# Construction of Families of Permutation Trinomials over Finite Fields

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# Finite Fields

## Definition

*A **finite field** $\mathbb{F}_q$ is a field with $q = p^r$ elements where $p$ is prime.*

## Example

$$\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$$

**Addition:**
$2 + 2 = 4$
$4 + 4 = 8$
$\pmod 5 = 3$

**Multiplication:**
$2 \cdot 2 = 4$
$4 \cdot 4 = 16$
$\pmod 5 = 1$

# Value Sets

## Definition

*Let $f(x)$ be a polynomial defined over a finite field $\mathbb{F}_q$. Then the* ***value set*** *of $f$ is defined as $V(f) = \{f(a) \mid a \in \mathbb{F}_q\}$*

## Example

Consider $f(x) = x^2$ defined over $\mathbb{F}_5$.

Note: $f(0) = 0, f(1) = 1, f(2) = 4, f(3) = 4, f(4) = 1$

$V(f) = \{0, 1, 4\}$.

# Permutation Polynomials

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Let $f(x) = x^2$ over $\mathbb{F}_5$. We have that $V(f) = \{0, 1, 4\}$ so $f(x)$ is not a permutation polynomial over $\mathbb{F}_5$.

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# Permutation Polynomials

- Everyting is known about Permutation Monomials
- Permutation Binomials have been studied extensively
- The next case is to study Permutation Trinomials

# Permutation trinomials of the form $X^{\frac{q+1}{2}} + aX^{\frac{q-1}{d}+1} + bX$

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| $p$ | $N(p,3)$ | $N(p,4)$ | $N(p,6)$ | $p$ | $N(p,3)$ | $N(p,4)$ | $N(p,6)$ |
|---|---|---|---|---|---|---|---|
| 13 | – | 8 | 18 | 61 | 60 | 304 | 30 |
| 17 | – | 16 | – | 67 | 78 | – | 108 |
| 19 | 0 | – | 0 | 73 | 54 | 440 | 54 |
| 29 | – | 48 | – | 79 | 96 | – | 48 |
| 31 | 0 | – | 18 | 89 | – | 680 | – |
| 37 | 12 | 132 | 12 | 97 | 174 | 840 | 102 |
| 41 | – | 140 | – | 101 | – | 940 | – |
| 43 | 48 | – | 36 | 103 | 162 | – | 72 |
| 53 | – | 244 | – | | | | |

# Our Polynomial

Let $d_1, d_2 \in \mathbb{N}$ such that $d_1 \mid (q-1)$ y $d_2 \mid (q-1)$. We are interested in the polynomial:

$$f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$$

with $a, b \in \mathbb{F}_q^{\times}$.

# Problem

## Our Problem

*Study the value set of polynomials of the form*

$$f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$$

*and determine conditions in $a, b$ such that they are permutation polynomials.*

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# The class of equivalence $[a, b]$

$$f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$$

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$$h = 1{:}(2^2, 2^3) \sim (2^4, 2^9)$$

# The class of equivalence $[a, b]$

$$f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$$

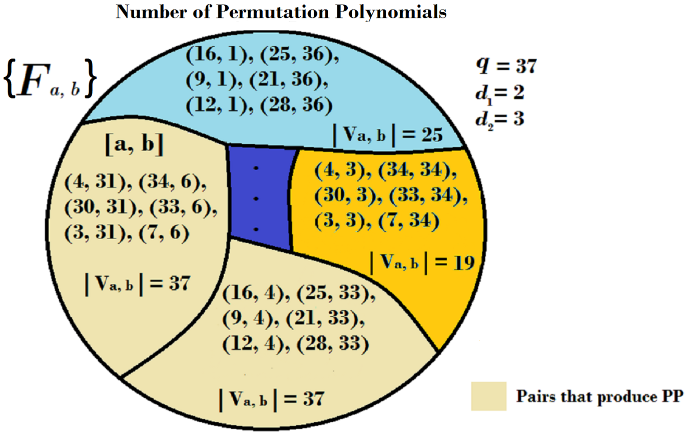
**Lemma**

*The relation $\sim$ defined previously is an equivalence relation.*

$f_{a,b}$ with equivalence classes:

$$[f_{a,b}] = [f_{\alpha^i,\alpha^j}] = \{f_{a',b'} \mid (a,b) \sim (a',b')\}$$

# Polynomial Results



# Value set correspondence

$$f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$$

**Theorem**

*Suppose that $f_{a,b} \sim f_{a',b'}$ then $|V(f_{a,b})| = |V(f_{a',b'})|$.*

**Example**

Let $q = 13, d_1 = 2, d_2 = 3, a = 4, b = 8$. Since $(2^2, 2^3) \sim (2^4, 2^9)$ we have that $|V(f_{2^2,2^3})| = |V(f_{2^4,2^9})| = 7$

# Permutation Polynomial correspondence

$$f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$$

## Corollary

*Suppose that $f_{a,b}$ is a permutation polynomial and $f_{a,b} \sim f_{a',b'}$, then $f_{a',b'}$ is also a permutation polynomial.*

# Size of equivalence classes

$$f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$$

## Proposition

*$|[f_{a,b}]| = lcm(d_1, d_2)$ where $lcm(x, y)$ is the least common multiple of $x$ and $y$.*

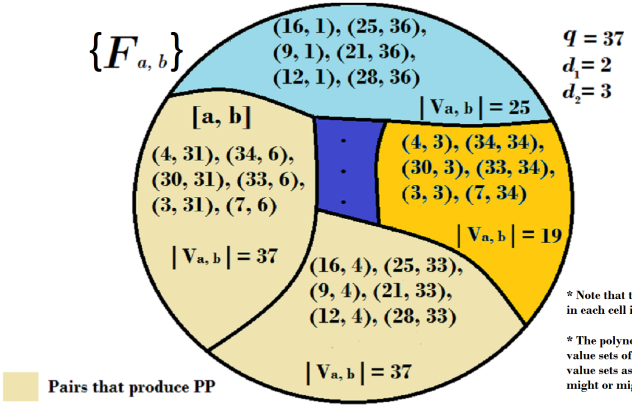
## Example

Let $q = 13, d_1 = 2, d_2 = 3, a = 4, b = 8$. Note that $lcm(2, 3) = 6$
These are the elements of $(a, b)$:

$$\begin{matrix} (2^2, 2^3) & (2^4, 2^9) & (2^6, 2^3) & (2^8, 2^9) & (2^{10}, 2^3) & (2^{12}, 2^9) & (2^2, 2^3) \\ (4, 8) & (3, 5) & (12, 8) & (9, 5) & (10, 8) & (1, 5) & (4, 8) \end{matrix}$$

# Polynomials Results

**Number of Permutation Polynomials**




* Note that the number of polynomials in each cell is $6 = lcm(2, 3)$

* The polynomials within each cell have value sets of the same size. The size of the value sets associated to different cells might or might not be equal.

# Polynomial Results

## Proposition

*The number of polynomials of the form $f_{a,b}(X)$ with $|V(f_{a,b})| = n$ is a multiple of $\mathrm{lcm}(d_1, d_2)$*

## Corollary

*The number of permutation polynomials of the form $f_{a,b}(X)$ is a multiple of $\mathrm{lcm}(d_1, d_2)$*

# Future Work

- Find necessary and sufficient conditions such that $V(f_{a,b}) = \mathbb{F}_q$
- Generalize results to polynomials with more terms and with exponents not divisors of $q-1$:

$$f_{a,b}(X) = X^r(X^{d_1} + aX^{d_2} + b)$$