CS391 Assignment 3

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Exercise 1

1) The row sums of a transition matrix are equal 1 because the total probability to transition somewhere is equal to 1. Because the matrix is symmetric the row sums is also equal to the column sums. But this means that the uniform distribution (a vector all of all ones, re-scaled to add up to one) is a stationary distribution. Since the chain is ergodic, there is only one stationary distribution.

2) The stationary measure of a vertex is proportional to its degree. Let **G** be a graph with e edges. Then for any vertex the stationary probability measure is $\pi(i) = \frac{d_i}{2|e|}$. The reason the stationary probability distribution is equal to π is because for a random walk on an undirected graph the probability of reaching any given node is the total number of ways to reach that specific node can be reached divided by the total number of ways to reach any given node. Which is why π is equal to degree of i which is the number of ways to reach node i divided by 2 times the number of edges which is the total number of degrees.

3) This problem is the same random walk as problem 1.2 except instead of basing the probabilities off of the degrees of each node over the total number of degrees in the graph we use the edge weights. Each edge weight represents the probability of that edge being reached. Therefore the probability of reaching node i is doing to be the sum of all the edge weights that lead to node i, over 2 times the total edge weights since each edge must be counted twice to account for the fact that in undirected graphs each edge represents a degree for both nodes connected.

Exercise 2

1) We have P which is a stochastic matrix. If $P' = \alpha P + (1-\alpha)UV^T$ where u is a vector of all 1's and v is a uniform vector (all entries have value 1/n).

$$P' = \alpha P + (1-\alpha)\begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \\ \dots & \dots & \dots & \dots \\ \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \end{bmatrix}$$

Given that $0 < \alpha < 1$ then αP is a stochastic matrix that is being multiplied by a scalar therefore every row will equal to the scalar α.

$(1-\alpha)\begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \\ \dots & \dots & \dots & \dots \\ \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \end{bmatrix}$ is also a stochastic matrix multiplied by a scalar, the value is the complement if α therefore the rows will sum up to $1-\alpha$. The sum of both will be 1 which makes P' a stochastic matrix.

2) $y = xP' = \alpha x P + (||x|| - ||\alpha x P||)V^T$ and $P' = \alpha P + (1-\alpha)UV^T$

$x(\alpha P + (1-\alpha)UV^T) = \alpha x P + (||x|| - ||\alpha x P||)V^T$

$xP' = \alpha x P + (1-\alpha)V^T$

Exercise 3

$$M = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

1) If $x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ then the stationary distribution is $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$$

If $x = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ then the stationary distribution is $\begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

$$\begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

2) I don't know

Exercise 4 I don't know

Exercise 5

1) If M has eigenvalues of $\lambda_1, \lambda_2, ..., \lambda_n$ with eigenvectors of $e_1, ..., e_n$.

Assume M^t has eigenvalues of $\lambda_1^t, \lambda_2^t, ..., \lambda_n^t$ and $\mathbf{X}$ is just a vector. Then

$\mathbf{X}M^t = \mathbf{X}M^{t-1}M$

$\mathbf{X}M^t = (\Sigma \alpha_i e_i)M^{t-1}M$

$\mathbf{X}M^t = (\Sigma \alpha_i e_i M^{t-1})M$

$\mathbf{X}M^t = (\Sigma \alpha_i \lambda_i^t e_i)$

2) We want to show that $||\mathbf{X}M^T - \pi||_2 \leq \sqrt{n}\lambda_2^t$. M has eigenvalues and vectors similar to the previous problem. $\pi = \frac{1}{n}1$

$\mathbf{X}M^t = (\Sigma \alpha_i \lambda_i^t e_i) - \alpha_1 e_1 + \sum_{i=2}^n$

$||\sum_{i=2}^n -\lambda_i^t e_i^t|| \leq \lambda_2^t$