Let α be a primitive element of $\mathbb{F}_p$ and $F(X) = X^{\frac{p-1}{2}+1} + aX^{\frac{p-1}{3}+1} + bX$ be a polynomial over $\mathbb{F}_p$. We are going to use that $\alpha^{\frac{p-1}{2}} = -1$, $\alpha^{k(p-1)} = 1$.

(1)

$$\begin{aligned} F(\alpha^{3k}) &= \alpha^{3k}(\alpha^{\frac{3k(p-1)}{2}} + a\alpha^{\frac{3k(p-1)}{3}} + b) \\ &= \alpha^{3k}(\alpha^{\frac{k(p-1)}{2}} + a + b) \\ &\alpha^{3k}((-1)^k + a + b) \end{aligned}$$

(2)

$$\begin{aligned} F(\alpha^{3k+1}) &= \alpha^{3k+1}(\alpha^{\frac{(k+1)(p-1)}{2}} + a\alpha^{\frac{(p-1)}{3}} + b) \\ &= \alpha^{3k+1}(\alpha^{\frac{(k+1)(p-1)}{2}} + a\alpha^{\frac{p-1}{3}} + b) \\ &= \alpha^{3k+1}((-1)^{k+1} + a\alpha^{\frac{p-1}{3}} + b) \end{aligned}$$

(3)

$$\begin{aligned} F(\alpha^{3k+2}) &= \alpha^{3k+2}(\alpha^{\frac{(k+2)(p-1)}{2}} + a\alpha^{\frac{(2(p-1))}{3}} + b) \\ &= \alpha^{3k+2}(\alpha^{\frac{(k+2)(p-1)}{2}} + a\alpha^{\frac{2(p-1)}{3}} + b) \\ &= \alpha^{3k+2}((-1)^k + a\alpha^{\frac{2(p-1)}{3}} + b) \end{aligned}$$

Note that $l_1 = (-1)^k + a + b, l_2 = (-1)^{k+1} + a\alpha^{\frac{p-1}{3}} + b, l_3 = (-1)^k + a\alpha^{\frac{2(p-1)}{3}} + b$ are constants.

Example 1. Note 22 is a primitive root of $\mathbb{F}_{31}$. We have $F(22^{3k}) = \{15, 8, 27, 2, 30, 16, 23, 4, 29, 1\}$, $F(22^{3k+1}) = \{22, 20, 21, 5, 13, 9, 11, 10, 26, 18\}$ and $F(22^{3k+2}) = \{19, 6, 28, 17, 7, 12, 25, 3, 14, 24\}$. Also $22^{\frac{31-1}{3}} = 5$, $22^{\frac{2(31-1)}{3}} = 25$. We have $l_1 = (-1)^k + a + b, l_2 = (-1)^{k+1} + 5a + b, l_3 = (-1)^k + 25a + b$. Therefore $F(22^{3k}) = 22^{3k}((-1)^k + a + b), F(22^{3k+1}) = 22^{3k+1}((-1)^{k+1} + 5a + b), F(22^{3k+2}) = 22^{3k+2}((-1)^k + 25a + b))$. The number of PP of type F in $\mathbb{F}_{31}$ is 0.

Example 2. We consider the polynomial $F(X) = X^{19} + aX^{13} + bX$ over $\mathbb{F}_{37}$. In this case F is PP of $\mathbb{F}_{37}$ for

$[a, b] \in \{[3, 31], [4, 31], [7, 6], [9, 4], [12, 4], [16, 4], [21, 33], [25, 33], [28, 33], [30, 31][33, 6], [34, 6]\}$.

2 We consider the polynomial $F(X) = X^{19} + aX$ over $\mathbb{F}_{37}$. In this case F is PP of $\mathbb{F}_{37}$ for

$$a \in \{7, 8, 10, 11, 14, 16, 18, 19, 21, 23, 26, 27, 29, 30, 35\}.$$

Recall $a^2 - 1$ is quadratic residue. We consider the polynomial $F(X) = X^{13} + aX$ over $\mathbb{F}_{37}$. In this case F is PP of $\mathbb{F}_{37}$ for

$$a \in \{9, 12, 16\}.$$

5 is a primitive root of $\mathbb{F}_{37}$. Then

$$F(5^{3k}) = \{1, 14, 11, 6, 10, 29, 36, 23, 26, 31, 27, 8\}.$$

Then

$$F(5^{3k+1}) = \{5, 33, 18, 30, 13, 34, 32, 4, 19, 7, 24, 3\}.$$

Then
$$F(5^{3k+2}) = \{25, 17, 16, 2, 28, 22, 12, 20, 21, 35, 9, 15\}.$$
We have $A_1(a,b) = \{5^{3k}((-1)^k+a+b)\}$, $A_2(a,b) = \{5^{3k+1}((-1)^k+10a+b)\}$, $A_3(a,b) = \{5^{3k+2}((-1)^k + 26a + b)\}$. Find (a,b) such that $A_i(a,b) \cap A_j(a,b) = \{\}$.

Let $h = \alpha^{\frac{p-1}{3}}$. We need to find the pairs (k,k') such that
$$\alpha^{3k}((-1)^k + a + b) = \alpha^{3k'+1}((-1)^{k'+1} + ha + b)$$
$$(\alpha^{k-k'})^3 = \alpha\left(\frac{(-1)^{k'+1} + ha + b}{(-1)^k + a + b}\right),$$

$$\alpha^{3k}((-1)^k + a + b) = \alpha^{3k'+2}((-1)^{k'} + h^2a + b)$$
$$(\alpha^{k-k'})^3 = \alpha^2\left(\frac{(-1)^{k'} + h^2a + b}{(-1)^k + a + b}\right),$$

and
$$\alpha^{3k+1}((-1)^{k+1} + ha + b) = \alpha^{3k'+2}((-1)^k + h^2a + b)$$
$$(\alpha^{k-k'})^3 = \alpha\left(\frac{(-1)^{k'} + h^2a + b}{(-1)^{k+1} + ha + b}\right),$$

We obtain that
$$1 = h\left(\frac{(-1)^{k'+1} + ha + b}{(-1)^k + a + b}\right)^{\frac{p-1}{3}},$$
$$1 = h^2\left(\frac{(-1)^{k'} + h^2a + b}{(-1)^k + a + b}\right)^{\frac{p-1}{3}},$$

and
$$1 = h\left(\frac{(-1)^{k'+1} + h^2a + b}{(-1)^k + ha + b}\right)^{\frac{p-1}{3}}.$$

Hence necessaries conditions are
$$h^2 \neq \left(\frac{(-1)^{k'+1} + ha + b}{(-1)^k + a + b}\right)^{\frac{p-1}{3}},$$
$$h \neq \left(\frac{(-1)^{k'+1} + h^2a + b}{(-1)^k + a + b}\right)^{\frac{p-1}{3}},$$

and
$$h^2 \neq \left(\frac{(-1)^{k'+1} + h^2a + b}{(-1)^k + ha + b}\right)^{\frac{p-1}{3}}.$$

Let η be the legendre character, i.e., $\eta(a) = a^{\frac{p-1}{2}}$. Then

(1)
$$\eta(\alpha^{3k}((-1)^k + a + b)) = \eta(\alpha^{3k'+1}((-1)^{k'+1} + ha + b))$$
$$(-1)^k\eta((-1)^k + a + b) = (-1)^{k'+1}\eta((-1)^{k'+1} + ha + b))$$

(2)

$$\eta(\alpha^{3k}((-1)^k + a + b)) = \eta(\alpha^{3k'+2}((-1)^{k'+1} + h^2a + b))$$
$$(-1)^k\eta((-1)^k + a + b) = (-1)^{k'}\eta((-1)^{k'+1} + h^2a + b))$$

(3)

$$\eta(\alpha^{3k+1}((-1)^k + ha + b)) = \eta(\alpha^{3k'+2}((-1)^{k'+1} + h^2a + b))$$
$$(-1)^{k+1}\eta((-1)^k + ha + b) = (-1)^{k'}\eta((-1)^{k'+1} + h^2a + b))$$