Let $p \equiv 1 \bmod 3$. Let $F(X) = X^{\frac{p+1}{2}} + aX^{\frac{p+5}{6}} + bX$ be a polynomial over $\mathbb{F}_p$. Let α be a primitive root in $\mathbb{F}_p$.
Notice that $F(X) = X(X^{\frac{p-1}{2}} + aX^{\frac{p-1}{6}} + b)$. We will use the approach of considering the $\alpha^{6k+r}, r = 0, ..., 5$. This divides $F(X)$ into 6 classes:

- $F(\alpha^{6k}) = \alpha^{6k}(1 + a + b)$
- $F(\alpha^{6k+1}) = \alpha^{6k+1}(1 + a\alpha^{\frac{p-1}{6}} + b)$
- $F(\alpha^{6k+2}) = \alpha^{6k+2}(1 + a\alpha^{\frac{p-1}{3}} + b)$
- $F(\alpha^{6k+3}) = \alpha^{6k+3}(-1 - a + b)$
- $F(\alpha^{6k+4}) = \alpha^{6k+4}(1 + a\alpha^{2\frac{p-1}{3}} + b)$
- $F(\alpha^{6k+5}) = \alpha^{6k+5}(1 + a\alpha^{5\frac{p-1}{6}} + b)$

These classes should be disjoint, to find conditions on a and b we equate these classes (with different exponents of 6).

Class $6k$

- Class $6l + 1$

  $\alpha^{6k}(1 + a + b) = \alpha^{6l+1}(1 + a\alpha^{\frac{p-1}{6}} + b)$

  $\alpha^{6(k-l)}(1 + a + b) = \alpha(1 + a\alpha^{\frac{p-1}{6}} + b)$

  $\alpha^{6(k-l)} = \alpha\frac{(1+a\alpha^{\frac{p-1}{6}}+b)}{(1+a+b)}$

  From this we know that a and b must satisfy $\alpha^{6m} \neq \alpha\frac{(1+a\alpha^{\frac{p-1}{6}}+b)}{(1+a+b)}$

- Class $6l + 2$

  $\alpha^{6k}(1 + a + b) = \alpha^{6l+2}(1 + a\alpha^{\frac{p-1}{3}} + b)$

  $\alpha^{6(k-l)}(1 + a + b) = \alpha^2(1 + a\alpha^{\frac{p-1}{3}} + b)$

  $\alpha^{6(k-l)} = \alpha^2\frac{(1+a\alpha^{\frac{p-1}{3}}+b)}{(1+a+b)}$

From this we know that a and b must satisfy $\alpha^{6m} \neq \alpha^2 \frac{(1+a\alpha^{\frac{p-1}{3}}+b)}{(1+a+b)}$

- Class $6l+3$

$\alpha^{6k}(1+a+b) = \alpha^{6l+3}(-1-a+b)$

$\alpha^{6(k-l)}(1+a+b) = \alpha^3(-1-a+b)$

$\alpha^{6(k-l)} = \alpha^3 \frac{(-1-a+b)}{(1+a+b)}$

From this we know that a and b must satisfy $\alpha^{6m} \neq \alpha^3 \frac{(-1-a+b)}{(1+a+b)}$

- Class $6l+4$

$\alpha^{6k}(1+a+b) = \alpha^{6l+4}(1+a\alpha^{2\frac{p-1}{3}}+b)$

$\alpha^{6(k-l)}(1+a+b) = \alpha^4(1+a\alpha^{2\frac{p-1}{3}}+b)$

$\alpha^{6(k-l)} = \alpha^4 \frac{(1+a\alpha^{2\frac{p-1}{3}}+b)}{(1+a+b)}$

From this we know that a and b must satisfy $\alpha^{6m} \neq \alpha^4 \frac{(1+a\alpha^{2\frac{p-1}{3}}+b)}{(1+a+b)}$

- Class $6l+5$

$\alpha^{6k}(1+a+b) = \alpha^{6l+5}(1+a\alpha^{5\frac{p-1}{6}}+b)$

$\alpha^{6(k-l)}(1+a+b) = \alpha^5(1+a\alpha^{5\frac{p-1}{6}}+b)$

$\alpha^{6(k-l)} = \alpha^5 \frac{(1+a\alpha^{5\frac{p-1}{6}}+b)}{(1+a+b)}$

From this we know that a and b must satisfy $\alpha^{6m} \neq \alpha^5 \frac{(1+a\alpha^{5\frac{p-1}{6}}+b)}{(1+a+b)}$