Math 320 Homework 3.7

Chris Rytting

October 28, 2015

3.39

$$E(\hat{p})E\left[\frac{1}{Nn}\sum_{i=1}^{N}X_i\right] = \frac{1}{Nn}\sum_{i=1}^{N}E[X_i] = \frac{Nnp}{Nn} = p$$

3.40

We have that

$$L(\lambda) = e^{-\lambda n}\frac{\lambda^{\sum x_i}}{\prod(x_i)!}$$

Now, differentiating and setting equal to zero, we have

$$\frac{ne^{-\lambda n}\lambda^{n\bar{x}}}{\prod(x_i)!} + \frac{\bar{x}ne^{-\lambda n}\lambda^{n\bar{x}-1}}{\prod(x_i)!} = 0$$
$$\implies n\bar{x}\lambda^{n\bar{x}-1} = n\lambda^{n\bar{x}}$$
$$\implies \lambda = \bar{x} = \frac{\sum_{i=1}^{n}}{n}$$

3.41

We have that

$$L(\lambda) = \prod \lambda e^{-\lambda x}$$
$$= \lambda^n e^{-\lambda\bar{x}n}$$

Now, differentiating and setting equal to zero, we have

$$n\lambda^{n-1} - \bar{x}n\lambda^n = 0$$
$$\implies \frac{n\lambda^{n-1}}{\lambda^{n-1}} = \frac{\bar{x}n\lambda^n}{\lambda^{n-1}}$$
$$\implies \lambda = \frac{1}{\bar{x}} = \frac{n}{\sum_{i=1}^{n}}$$

3.42

The maximum likelihood estimator $\hat{p}$ would be one as we see in Example 3.7.10. This would be misleading because the probability/mean p is likely not equal to one. However, the key words here are "a small number of draws". Therein lies the weakness of MLE, for if there is a small number of draws, then we can get misleading results.

3.43

For $k = 1$, $k < n$, $P(M \leq k) = 0$. This is true because upon drawing n, we have that $M > 1$, because the next draw will necessarily be greater than 1.

Suppose now that $k - 1 < n$, $P(M \leq k - 1) = 0$. Without loss of generality, we can assum that on the first $k - 1$ draws, we draw the first $k - 1$ values. The minimum M value possible on the k^{th} draw is k. However, we know that $k < n$, so we must draw at least one more, and the $k+1$ draw maximum value will be greater than k.

Therefore, we have that $P(M \leq k) = 0$ for all $k < n$.

As for finding $P(M \leq k) \quad k > n$, note that for any draw, for $M \leq k$, every draw must come from beneath k. The total number of draws possible fitting that circumstance can be expressed by $C(n, k)$. In our sample space there are $C(b, n)$ total possible draws.

Therefore, we have that

$$P(M \leq k) = \frac{\binom{k}{n}}{\binom{b}{n}}$$

As for the probability of equality,

$$\begin{aligned}
P(M = k) &= P(M \leq k - 1) - P(M \leq k) \\
&= \frac{\binom{k}{n}}{\binom{b}{n}} - \frac{\binom{k-1}{n}}{\binom{b}{n}} \\
&= \frac{\binom{k}{n} - \binom{k-1}{n}}{\binom{b}{n}} \\
&\text{by Pascal's identity} \\
&= \frac{\binom{k-1}{n-1}}{\binom{b}{n}}
\end{aligned}$$

Which is the desired result.