CS391 Assignment 2

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March 2, 2020

Exercise 1

a) If the data comes from a Gaussian Distribution we can use the Z-scores to find outliers. Anything that falls 3 standard deviation away from the mean is an outlier.

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[20.104721203414197, -2.395147527343992, -10.586591188141448, -8.07520595960352, 20.08073646187272, -3.02210643273962
28, 21.111968065340623, 11.5578708791414, -1.6009047634840101, 21.700777435927332, 11.395997104576299, 11.35797838900
0063, 20.04886141800474, -10.9700334130475, -1.6772466232641587, -10.072341609825347, -9.586953797092942]
There are 17 outliers
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b) I don't know

Exercise 2

a) For a matrix M with vectors u and v, it follows that: $Mu \cdot v = u \cdot M^T v$

Now if M is symmetric and the vectors are eigenvectors of M with corresponding distinct eigenvalues λ_1 and λ_2, such that $Mu = \lambda_1 u$ and $Mv = \lambda_2 v$ then we obtain the following:

$\lambda_1(u \cdot v) = (\lambda_1 u \cdot v) = (Mu \cdot v) = (u \cdot M^T v) = (u \cdot Mv) = (u \cdot \lambda_2 v) = \lambda_2(u \cdot v)$

Thus, $(\lambda_1 - \lambda_2)(u \cdot v) = 0$. Since $(\lambda_1 - \lambda_2) \neq 0$, it must mean that $(u \cdot v) = 0$. Making the eigenvectors orthogonal.

b) If matrix A is symmetric it follows that $A = A^T$, we also know that $(AB)^T = B^T A^T$ for any matrix A and B. Therefore:

$(AA^T)^T = (A^T)^T A^T = AA^T \implies AA^T$ is symmetric. Since by definition a symmetric matrix is equal to its transpose.

c) To begin we know that a symmetric matrix is always diagonizable. Let $A = U\Sigma V^T$ be the SVD. Then $AA^T = U\Sigma^2 U^T$ and $A^T A = V\Sigma^2 V^T$. Since in part (a) we proved that U and V are orthogonal. The inverse of an orthogonal matrix is equal to it's inverse. $A^T A = I$, we have $(A^T A)A^{-1} = IA^{-1} = A^{-1}$. Matrix multiplication is associative $(A^T A)A^{-1} = A^T(AA^{-1}) \implies A^T = A^{-1}$. The same would apply for U and V.

d) $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$

First we need to find AA^T

$$AA^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

$det(AA^T - \lambda I) = \lambda^2 - 34\lambda + 225 = (\lambda - 25)(\lambda - 9)$. The eigenvalues of $A^T A$ are 25, 9, 0.

$\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$

For $\lambda = 25$, $A^T A - 25I = \begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix}$ which row-reduces to $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies$

$v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$

For $\lambda = 9$, $A^T A - 9I = \begin{bmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{bmatrix}$ which row-reduces to $\begin{bmatrix} 1 & 0 & \frac{-1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix} \implies$

$v_2 = \begin{bmatrix} \frac{1}{\sqrt{18}} \\ \frac{-1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \end{bmatrix}$

For the last vector we need $v_2^T v_3 = 0 \implies v_3 = \begin{bmatrix} \frac{2}{\sqrt{3}} \\ \frac{-2}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{bmatrix}$

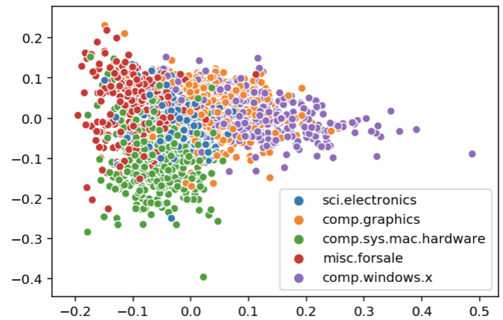
To get you we just need find the vectors using $u_i = \frac{1}{\sqrt{\lambda}} A v_i \implies U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

The SVD of A $= U\Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{18}} & \frac{-1}{\sqrt{18}} & \frac{4}{\sqrt{18}} \\ \frac{2}{\sqrt{3}} & \frac{-2}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$

Exercise 3

I limited to my graph to a rank of 25. In class we discussed that when looking at the singular value graph we want to choose a value of K at around the knee. I approximate my K to be 25 when look at the singular value graph.

I picked the 0.6 as the limit for outliers is because most data points fall between 1 and 1. Any values below 0.6 seems to be an extremity. With this there are 14 outliers.

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[   61    77   360   508  1008  1214  1560  1662  1867  1986  2183  2329  2354  2426]
14
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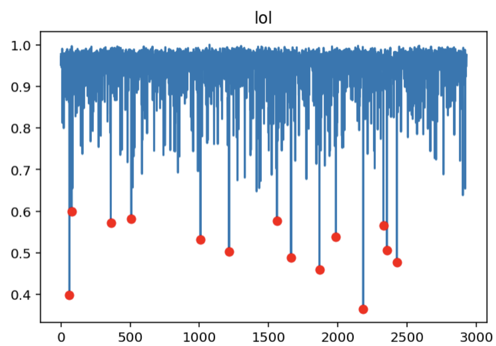
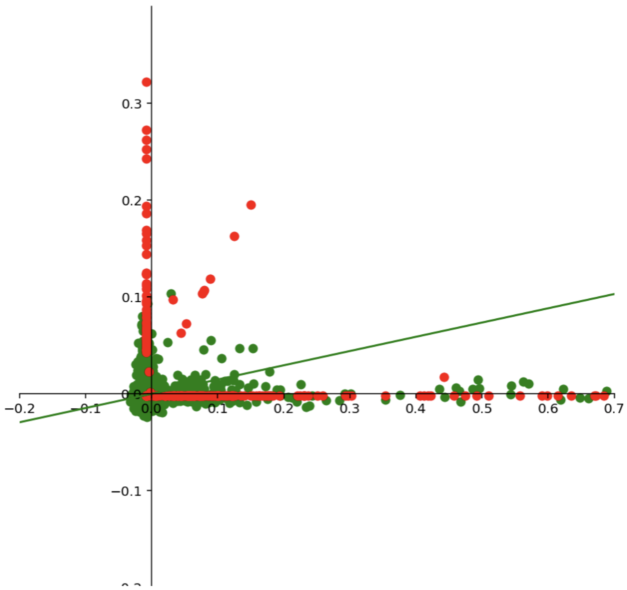


The graph above shows where a lot of the points are clustered. Anything below 0.6 is declared an outlier.

The graph below show a scatter plot of the data points with an estimated best fine line. Points that are really far away from the green line are considered outliers.


0.3
0.2
0.1
0.0
−0.1
−0.2
−0.1
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7