Construction of Families of Permutation Trinomials over Finite Fields

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Table of Contents

Table of Contents

Finite Fields

Definition

*A **finite field** $\mathbb{F}_q$ is a field with $q = p^r$ elements where p is prime.*

Example

$$\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$$

Addition:
$2 + 2 = 4$
$4 + 4 = 8 \pmod 5 = 3$

Multiplication:
$2 \cdot 2 = 4$
$4 \cdot 4 = 16 \pmod 5 = 1$

Value Sets

Definition

Let $f(x)$ be a polynomial defined over a finite field $\mathbb{F}_q$. Then the ***value set*** *of f is defined as $V(f) = \{f(a) \mid a \in \mathbb{F}_q\}$*

Example

Consider $f(x) = x^2$ defined over $\mathbb{F}_5$.

Note: $f(0) = 0, f(1) = 1, f(2) = 4, f(3) = 4, f(4) = 1$

$V(f) = \{0, 1, 4\}$.

Permutation Polynomials

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Example

Let $f(x) = x^2$ over $\mathbb{F}_5$. We have that $V(f) = \{0, 1, 4\}$ so $f(x)$ is not a permutation polynomial over $\mathbb{F}_5$.

Primitive Roots

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Table of Contents

Permutation Polynomials

- Everyting is known about Permutation Monomials
- Permutation Binomials have been studied extensively
- The next case is to study Permutation Trinomials

Permutation trinomials of the form $X^{\frac{q+1}{2}} + aX^{\frac{q-1}{d}+1} + bX$

$$f_{a,b} = X\left(X^{\frac{p-1}{2}} + aX^{\frac{p-1}{d}} + b\right)$$

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p	$N(p,3)$	$N(p,4)$	$N(p,6)$	p	$N(p,3)$	$N(p,4)$	$N(p,6)$
13	–	8	18	61	60	304	30
17	–	16	–	67	78	–	108
19	0	–	0	73	54	440	54
29	–	48	–	79	96	–	48
31	0	–	18	89	–	680	–
37	12	132	12	97	174	840	102
41	–	140	–	101	–	940	–
43	48	–	36	103	162	–	72
53	–	244	–				

Our Polynomial

Let $d_1, d_2 \in \mathbb{N}$ such that $d_1 \mid (q-1)$ y $d_2 \mid (q-1)$. We are interested in the polynomial:

$$f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$$

with $a, b \in \mathbb{F}_q^{\times}$.

Problem

Our Problem

Study the value set of polynomials of the form

$$f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$$

and determine conditions in a, b such that they are permutation polynomials.

Table of Contents

The class of equivalence $[a, b]$

$$f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$$

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$$b' = \alpha^{j+(\frac{q-1}{d_1})}$$

The class of equivalence $[a, b]$

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$$(2^2, 2^3) \sim (a', b') \iff a' = 2^{2+(6-4)}, b' = 2^{3+6}$$

$$(2^2, 2^3) \sim (2^4, 2^9)$$

The class of equivalence $[a, b]$

$$f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$$

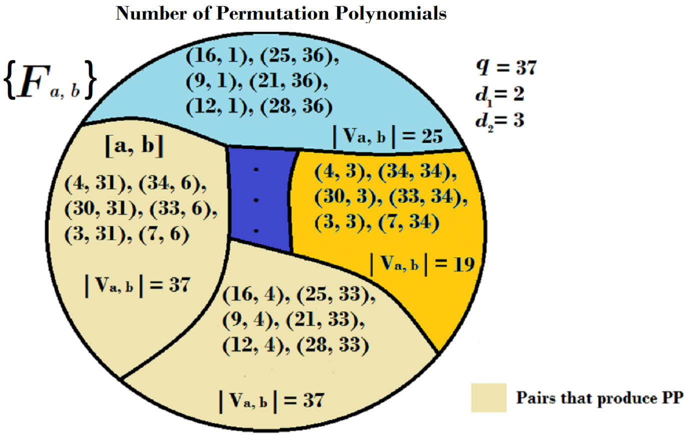
Lemma

The relation $\sim$ defined previously is an equivalence relation.

$f_{a,b}$ with equivalence classes:

$$[f_{a,b}] = [f_{\alpha^i,\alpha^j}] = \{f_{a',b'} \mid (a, b) \sim (a', b')\}$$

Polynomial Results



Value set correspondence

$$f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$$

Theorem

Suppose that $f_{a,b} \sim f_{a',b'}$ then $|V(f_{a,b})| = |V(f_{a',b'})|$.

Example

Let $q = 13, d_1 = 2, d_2 = 3, a = 4, b = 8$. Since $(2^2, 2^3) \sim (2^4, 2^9)$ we have that $|V(f_{2^2,2^3})| = |V(f_{2^4,2^9})| = 7$

Permutation Polynomial correspondence

$$f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$$

Corollary

Suppose that $f_{a,b}$ is a permutation polynomial and $f_{a,b} \sim f_{a',b'}$, then $f_{a',b'}$ is also a permutation polynomial.

Size of equivalence classes

$$f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$$

Proposition

$|[f_{a,b}]| = lcm(d_1, d_2)$ where $lcm(x, y)$ is the least common multiple of x and y.

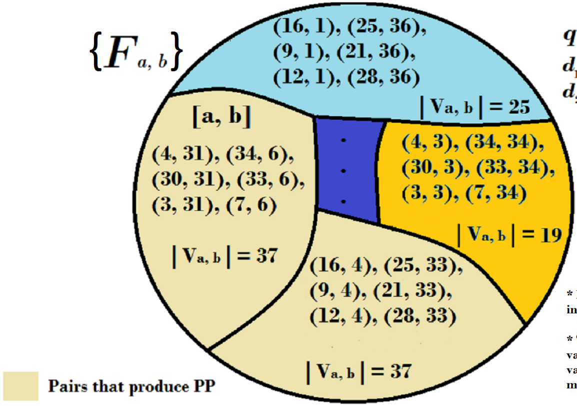
Example

Let $q = 13, d_1 = 2, d_2 = 3, a = 4, b = 8$. Note that $lcm(2, 3) = 6$
These are the elements of (a, b):

$$\begin{array}{ccccccc} (2^2, 2^3) & (2^4, 2^9) & (2^6, 2^3) & (2^8, 2^9) & (2^{10}, 2^3) & (2^{12}, 2^9) & (2^2, 2^3) \\ (4, 8) & (3, 5) & (12, 8) & (9, 5) & (10, 8) & (1, 5) & (4, 8) \end{array}$$

Polynomials Results

Number of Permutation Polynomials



$q = 37$
$d_1 = 2$
$d_2 = 3$

* Note that the number of polynomials in each cell is $6 = lcm(2, 3)$

* The polynomials within each cell have value sets of the same size. The size of the value sets associated to different cells might or might not be equal.

Polynomial Results

Proposition

The number of polynomials of the form $f_{a,b}(X)$ with $|V(f_{a,b})| = n$ is a multiple of $lcm(d_1, d_2)$

Corollary

The number of permutation polynomials of the form $f_{a,b}(X)$ is a multiple of $lcm(d_1, d_2)$

Future Work

- Find necessary and sufficient conditions such that $V(f_{a,b}) = \mathbb{F}_q$
- Generalize results to polynomials with more terms and with exponents not divisors of $q-1$:

$$f_{a,b}(X) = X^r(X^{d_1} + aX^{d_2} + b)$$