Construction of Families of Permutation Trinomials over Finite Fields

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Finite Fields

Definition

*A **finite field** $\mathbb{F}_q$, $q = p^r$, p prime, is a field with $q = p^r$ elements.*

Example

$\mathbb{F}_7 = 0, 1, 2, 3, 4, 5, 6$

Polynomials in Finite Fields

Definition

Let $f(x)$ be a polynomial defined over a finite field $\mathbb{F}_q$. This is $f : \mathbb{F}_q \to \mathbb{F}_q$.

Example

Consider $f(x) = x + 3$ over $\mathbb{F}_5$. The domian of f is $\{0, 1, 2, 3, 4\}$.

Value Sets

Definition

*Let $f(x)$ be a polynomial defined over a finite field $\mathbb{F}_q$. Then the **value set** of f is defined as $V(f) = \{f(a) \mid a \in \mathbb{F}_q\}$*

Example

Consider $f(x) = x^2$ defined over $\mathbb{F}_5$. Note:
$f(0) = 0, f(1) = 1, f(2) = 4, f(3) = 4, f(4) = 1$, so
$V(f) = \{0, 1, 4\}$.

Permutation Polynomials

Definition

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Example

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Let $f(x) = x^2$ over $\mathbb{F}_5$. We have that $V(f) = \{0, 1, 4\}$ so $f(x)$ is not a permutation polynomial over $\mathbb{F}_5$.

Primitive Roots

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Consider $\mathbb{F}_7$. Since $3^1 = 3, 3^2 = 2, 3^3 = 6, 3^4 = 4$, $3^5 = 5, 3^6 = 1$, 3 is a primitive root of $\mathbb{F}_7$.

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Example

Consider $\mathbb{F}_7$. Since $2^1 = 2, 2^2 = 4, 2^3 = 1, 2^4 = 2$, $2^5 = 4, 2^6 = 1$, 2 is not a primitive root of $\mathbb{F}_7$.

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Our Polynomial

Let $d_1, d_2 \in \mathbb{F}_q$ such that $d_1 \mid (q-1)$ y $d_2 \mid (q-1)$. We are interested in the polynomial:

$$f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$$

with $a, b \in \mathbb{F}_q^\times$.

Denote the value set of this polynomial $V(f_{a,b})$.

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Problem

Our Problem

Study the value set of polynomials of the form $f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$ *and determine conditions in* a, b *such that they are permutation polynomials.*

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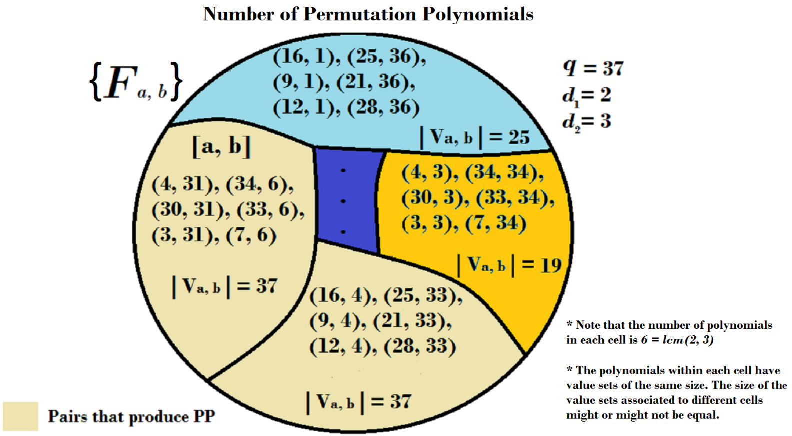
Polynomials Results



The class of equivalence $[a, b]$

$$f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$$

Let $a = \alpha^i, b = \alpha^j$, α a primitive root in $\mathbb{F}_q$ and $\sim$ the relation defined as $(a, b) \sim (a', b')$
$\iff a' = \alpha^{i+h(\frac{q-1}{d_1} - \frac{q-1}{d_2})}, b' = \alpha^{j+h(\frac{q-1}{d_1})}$

Example

Let $q = 13, d_1 = 2, d_2 = 3$, then we have $\alpha = 2$ and take $a = 4 = 2^2, b = 8 = 2^3$. Now $(a, b) \sim (a', b')$ if and only if $a' = \alpha^{2+2h}, b' = \alpha^{3+6h}$. For example $(a, b) \sim (3, 5)$

The class of equivalence $[a, b]$

Lemma

The relation $\sim$ defined above is an equivalence relation.

The previous relation induces an equivalence relation in the set of trinomials $f_{a,b}$ with equivalence classes:

$$[f_{a,b}] = [f_{\alpha^i,\alpha^j}] = \left\{ f_{a',b'} \mid (a, b) \sim (a', b') \right\}$$

Value set correspondence

$$f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$$

Theorem

Suppose that $f_{a,b} \sim f_{a',b'}$ then $|V(f_{a,b})| = |V(f_{a',b'})|$.

Example

Let $q = 13, d_1 = 2, d_2 = 3, a = 4, b = 8$. Since $(4,8) \sim (3,5)$ we have that $|V(f_{4,8})| = |V(f_{3,5})|$

Size of equivalence classes

$$f_{a,b}(X) = X^r(X^{\frac{q-1}{d_1}} + aX^{\frac{q-1}{d_2}} + b)$$

Proposition

$|[f_{a,b}]| = lcm(d_1, d_2)$ where $lcm(x, y)$ is the least common multiple of x and y.

Example

Let $q = 13, d_1 = 2, d_2 = 3, a = 4, b = 8$. Note that $lcm(2,3) = 6$
These are the elements of (a, b):

$$(4,8),(3,5),(12,8),(9,5),(10,8),(1,5),(4,8)$$

Polynomial Results

Proposition

The number of polynomials of the form $f_{a,b}(X)$ with $|V(f_{a,b})| = n$ is a multiple of $lcm(d_1, d_2)$

Corollary

The number of permutation polynomials of the form $f_{a,b}(X)$ is a multiple of $lcm(d_1, d_2)$

Future Work

- Find necessary and sufficient conditions such that $V(f_{a,b}) = \mathbb{F}_q$
- Collect data on number of permutation polynomials of the form $f_{a,b}$ for different values of d_1 and d_2 and compare results with number of permutation polynomials.