On a Class of Permutation Polynomials over Finite Fields

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January 21, 2014

Abstract

Given $q = p^r$, d_1 and d_2, we construct partitions of polynomials of the form $F_{a,b}(X) = X\left(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b\right)$, where $a, b \in \mathbb{F}_q^*$, that have value sets of the same cardinality. As a consequence we provide families of permutation polynomials and of polynomials with small value sets.

1 Introduction

Permutation polynomials over finite fields are important in many applications, for example in cryptography. Binomials that produce permutations have been studied extensively. The next case to be studied are trinomials. We want to provide families of polynomials that are rich in permutation polynomials. We have found that within the family of polynomials of the form

$$F_{a,b}(X) = X(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$$

where $d_1|(q-1)$ and $d_2|(q-1)$ there are many permutation polynomials. We want to find conditions in a, b that guarantee that $F_{a,b}(X)$ is a permutation polynomial and count how many permutation polynomials exist in each family.

An example of applications of permutation polynomials over finite fields are RSA-type cryptosystems. In some of these systems secret messages are encoded as elements of a field $\mathbb{F}_q$ with a sufficiently large q. The encryption operator used for these systems is a permutation of the field $\mathbb{F}_q$ and needs to be efficiently computable. It is easy to see that expressing this operator in terms of a permutation polynomial is simple and efficient.

The rest of this report goes as follows: In the preliminaries section we present definitions pertinent to our work. Afterwards, in the following section we present our results related to the value sets of our polynomial $F_{a,b}(X)$. In the next section we talk about our ongoing work and current open problems.

2 Preliminaries

Definition 2.1. A **permutation** of a set A is an ordering of the elements of A. A function $f : A \to A$ gives a permutation of A if and only if f is one to one and onto.

We are interested in studying polynomials that permute the elements of a finite field.

Definition 2.2. A **finite field** $\mathbb{F}_q$, $q = p^r$, p prime, is a field with $q = p^r$ elements.

Example 2.3. $\mathbb{F}_7 = 0, 1, 2, 3, 4, 5, 6$

An important property across our work is the existence of a primitive root for a finite field. We use primitive roots in many of our proofs.

Definition 2.4. A **primitive root** $\alpha \in \mathbb{F}_q$ is a generator for the multiplicative group $\mathbb{F}_q^*$

Example 2.5. Consider the finite field $\mathbb{F}_7$. We have that: $3^1 = 3, 3^2 = 2, 3^3 = 6, 3^4 = 4, 3^5 = 5, 3^6 = 1$, so 3 is a primitive root of $\mathbb{F}_7$.

Example 2.6. Consider $\mathbb{F}_7$. Since $2^1 = 2, 2^2 = 4, 2^3 = 1, 2^4 = 2, 2^5 = 4, 2^6 = 1$, 2 is not a primitive root of $\mathbb{F}_7$.

Given a polynomial defined over a finite field, we study the image of this polynomial. We call this image the value set of the polynomial.

Definition 2.7. Let $f(x)$ be a polynomial defined over a finite field $\mathbb{F}_q$. Then the **value set** of f is defined as $V(f) = Im(f) = \{f(a) \mid a \in \mathbb{F}_q\}$

Example 2.8. Consider $f(x) = x^2$ defined over $\mathbb{F}_5$. Note: $f(0) = 0, f(1) = 1, f(2) = 4, f(3) = 4, f(4) = 1$, so $V_f = \{0, 1, 4\}$.

Note that a polynomial $f(x)$ defined over $\mathbb{F}_q$ is a permutation polynomial if and only if $V(f) = \mathbb{F}_q$.

Example 2.9. Consider the polynomial $f(x) = x + 3$ defined over $\mathbb{F}_7$. We have that $f(0) = 3, f(1) = 4, f(2) = 5, f(3) = 6, f(4) = 0, f(5) = 1, f(6) = 2$, so $f(x)$ is a permutation polynomial over $\mathbb{F}_7$

Example 2.10. Let $f(x) = x^2$ over $\mathbb{F}_5$. We have that $V_f = \{0, 1, 4\}$ so $f(x)$ is not a permutation polynomial over $\mathbb{F}_5$.

Recall now that we are interested in studying the family of polynomials $F_{a,b}(X) = X(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$ defined over $\mathbb{F}_q$ where $d_1 | (q-1)$, $d_2 | (q-1)$ and $a, b \in \mathbb{F}_q^*$. Specifically, we are interested in studying the value set of $F_{a,b}(X)$ given a pair of coefficients (a, b).

3 Motivation

Binomials that produce permutations of finite fields have been studied extensively. The next case to be studied are trinomials. We have found that within the family of polynomials of the form $F_{a,b}(X) = X\left(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b\right)$, there are many permutation polynomials. We want to find conditions in a, b that guarantee that $F_{a,b}(X)$ is a permutation polynomial and count how many permutation polynomials exist in each family.

4 Applications

- The encryption operator of some cryptosystems is a permutation of a finite field $\mathbb{F}_q$ and needs to be efficiently computable. Expressing this operator in terms of a polynomial is simple and efficient.
- Polynomials with minimal value sets are related with curves with a large number of rational points.

5 Results

Given q, d_1 and d_2 the value set of a particular polynomial $F_{a,b}(x)$ is characterized by the pair of coefficients (a, b). In order to simplify notation we use the following definition for value sets.

Definition 5.1. Let $d_1, d_2 \in \mathbb{F}_q$ be such that $d_1 \mid q$ and $d_2 \mid q$. We define the polynomial $F_{a,b}(x) = x(x^{\frac{q-1}{d_1}} + ax^{\frac{q-1}{d_2}} + b)$ with $a, b \in \mathbb{F}_q^*$ and $V(F_{a,b}) = Im(F_{a,b}(x))$.

For convenience we define a relation in $\mathbb{F}_q^* \times \mathbb{F}_q^*$ between pairs of coefficients (a, b) of $F_{a,b}$. This will allow us to express our results in a simplified notation.

Definition 5.2. Let $a = \alpha^i, b = \alpha^j$ and $\sim$ be the relation in $\mathbb{F}_q^* \times \mathbb{F}_q^*$ defined by $(a, b) \sim (a', b') \iff a' = \alpha^{i+h(\frac{q-1}{d_1}-\frac{q-1}{d_2})}, b' = \alpha^{j+h(\frac{q-1}{d_1})}$, where $h \in \mathbb{Z}$

Example 5.3. Let $q = 13, d_1 = 2, d_2 = 3$, then we have $\alpha = 2$ and take $a = 4 = 2^2, b = 8 = 2^3$. Now $(a, b) \sim (a', b')$ if and only if $a' = \alpha^{2+2h}, b' = \alpha^{3+6h}$. In particular $(2, 8) \sim (3, 5)$

Our first result provides a solid foundation for our work.

Lemma 5.4. *The relation $\sim$ defined in Def 5.2 is an equivalence relation in $\mathbb{F}_q^* \times \mathbb{F}_q^*$.*

Proof. 1. Let $a = \alpha^i$, $b = \alpha^j$ and choose $h = 0$. Then $a' = \alpha^{i+0(\frac{q-1}{d_1}-\frac{q-1}{d_2})} = \alpha^i = a$ and $b' = \alpha^{j+0(\frac{q-1}{d_1})} = \alpha^j = b$. Therefore $(a, b) \sim (a, b)$ and the relation is reflexive.

2. Let $a = \alpha^i$, $b = \alpha^j$, $a' = \alpha^{i+h(\frac{q-1}{d_1}-\frac{q-1}{d_2})}$ y $b' = \alpha^{j+h(\frac{q-1}{d_1})}$ then $(a,b) \sim (a',b')$. We want to find l such that $a = \alpha^{i+h(\frac{q-1}{d_1}-\frac{q-1}{d_2})+l(\frac{q-1}{d_1}-\frac{q-1}{d_2})}$ y $b = \alpha^{j+h(\frac{q-1}{d_1})+l(\frac{q-1}{d_1})}$. Choose $l = d_1d_2 - h$, then we obtain: $\alpha^{i+d_1d_2(\frac{q-1}{d_1}-\frac{q-1}{d_2})} = \alpha^i = a$ and $\alpha^{j+d_1d_2(\frac{q-1}{d_1})} = \alpha^j = b$. Therefore $(a',b') \sim (a,b)$ and the relation is symmetric.

3. Suppose that $a = \alpha^i$, $b = \alpha^j$, $a' = \alpha^{i+h(\frac{q-1}{d_1}-\frac{q-1}{d_2})}$, $b' = \alpha^{j+h(\frac{q-1}{d_1})}$, $a'' = \alpha^{i+h(\frac{q-1}{d_1}-\frac{q-1}{d_2})+l(\frac{q-1}{d_1}-\frac{q-1}{d_2})}$, $b'' = \alpha^{j+h(\frac{q-1}{d_1})+l(\frac{q-1}{d_1})}$. Therefore $(a,b) \sim (a',b')$ and $(a',b') \sim (a'',b'')$. Note that $a'' = \alpha^{i+(h+l)(\frac{q-1}{d_1}-\frac{q-1}{d_2})}$, $b'' = \alpha^{j+(h+l)(\frac{q-1}{d_1})}$, therefore $(a,b) \sim (a'',b'')$ and the relation is transitive.

In conclusion the relation is an equivalence relation.

□

The equivalence relation defined in Def 5.2 induces and equivalence relation in the set of polynomials of the form $F_{a,b}(X) = X(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$ with equivalence classes $[F_{a,b}] = [F_{\alpha^i,\alpha^j}] = \left\{ F_{a',b'} | a' = \alpha^{i+h(\frac{q-1}{d_1}-\frac{q-1}{d_2})}, b' = \alpha^{j+h(\frac{q-1}{d_1})} \right\}$

Given that our relation $\sim$ is an equivalence relation, we study the value set $V(F_{a,b})$ in the context of the equivalence class $[F_{a,b}]$. Our next lemma states that all polynomials belonging to the same equivalence class have value sets of the same size.

Lemma 5.5. *Suppose that $F_{a,b} \sim F_{a',b'}$ where $\sim$ is the relationship defined in Definition 5.2. Then $|V(F_{a,b})| = |V(F_{a',b'})|$*

Proof. Let α be a primitive root of the finite field $\mathbb{F}_q$.

$$F_{a',b'}(\alpha^{k+1}) = \alpha^{k+1}((\alpha^{k+1})^{\frac{q-1}{d_1}} + \alpha^{i+\frac{q-1}{d_1}-\frac{q-1}{d_2}}(\alpha^{k+1})^{\frac{q-1}{d_2}} + \alpha^{j+\frac{q-1}{d_1}})$$

$$= \alpha^{k+1}((\alpha^k)^{\frac{q-1}{d_1}} \cdot \alpha^{\frac{q-1}{d_1}} + \alpha^i \cdot \frac{\alpha^{\frac{q-1}{d_1}}}{\alpha^{\frac{q-1}{d_2}}}(\alpha^k)^{\frac{q-1}{d_2}} \cdot \alpha^{\frac{q-1}{d_2}} + \alpha^j \cdot \alpha^{\frac{q-1}{d_1}})$$

$$= \alpha^{\frac{q-1}{d_1}+1} \cdot \alpha^k((\alpha^k)^{\frac{q-1}{d_1}} + \alpha^i(\alpha^k)^{\frac{q-1}{d_2}} + \alpha^j)$$

$$= C \cdot F_{a,b}(\alpha^k), \text{where } C = \alpha^{\frac{q-1}{d_1}+1}$$

In general for each element $\alpha^{\frac{q-1}{d_1}+1}F_{a,b}(\alpha^k)$ there will be a corresponding element $F_{a',b'}(\alpha^{k+1})$ where $a' = \alpha^{i+h(\frac{q-1}{d_1}-\frac{q-1}{d_2})}$ y $b' = \alpha^{j+h(\frac{q-1}{d_1})}$. We can then define a function $f : V(F_{a',b'}) \to \alpha^{\frac{q-1}{d_1}}V(F_{a,b})$ by $f(F_{a',b'}(\alpha^{k+1})) = \alpha^{\frac{q-1}{d_1}+1}F_{a,b}(\alpha^k)$. We prove that f is one to one.

Suppose that $f(F_{a',b'}(\alpha^{k_1+1})) = f(F_{a',b'}(\alpha^{k_2+1}))$ where $k_1, k_2 \in \mathbb{F}_q$. Consider $f(F_{a',b'}(\alpha^{k_1+1}))$

$$= f(\alpha^{k_1+1}((\alpha^{k_1+1})^{\frac{q-1}{d_1}} + \alpha^i(\alpha^{k_1+1})^{\frac{q-1}{d_2}} + \alpha^j))$$

$$= \alpha^{\frac{q-1}{d_1}+1}(\alpha^{k_1}((\alpha^{k_1})^{\frac{q-1}{d_1}} + \alpha^i(\alpha^{k_1})^{\frac{q-1}{d_2}} + \alpha^j))$$

$$= \alpha^{k_1+1}(\alpha^{\frac{q-1}{d_1}}((\alpha^{k_1})^{\frac{q-1}{d_1}} + \alpha^i(\alpha^{k_1})^{\frac{q-1}{d_2}} + \alpha^j))$$

$$= \alpha^{k_1+1}((\alpha^{k_1+1})^{\frac{q-1}{d_1}} + \alpha^{i+\frac{q-1}{d_1}-\frac{q-1}{d_2}+\frac{q-1}{d_2}}(\alpha^{k_1})^{\frac{q-1}{d_2}} + \alpha^{j+\frac{q-1}{d_1}})$$

$$= \alpha^{k_1+1}((\alpha^{k_1+1})^{\frac{q-1}{d_1}} + \alpha^{i+\frac{q-1}{d_1}-\frac{q-1}{d_2}}(\alpha^{k_1+1})^{\frac{q-1}{d_2}} + \alpha^{j+\frac{q-1}{d_1}})$$

$$= F_{a',b'}(\alpha^{k_1+1})$$

Then consider $f(F_{a',b'}(\alpha^{k_2+1}))$

$$= f(\alpha^{k_2+1}((\alpha^{k_2+1})^{\frac{q-1}{d_1}} + \alpha^i(\alpha^{k_2+1})^{\frac{q-1}{d_2}} + \alpha^j))$$

$$= \alpha^{\frac{q-1}{d_1}+1}(\alpha^{k_2}((\alpha^{k_2})^{\frac{q-1}{d_1}} + \alpha^i(\alpha^{k_2})^{\frac{q-1}{d_2}} + \alpha^j))$$

$$= \alpha^{k_2+1}(\alpha^{\frac{q-1}{d_1}}((\alpha^{k_2})^{\frac{q-1}{d_1}} + \alpha^i(\alpha^{k_2})^{\frac{q-1}{d_2}} + \alpha^j))$$

$$= \alpha^{k_2+1}((\alpha^{k_2+1})^{\frac{q-1}{d_1}} + \alpha^{i+\frac{q-1}{d_1}-\frac{q-1}{d_2}+\frac{q-1}{d_2}}(\alpha^{k_2})^{\frac{q-1}{d_2}} + \alpha^{j+\frac{q-1}{d_1}})$$

$$= \alpha^{k_2+1}((\alpha^{k_2+1})^{\frac{q-1}{d_1}} + \alpha^{i+\frac{q-1}{d_1}-\frac{q-1}{d_2}}(\alpha^{k_2+1})^{\frac{q-1}{d_2}} + \alpha^{j+\frac{q-1}{d_1}})$$

$$= F_{a',b'}(\alpha^{k_2+1})$$

In conclusion $F_{a',b'}(\alpha^{k_1+1}) = F_{a',b'}(\alpha^{k_2+1})$ therefore f is a injective function. Since the sets $V(F_{a,b}), V(F_{a',b'})$ are finite the function f is a bijection and $|V(F_{a,b})| = |V(F_{a',b'})|$.

□

Example 5.6. Let $q = 13, d_1 = 2, d_2 = 3, a = 4, b = 8$. Since $(4,8) \sim (3,5)$ we have that $|V(F_{4,8})| = |V(F_{3,5})|$. In fact $V(F_{4,8}) = \{0,1,2,3,10,11,12\}$, $V(F_{3,5}) = \{0,2,4,6,7,9,11\}$. Note that even though the sizes are equal the value sets are not.

Recall that our family of polynomials $F_{a,b}(X) \in \mathbb{F}_q[x]$ depends on a specific choice of d_1 and d_2. We want to know the number of polynomials in each equivalence class. The next lemma gives an exact number for any class of equivalence $[F_{a,b}]$ in terms of d_1 and d_2.

Lemma 5.7. *$|[F_{a,b}]| = lcm(d_1, d_2)$.*

Proof. Suppose that $a = \alpha^i$, $b = \alpha^j$. Note that we can obtain the elements of $[F_{a,b}]$ applying the transformation $f : (a, b) \to (a \cdot \alpha^{(\frac{q-1}{d_1} - \frac{q-1}{d_2})}, b \cdot \alpha^{(\frac{q-1}{d_1})})$ multiple times. Now note that:

We obtain a set of coefficients

$$[F_{a,b}] = \Big\{(\alpha^i, \alpha^j), (\alpha^{i+(\frac{q-1}{d_1} - \frac{q-1}{d_2})}, \alpha^{j+(\frac{q-1}{d_1})}), (\alpha^{i+2(\frac{q-1}{d_1} - \frac{q-1}{d_2})}, \alpha^{j+2(\frac{q-1}{d_1})}), \Big\}$$

$$\cup \Big\{..., (\alpha^{i+h(\frac{q-1}{d_1} - \frac{q-1}{d_2})}, \alpha^{j+h(\frac{q-1}{d_1})}) = (\alpha^i, \alpha^j)\Big\}$$

Therefore applying the transformation $lcm(d_1, d_2)$ times, we obtain a chain of elements in $[a, b]$. Now suppose that $c < lcm(d_1, d_2)$ such that $\alpha^{i+c(\frac{q-1}{d_1} - \frac{q-1}{d_2})} = \alpha^i$ and $\alpha^{j+c(\frac{q-1}{d_1})} = \alpha^j$. This implies that implica $\alpha^{c(\frac{q-1}{d_1} - \frac{q-1}{d_2})} = 1$, then $\alpha^{c(\frac{q-1}{d_1}) - c(\frac{q-1}{d_2})} = 1$, this is only possible if c is a multiple of d_1 and d_2 but $c < lcm(d_1, d_2)$ and $lcm(d_1, d_2)$ is the smallest element such that this occurs. Therefore the amount of elements in the equivalence relation $[a, b]$ is equal to $lcm(d_1, d_2)$.

□

The construction in Proof 5 gives a way to construct a set of $lcm(d_1, d_2)$ polynomials $F_{a',b'}$ with $|V(F_{a',b'})| = |V(F_{a,b})|$. In particular if $F_{a,b}$ is a permutation polynomial of $\mathbb{F}_q$ we can construct $lcm(d_1, d_2)$ other permutation polynomials of $\mathbb{F}_q$.

Example 5.8. Let $q = 13, d_1 = 2, d_2 = 3, a = 4, b = 8$. Note that $lcm(2, 3) = 6$ These are the elements of $F_{a,b}$:

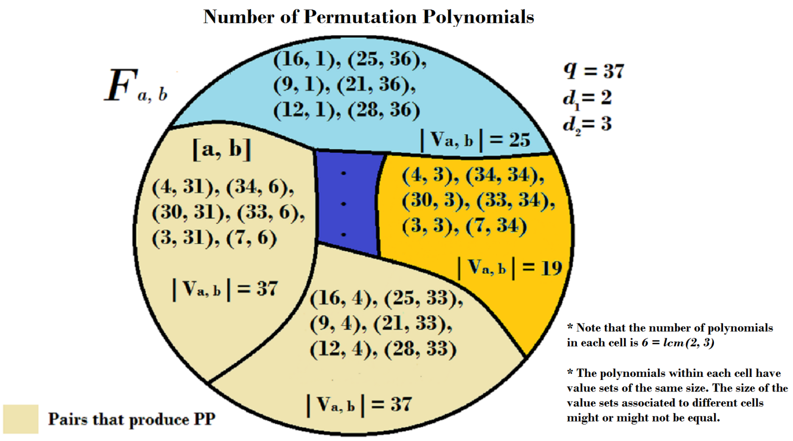
$$F_{4,8}, F_{3,5}, F_{12,8}, F_{9,5}, F_{10,8}, F_{1,5}, F_{4,8}$$

Finally, our last result uses all of our previous lemmas to provide information about the number of polynomials of the form $F_{a,b}(X)$ with a specific size of value set.

Proposition 5.9. The number of polynomials $F_{a',b'}(x)$ with $|V_{a,b}| = n$ is a multiple of $lcm(d_1, d_2)$

A specific case of the previous proposition is the case where $|V_{a,b}| = q$ that is, when we have permutation polynomials

Corollary 5.10. The number of permutation polynomials $F_{a',b'}(x)$ is a multiple of $lcm(d_1, d_2)$



* Note that the number of polynomials in each cell is *6 = lcm(2, 3)*

* The polynomials within each cell have value sets of the same size. The size of the value sets associated to different cells might or might not be equal.

6 Ongoing work

- Study our results on the family of polynomials of the form $F_{a,b}(X) = X^m(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$
- Find necessary and sufficient conditions such that $V_{a,b} = \mathbb{F}_q$ and $V_{a,b}$ is of minimal cardinality.
- Collect data on number of permutation polynomials of this form for different values of d_1 and d_2 and compare results with number of permutation binomials.

7 Acknowledgements

This work has been supported by a grant from the Center of Undergraduate Research in Mathematics (CURM) from Brigham Young University. Travel funds were provided by the MAA and the University of Puerto Rico, Rio Piedras.

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