580 Optimization Homework

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Exercise 1 (i)

Let X be a vector space. Let $x, y \in X \quad \alpha \in \mathscr{R}$. Note that

$$\alpha x + (1-\alpha)y \in X$$

by properties 7 and 8 of vector spaces $\implies$ X is convex.

Exercise 1 (ii)

Assume to the contrary that X is finite and consists of n elements

$$\text{s.t.} X = \{x_1, x_2, \cdots, x_n\}$$

Now let $x' = x_1 + x_2 + \cdots + x_n$. By properties 7 and 8 $x' \in X$, which implies that there are more than n elements in X, and we have a contradiction.

Exercise 2

Note that $||y|| = ||(y-x)+x|| \leq ||y-x|| + ||x||$
$\implies ||y|| - ||x|| \leq ||y-x||$

Exercise 3

In order for $f(x)$ to be linear, we need to show that $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$. Note that

$$f(\alpha x + \beta y) = a'(\alpha x + \beta y) = a'\alpha x + a'\beta y = \alpha(a'x) + \beta(a'y) = \alpha f(x) + \beta f(y)$$

Exercise 4

Let

$$\alpha = (\frac{|x_i|}{||x||_p})^p \quad \beta = (\frac{|a_i|}{||a||_q})^q \quad \lambda = \frac{1}{p} \quad (1-\lambda) = \frac{1}{q}$$

Then by the Holder inequality, we have

$$((\frac{|x_i|}{||x||_p})^p)^{\frac{1}{p}}((\frac{|a_i|}{||a||_q})^q)^{\frac{1}{q}} = (\frac{|x_i|}{||x||_p})(\frac{|a_i|}{||a||_q}) = \frac{|x_i||a_i|}{||x||_p||a||_q} \leq \frac{1}{p}(\frac{|x_i|}{||x||_p})^p + \frac{1}{q}(\frac{|a_i|}{||a||_q})^q$$

which is the desired result.

For the last part, we will sum up all the elements of both sides of the last inequality, such that

$$\sum_{i=1}^{\infty} \frac{|a_i x_i|}{||x||_p||a||_q} \leq \sum_{i=1}^{\infty}(\frac{1}{p}(\frac{|x_i|}{||x||_p})^p + \frac{1}{q}(\frac{|a_i|}{||a||_q})^q)$$

$$\frac{\sum_{i=1}^{\infty}|a_i x_i|}{||x||_p||a||_q} \leq (\frac{1}{p}(\frac{\sum_{i=1}^{\infty}|x_i|}{||x||_p})^p + \frac{1}{q}(\frac{\sum_{i=1}^{\infty}|a_i|}{||a||_q})^q)$$

$$\frac{\sum_{i=1}^{\infty}|a_i x_i|}{||x||_p||a||_q} \leq (\frac{1}{p}(\frac{\sum_{i=1}^{\infty}|x_i|}{||x||_p})^p + \frac{1}{q}(\frac{\sum_{i=1}^{\infty}|a_i|}{||a||_q})^q)$$

Now, since $||x||_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$ and $||a||_q = (|a_1|^q + |a_2|^q + \cdots + |a_n|^q)^{\frac{1}{q}}$, Note that $(||x||_p)^p = |x_1|^p + |x_2|^p + \cdots + |x_n|^p$ and $(||a||_q)^q = |a_1|^q + |a_2|^q + \cdots + |a_n|^q$, so we have

$$\frac{\sum_{i=1}^{\infty}|a_i x_i|}{||x||_p||a||_q} \leq (\frac{1}{p}(\frac{\sum_{i=1}^{\infty}|x_i|^p}{\sum_{i=1}^{\infty}|x_i|^p}) + \frac{1}{q}(\frac{\sum_{i=1}^{\infty}|a_i|^q}{\sum_{i=1}^{\infty}|a_i|^q}))$$

$$\frac{\sum_{i=1}^{\infty}|a_i x_i|}{||x||_p||a||_q} \leq \frac{1}{p} + \frac{1}{q} = 1$$

which is the desired result.

Exercise 5

We want to show that $||ax||_1 \leq ||a||_q||x||_p$. Note that where $p = 1, q = 1$,

$$||ax||_1 \leq ||a||_1||x||_\infty$$

$$||ax||_1 \leq ||a||_1 \max\{x\}$$

(Let $x^* = \max\{x\}$)

$$||ax||_1 \leq ||a||_1 x*$$

$$||ax||_1 \leq ||ax*||_1$$

$$\sum_{i=1}^{\infty} a_i x_i \leq \sum_{i=1}^{\infty} a_i x^*$$

Since the left side is the sum of the products of the elements of a_i and x_i where $x_i \leq x* \quad \forall i$, it should be clear that this inequality holds, suggesting that

$$||ax||_1 \leq ||a||_1||x||_\infty$$

Exercise 6

We know that

$$\sum_{i=1}^{\infty} |x_i|^p < \infty$$

by the definition of ℓ_p. Therefore, x contains a finite number, call it n, of elements $y_i \in x$ s.t. $y_i \geq 1 \quad \forall i$, and an infinite number, call it m, of elements $x_i \in x$ s.t. $x_i < 1 \quad \forall i$, else the summation of elements of x would not converge for $p \in [1, \infty)$.

$$S = y_1, y_2, \cdots, y_n$$

$$J = x_1, x_2, \cdots, x_m$$

$$N = \sum_{i \in S} |y_i|^p$$

$$M = \sum_{i \in J} |x_i|^p$$

Now, notice that

$$\sum_{i=1}^{\infty} |x_i|^p = M + N \text{ where } M < \infty, N < \infty$$

Now, since N is a finite sum made up of finite terms, if we let $N' = \sum_{i \in S} |y_i|^{p'} \quad M' = \sum_{i \in J} |x_i|^{p'}$ where $p' \geq p$, we know that $N' < \infty$. Since M is a sum of elements that converge to a finite number when raised to p, we know that M' will converge even faster to a finite number since every $x_i \in J$ is less than one, raised to a number that is greater than p, namely p'.

Exercise 7

We know that since $\langle \cdot, \cdot \rangle$ is an inner product, it behaves in the following way: For $x, y \in X$, since

$$\langle x, x \rangle \geq 0$$

$$\langle x, x \rangle > 0 \text{ if and only if } x = 0$$

We have that

$$\sqrt{\langle x, x \rangle} > 0 \text{ if } x \neq 0$$

Also, since

$$\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$$

and

$$\langle x, y \rangle = \langle y, x \rangle$$

We know that

$$\begin{aligned}\sqrt{\langle \alpha x, \alpha x\rangle} &= \sqrt{\alpha\langle x, \alpha x\rangle}\\ &= \sqrt{\alpha\langle \alpha x, x\rangle}\\ &= \sqrt{\alpha^2\langle x, x\rangle}\\ &= \alpha\sqrt{\langle x, x\rangle}\end{aligned}$$

So we know that

$$||\alpha x|| = |\alpha|||x|| \forall x \in X$$

Finally, we know that

$$\sqrt{\langle x+y, x+y\rangle} \leq \sqrt{\langle x, x\rangle} + \sqrt{\langle y, y\rangle}$$

is equivalent to showing

$$\begin{aligned}\langle x+y, x+y\rangle &\leq \langle x, x\rangle + \langle y, y\rangle + 2\sqrt{\langle x, x\rangle\langle y, y\rangle}\\ \langle x, x+y\rangle + \langle y, x+y\rangle &\leq \langle x, x\rangle + \langle y, y\rangle + 2\sqrt{\langle x, x\rangle\langle y, y\rangle}\\ \langle x, x\rangle + \langle y, y\rangle + 2\langle x, y\rangle &\leq \langle x, x\rangle + \langle y, y\rangle + 2\sqrt{\langle x, x\rangle\langle y, y\rangle}\\ 2\langle x, y\rangle &\leq 2\sqrt{\langle x, x\rangle\langle y, y\rangle}\\ \langle x, y\rangle &\leq \sqrt{\langle x, x\rangle\langle y, y\rangle}\\ &\text{By Cauchy-Shwarz we have}\\ \langle x, y\rangle &\leq \sqrt{\langle x, x\rangle}\sqrt{\langle y, y\rangle}\end{aligned}$$

And so we have that

$$\sqrt{\langle x+y, x+y\rangle} \leq \sqrt{\langle x, x\rangle} + \sqrt{\langle y, y\rangle} \quad \forall x, y \in X$$

Which is the desired result.

Exercise 8

(i)

$$\mathscr{L} = -\frac{1}{2}(x-2)^2 + \lambda_1(x-0) + \lambda_2(4-x)$$

implies the following first order conditions and solution:

$$\begin{aligned}&-(x-2)^2 + \lambda_1 - \lambda_2\\ &\lambda_1 x = 0\\ &\lambda_2(4-x) = 0\end{aligned}$$

and the following solution:

$$x = 2$$

(ii)

$$\mathscr{L} = -(x-2)^2 - (y-2)^2 + \lambda_1(x-0) + \lambda_2(y-0) + \lambda_3(xy-4)$$

which yields the following first order conditions

$$\begin{aligned}
&-2(x-2) + \lambda_1 + \lambda_3 y = 0 \\
&-2(y-2) + \lambda_2 + \lambda_3 x = 0 \\
&\lambda_1 x = 0 \\
&\lambda_2 y = 0 \\
&xy = 4
\end{aligned}$$

and the following solution:

$$x = 2, y = 2$$

(iii)

$$\mathscr{L}(\sum_{t=0}^{\infty} \beta^t \log(x(t))) + \lambda_1(w(0) - x(0)) + \sum_{i=1}^{\infty} \gamma_i(w(i-1) - x(i-1))$$

However, as this is an infinite sum, there are infinite constraints and a finite number of variables and therefore no solution.

(iv)

$$\mathscr{L} = x + \log(y) + \lambda_1(w - ax - by) + \lambda_2(x - 0) + \lambda(y - 0)$$

which yields the following first order conditions

$$\begin{aligned}
&1 - \lambda_1 a + \lambda_2 = 0 \\
&\frac{1}{y} - \lambda_1 b = 0 \\
&w - ax - by = 0 \\
&\lambda x_2 = 0
\end{aligned}$$

We know $y > 0 \implies \lambda_3 = 0$. Now consider two cases $\lambda_2 > 0, \lambda_2 = 0$,

$$\lambda_2 > 0$$

$$\begin{aligned}
x &= 0 \\
y &= \frac{w}{b}
\end{aligned}$$

$$\begin{aligned}
\lambda_2 &= 0 \\
x &= \frac{w}{a} - 1 \\
w &= a(x+1) \\
y &= \frac{a}{b} \\
1 &= \lambda_1 a \\
\frac{1}{y} - \frac{1}{a} b &= 0
\end{aligned}$$