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Statistics 140
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Homework 9

1. (a) The probability that $|Z| \in dy$ is twice the probability that $Z \in dy$ because there are two values of y that yield $|Z|$. Therefore,

$$f_{|Z|}(y) = 2\phi(y)$$

(b)

$$E(|Z|) = \int_0^\infty y \cdot 2\phi(y)\ dy = \frac{2}{\sqrt{2\pi}} \int_0^\infty ye^{-\frac{1}{2}y^2}\ dy = \frac{2}{\sqrt{2\pi}} \left(\lim_{b \to \infty} -e^{-\frac{1}{2}y^2} \Big|_0^b \right)$$

$$= \frac{2}{\sqrt{2\pi}} \left[\lim_{b \to \infty} \left(-e^{-\frac{1}{2}b^2} + 1 \right) \right] = \frac{2}{\sqrt{2\pi}} \cdot 1 = \frac{2}{\sqrt{2\pi}}$$

(d) Let $g(x) = e^x$; $Y = g(X)$. The change of variable formula says that $f_Y(y) = \frac{f_X(x)}{\frac{d}{dx}g(x)}$ evaluated at $x = g^{-1}(y)$. The inverse of g is

$$e^{g^{-1}(x)} = x \Rightarrow g^{-1}(x) = \log(x)$$

and the derivative of g is

$$g'(x) = e^x$$

Because X is normal (μ, σ^2), its pdf is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Plugging these into the change of variable formula,

$$f_Y(y) = \frac{f_X(\log(y))}{g'(\log(y))} = \frac{1}{e^{\log(y)}} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{log(y)-\mu}{\sigma}\right)^2} = \frac{1}{y} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{log(y)-\mu}{\sigma}\right)^2}$$

2. (a) Converting Θ to rectangular coordinates (X, Y) gives $\Theta = \arctan\left(\frac{Y}{X}\right)$. Because we want to know the value of Y given that $X = 1$, substitute 1 in for X in the formula for Θ:

$$\Theta = \arctan(Y) \Rightarrow Y = \tan(\Theta) = g(\Theta)$$

Because Θ is uniform $(-\frac{\pi}{2}, \frac{\pi}{2})$, its pdf is $f_\Theta(\theta) = \frac{1}{\frac{\pi}{2}-(-\frac{\pi}{2})} = \frac{1}{\pi}$. Using the change of variable formula and the fact that $g'(\theta) = \sec^2(\theta)$ and $g^{-1}(y) = \arctan(y)$,

$$f_Y(y) = \frac{f_\Theta(\theta)}{\frac{d}{d\theta}g(\theta)}, \text{where } \theta = g^{-1}(y)$$

$$f_Y(y) = \frac{1}{\pi\sec^2(\arctan(y))}$$

Using the fact that $\tan\theta = y$ and $\sec^2\theta = 1 + \tan^2\theta$,

$$f_Y(y) = \frac{1}{\pi\sec^2(\arctan(y))} = \frac{1}{\pi(1+\tan^2\theta)} = \frac{1}{\pi(1+y^2)}$$

(c) As in 1(a), the probability that $|Y| \in dy$ is double because there are two inputs that yield the same output, so

$$E(|Y|) = \int_0^\infty 2y\frac{1}{\pi(1+y^2)}\,dy = \frac{2}{\pi}\int_0^\infty \frac{y}{1+y^2}\,dy = \frac{2}{\pi}\left[\frac{1}{2}\log(1+y^2)\right]_0^\infty$$

The absolute value bars are not needed in the log because $1 + y^2$ is always positive. Because the log function approaches ∞ and ∞ divided by a constant approaches ∞,

$$E(|Y|) = \frac{1}{\pi}\lim_{b\to\infty}(\log(1+b^2) - 0) = \infty$$

(d) The graph is a plot of the sample mean of i.i.d. Cauchy RVs as the sample size, n, approaches a large N. The x-axis tells the number of Cauchy RVs in the sample, n, and the y-axis tells what the sample mean is. The Weak Law of Large Numbers says that the sample mean of an i.i.d. sample should converge to the population mean as the sample size approaches ∞. Because the expectation of a Cauchy RV is undefined, there is no value for the sample mean to converge to, which is why the value that the graph approaches changes each time the graph is plotted for a fixed value of N. The case of Cauchy RVs is one in which the Weak Law of Large Numbers does not apply.

3. (a) $P(Y > 2X)$ is the double integral over the region where $0 < x < 1$ and $2x < y < 1$ in the plane. The integral is then

$$P(Y > 2X) = \int_0^1 \int_0^{\frac{y}{2}} 90(y-x)^2 \, dx \, dy = \int_0^1 \left[-10(y-x)^9\right]_0^{\frac{y}{2}} \, dy = \int_0^1 \left(-10\left(\frac{y}{2}\right)^9 + 10y^9\right) \, dy$$

$$= \left[-\left(\frac{1}{2}\right)^9 y^{10} + y^{10}\right]_0^1 = -\frac{1}{2^9} + 1 + 0 - 0 = \frac{511}{512}$$

(b) The marginal density of X is the integral of $f(x,y)$ over all possible values of Y, or

$$f_X(x) = \int_0^1 90(y-x)^8 \, dy = 10(y-x)^9\big|_0^1 = 10(1-x)^9 - (-10x^9) = 10(1-x)^9 + 10x^9$$

(c) *The joint density f above is the joint density of the* ***maximum*** *and* ***minimum*** *of ten independent uniform (0,1) random variables.* This is because $\frac{(y-x)^{10}}{(1-0)^{10}} = (y-x)^{10}$ is the cdf of 10 uniform (0,1) RVs being on the interval $[x,y]$, and $\frac{\partial^2}{\partial xy} F = 90(y-x)^8$ is the pdf of that probability.

4. (a) Because $P(Z = z) = 0$ for any continuous RV Z, the survival function of an exponential RV can be used when working with minima (because $P(X > 0) = P(X \geq 0)$ for a continuous RV). The survival functions of X and Y are

$$S_X(x) = e^{-\lambda x} \quad S_Y(y) = e^{-\mu y}$$

The probability $P(W \leq w)$ is the probability that the minimum of X and Y is no more than w, and can be rewritten to utilize the survival functions of X and Y:

$$\begin{aligned} P(W \leq w) &= 1 - P(W \geq w)) \\ &= 1 - P(X \geq w, Y \geq w) \\ &= 1 - P(X \geq w)P(Y \geq w) \qquad \text{because } X \text{ and } Y \text{ are independent} \\ &= 1 - S_X(w)S_Y(w) \\ &= 1 - e^{-\lambda w}e^{-\mu w} \\ &= 1 - e^{-(\lambda+\mu)w} \end{aligned}$$

The above probability is the cdf of W, so its derivative is the density of W:

$$f_W(w) = \frac{d}{dw}(1 - e^{-(\lambda+\mu)w}) = (\lambda + \mu)e^{-(\lambda+\mu)w}$$

Therefore, W has the exponential $(\lambda + \mu)$ distribution.

(b) From Section 17.2, the probability that $X > Y$ is $\frac{\mu}{\lambda+\mu}$. Using the linear transformation $Y \mapsto cY$, cY has the exponential $(\frac{\mu}{c})$ distribution (by the results from Section 16.1), and the probability becomes

$$P(X > cY) = \frac{\frac{\mu}{c}}{\lambda + \frac{\mu}{c}} = \frac{\mu}{c\lambda + \mu}$$

(c) As previously stated, $P\left(\frac{X}{Y} < c\right) = P\left(\frac{X}{Y} \leq c\right)$ because X and Y are continuous RVs. Using this fact, the cdf of $\frac{X}{Y}$ is

$$P\left(\frac{X}{Y} \leq c\right) = 1 - P\left(\frac{X}{Y} > c\right) = 1 - P(X > cY) = 1 - \frac{\mu}{c\lambda + \mu} = \frac{c\lambda}{c\lambda + \mu}$$

5. (a) Using integration by parts,

$$\Gamma(r+1) = \int_0^\infty t^{r+1-1}e^{-t}\,dt = \int_0^\infty t^r e^{-t}\,dt = -t^r e^{-t}\Big|_0^\infty - \int_0^\infty -e^{-t} r t^{r-1}\,dt$$

$$= \lim_{b\to\infty}(-b^r e^{-b} - 0) + r\Gamma(r) = r\Gamma(r)$$

(b) Let $P(n)$ be the proposition that $\Gamma(n) = (n-1)!$. For $n = 2$, $\Gamma(2) = 1 \cdot \Gamma(1)$ by part (a) and $\Gamma(1)$ is

$$\Gamma(1) = \int_0^\infty t^0 e^{-t}\,dt = \int_0^\infty e^{-t}\,dt = -e^{-t}\Big|_0^\infty = \lim_{b\to\infty} -e^{-b} + 1 = 0 + 1 = 1$$

Therefore, $\Gamma(2) = 1 = 1! = (2-1)!$, proving the base case. Now assume that $P(n)$ is true for any positive integer n. $\Gamma(n+1) = n\Gamma(n)$ by part (a), and because $P(n)$ is true,

$$\Gamma(n+1) = n\Gamma(n) = n \cdot (n-1)! = n! = ((n+1)-1)!$$

Therefore, $P(n)$ being true implies that $P(n+1)$ is true, and the statement is proven by induction.

(c) For this problem, assume $t > 0$. Let $g(x) = \frac{1}{\lambda}x$, such that $g'(x) = \frac{1}{\lambda}$ and $g^{-1}(x) = \lambda x$. Using the change of variable formula,

$$f_Y(t) = \frac{f_X(x)}{g'(x)}, \text{ evaluated at } x = g^{-1}(t)$$

$$f_Y(t) = \frac{\frac{1}{\Gamma(r)}(\lambda t)^{r-1}e^{-\lambda t}}{\frac{1}{\lambda}} = \lambda\frac{\lambda^{r-1}}{\Gamma(r)}t^{r-1}e^{-\lambda t} = \frac{\lambda^r}{\Gamma(r)}t^{r-1}e^{-\lambda t}$$

Using the piecewise notation,

$$f_Y(t) = \begin{cases} \frac{\lambda^r}{\Gamma(r)}t^{r-1}e^{-\lambda t}, & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

(d) From section 16.3, if Z has the standard normal density, then the density of $W = Z^2$ is

$$f_W(w) = \frac{1}{\sqrt{2\pi}}w^{-\frac{1}{2}}e^{-\frac{1}{2}w} = \frac{\left(\frac{1}{2}\right)^{\frac{1}{2}}}{\sqrt{\pi}}w^{-\frac{1}{2}}e^{-\frac{1}{2}w}$$

Using the formula from part (c), W has the gamma $\left(\frac{1}{2}, \frac{1}{2}\right)$, and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ (from the denominator of the constant of integration). Using the result from part (a),

$$\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{3}{2}\sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\,\Gamma\left(\frac{3}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{15}{4}\sqrt{\pi}$$

6. (a) From $f_Y(t)$ in 5(c),

$$\int_0^\infty t^{r-1}e^{-\lambda t}\,dt = \frac{\Gamma(r)}{\lambda^r}\int_0^\infty f_Y(t)\,dt$$

Because f_Y is a density, its integral over the domain of t must be 1, so

$$\int_0^\infty t^{r-1}e^{-\lambda t}\,dt = \frac{\Gamma(r)}{\lambda^r}\int_0^\infty f_Y(t)\,dt = \frac{\Gamma(r)}{\lambda^r}$$

(b) The expectation of T, again using f_Y from 5(c), is

$$E(T) = \int_0^\infty t f_Y(t)\,dt = \int_0^\infty \frac{\lambda^r}{\Gamma(r)} t \cdot t^{r-1}e^{-\lambda t}\,dt = \int_0^\infty \frac{\lambda^r}{\Gamma(r)} t^r e^{-\lambda t}\,dt = \frac{\lambda^r}{\Gamma(r)}\int_0^\infty t^r e^{-\lambda t}\,dt$$

Using the result from part (a) to simplify,

$$E(T) = \frac{\lambda^r}{\Gamma(r)}\int_0^\infty t^r e^{-\lambda t}\,dt = \frac{\lambda^r}{\Gamma(r)} \cdot \frac{\Gamma(r+1)}{\lambda^{r+1}} = \frac{r\Gamma(r)}{\Gamma(r)\lambda} = \frac{r}{\lambda}$$

(c) Again using f_Y from 5(c) and the result of part(a),

$$E(T^2) = \int_0^\infty t^2 f_Y(t)\,dt = \int_0^\infty \frac{\lambda^r}{\Gamma(r)} t^2 \cdot t^{r-1}e^{-\lambda t}\,dt = \int_0^\infty \frac{\lambda^r}{\Gamma(r)} t^{r+1} e^{-\lambda t}\,dt = \frac{\lambda^r}{\Gamma(r)}\int_0^\infty t^{r+1} e^{-\lambda t}\,dt$$

$$= \frac{\lambda^r}{\Gamma(r)} \cdot \frac{\Gamma(r+2)}{\lambda^{r+2}} = \frac{r(r+1)\Gamma(r)}{\Gamma(r)\lambda^2} = \frac{r(r+1)}{\lambda^2}$$

Using the computational formula for variance gives

$$Var(T) = E(T^2) - E(T)^2 = \frac{r(r+1)}{\lambda^2} - \left(\frac{r}{\lambda}\right)^2 = \frac{r(r+1) - r^2}{\lambda^2} = \frac{r}{\lambda^2}$$

(d) The gamma $(1, \lambda)$ distribution has pdf

$$\frac{\lambda^1}{\Gamma(1)} t^{1-1}e^{-\lambda t} = \lambda e^{-\lambda t}$$

because $\Gamma(1) = 1$. This means that the gamma $(1, \lambda)$ distribution seems to also be the exponential (λ) distribution. If T has the gamma $(1, \lambda)$ distribution and S has the exponential (λ) distribution,

$$E(T) = \frac{1}{\lambda} = E(S)$$

$$Var(T) = \frac{1}{\lambda^2} = Var(S)$$

These results are consistent, and therefore gamma $(1, \lambda)$ distribution is equal to the exponential (λ) distribution.