Distribution Fitting for the Lazy Scientist

or, All you wanted to know about maximum likelihood estimation but were too afraid to ask.

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Outline

1. Distribution Fitting: What, Why, and When?

2. Fitting the distribution with MLE

3. Goodness of fit

Regression Example:

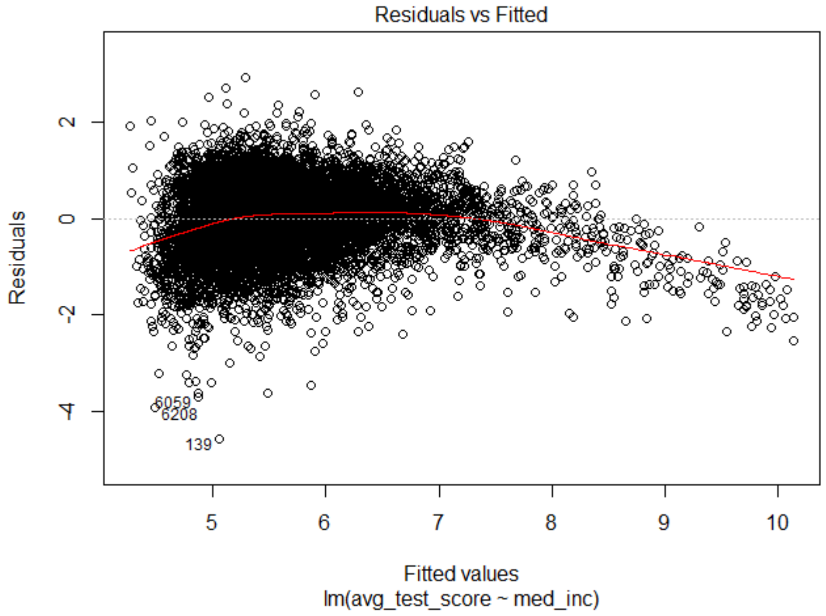
Is income in a school district related to students' test scores in that district?

Variables: average test score, median income (1000's)

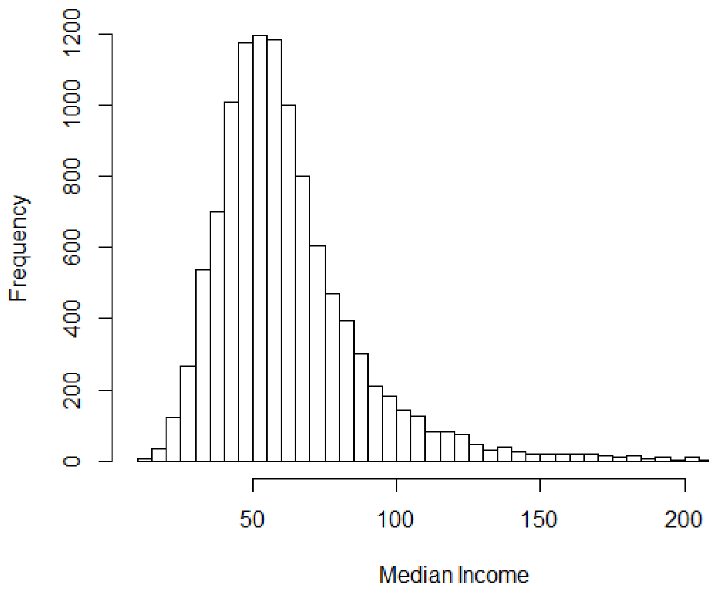
```
lm(formula = avg_test_score ~ med_inc, data = dat)

              Estimate   Std. Error     t value    Pr(>|t|)
(Intercept)    3.9295      0.0168        233.7      <2e-16 ***
med_inc        0.0272      0.0002        109.6      <2e-16 ***

Residual standard error: 0.6962 on 11052 degrees of freedom
Multiple R-squared:  0.5209,  Adjusted R-squared:  0.5209
```

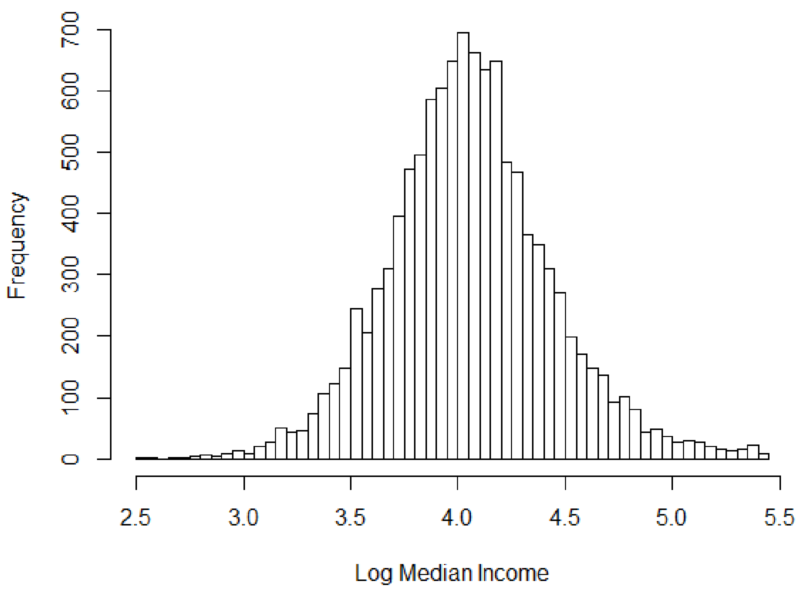
estimated parameters

How confident are we?



Is median income:





Normal?

Additive

Log-Normal?

Multiplicative

1. What are the parameters of each distribution?
2. Which distribution is better?

These represent different processes!

Fit the distribution

"I have a good model for how my data was generated, and want to know the parameters."

- Poisson process
- Power law: Find α in $x^{-\alpha}$.
- Network community detection: SBM

Check whether it fits well enough

"I want to see *which* distribution fits better"

OR

"I want to *test* whether my model fits the data."

It usually doesn't!

How to do MLE:

1. Choose a model

2. Find the log-likelihood function

3. Find the parameters that maximize the log-likelihood
    - Calculate estimators by hand (or look them up)
    - Computationally

Fitting the Distribution

model with parameters $\vec{\theta}$

independent data points $\vec{x} = \{x_1, x_2, x_3, \dots, x_n\}$

Likelihood of a given model

$$\mathcal{L}\left(\vec{\theta} \mid \vec{x}\right) = \mathrm{P}(\vec{\mathrm{x}} \mid \vec{\theta})$$
$$= \prod_i P\left(x_i \middle| \vec{\theta}\right)$$

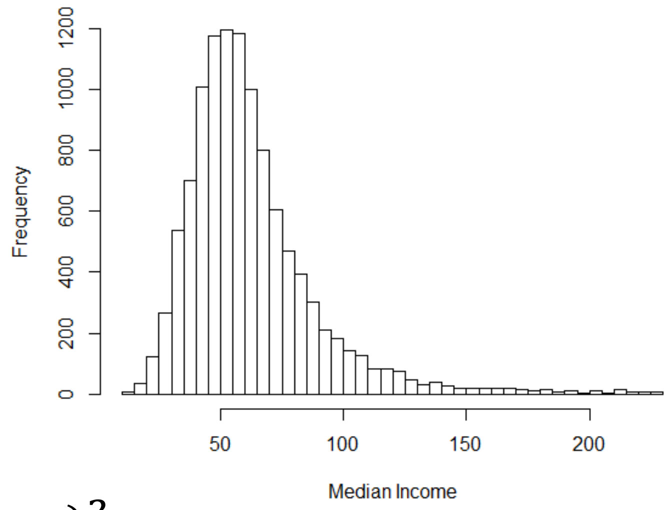
very small

Log-likelihood

$$\log \mathcal{L} = \sum_i \log P\left(x_i \middle| \vec{\theta}\right)$$

maximize this

Example of MLE



model: normal $\mathrm{P}(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{\frac{-(\mathrm{x}-\mu)^2}{2\sigma^2}}\ dx$

parameters: μ, σ

$$\mathcal{L} = \prod_i P(x_i|\mu,\sigma) = \prod_i \frac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{\frac{-(x_i-\mu)^2}{2\sigma^2}}$$

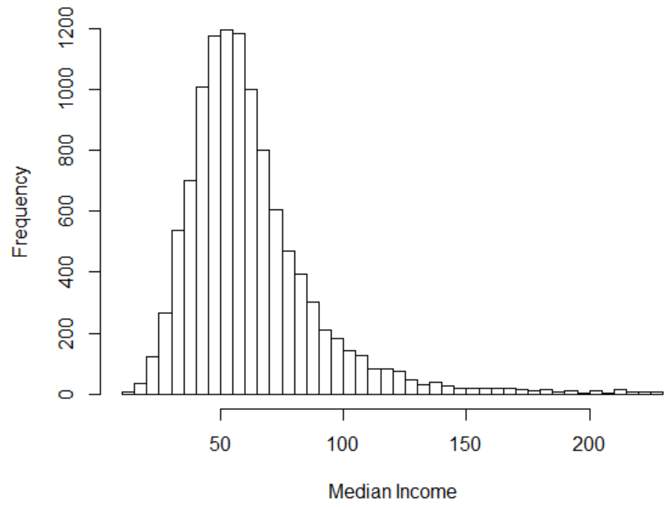
$$\log\mathcal{L} = \sum_i -\log(\sigma) - \log\left(\sqrt{2\pi}\right) + \frac{-(x_i-\mu)^2}{2\sigma^2}$$

$$= -n\log(\sigma) - n\log\left(\sqrt{2\pi}\right) - \sum_i \frac{(x_i-\mu)^2}{2\sigma^2}$$

Example of MLE

Goal: Find μ, σ that maximize

$$\log \mathcal{L}\,(\mu,\sigma) \;= -n\log(\sigma) - n\log\left(\sqrt{2\pi}\right) - \textstyle\sum_i \frac{(x_i-\mu)^2}{2\sigma^2}$$



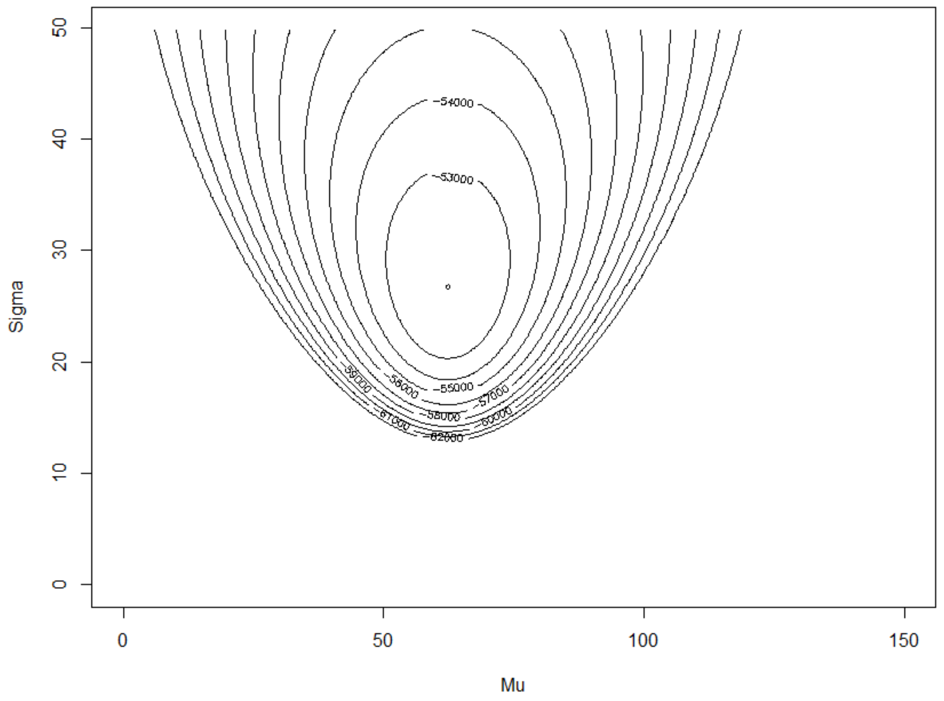
<u>Analytically</u>

$$\frac{d}{d\mu}\log\mathcal{L} = 0$$

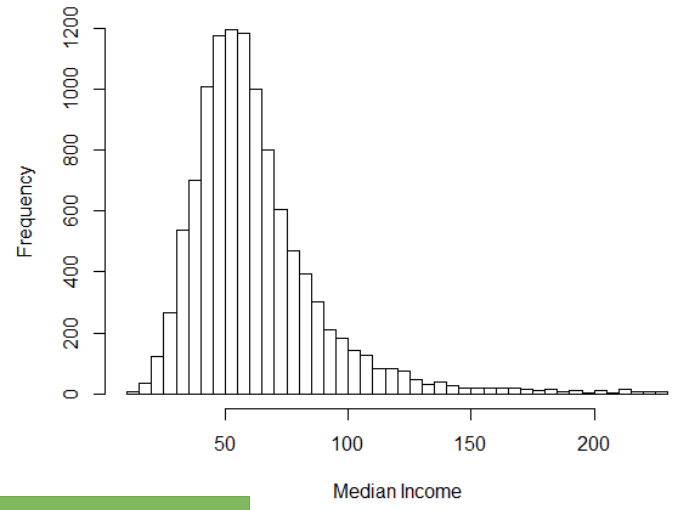
$$\frac{d}{d\sigma}\log\mathcal{L} = 0$$

let's do this!

<u>Computationally</u>



Example of MLE



Analytically

model: normal $\quad P(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$

parameters: μ, σ

$$\left.\begin{aligned} \hat{\mu} &= \frac{1}{n}\sum_i x_i = \bar{x} \\ \hat{\sigma}^2 &= \frac{1}{n}\sum_i [x_i - \hat{\mu}]^2 = \overline{(x^2)} - \bar{x}^2 \end{aligned}\right\}$$

maximize likelihood estimators for the normal distribution

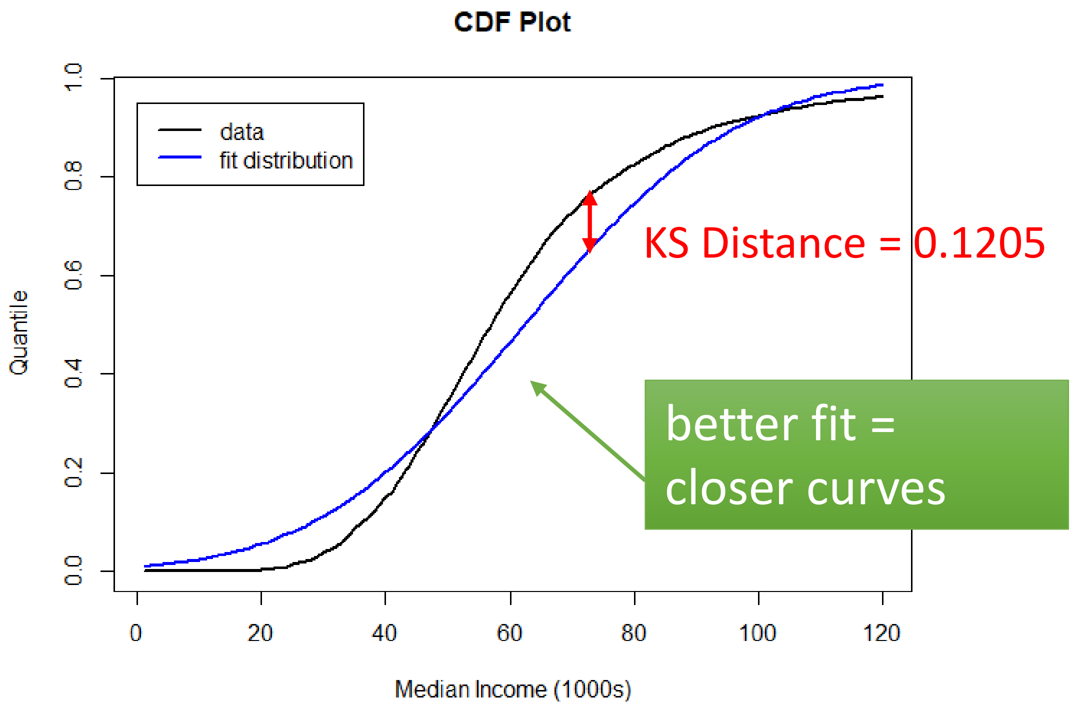
For our data: $\bar{x} = 62.34$, $\overline{(x^2)}$=4600

$\hat{\mu} = 62.34$

$\hat{\sigma} = 26.71$

$\log \mathcal{L}(\hat{\mu}, \hat{\sigma}) = -51999$

Measuring Goodness of Fit



<u>Some Goodness of Fit Statistics</u>

Kolmogorov-Smirnov (KS) Distance – Maximum distance between CDFs of data and best-fit distribution

Akaike Information Criterion (AIC) = 2(# of parameters) – 2(log-likelihood)

Bayesian Information Criterion (BIC) = $\ln(n)$(# of parameters) – 2(log-likelihood)

YOU CAN NOT COMPARE LOG-LIKELIHOODS OF NON-NESTED MODELS. Use AIC instead

better fit =
SMALLER statistics!

Measuring Goodness of Fit

“I want to see *which* distribution fits better”

1. Fit multiple distributions
2. Compare statistics and look at graphs
3. Choose the best distribution (or none)

“I want to *test* whether my model fits the data.”

"I want to *test* whether my model fits the data."

H_0: Your data is a set of random draws from this probability distribution
H_A: Your data is *not* drawn from this distribution

Strategy:

1. Fit the distribution
2. Generate synthetic distributions of the same size as your data using the probability distribution. Store their KS distance
3. Create a sampling distribution of KS
4. Calculate a p-value

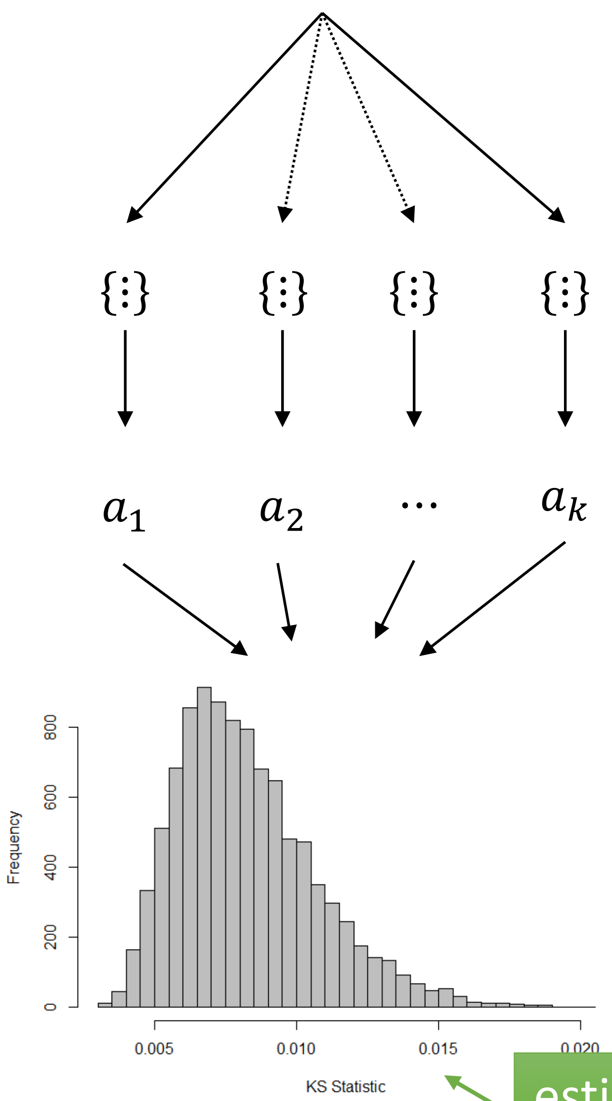
the distribution that we already fit to the data

$$\mathcal{N}(62.34, 26.71)$$

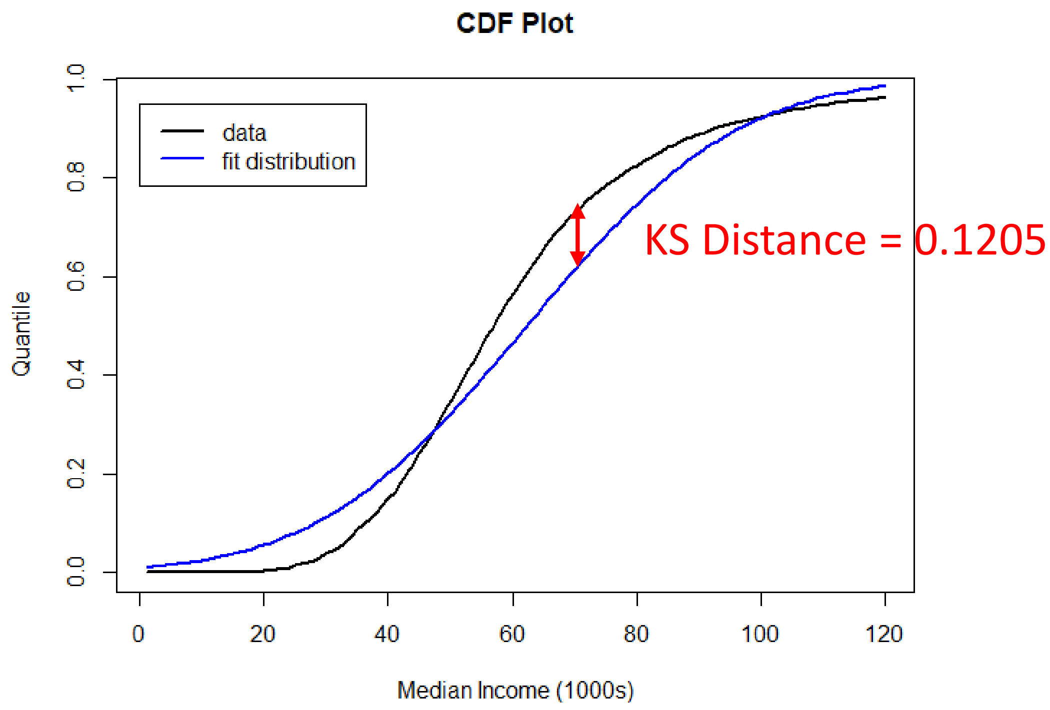
simulated samples n=11054

KS distances

sampling distribution



estimated sampling distribution ASSUMING H_0



Compare our test statistic (KS = 0.1205) to the sampling distribution

p-value = 1

Reject H_0

Our data was not drawn from a normal distribution

Your Turn

1. Download the data
2. Use MLE to fit the following distributions to the data:

log-normal

$$\mathrm{P}(x|\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \mathrm{e}^{\frac{-(\log(\mathrm{x}) - \mu)^2}{2\sigma^2}} \, dx$$

$$\hat{\mu} = \frac{1}{n}\sum_i \log(x_i)$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_i (\log(x_i) - \hat{\mu})^2$$

power law with $x_{min} = 1$

$$P(x|\alpha) = \frac{\alpha-1}{x_{min}}\left(\frac{x}{x_{min}}\right)^{-\alpha}$$

$$\hat{\alpha} = ???$$

3. Compare the fits of the three distributions (normal, log-normal, power law), and decide which fits the best