Math 344 Homework 2.6

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2.32

Let our elementary matrices be as follows:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3.5 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

We see that:

$$U = E_3E_2E_1A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0.5 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 0 & -3 & 6 \\ 0 & 0 & -9 \end{bmatrix}$$

Also, we have that

$$A = (E_3E_2E_1)^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 3.5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 0 & -3 & 6 \\ 0 & 0 & -9 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}$$

2.33 (i)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathscr{N}(A) = span \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

2.33 (ii)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathscr{N}(A) = span \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + span \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

2.33 (iii)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathscr{N}(A) = span \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

2.33 (iv)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathscr{N}(A) = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Solution: None, because $0x_1 + 0x_2 + 0x_3 = 1$ is not possible.

2.33 (v)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Null space empty and solution is

$$x = \begin{bmatrix} -.7625 \\ .025 \\ .57083 \end{bmatrix}$$

2.34 (i)

Given these two elements $e_j = e_{j,1}, e_{j,2}, \cdots, e_{j,n} \quad e_i = e_{i,1}, e_{i,2}, \cdots, e_{i,n}$ of the standard basis for $\mathbb{R}^n$, which are both $n \times 1$, we have that this operation $e_i e_j^T$ will yield an $n \times n$ matrix E, where $E_{1,1} = e_{j,1}e_{i,1}, E_{1,2} = e_{j,2}e_{i,1}, \cdots, E_{m,k} = e_{j,k}e_{i,m}$. Now, since the only nonzero entry of e_j is 1 at entry j and the only nonzero entry of e_i is 1 at entry i, it follows that every entry will be either the product of zero and zero or zero and 1, which are both zero, except for one, in which both $e_{j,j} = 1 \quad e_{i,i} = 1 \implies E_{i,j} = 1$.

2.34 (ii)

$$
\begin{aligned}
(I - a\mathbf{u}\mathbf{v}^T)^{-1} &= \left(I - \frac{a\mathbf{u}\mathbf{v}^T}{a\mathbf{v}^T\mathbf{u} - 1}\right) \\
(I - a\mathbf{u}\mathbf{v}^T)^{-1}(I - a\mathbf{u}\mathbf{v}^T) &= \left(I - \frac{a\mathbf{u}\mathbf{v}^T}{a\mathbf{v}^T\mathbf{u} - 1}\right)(I - a\mathbf{u}\mathbf{v}^T) \\
I &= II - a\mathbf{u}\mathbf{v}^T - I\frac{a\mathbf{u}\mathbf{v}^T}{a\mathbf{v}^T\mathbf{u} - 1} + \frac{a\mathbf{u}\mathbf{v}^T}{a\mathbf{v}^T\mathbf{u} - 1}(a\mathbf{u}\mathbf{v}^T) \\
I &= I - a\mathbf{u}\mathbf{v}^T - \frac{a\mathbf{u}\mathbf{v}^T}{a\mathbf{v}^T\mathbf{u} - 1} + \frac{a\mathbf{u}\mathbf{v}^T a\mathbf{u}\mathbf{v}^T}{a\mathbf{v}^T\mathbf{u} - 1} \\
0 &= \frac{-a\mathbf{u}\mathbf{v}^T(a\mathbf{v}^T\mathbf{u} - 1) - a\mathbf{u}\mathbf{v}^T + a\mathbf{u}\mathbf{v}^T a\mathbf{u}\mathbf{v}^T}{a\mathbf{v}^T\mathbf{u} - 1} \\
0 &= \frac{a\mathbf{u}\mathbf{v}^T - a\mathbf{u}\mathbf{v}^T a\mathbf{v}^T\mathbf{u} - a\mathbf{u}\mathbf{v}^T + a\mathbf{u}\mathbf{v}^T a\mathbf{u}\mathbf{v}^T}{a\mathbf{v}^T\mathbf{u} - 1} \\
0 &= \frac{a\mathbf{u}\mathbf{v}^T a\mathbf{u}\mathbf{v}^T - a\mathbf{u}\mathbf{v}^T a\mathbf{v}^T\mathbf{u}}{a\mathbf{v}^T\mathbf{u} - 1} \\
0 &= \frac{a^2(\mathbf{u}\mathbf{v}^T\mathbf{u}\mathbf{v}^T - \mathbf{u}\mathbf{v}^T\mathbf{v}^T\mathbf{u})}{a\mathbf{v}^T\mathbf{u} - 1} \\
0 &= \frac{a^2(\mathbf{v}^T\mathbf{u})(\mathbf{u}\mathbf{v}^T - \mathbf{u}\mathbf{v}^T)}{a\mathbf{v}^T\mathbf{u} - 1} \\
0 &= \frac{a^2(\mathbf{v}^T\mathbf{u})(0)}{a\mathbf{v}^T\mathbf{u} - 1} \\
0 &= \frac{0}{a\mathbf{v}^T\mathbf{u} - 1}
\end{aligned}
$$

2.35

Suppose that A doesn't fulfill the conditions of RREF. This entails that anywhere from one to all of the following conditions do not apply
i. The leading coefficient of each row is always strictly to the right of the leading coefficient of the row above it.
ii. All nonzero rows are above any zero rows.
iii. The leading coefficient of every row is equal to one.
iv. The leading coefficient of every row is the only nonzero entry in its column. Now, i and ii can be corrected by left-multiplying A by a Type I elementary matrix.
iii can be corrected by multiplying a leading coefficient not equal to one by an α, namely its inverse, by left-multiplying A by a Type II elementary matrix. And
iv can be corrected by left-multiplying A by a Type III elementary matrix in order to turn every other entry of a column to a zero excepting the column's leading coefficient. At this point, we have left-multiplied A by an arbitrary number of elementary matrices

to attain a new RREF matrix

$$B = E_k E_{k-1} \cdots E_1 A$$

which according to the definition of row equivalence, is row equivalent to A.

2.36

We know by Proposition 2.6.2 that all elementary matrices are invertible. Note that for a matrix A that is row equivalent to a matrix B, by the definition of row equivalence we have the following:

$$A = E_1 E_2 \dots E_k B \implies B = E_k^{-1} E_{k-1}^{-1} \dots E_1^{-1} A$$

where $E_i \quad i = 0, 1, \dots, k$ is a sequence of elementary matrices.
It follows that the inverse of each elementary matrix simply undoes the operation performed on A in order to yield B and vice-versa. Therefore, if E is an elementary matrix, then E^{-1} is an elementary matrix as well. Now we need to show, for row equivalence:

Reflexivity:

$$A = IA$$

where I is the identity matrix (an elementary matrix) and therefore row equivalent to itself, implying reflexivity.

Symmetry:

Suppose a matrix A is row equivalent to a matrix B, then since elementary matrices are invertible, we have that

$$A = E_1 E_2 \dots E_k B$$
$$\implies B = E_k^{-1} E_{k-1}^{-1} \dots E_1^{-1} A$$

where $E_i \quad i = 0, 1, \dots, k$ is a sequence of elementary matrices, implying that B is row equivalent to A, implying symmetry.

Transitivity:

Suppose a matrix A is row-equivalent to a matrix B, and B is row equivalent to a matrix C

$$A = E_1 E_2 \dots E_n B$$
$$B = E'_1 E'_2 \dots E'_m C$$

where $E_i \quad i = 0, 1, \dots, n \quad E'_j \quad j = 0, 1, \dots, m$ are two sequences of elementary matrices. Then we have

$$A = E_1 E_2 \dots E_n E'_1 E'_2 \dots E'_m C$$

Which implies that A is row equivalent to C, implying transitivity.

2.37

L from 2.17 is as follows:

$$L = \begin{bmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 2 \\ 0 & 0 & \alpha \end{bmatrix}$$

In this case $\alpha = 1$ giving the derivative operator as follows

$$L = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\implies L^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Uniqueness is implied by the fact that$\mathscr{N}(L) = \mathbf{0}$ and $\mathscr{N}(L^{-1}) = \mathbf{0}$.

2.38

Consider the following basis for this space

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x^2 & 0 \\ 0 & 0 & 0 & x^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Performing the linear transformation on each element of the basis, we have

$$\begin{aligned} (x-1)0 &= 0 \\ (x-1)1 &= x - 1 \\ (x-1)2x &= 2x^2 - 2x \\ (x-1)3x^2 &= 3x^3 - 3x^2 \end{aligned}$$

yielding the transformation matrix:

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

And we have the following:

$$\mathscr{N}(A) = \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \quad \mathscr{R}(A) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$