} \cdot \frac{1}{98}$.

36. There are 5 (2-6, 3-5, 4-4) possible rolls that add to 8 with 2 dice, and 6^2 total possible rolls. With 3 dice, there are 21 (1-1-6, 1-2-5, 1-3-4, 2-3-3, 2-2-4) possible rolls that add up to 8, with 6^3 total possible rolls. Since $\frac{5}{36} > \frac{21}{216}$, the first scenario is more likely.

Section 7.2

12.

$p(E) = 0.8$ and $p(F) = 0.6$	Premise
$p(E \cup F) = 0.8 + 0.6 - p(E \cap F)$	$\because\ p(E \cup F) = p(E) + p(F) - p(E \cap F)$
$p(E \cup F) \geq p(E)$	$\because\ p(F) > 0$
$p(E \cup F) \geq 0.8$	$\because\ p(E) = 0.8$
$1 \geq 0.8 + 0.6 - p(E \cap F)$	Probability must be between 0 and 1
$p(E \cap F) \geq 0.4$	Rearrange above inequality

16.

E and F are independent	Premise
$p(E \cap F) = p(E)\,p(F)$	Definition of independence
$p\left(\overline{E} \cap \overline{F}\right) = p\left(\overline{E \cup F}\right)$	De Morgan's Laws
$p\left(\overline{E \cup F}\right) = 1 - p(E \cup F)$	$\because p\left(\overline{E}\right) = 1 - p(E)$
$p\left(\overline{E \cup F}\right) = 1 - p(E) - p(F) + p(E \cap F)$	$\because\ p(E \cup F) = p(E) + p(F) - p(E \cap F)$
$p\left(\overline{E \cup F}\right) = (1 - p(E))(1 - p(F))$	Rearrange above equation
$p\left(\overline{E} \cap \overline{F}\right) = p\left(\overline{E}\right) p\left(\overline{F}\right)$	Follows from lines 3 and 5
$\therefore\ \overline{E}$ and $\overline{F}$ are independent	Definition of independence

22. (a) In every 400-year cycle, there are $(3 \cdot 24) + 25 = 97$ leap years (based on the restrictions in the question). This means that in a 400-year cycle, there are $(97 \cdot 366) + (303 \cdot 365) = 146097$ days. Since there are 97 Feb 29's (1 in each leap year), the probability of event L, which is a birthday occurring on Feb 29, is $p(L) = \frac{97}{146097}$, assuming all birthdays are independent and every birthday is an equally likely outcome. The probability of any other day occurring (denoted event M) is $p(M) = \frac{400}{146097}$, since each day occurs 400 times (once every year) in the cycle.

    (b) There are $C(365, n)$ possible ways for everyone to have a different birthday in a group of n people. Because of this, the probability that there are n different birthdays (denoted event N) is $p(N) = \frac{1}{C(365,n)}$. Therefore, the probability of at least 2 people having the same birthday is $p\left(\overline{N}\right) = 1 - p(N) = 1 - \frac{1}{C(365,n)}$.

24. If the first flip is T, then the only combination of 5 flips that renders 4 heads is THHHH; since there are 2^5 possible combinations of flips, the probability is $p(E \cap F) = \frac{1}{2^5}$. Then $p(F) = \frac{1}{2}$, since there are two options for the first flip. This means that the conditional probability is $p(E \mid F) = \frac{1}{16}$.

30. (a) If the two bits are equally likely, then the probability of no zeroes in the bit string is $\frac{1}{2^{10}}$.

    (b) If $p(1) = 0.6$, then $p(0) = 0.4$. Then the probability of a 10-bit string with no zeroes is $C(10, 10)\, p^{10} q^{10-10} = 1 \cdot 0.6^{10} \cdot 0.4^0 = 0.6^{10}$.

    (c) If $p(1) = \frac{1}{2^i}$ for $i = 1, 2, \cdots, 10$, then $p(0) = 1 - \frac{1}{2^i}$ for $i = 1, 2, \cdots, 10$. This means that the probability of the desired bit string is $1/\left(\prod_{i=1}^{10} 2^i\right)$.

Section 7.3

2. Note that $p\left(\overline{E}\right) = 1 - p(E) = \frac{1}{3}$ and $p(E \cap F) = p(F \mid E) \cdot p(E) = \frac{5}{12}$. Also, $p\left(F \mid \overline{E}\right) = p(F) - p(E \cap F) = \frac{1}{3}$. Now:

    $$p(E \mid F) = \frac{p(F|E)\; p(E)}{p(F|E)\; p(E) + p\left(F|\overline{E}\right)\; p\left(\overline{E}\right)} = \frac{\frac{5}{8} \cdot \frac{2}{3}}{\frac{5}{8} \cdot \frac{2}{3} + \frac{1}{3}} = \frac{5}{9}$$

4. Let E be the event that Ann picked an orange ball and F be the event that Ann picked from the second box. Note that $p(F) = \frac{1}{2}$, $p\left(\overline{F}\right) = \frac{1}{2}$, $p(E \mid F) = \frac{5}{11}$, and $p\left(E \mid \overline{F}\right) = \frac{3}{7}$. Then, by Bayes' Theorem, $p(F \mid E) = \frac{\frac{5}{11} \cdot \frac{1}{2}}{\frac{5}{11} \cdot \frac{1}{2} + \frac{3}{7} \cdot \frac{1}{2}} = \frac{35}{68}$.

6. Let F be the event that a player takes steroids and E be the event that they test positive. Then, $p(E \mid F) = 0.98$, $p\left(E \mid \overline{F}\right) = 0.12$, and $p(F) = 0.05$. Note that $p\left(\overline{F}\right) = 0.95$. Then, $p(F \mid E) = \frac{p(E|F)\; p(F)}{p(E|F)\; p(F) + p\left(E|\overline{F}\right)\; p\left(\overline{F}\right)} = \frac{0.98 \cdot 0.05}{0.98 \cdot 0.05 + 0.12 \cdot 0.95}$

    ≈ 0.3006.

12. Let E denote the event that a 0 is sent, F_0 denote that a 0 is received, and F_1 denote that a 1 is received. The question gives the following values: $p(E) = \frac{2}{3}$, $p(\overline{E}) = \frac{1}{3}$, $p(F_0 \mid E) = 0.9$, and $p(F_1 \mid \overline{E}) = 0.8$.

    (a) Note that $p(F_0 \mid \overline{E}) = 1 - p(F_1 \mid \overline{E}) = 0.2$ because if a 1 is sent $(\overline{E})$, it will either be received correctly $(p(F_1 \mid \overline{E}))$ or incorrectly $(p(F_0 \mid \overline{E}))$ with no other possible outcomes. The probability that a 0 is received (denoted event G) is the sum of the probability that a 0 is sent and received correctly and that a 1 is sent and received incorrectly; therefore:
    $p(G) = p(F_0 \mid E)\ p(E) + p(F_0 \mid \overline{E})\ p(\overline{E}) \approx 0.667$.

    (b) $p(E \mid F_0) = \frac{p(F_0|E)\ p(E)}{p(F_0|E)\ p(E)\ +\ p(F_0|\overline{E})\ p(\overline{E})} = \frac{0.9 \cdot \frac{2}{3}}{0.9 \cdot \frac{2}{3} + 0.2 \cdot \frac{1}{3}} = 0.9$

16. Let E denote the event that Ramesh is late, F_1 he arrives by car, F_2 he arrives by bus, and F_3 he arrives by bicycle. The question gives the probabilities: $p(E \mid F_1) = 0.5$, $p(E \mid F_2) = 0.2$, and $p(E \mid F_3) = 0.05$.

    (a) Ramesh's boss assumes that $p(F_1) = p(F_2) = p(F_3) = \frac{1}{3}$. He wants to know $p(F_1 \mid E)$, which Bayes' Theorem says is this:
    $p(F_1 \mid E) = \frac{p(E|F_1)\ p(F_1)}{p(E|F_1)\ p(F_1)\ +\ p(E|F_2)\ p(F_2) +\ p(E|F_3)\ p(F_3)} = \frac{0.5 \cdot \frac{1}{3}}{0.5 \cdot \frac{1}{3} + 0.2 \cdot \frac{1}{3} + 0.05 \cdot \frac{1}{3}} = \frac{2}{3}$

    (b) Now $p(F_1) = 0.3$, $p(F_2) = 0.1$, and $p(F_3) = 0.6$. Bayes' Theorem now says:
    $p(F_1 \mid E) = \frac{0.5 \cdot 0.3}{0.5 \cdot 0.3 + 0.2 \cdot 0.1 + 0.05 \cdot 0.6} = 0.75$

Section 7.4

4. Because this is a Bernoulli trial, the value of the random variable X (the number of heads) is $np = 10$.

10. Let X denote the number of flips of the coin. Note that $p(T) = p(H) = 0.5$.
    $\forall 2 \leq n \leq 6$, $n - 2$ of the first $n - 1$ flips must be H, meaning that there are $n - 1$ possible outcomes ($\because$ the first T can be in any 1 of $n - 1$ positions, with the rest being H's; the last position in the string of length n must be a T, because otherwise the string would have ended already and would thus have been counted in another value of $X = n$). There are five cases to consider, for the five possible values of $X = n$:

    $p(X = 2) = p(T)^2 = 0.5^2 = 0.25$

    $p(X = 3) = 2\ p(T)^2\ p(H) = 2 \cdot 0.5^2 \cdot 0.5 = 0.25$

    $p(X = 4) = 3\ p(T)^2\ p(H)^2 = 0.1875$

    $p(X = 5) = 4\ p(T)^2\ p(H)^3 = 0.125$

In the case of $X = 6$, note that either 4 *or* 5 of the first 5 flips must be H, because there can be at most 1 T among the first 5 flips, and we stop flipping at 6 flips regardless if it's H or T. Hence:

$p(X = 6) = 5\, p(T)\ p(H)^4 + p(H)^5 = 0.15625 + 0.03125 = 0.1875$

$\therefore\ E(X) = 0.25 \cdot 2 + 0.25 \cdot 3 + 0.1875 \cdot 4 + 0.125 \cdot 5 + 0.1875 \cdot 6 = 3.75$

16. If the coins are fair, then $p(H) = p(T) = 0.5$ and there are 4 possible outcomes (HH, TT, TH, HT). The probabilities of X and Y for their 3 possible values are:

$$\begin{array}{cc} p(X=0) = 0.25 & p(Y=0) = 0.25 \\ p(X=1) = 0.5 & p(Y=1) = 0.5 \\ p(X=2) = 0.25 & p(Y=2) = 0.25 \end{array}$$

The definition of independence is:

$$\forall r \in \mathbb{R}\, (p(X = r_1 \wedge Y = r_2) = p(X = r_1)\ p(Y = r_2)).$$

Now, assume that X and Y are independent and that $X = 0$ and $Y = 1$. Since they are independent, $p(X = 0 \wedge Y = 1) = 0.25 \cdot 0.5 = 0.125$. However, it is not possible for there to be 0 heads and only 1 tail; the only outcome with $X = 0$ is TT, for which $Y = 2$. $\therefore\ p(X = 0 \wedge Y = 1) = 0$. This is a contradiction, and thus X and Y are not independent.

24. Let I_{A1} denote I_A when A occurs ($\therefore\ I_{A1} = 1$) and I_{A2} denote I_A when $\overline{A}$ occurs ($\therefore\ I_{A2} = 0$). Note that the expectation of I_A is the sum of its possible values times the probability of the event occurring, or $E(I_A) = I_{A1} \cdot p(A) + I_{A2} \cdot p\left(\overline{A}\right)$. Substituting in the values of I_{A1} and I_{A2} gives $E(I_A) = 1 \cdot p(A) + 0 = p(A)$. Therefore, the expectation of the indicator random variable is equal to the probability of the event's occurrence, and the statement is proven.

26. The expectation of X is $E(X) = \sum\limits_{s \in S} X(s)\, p(s)$. Note that $p(A_k) = p(X \geq k) = p(X = k) + p(X \geq k+1) = p(X = k) + p(A_{k+1})$. Note that as k increases, the probability $p(X = k)$ occurs in the sequence $\sum\limits_{k=1}^{\infty} p(A_k)$ exactly k times. This means the sum can be rewritten as $\sum\limits_{k=1}^{\infty} k\, p(X = k) = E(X)$. The statement is proven.

30. X and Y are independent

$E\left(XY\right) = E\left(X\right)E\left(Y\right)$

$V\left(XY\right) = E\left(X^2Y^2\right) - E\left(XY\right)^2$

$V\left(XY\right) = E\left(X^2\right)E\left(Y^2\right) - E\left(X\right)^2 E\left(Y\right)^2$

$V\left(XY\right) = E\left(X^2\right)E\left(Y^2\right) - V\left(X\right)V\left(Y\right) - E\left(X\right)^2 E\left(Y\right)^2 + V\left(X\right)V\left(Y\right)$

$V\left(XY\right) = E\left(X^2\right)E\left(Y^2\right) - \left(E\left(X^2\right) - E\left(X\right)^2\right)\left(E\left(Y^2\right) - E\left(Y\right)^2\right) -$
$E\left(X\right)^2 E\left(Y\right)^2 + V\left(X\right)V\left(Y\right)$

$\therefore\ V\left(XY\right) = E\left(X\right)^2 V\left(Y\right) + E\left(Y\right)^2 V\left(X\right) + V\left(X\right)V\left(Y\right)$