Math 344 Homework 2.3

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2.14

$$L(2e_1, -3e_2) = -4e_1 + 7e_2$$
$$L^2(2e_1, -3e_2) = 10e_1 - 15e_2$$

$$L = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} L^2 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

2.15

(ii)

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(iv)

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 4 & -6 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

2.16

For the transformation $p(x) \to p(x) + 4p'(x)$ we have the matrix representation:

$$L\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\1\end{bmatrix} + 4\begin{bmatrix}0\\2\\0\end{bmatrix} = \begin{bmatrix}1\\8\\1\end{bmatrix}$$
$$L\left(\begin{bmatrix}-1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\1\\0\end{bmatrix} + 4\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}3\\1\\0\end{bmatrix}$$
$$L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix} + 4\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

Ultimately giving:

$$\begin{bmatrix}1 & 3 & 1\\8 & 1 & 0\\1 & 0 & 0\end{bmatrix}$$

2.17

Let the matrix representation of S be given by

$$S = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$L(s_1) = L\Big(\begin{bmatrix}1\\0\\0\end{bmatrix}\Big) = \Big(\begin{bmatrix}\alpha\\0\\0\end{bmatrix}\Big)$$

$$L(s_2) = L\Big(\begin{bmatrix}0\\1\\0\end{bmatrix}\Big) = \Big(\begin{bmatrix}\alpha\\0\\0\end{bmatrix}\Big)$$

$$L(s_3) = L\Big(\begin{bmatrix}1\\\alpha\\0\end{bmatrix}\Big) = \begin{bmatrix}0\\2\\\alpha\end{bmatrix}\Big)$$
$$\implies L = \begin{bmatrix}\alpha & 1 & 0\\0 & \alpha & 2\\0 & 0 & \alpha\end{bmatrix}$$

2.18

$$
\begin{aligned}
s_1 = e^{i\theta} &= \cos(\theta) + i\sin(\alpha) = t_1 + t_2 \\
s_2 = e^{-i\theta} &= \cos(\theta) - i\sin(\alpha) = t_1 - t_2 \\
t_1 = \cos(\theta) = \frac{1}{2}(\cos(\theta) + i\sin(\theta)) + \frac{1}{2}(\cos(\theta) - i\sin(\theta)) &= \frac{1}{2}s_1 + \frac{1}{2}s_2 \\
t_2 = i\sin(\theta) = \frac{1}{2}(\cos(\theta) + i\sin(\theta)) - \frac{1}{2}(\cos(\theta) - i\sin(\theta)) &= \frac{1}{2}s_1 - \frac{1}{2}s_2
\end{aligned}
$$

(i)

$$\implies C_{TS} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

(ii)

$$\implies C_{ST} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(iii)

$$C_{ST}^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

2.19(i)

For the derivative operator $D : V \to V$:

We know that:

$$
\begin{aligned}
\frac{d}{d\theta}\left(e^{i\theta}\right) = ie^{i\theta} \to D\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} i \\ 0 \end{bmatrix} \\
\frac{d}{d\theta}\left(e^{-i\theta}\right) = -ie^{-i\theta} \to D\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -i \end{bmatrix}
\end{aligned}
$$

So,

$$D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

2.19(ii)

We know that:

$$\frac{d}{d\theta}(\cos(\theta)) = -\sin(\theta) \to D\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\i\end{bmatrix}$$
$$\frac{d}{d\theta}(i\sin(\theta)) = i\cos(\theta) \to D\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}i\\0\end{bmatrix}$$

So,

$$D = \begin{bmatrix}0 & i\\ i & 0\end{bmatrix}$$

2.19(iii)

We first want apply our linear transformation and then change our basis to T, this can be done using the transformation matrix for D in terms of S, and then applying our transition matrix from S to T. So if we multiply this transformation matrix and transition matrix, this will yield the desired result.

$$\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}\begin{bmatrix}0 & i\\ i & 0\end{bmatrix} = \begin{bmatrix}i & -i\\ i & i\end{bmatrix}$$

2.20

We want to show that for the linear transformation $A = [a_1, a_2, \ldots, a_n]$, it is true that $\mathscr{R}(A) = \text{span}\{a_1, a_2, \ldots, a_n\}$.

$\Rightarrow$
Let us consider $b \in \mathscr{R}(A)$. Then the following is also true that $\exists\, x$s.t. $A(x) = b$. Since x can be expressed $x = x_1, x_2, \ldots, x_n$, we can also express b as $b = x_1a_1 + x_2a_2 + \cdots + x_na_n$. Thus we can see that b is represented as a linear combination of the columns of A and therefore is also in $\mathscr{R}(A) = \text{span}\{a_1, a_2, \ldots, a_n\}$.

$\Leftarrow$
Now let us consider $b \in \text{span}\{a_1, a_2, \ldots, a_n\}$. Then it is true that b is a linear combination of the elements of $\text{span}\{a_1, a_2, \ldots, a_n\}$ and therefore can be represented as $b = x_1a_1 + x_2a_2 + \cdots + x_na_n$, and we can say that $\exists\, x$ s.t. $b = A(x)$. Showing that $b \in \mathscr{R}(A)$.

Since every element of $\mathscr{R}(A)$ is in $\text{span}\{a_1, a_2, \ldots, a_n\}$, and every element of $\text{span}\{a_1, a_2, \ldots, a_n\}$ is in $\mathscr{R}(A)$, it is true that $\mathscr{R}(A) = \text{span}\{a_1, a_2, \ldots, a_n\}$.