Math 344 Homework 4.6

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4.33

We know that $A^H A$'s eigenvalues are the squared singular values of A, and since $(A^H A)^H = (A^H A)$, the same thing applies for A^H. Now, we know that

$$p_A(z) = \det(A - zI) = (\lambda_1 - z)(\lambda_2 - z)\dots(\lambda_n - z)$$
$$\implies p_A(0) = \det(A) = (\lambda_1)(\lambda_2)\dots(\lambda_n) = \Pi_{i=1}^n \lambda_i$$

This gives us that

$$\begin{aligned}|\det(A^H A)| &= |\Pi_{i=1}^n \lambda_i| \\ &= |\Pi_{i=1}^n \sigma_{Ai}^2| \\ &= |\Pi_{i=1}^n \sigma_{A^H i}^2|\end{aligned}$$

Since these are equal, we will denote them

$$= |\Pi_{i=1}^n \sigma_i^2|$$

As for the other side of the equality, we have that

$$\det(A) = \Pi_{i=1}^n \lambda_i = \det(A^H)$$

and so the left side of the equality turns into

$$\begin{aligned}|\det(A^H A)| &= |\det(A)\det(A^H)| \\ &= |\det(A)|^2\end{aligned}$$

and we have finally that

$$\begin{aligned}|\det(A)|^2 &= |\Pi_{i=1}^n \sigma_i^2| \\ \implies |\det(A)| &= |\Pi_{i=1}^n \sigma_i| \\ \implies |\det(A)| &= \Pi_{i=1}^n \sigma_i\end{aligned}$$

since the singular values of A are positive and real. Therefore the equality holds and we have the desired result.

4.34

If we let

$$A = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

whose eigenvalues are $\lambda = -i, i$ whose determinant $\det(A) = 1$
Therefore,

$$A^H A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

whose eigenvalue is 1. Therefore, we have that the singular value is 1.

4.35

As the first r columns of U and V from the Singular Value Decomposition of A yield U_1 and V_1, Σ_1 is the $r \times r$ diagonal matrix with the diagonals as the non-zero eigenvalues of $A^H A$, we have the following

$$U_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix} \quad \Sigma_1 = \begin{bmatrix} \sqrt{15} \end{bmatrix} \quad V_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{bmatrix}$$

Then we have, by the definition of the Moore-Penrose Inverse,

$$A^\dagger = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{15}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{1}{15} & 0 & \frac{2}{15} \\ \frac{1}{15} & 0 & \frac{2}{15} \\ \frac{1}{15} & 0 & \frac{2}{15} \\ 0 & 0 & 0 \end{bmatrix}$$

Now, we have that

$$A^H A = \begin{bmatrix} 5 & 5 & 5 & 0 \\ 5 & 5 & 5 & 0 \\ 5 & 5 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad A^\dagger A = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrices are scalar multiples of one another.

4.36

$$A = U_1 \Sigma_1 V_1^H \quad A^\dagger = V_1 \Sigma_1^{-1} U_1^H$$

So we have that

4.36 (i)

$$
\begin{aligned}
AA^\dagger A &= U_1\Sigma_1V_1^HV_1\Sigma_1^{-1}U_1^HU_1\Sigma_1V_1^H \\
&= U_1\Sigma_1I\Sigma_1^{-1}I\Sigma_1V_1^H \\
&= U_1II\Sigma_1V_1^H \\
&= U_1\Sigma_1V_1^H \\
&= A
\end{aligned}
$$

4.36 (ii)

$$
\begin{aligned}
A^\dagger AA^\dagger &= V_1\Sigma_1^{-1}U_1^HU_1\Sigma_1V_1^HV_1\Sigma_1^{-1}U_1^H \\
&= V_1\Sigma_1^{-1}I\Sigma_1I\Sigma_1^{-1}U_1^H \\
&= V_1I\Sigma_1^{-1}U_1^H \\
&= A^\dagger
\end{aligned}
$$

4.36 (iii)

$$
\begin{aligned}
(AA^\dagger)^H &= (U_1\Sigma_1V_1^HV_1\Sigma_1^{-1}U_1^H)^H \\
&= U_1(\Sigma_1^{-1})^HV_1^HV_1(\Sigma_1)^HU_1^H \\
&= U_1(\Sigma_1^{-1})^HI(\Sigma_1)^HU_1^H \\
&= U_1(\Sigma_1\Sigma_1^{-1})^HU_1^H \\
&= U_1U_1^H \\
&= I
\end{aligned}
$$

and we have that

$$
\begin{aligned}
AA^\dagger &= U_1\Sigma_1V_1^HV_1\Sigma_1^{-1}U_1^H \\
&= U_1\Sigma_1I\Sigma_1^{-1}U_1^H \\
&= U_1IU_1^H \\
&= U_1U_1^H \\
&= I
\end{aligned}
$$

which is equivalent to the first expression.

4.36 (iv)

$$\begin{aligned}
A^{\dagger}A &= V_1\Sigma_1^{-1}U_1^H U_1\Sigma_1 V_1^H \\
&= V_1(\Sigma_1)^H(U_1)^H U_1(\Sigma_1^{-1})^H V_1^H \\
&= V_1(\Sigma_1)^H I(\Sigma_1^{-1})^H V_1^H \\
&= V_1(\Sigma_1^{-1}\Sigma_1)^H V_1^H \\
&= V_1(I)^H V_1^H \\
&= V_1 V_1^H \\
&= I
\end{aligned}$$

and we have that

$$\begin{aligned}
A^{\dagger}A &= V_1\Sigma_1^{-1}U_1^H U_1\Sigma_1 V_1^H \\
&= V_1\Sigma_1^{-1} I\Sigma_1 V_1^H \\
&= V_1 I V_1^H \\
&= I
\end{aligned}$$

which is equivalent to the first expression.

4.36 (v)

$$AA^{\dagger} = U_1 U_1^H$$

We know, though, that the first r columns of U_1 form a basis for $\mathscr{R}(A)$, so it follows that anything's projection onto $\mathscr{R}(A)$ can be expressed in terms of this basis.

4.36 (vi)

$$A^{\dagger}A = V_1 V_1^H$$

We know, though, that the first r columns of V_1 form a basis for $\mathscr{R}(A^H)$, so it follows that anything's projection onto $\mathscr{R}(A^H)$ can be expressed in terms of this basis.

4.37

Note

$$||\Delta||_2 = || - \sum_{i=s+1}^{r} \sigma_i u_i v_i^H ||_2$$
$$\geq \sigma_{s+1}$$

by Schmidt-Eckart-Young-Mirsky

$$= \inf_{\text{rank(B) = S}} ||A - B||_2$$
$$= \inf_{\text{rank(B) = S}} || \sum_{i=1}^{r} \sigma_i u_i v_i^H - \sum_{i=1}^{s} \sigma_i u_i v_i^H ||_2$$
$$= \inf_{\text{rank(B) = S}} || \sum_{i=s+1}^{r} \sigma_i u_i v_i^H ||_2$$
$$= \inf_{\text{rank(B) = S}} || - \sum_{i=s+1}^{r} \sigma_i u_i v_i^H ||_2$$

Which holds, so we have the inequality thanks to the infimum.

Now note

$$|| - \sum_{i=s+1}^{r} \sigma_i u_i v_i^H ||_F \geq \left(\sum_{k=s+1}^{r} \sigma_k^2 \right)^{1/2}$$

by Schmidt-Eckart-Young-Mirsky

$$= \inf ||A - B||_F$$
$$= \inf || \sum_{i=s+1}^{r} ||_F$$
$$= \inf || - \sum_{i=s+1}^{r} ||_F$$

which also holds, so we have the desired inequality thanks to the infimum.

4.38

Note that

$$
\begin{aligned}
(I - A\Delta^{\diamond})\mathbf{u}_1 &= (I - U\Sigma V^H \frac{1}{\sigma_1}\mathbf{v}_1\mathbf{u}_1^H)\mathbf{u}_1 \\
&= \mathbf{u}_1 - U\Sigma V^H \frac{1}{\sigma_1}\mathbf{v}_1\mathbf{u}_1^H\mathbf{u}_1 \\
&= \mathbf{u}_1 - U\frac{1}{\sigma_1}\Sigma V^H\mathbf{v}_1 \\
&= \mathbf{u}_1 - U\frac{1}{\sigma_1}\Sigma \left[\mathbf{v}_1|\mathbf{v}_2|\dots|\mathbf{v}_n\right]^H \mathbf{v}_1
\end{aligned}
$$

Now, we know that $\mathbf{v}_i$ are orthonormal for all i, so we have

$$
\begin{aligned}
&= \mathbf{u}_1 - U\frac{1}{\sigma_1}\Sigma \left[\mathbf{v}_1|\mathbf{v}_2|\dots|\mathbf{v}_n\right]^H \mathbf{v}_1 \\
&= \mathbf{u}_1 - U\frac{1}{\sigma_1}\Sigma \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}
\end{aligned}
$$

We also know that because of the format of the matrix Σ:

$$
\begin{aligned}
&= \mathbf{u}_1 - U\frac{1}{\sigma_1}\Sigma \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} \\
&= \mathbf{u}_1 - U\frac{1}{\sigma_1} \begin{bmatrix}\sigma_1\\0\\\vdots\\0\end{bmatrix} \\
&= \mathbf{u}_1 - U \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} \\
&= \mathbf{u}_1 - \mathbf{u}_1 = \mathbf{0}
\end{aligned}
$$

4.39

$$
\|A\|_2\|\Delta\|_2 = \sigma_1\|\Delta\|_2 < \sigma_1\frac{1}{\sigma_1} = 1
$$

which is the desired result.

4.40 (i)

Let $\mathscr{R}(\Delta) \subset \mathscr{R}(A)$ and we have

$$\begin{aligned} A + \Delta &= (A^\dagger A + A^\dagger \Delta) \\ &= A^\dagger A(I + \Delta) \\ &= (I + A^\dagger \Delta) \end{aligned}$$

4.40 (ii)

Note, $\text{rank}(XY) < \text{rank}(X)$. Thus, $\exists \mathbf{v}$ s.t. $XY\mathbf{v} = 0$, but $X\mathbf{v} \neq 0$. Therefore $\mathbf{v} \in \mathscr{N}(Y)$, which is non-trivial, thus Y is not invertible.

4.40 (iii)

Suppose $A + \Delta$ has rank $s < r$. Thus, by (i) and (ii),

$$\text{rank}(A + \Delta) = \text{rank}(A(I + A^\dagger \Delta))$$

and by Thm. 4.6.7

$$\|\Delta\|_2 \geq \sigma_r$$