Construction of Families of Permutation Trinomials over Finite Fields

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Table of Contents

Table of Contents

Finite Fields

Definition

*A **finite field** $\mathbb{F}_q$, $q = p^r$, p prime, is a field with $q = p^r$ elements.*

Example

$$\mathbb{F}_7 = \{0, 1, 2, 3, 4, 5, 6\}$$

Addition:
$2 + 2 = 4$
$4 + 4 = 8$
$\pmod 7 = 1$

Multiplication:
$2 \cdot 2 = 4$
$4 \cdot 4 = 16$
$\pmod 7 = 2$

Polynomials in Finite Fields

Definition

Let $f(x)$ be a polynomial defined over a finite field $\mathbb{F}_q$. This is $f : \mathbb{F}_q \to \mathbb{F}_q$.

Example

Consider $f(x) = x + 3$ over $\mathbb{F}_5$. The domian of f is $\{0, 1, 2, 3, 4\}$.

Value Sets

Definition

*Let $f(x)$ be a polynomial defined over a finite field $\mathbb{F}_q$. Then the **value set** of f is defined as $V(f) = \{f(a) \mid a \in \mathbb{F}_q\}$*

Example

Consider $f(x) = x^2$ defined over $\mathbb{F}_5$. Note:
$f(0) = 0, f(1) = 1, f(2) = 4, f(3) = 4, f(4) = 1$, so
$V(f) = \{0, 1, 4\}$.

Permutation Polynomials

Definition

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Let $f(x) = x^2$ over $\mathbb{F}_5$. We have that $V(f) = \{0, 1, 4\}$ so $f(x)$ is not a permutation polynomial over $\mathbb{F}_5$.

Primitive Roots

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$$\begin{aligned} 3^1 &= 3 \\ 3^2 &= 2 \\ 3^3 &= 6 \\ 3^4 &= 4 \\ 3^5 &= 5 \\ 3^6 &= 1 \end{aligned}$$

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Table of Contents

Our Polynomial

Let $d_1, d_2 \in \mathbb{Z}$ such that $d_1 \mid (q-1)$ y $d_2 \mid (q-1)$. We are interested in the polynomial:

$$f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$$

with $a, b \in \mathbb{F}_q^\times$.

Denote the value set of this polynomial $V(f_{a,b})$.

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Problem

Our Problem

Study the value set of polynomials of the form $f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$ *and determine conditions in* a, b *such that they are permutation polynomials.*

Table of Contents

The class of equivalence $[a, b]$

$$f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$$

Let $a = \alpha^i, b = \alpha^j$, α a primitive root in $\mathbb{F}_q$ and $\sim$ the relation defined as $(a, b) \sim (a', b')$
$\iff a' = \alpha^{i+h(\frac{q-1}{d_1} - \frac{q-1}{d_2})}, b' = \alpha^{j+h(\frac{q-1}{d_1})}$

The class of equivalence $[a, b]$

Example

$q = 13, d_1 = 2, d_2 = 3, \alpha = 2$

The class of equivalence $[a, b]$

Example

$q = 13, d_1 = 2, d_2 = 3, \alpha = 2$
$a = 4 = 2^2, b = 8 = 2^3$.
Now $(2^2, 2^3) \sim (a', b')$ if and only if $a' = 2^{2+2h}, b' = 2^{3+6h}$.
For example $(2^2, 2^3) \sim (2^4, 2^9) = (3, 5)$.

The class of equivalence $[a, b]$

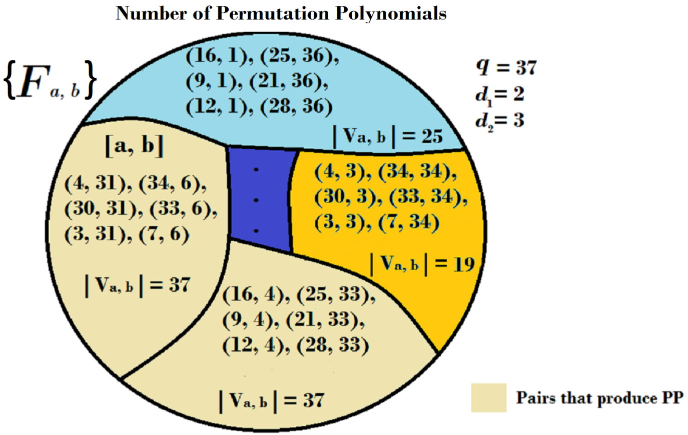
Lemma

The relation $\sim$ defined previously is an equivalence relation.

$f_{a,b}$ with equivalence classes:

$$[f_{a,b}] = [f_{\alpha^i,\alpha^j}] = \{f_{a',b'} \mid (a,b) \sim (a',b')\}$$

Polynomial Results



Value set correspondence

$$f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$$

Theorem

Suppose that $f_{a,b} \sim f_{a',b'}$ then $|V(f_{a,b})| = |V(f_{a',b'})|$.

Example

Let $q = 13, d_1 = 2, d_2 = 3, a = 4, b = 8$. Since $(2^2, 2^3) \sim (2^4, 2^9)$ we have that $|V(f_{2^2,2^3})| = |V(f_{2^4,2^9})|$

Size of equivalence classes

$$f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$$

Proposition

$|[f_{a,b}]| = lcm(d_1, d_2)$ where $lcm(x, y)$ is the least common multiple of x and y.

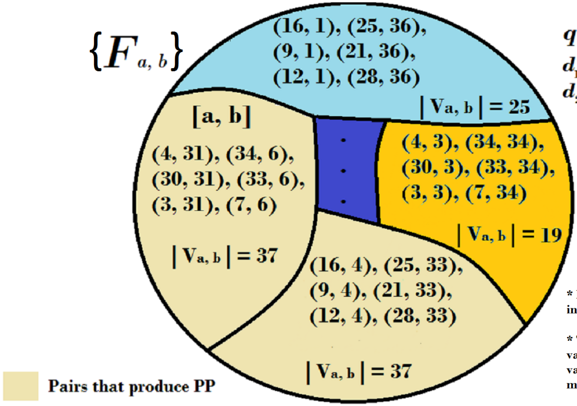
Example

Let $q = 13, d_1 = 2, d_2 = 3, a = 4, b = 8$. Note that $lcm(2, 3) = 6$ These are the elements of (a, b):

$$(2^2, 2^3), (2^4, 2^9), (2^6, 2^3), (2^8, 2^9), (2^10, 2^3), (2^{12}, 2^9), (2^2, 2^3)$$

Polynomials Results

Number of Permutation Polynomials



$q = 37$
$d_1 = 2$
$d_2 = 3$

* Note that the number of polynomials in each cell is $6 = lcm(2, 3)$

* The polynomials within each cell have value sets of the same size. The size of the value sets associated to different cells might or might not be equal.

Polynomial Results

Proposition

The number of polynomials of the form $f_{a,b}(X)$ with $|V(f_{a,b})| = n$ is a multiple of $lcm(d_1, d_2)$

Corollary

The number of permutation polynomials of the form $f_{a,b}(X)$ is a multiple of $lcm(d_1, d_2)$

Future Work

- Find necessary and sufficient conditions such that $V(f_{a,b}) = \mathbb{F}_q$
- Collect data on number of permutation polynomials of the form $f_{a,b}$ for different values of d_1 and d_2 and compare results with number of permutation polynomials.