MIDS W207
Applied Machine Learning

Fall 2022

Week 2

Regression Analysis

Regression analysis is a set of statistical methods used for the estimation of relationships between a dependent variable and one or more independent variables.

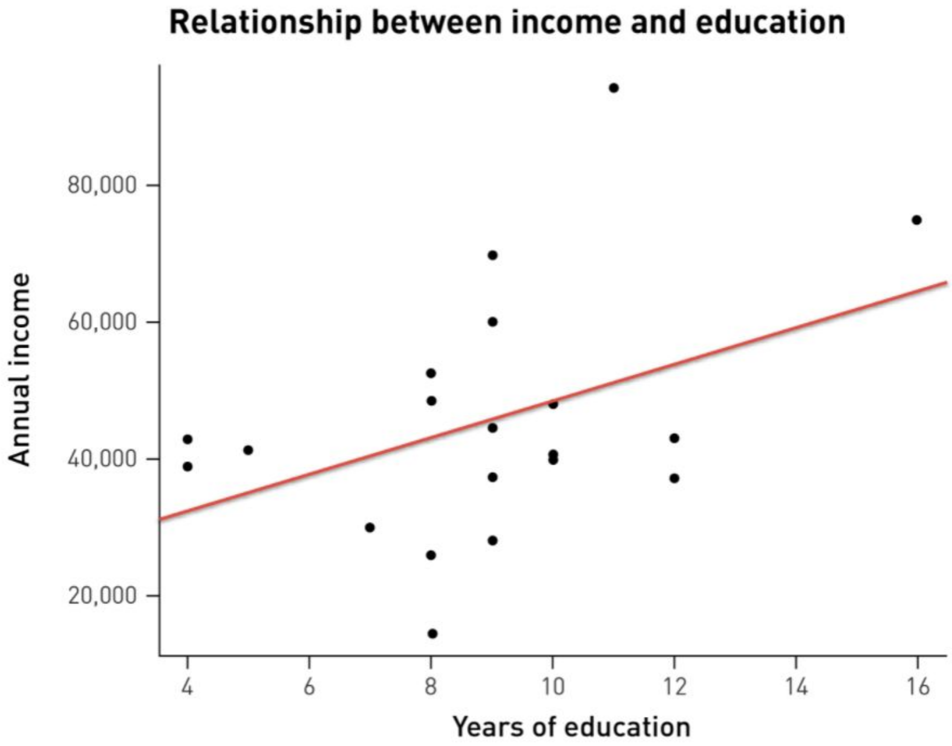
It can be utilized to assess the strength of the relationship between variables and for modeling the future relationship between them.

Applications

1. Forecasting
2. Capital Asset Pricing Model (CAPM)
3. Comparing with competition
4. Identifying problems
5. Reliable Source

$$\underbrace{Y_i}_{\text{Dependent Variable}} = \overbrace{\beta_0}^{\text{Constant/Intercept}} + \underbrace{\beta_1}_{\text{Slope/Coefficient}} \overbrace{X_i}^{\text{Independent Variable}}$$

Regression Analysis: Example



Regression Analysis: Notations

Subscript Notation

$$y_i = \beta_1 x_{i1} + \cdots + \beta_p x_{ip} + \varepsilon_i = x_i^T \beta + \varepsilon_i$$

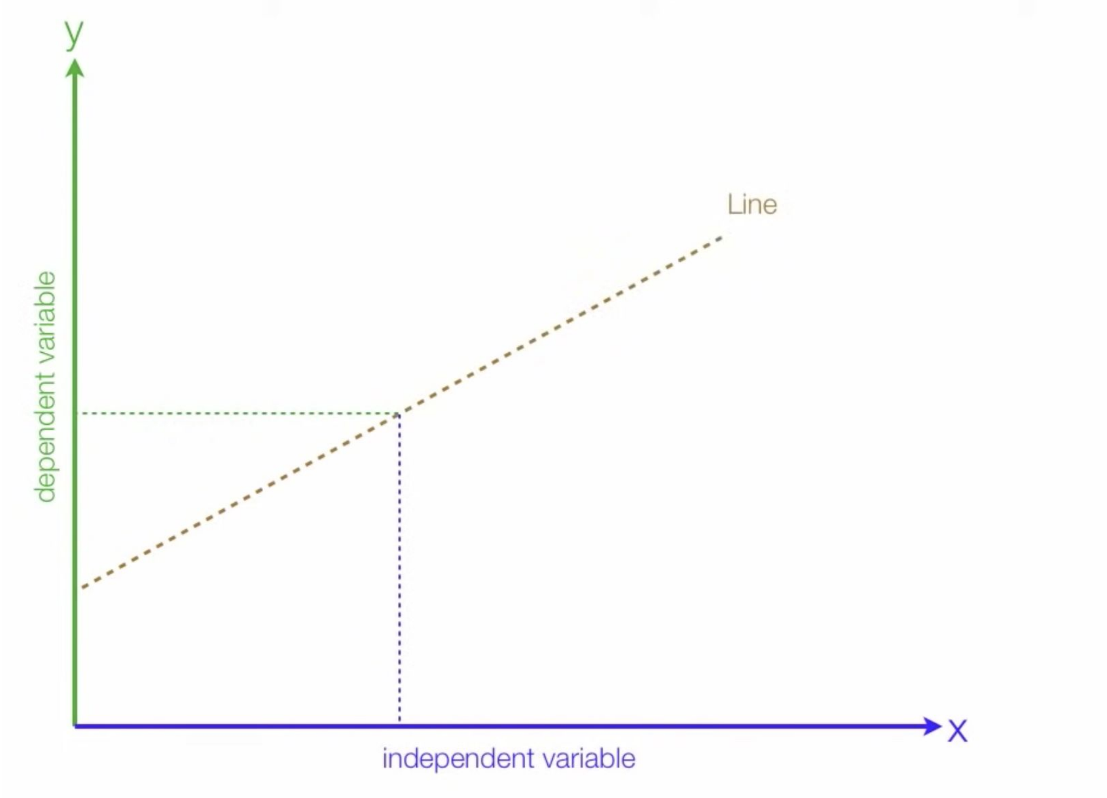
$$i = 1 ,\ldots, n$$

Matrix Notation

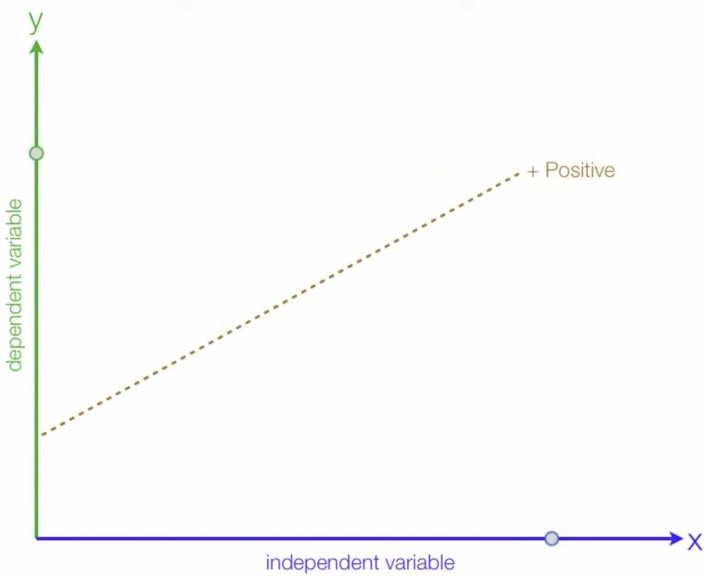
$$y = X\beta + \varepsilon$$

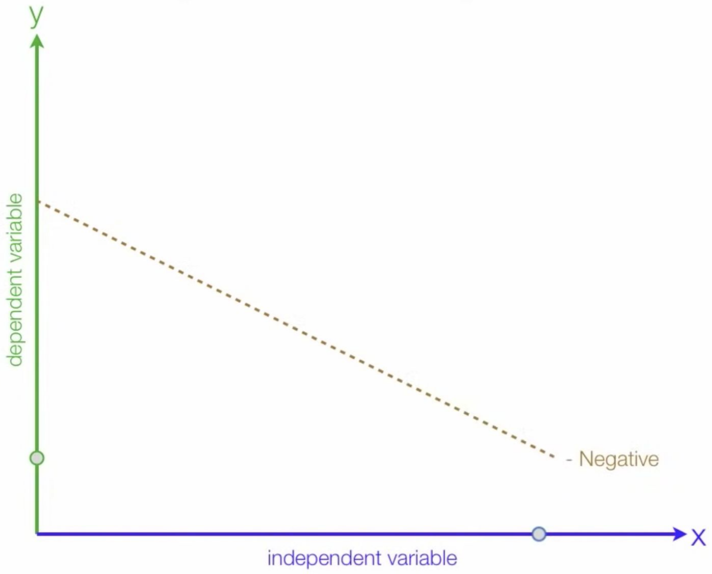
$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} \mathbf{x}_1^{\mathrm{T}} \\ \mathbf{x}_2^{\mathrm{T}} \\ \vdots \\ \mathbf{x}_n^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} x_{11} & \cdots & x_{1p} \\ x_{21} & \cdots & x_{2p} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{np} \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{pmatrix}, \quad \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

Regression Example

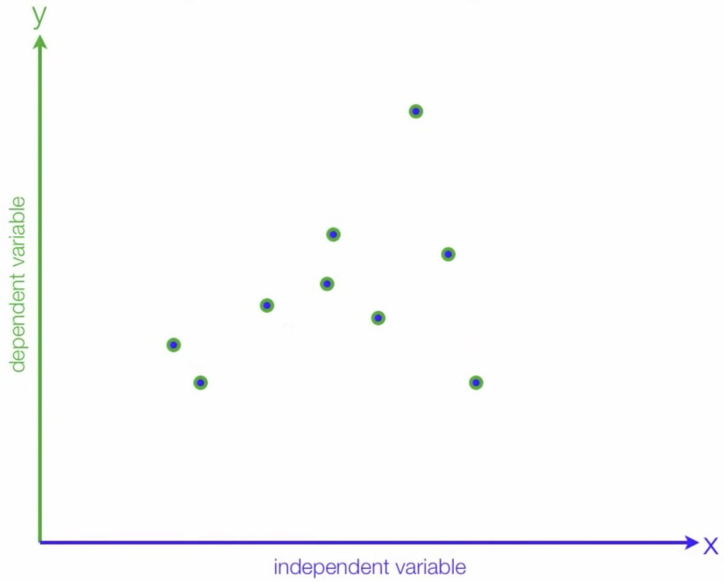


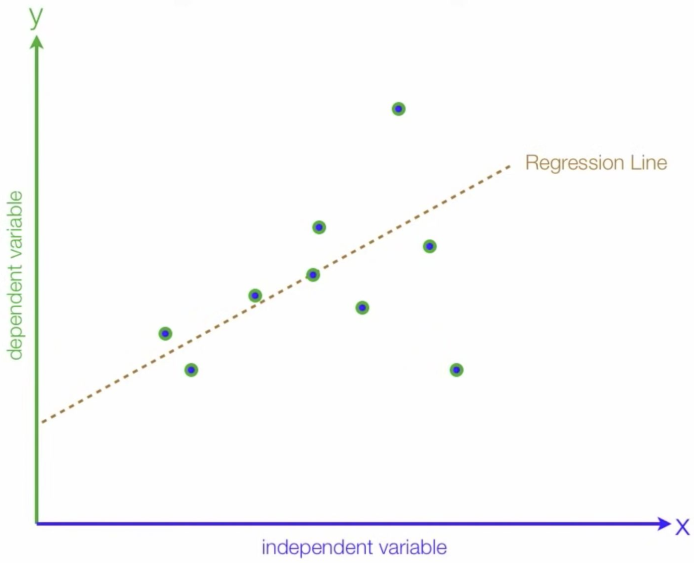
Regression Example



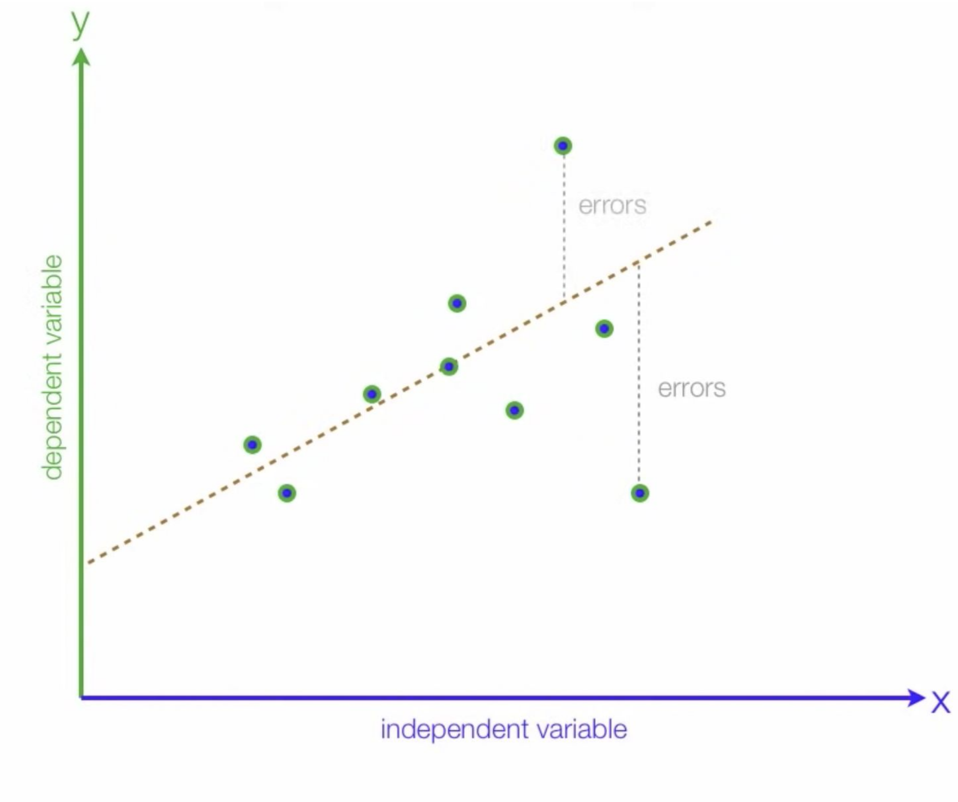


Regression Example

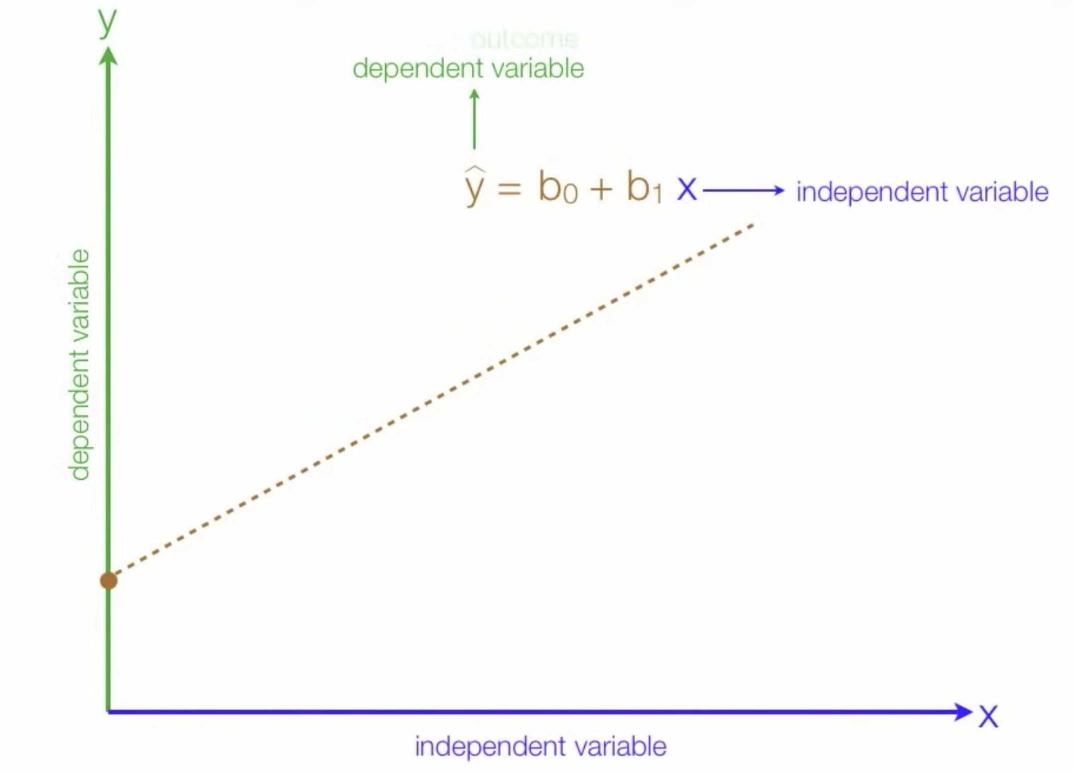




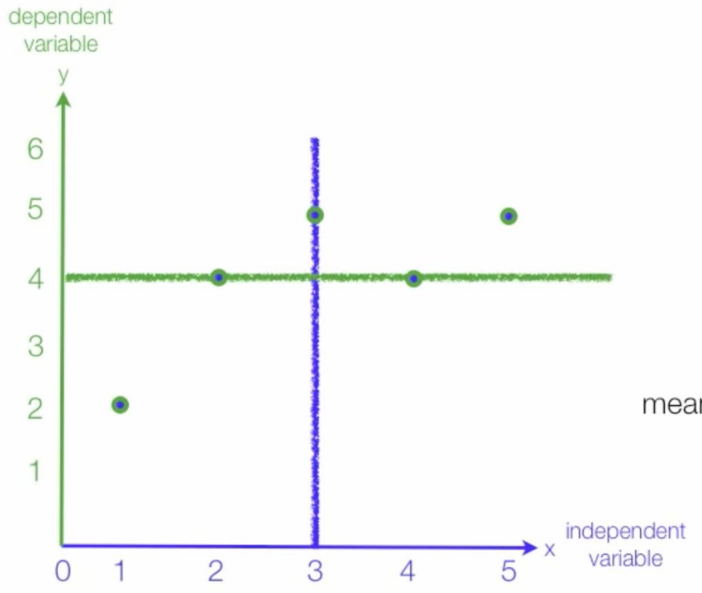
Regression Example



Regression Example

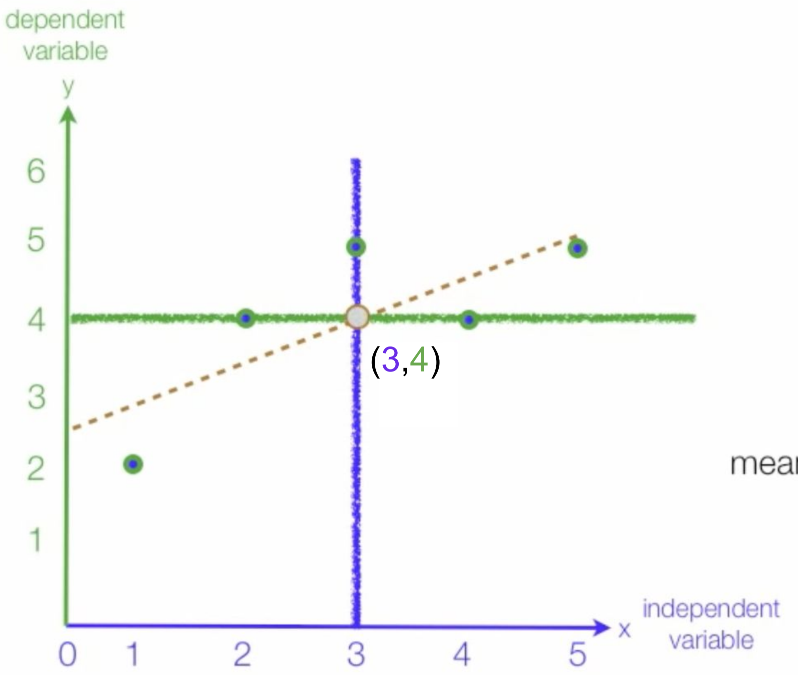


Regression Example



	independent variable	dependent variable
	x	y
	1	2
	2	4
	3	5
	4	4
	5	5
mean	3	4

Regression Example



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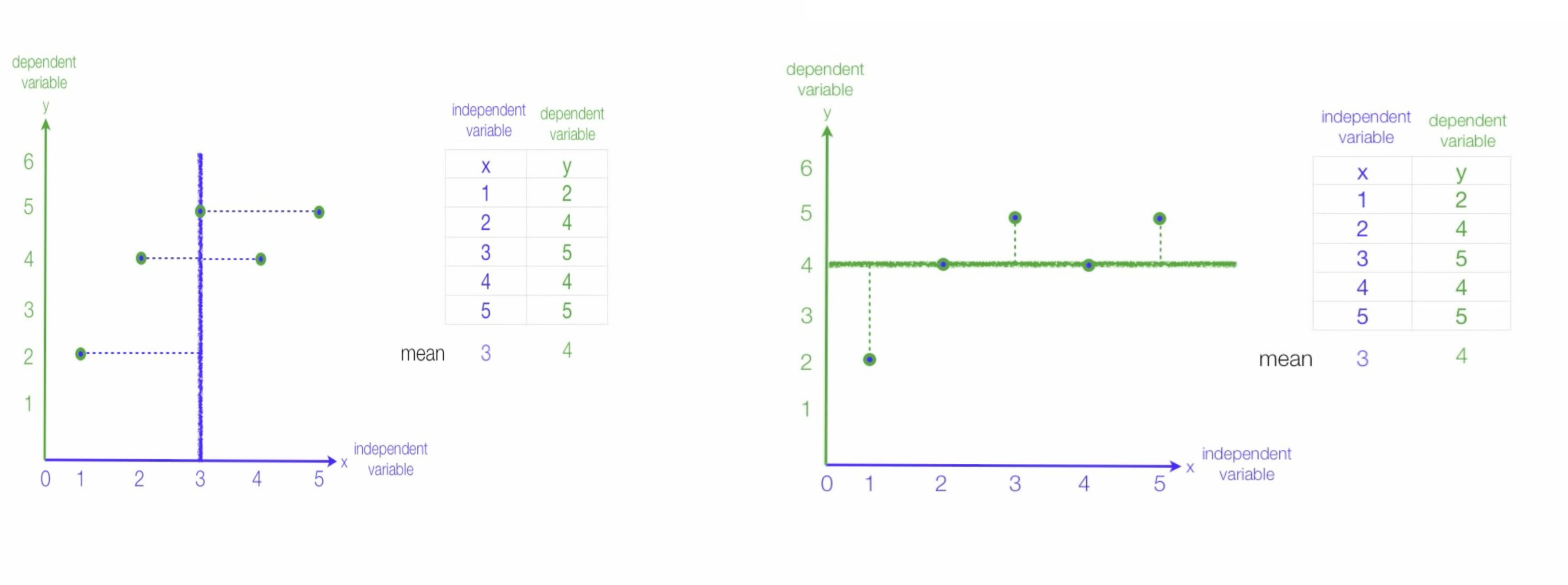
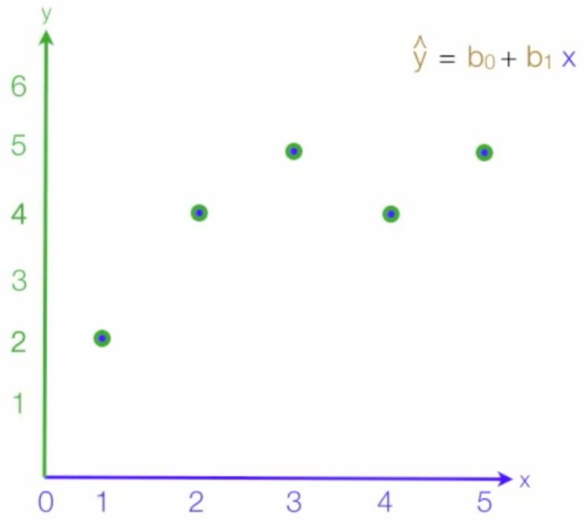
Regression Example

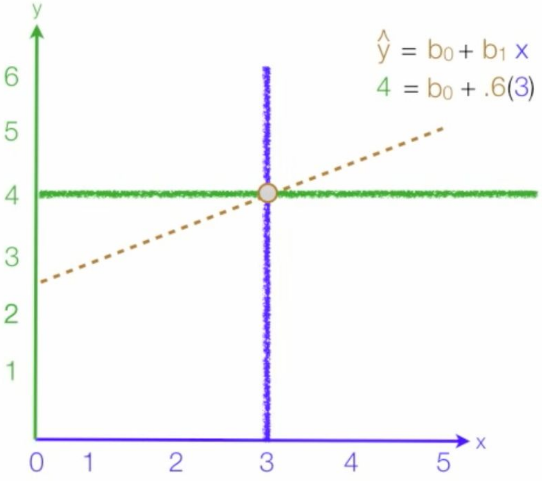


Regression Example



	x	y	$x - \bar{x}$	$y - \bar{y}$	$(x - \bar{x})^2$	$(x - \bar{x})(y - \bar{y})$
	1	2	-2	-2	4	4
	2	4	-1	0	1	0
	3	5	0	1	0	0
	4	4	1	0	1	0
	5	5	2	1	4	2
mean	3	4				

Regression Example



$\hat{y} = b_0 + b_1 x$

$4 = b_0 + .6(3)$

$b_0 = 2.2$

$b_1 = .6$

$\hat{y} = 2.2 + .6x$

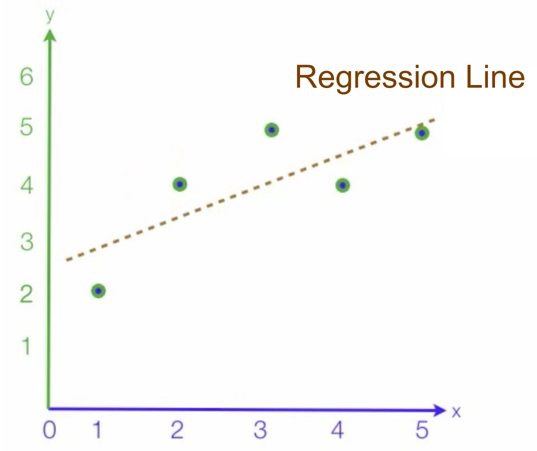
	x	y	$x - \bar{x}$	$y - \bar{y}$	$(x - \bar{x})^2$	$(x - \bar{x})(y - \bar{y})$
	1	2	-2	-2	4	4
	2	4	-1	0	1	0
	3	5	0	1	0	0
	4	4	1	0	1	0
	5	5	2	1	4	2
					10	6
mean	3	4				

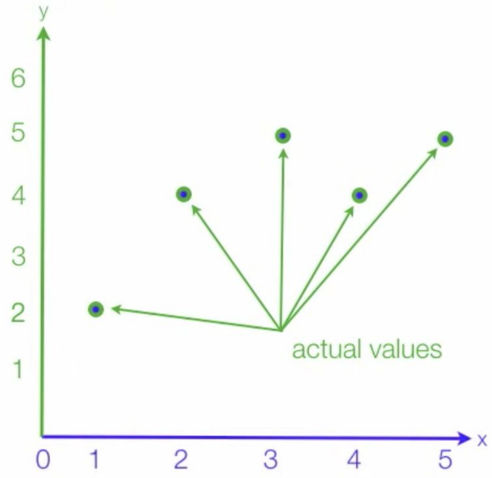
$$4 = b_0 + .6(3)$$

$$\begin{aligned} 4 &= b_0 + \cancel{1.8} \\ -1.8 & \qquad \cancel{-1.8} \\ \hline 2.2 &= b_0 \end{aligned}$$

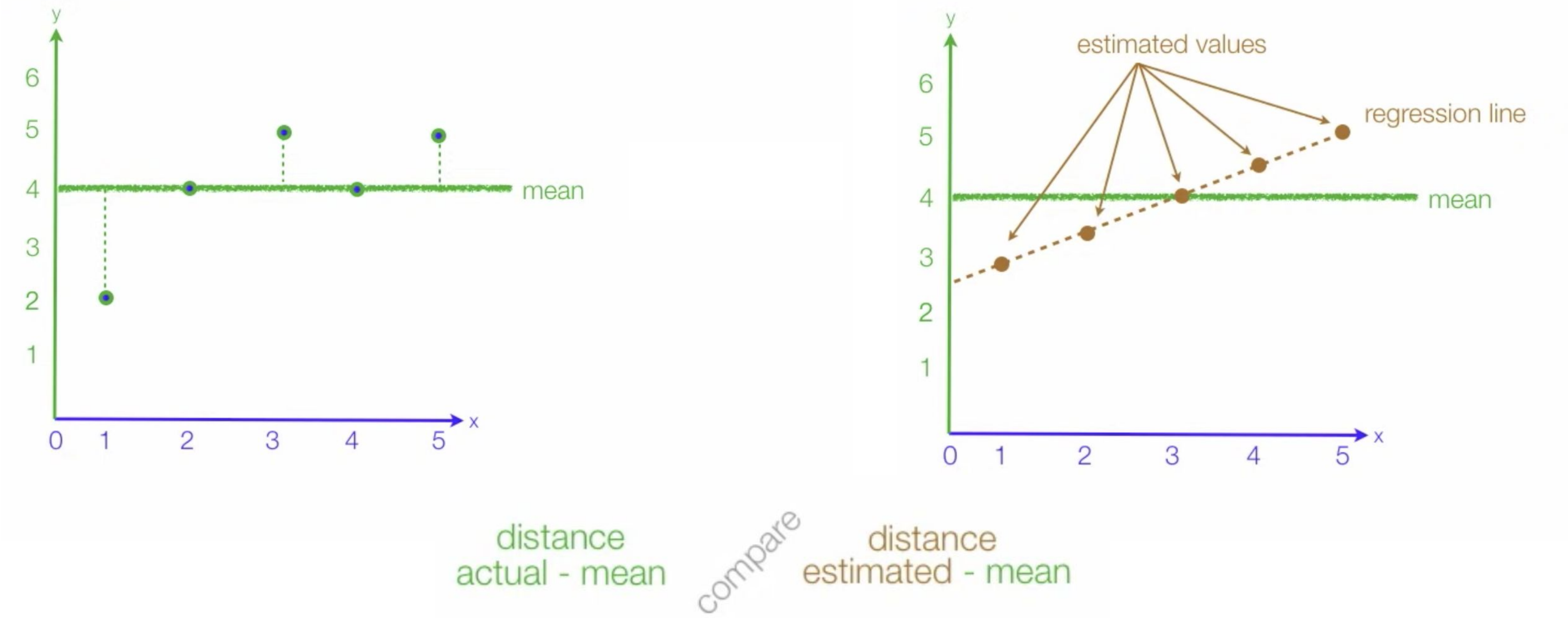
$$b_1 = \frac{6}{10} = .6 = \frac{\Sigma (x - \bar{x})(y - \bar{y})}{\Sigma (x - \bar{x})^2}$$

Regression Example: R-Squared

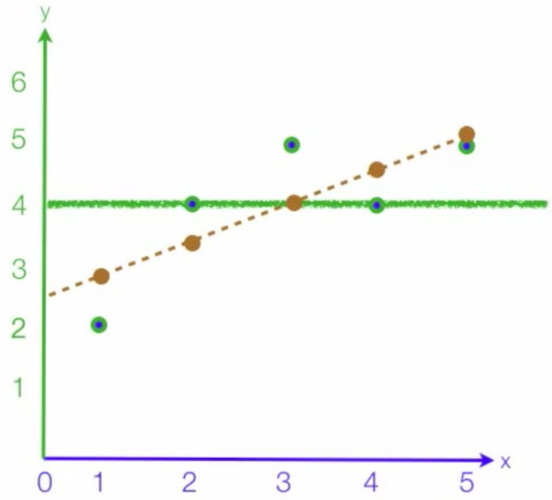




Regression Example: R-Squared



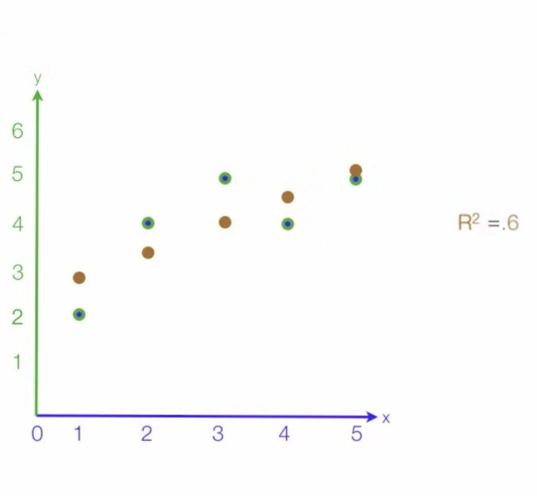
Regression Example: R-Squared

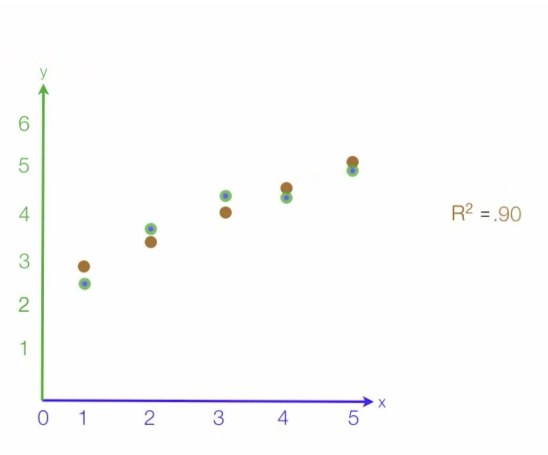


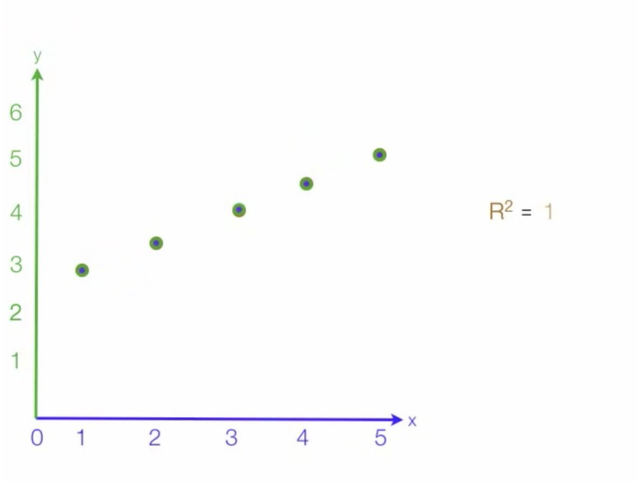
x	y	$y - \bar{y}$	$(y - \bar{y})^2$	$\hat{y}$	$\hat{y} - \bar{y}$	$(\hat{y} - \bar{y})^2$
1	2	-2	4	2.8	-1.2	1.44
2	4	0	0	3.4	-.6	.36
3	5	1	1	4	0	0
4	4	0	0	4.6	.6	.36
5	5	1	1	5.2	1.2	1.44
			6			3.6
mean	4					

$$R^2 = \frac{3.6}{6} = .6 = \frac{\Sigma(\hat{y} - \bar{y})^2}{\Sigma(y - \bar{y})^2}$$

Regression Example: R-Squared







Gradient Descent

Gradient Descent is an optimization algorithm for finding a local minimum of a differentiable function.

Gradient descent is simply used in machine learning to find the values of a function's parameters (coefficients) that minimize a cost function as far as possible.

It's based on a convex function and tweaks its parameters iteratively to minimize a given function to its local minimum.

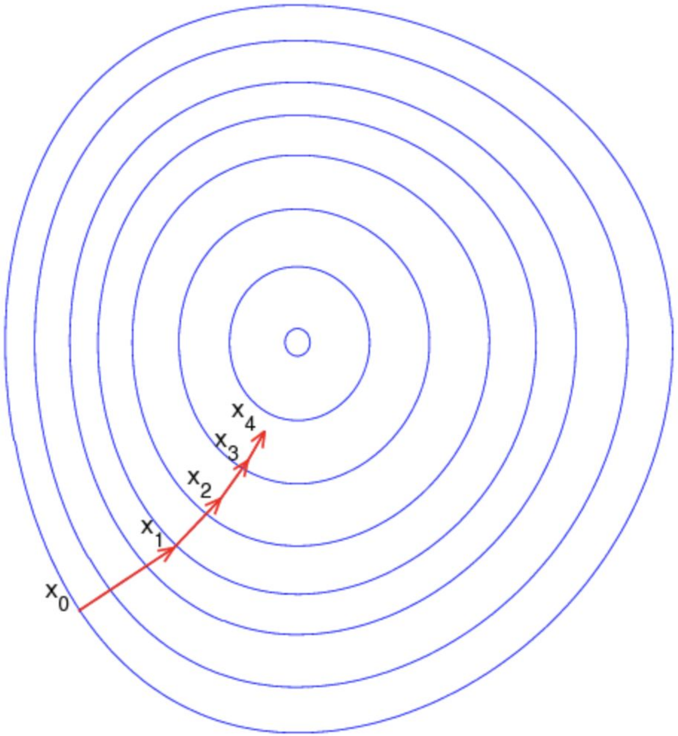
Gradient Descent

> **"A gradient measures how much the output of a function changes if you change the inputs a little bit." —Lex Fridman (MIT)**

A gradient is a derivative of a function that has more than one input variable.

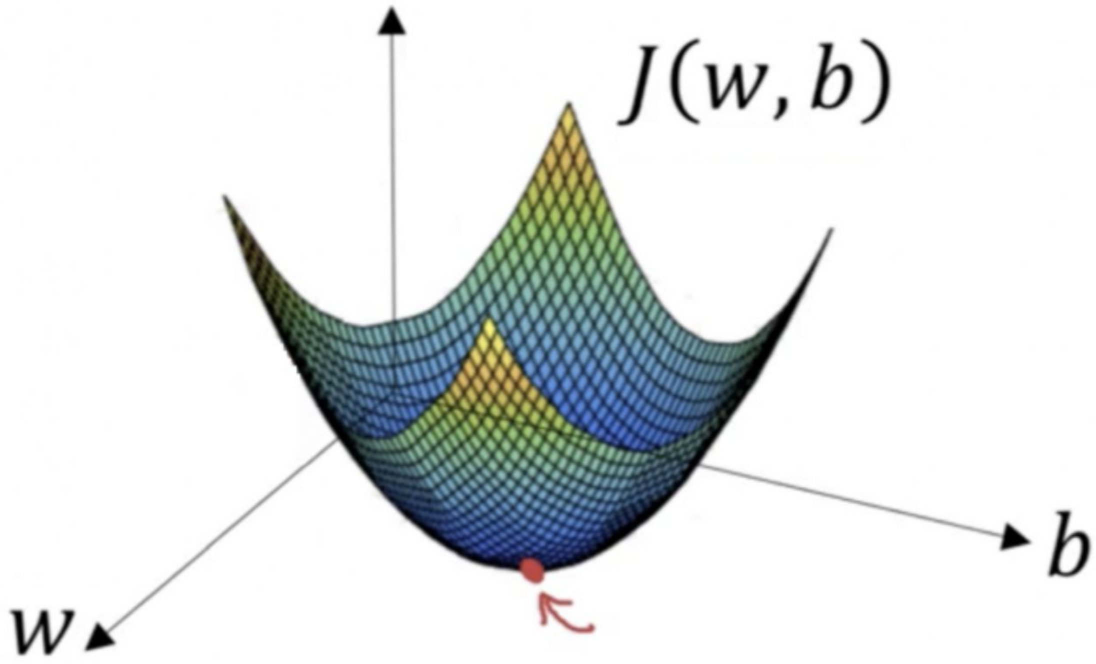
Known as the slope of a function in mathematical terms, the gradient simply measures the change in all weights with regard to the change in error.

Gradient Descent

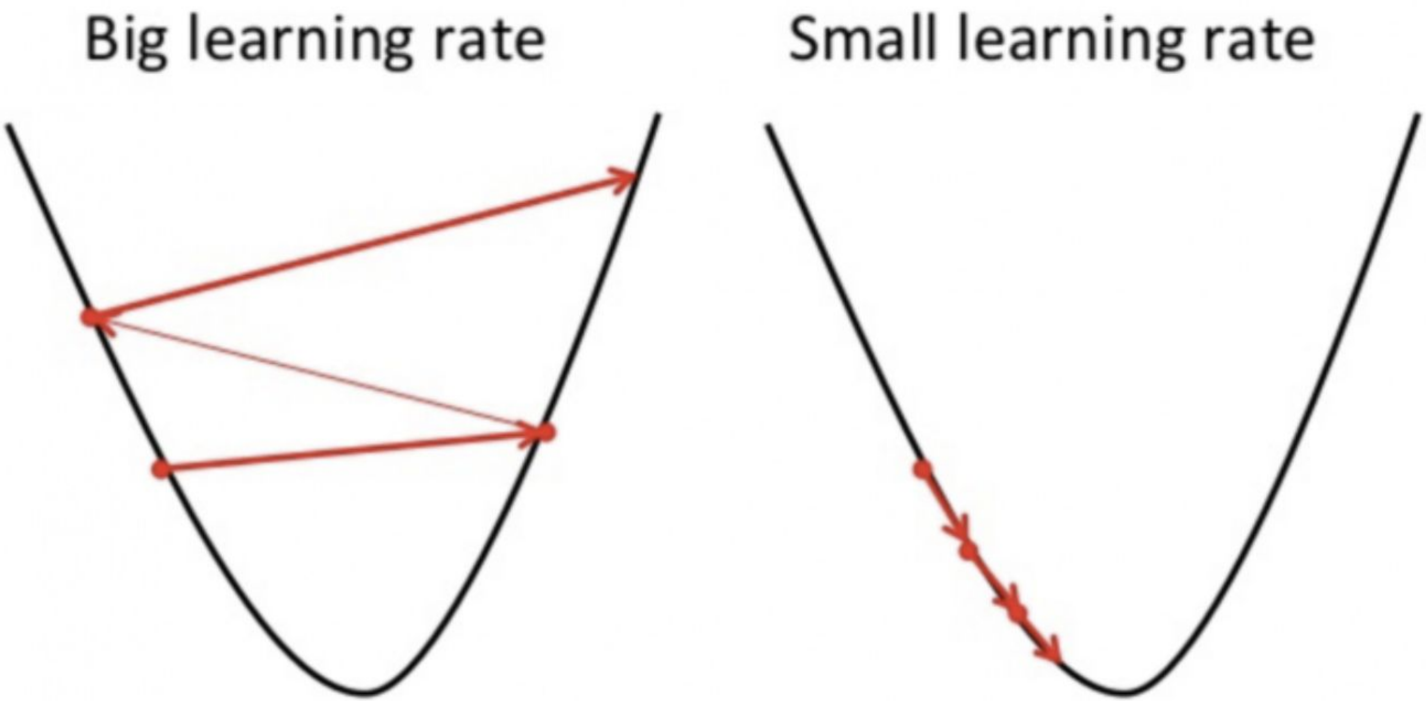


$$\mathbf{b} = \mathbf{a} - \gamma \ \nabla \ \mathbf{f}(\mathbf{a})$$

Gradient Descent: Analysis



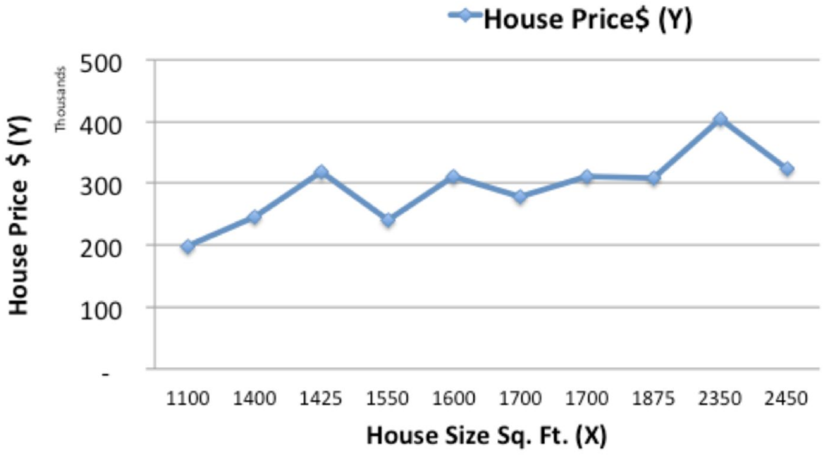
Gradient Descent: Learning Rate



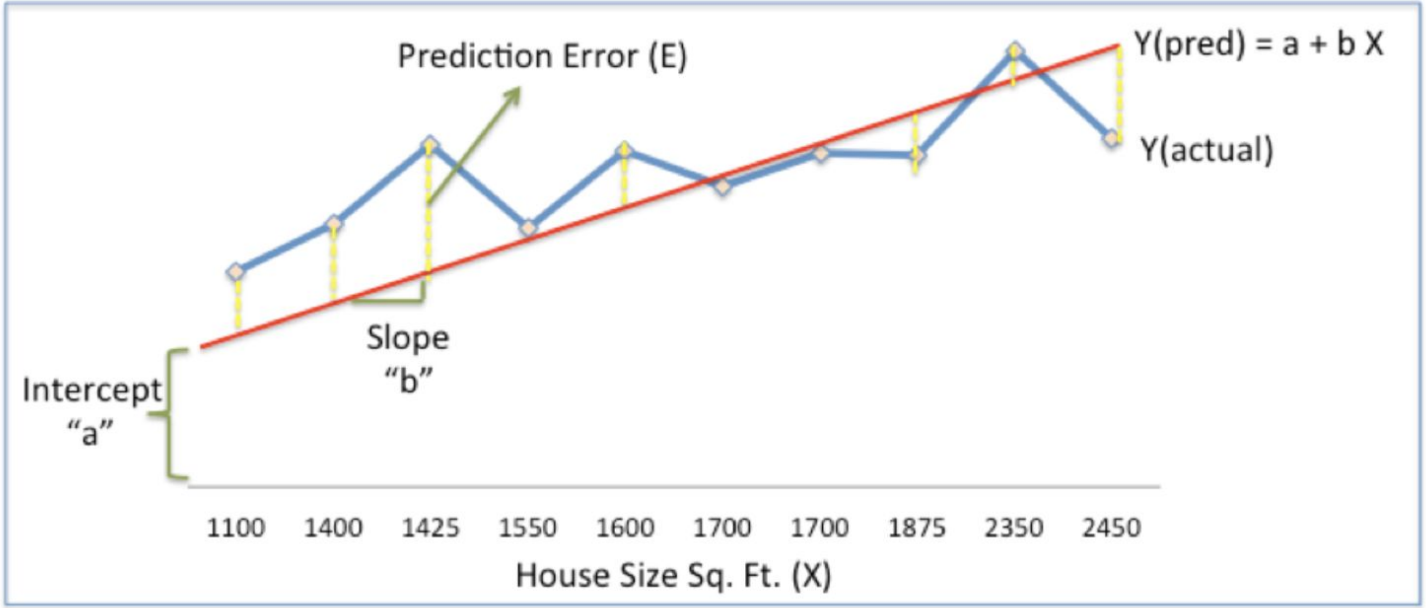
Gradient Descent: Example

House Size sq.ft (X)	1400	1600	1700	1875	1100	1550	2350	2450	1425	1700
House Price$ (Y)	245,000	312,000	279,000	308,000	199,000	219,000	405,000	324,000	319,000	255,000

Given its size (X), what will its price (Y) be?



Gradient Descent: Example



Sum of Squared Errors (SSE) = ½ Sum (Actual House Price – Predicted House Price)^2

= **½ Sum(Y – Ypred)^2**

Gradient Descent: Example

Step 1: Initialize the weights(a & b) with random values and calculate Error (SSE)

Step 2: Calculate the gradient i.e. change in SSE when the weights (a & b) are changed by a very small value from their original randomly initialized value. This helps us move the values of a & b in the direction in which SSE is minimized.

Step 3: Adjust the weights with the gradients to reach the optimal values where SSE is minimized

Step 4: Use the new weights for prediction and to calculate the new SSE

Step 5: Repeat steps 2 and 3 till further adjustments to weights doesn’t significantly reduce the Error

Gradient Descent: Example

HOUSING DATA	
House Size (X)	House Price (Y)
1,100	1,99,000
1,400	2,45,000
1,425	3,19,000
1,550	2,40,000
1,600	3,12,000
1,700	2,79,000
1,700	3,10,000
1,875	3,08,000
2,350	4,05,000
2,450	3,24,000

Normalize

Min-Max Standardization	
X (X-Min/Max-min)	Y (Y-Min/Max-Min)
0.00	0.00
0.22	0.22
0.24	0.58
0.33	0.20
0.37	0.55
0.44	0.39
0.44	0.54
0.57	0.53
0.93	1.00
1.00	0.61

Gradient Descent: Example

Step 1

a	b	X	Y	YP=a+bX	SSE=1/2(Y-YP)^2
0.45	0.75	0.00	0.00	0.45	0.101
		0.22	0.22	0.62	0.077
		0.24	0.58	0.63	0.001
		0.33	0.20	0.70	0.125
		0.37	0.55	0.73	0.016
		0.44	0.39	0.78	0.078
		0.44	0.54	0.78	0.030
		0.57	0.53	0.88	0.062
		0.93	1.00	1.14	0.010
		1.00	0.61	1.20	0.176
				Total SSE	**0.677**

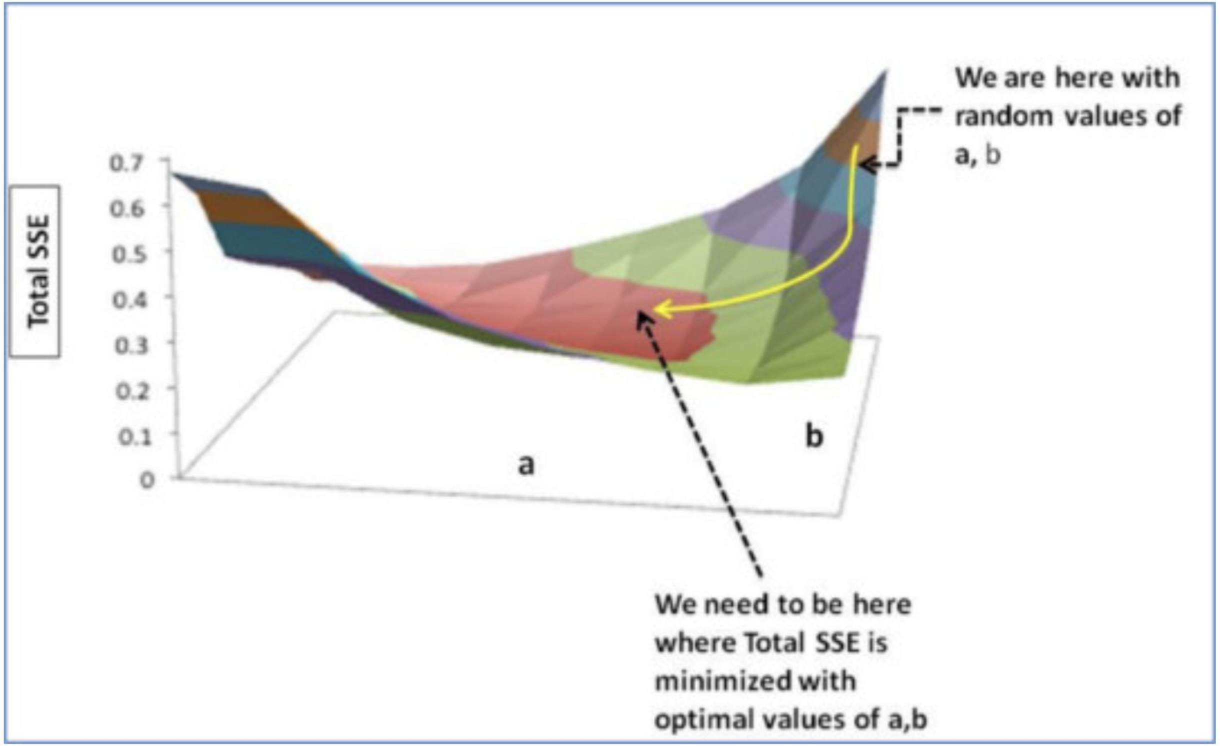
Gradient Descent: Example

Step 2

a	b	X	Y	YP=a+bX		SSE		∂SSE/∂a = -(Y-YP)	∂SSE/∂b = -(Y-YP)X
0.45	0.75	0.00	0.00	0.45		0.101		0.45	0.00
		0.22	0.22	0.62		0.077		0.39	0.09
		0.24	0.58	0.63		0.001		0.05	0.01
		0.33	0.20	0.70		0.125		0.50	0.17
		0.37	0.55	0.73		0.016		0.18	0.07
		0.44	0.39	0.78		0.078		0.39	0.18
		0.44	0.54	0.78		0.030		0.24	0.11
		0.57	0.53	0.88		0.062		0.35	0.20
		0.93	1.00	1.14		0.010		0.14	0.13
		1.00	0.61	1.20		0.176		0.59	0.59
					Total SSE	**0.677**	**Sum**	**3.300**	**1.545**

Gradient Descent: Example

Step 3



Gradient Descent: Example

Step 4

a	b	X	Y	YP=a+bX		SSE		∂SSE/∂a	∂SSE/∂b
0.42	0.73	0.00	0.00	0.42		0.087		0.42	0.00
		0.22	0.22	0.58		0.064		0.36	0.08
		0.24	0.58	0.59		0.000		0.01	0.00
		0.33	0.20	0.66		0.107		0.46	0.15
		0.37	0.55	0.69		0.010		0.14	0.05
		0.44	0.39	0.74		0.063		0.36	0.16
		0.44	0.54	0.74		0.021		0.20	0.09
		0.57	0.53	0.84		0.048		0.31	0.18
		0.93	1.00	1.10		0.005		0.10	0.09
		1.00	0.61	1.15		0.148		0.54	0.54
					Total SSE	0.553	Sum	2.900	1.350

Gradient Descent: In depth Analysis

Formula:

$$X = X - lr * \frac{d}{dX} f(X)$$

$Where,$

$X \quad = input$

$F(X) = output \ based \ on \ X$

$lr \quad = learning \ rate$

Gradient Descent: Single Variable

Cost Function

$$J(\theta) = \theta^2$$

Goal

$$\min J(\theta)$$

Update Function

$$\theta := \theta - \alpha * \frac{d}{d\theta} J(\theta)$$

Learning Rate

$$Learning\ Rate:$$

$$\alpha = 0.1$$

Gradient Descent: Single Variable

Updating Parameters

$$\theta := \theta - \alpha * \frac{d}{d\theta} J(\theta)$$

$$\theta := \theta - \alpha * 2\theta$$

$$\theta := \theta - 2\alpha\theta$$

$$\theta := 0.8 * \theta$$

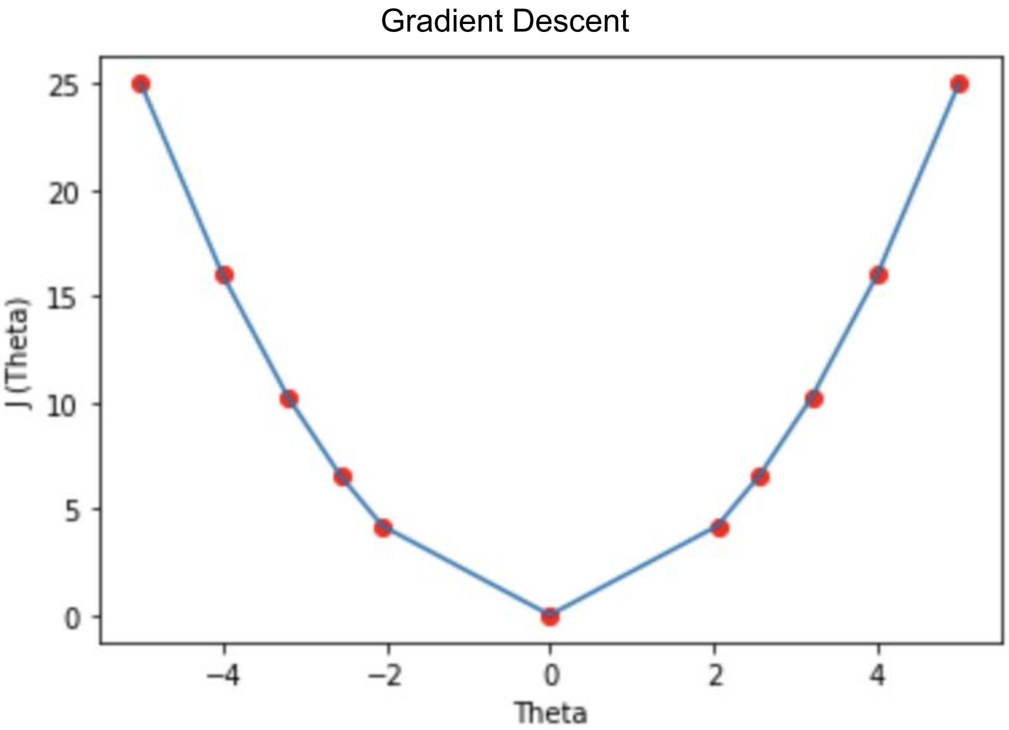
Table Generation

Gradient Descent: Single Variable

θ	J(θ)
5	25
4	16
3.2	10.24
2.56	6.55
2.04	4.19
|	|
|	|
0	0

θ	J(θ)
-5	25
-4	16
-3.2	10.24
-2.56	6.55
-2.04	4.19
|	|
|	|
0	0

Gradient Descent: Single Variable



Gradient Descent: Multiple Variables

Cost Function

$$J(\theta_1, \theta_2) = \theta_1{}^2 + \theta_2{}^2$$

Goal

$$\min \ J(\theta_1, \theta_2)$$

Update Function

$$\theta_1 := \theta_1 - \alpha * \frac{d}{d\theta_1} J(\theta_1, \theta_2)$$

$$\theta_2 := \theta_2 - \alpha * \frac{d}{d\theta_2} J(\theta_1, \theta_2)$$

Gradient Descent: Multiple Variables

Derivatives

$$\frac{d}{d\theta_1}J(\theta_1,\theta_2) = \frac{d}{d\theta_1}({\theta_1}^2 + {\theta_2}^2)$$

$$= \frac{d}{d\theta_1}({\theta_1}^2) + \frac{d}{d\theta_1}({\theta_2}^2)$$

$$= 2\theta_1 + 0$$

$$= 2\theta_1$$

$$\frac{d}{d\theta_2}J(\theta_1,\theta_2) = \frac{d}{d\theta_2}({\theta_1}^2 + {\theta_2}^2)$$

$$= \frac{d}{d\theta_2}({\theta_1}^2) + \frac{d}{d\theta_2}({\theta_2}^2)$$

$$= 0 + 2\theta_2$$

$$= 2\theta_2$$

Gradient Descent: Multiple Variables

Update Values

$$\theta_1 := \theta_1 - \alpha * 2\theta_1$$

$$\theta_1 := \theta_1 - 2\alpha\theta_1$$

$$\theta_2 := \theta_2 - \alpha * 2\theta_2$$

$$\theta_2 := \theta_2 - 2\alpha\theta_2$$

Learning Rate

$$Learning\ Rate:$$

$$\alpha = 0.1$$

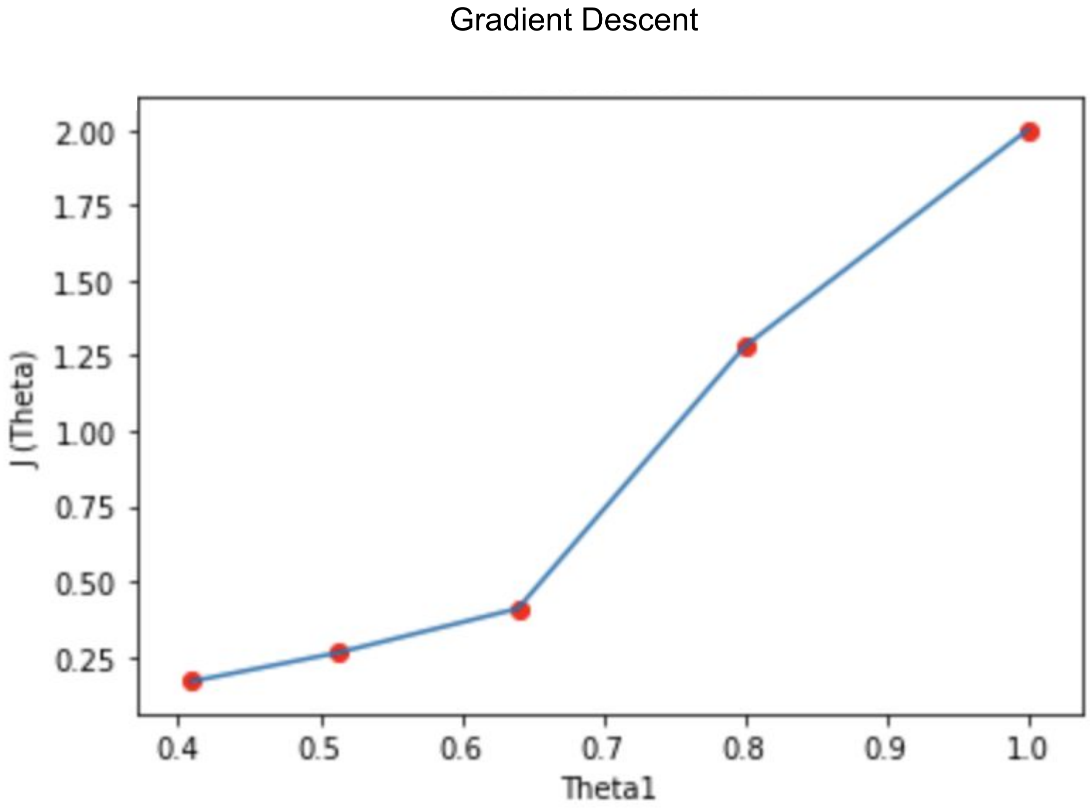
Gradient Descent: Multiple Variables

Table

θ1	θ2	J(θ)
1	1	2
0.8	0.8	1.28
0.64	0.64	0.4096
0.512	0.512	0.2621
0.4096	0.4096	0.1677
|	|	|
|	|	|
0	0	0

Gradient Descent: Multiple Variables

Graph



Gradient Descent: Types