1 Additions to the first observation of ionization cooling

1.1 Amplitude quantiles

Transverse 4D amplitudes distributed as

$$A_\perp \sim \epsilon_\perp \chi_4^2 = \frac{1}{4\epsilon_\perp^2} x e^{-x/2\epsilon_\perp}. \tag{1}$$

The α-quantile of the distribution, or α-amplitude, corresponds to

$$A_\alpha = \epsilon_\perp \chi_4^2(\alpha), \tag{2}$$

with $\chi_4^2(\alpha)$ the quantiles of the 4DOF chi-squared distribution. At $\alpha \simeq 9\,\%$, the probability content corresponds to that of the RMS ellipse and we naturally have $\chi_4^2(\alpha) = 1$ and $A_\alpha = \epsilon_\perp$.

For a distribution of covariance Σ, the volume of its RMS ellipse is simply $\mathcal{V} = \frac{1}{2}\pi^2|\Sigma|^{\frac{1}{2}}$. For a beam of emittance $\epsilon_\perp = |\Sigma|^{\frac{1}{4}}/m_\mu c$, the volume of the RMS ellipse thus reads

$$\mathcal{V} = \frac{1}{2}\pi^2 m^2 c^2 \epsilon_\perp^2. \tag{3}$$

The α-amplitude corresponds to an ellipse similar to the RMS ellipse of squared radius $A_\alpha/\epsilon_\perp$, so that the fractional emittance may be reconstructed as

$$\epsilon_\alpha = \frac{1}{2}\pi^2 m^2 c^2 A_\alpha^2. \tag{4}$$

The relative statistical uncertainty on the α-amplitude is identical to the chi-squared distribution quantile uncertainty, i.e.

$$\frac{\sigma_{A_\alpha}}{A_\alpha} = \epsilon_\perp \frac{\sigma_{\chi_4^2(\alpha)}}{\chi_4^2(\alpha)} = \sqrt{\frac{\alpha(1-\alpha)}{n}} \left[\chi_4^2(\alpha) f(4; \chi_4^2(\alpha))\right]^{-1}, \tag{5}$$

with $f(4; x)$ the 4DOF chi square distribution PDF in x. As the fractional emittance is proportional to the square of the α-amplitude, its statistical uncertainty satisfies

$$\frac{\sigma_{\epsilon_\alpha}}{\epsilon_\alpha} = 2\frac{\sigma_{A_\alpha}}{A_\alpha}. \tag{6}$$

1.2 kNN density estimator

Take a set of n points, $\{\boldsymbol{x}\}_{i=1}^n$, in 4 dimensions. To evaluate the density in a point $\boldsymbol{x}$, find the squared distances to the n points, i.e.

$$R_i^2 = (\boldsymbol{x} - \boldsymbol{x}_i)^T \Sigma^{-1} (\boldsymbol{x} - \boldsymbol{x_i}), \tag{7}$$

with Σ the covariance matrix. If Σ is set to unity, it is the Euclidean distance. Find the k closest points and the distance R_k to the farthest one of them. The density in $\boldsymbol{x}$ is evaluated as

$$\rho(\boldsymbol{x}) = \frac{k}{nV_k} = \frac{2k}{n\pi^2|\Sigma|^{\frac{1}{2}} R_k^4}, \tag{8}$$

with V_k the volume of the 4-ellipse of radius R_k. The optimal k in 4D is $\sqrt{n}$ (see dissertation).

Cannot compare 4D profiles, must find 1D aggregate, I propose the "density profiles", i.e. the quantiles of the inverse PDF. For a fraction α of the beam, find the density, ρ_α, above which lies a fraction α of the beam. This can be represented as a function of α to create the profiles.

The statistical uncertainty on the contours, ρ_α, have been showed to closely follow

$$\frac{\sigma_{\rho_\alpha}}{\rho_\alpha} = \left(\frac{4}{\alpha(1-\alpha)}\right)^{\frac{1}{4}} n^{-\frac{1}{2}}, \tag{9}$$

but a bootstrapped estimate is also provided in the code