Homework 1.4

Chris Rytting

September 11, 2015

1.20

We want to show Let $(\mathbf{x}+W),(\mathbf{y}+W)\in V/W \quad a,b\in\mathbb{F}$
Note that for
(v)

$$a\boxdot[(\mathbf{x}+W)\boxplus(\mathbf{y}+W)]=a\boxdot[(\mathbf{x}+\mathbf{y})+W]=a\boxdot(\mathbf{x}+W)\boxplus a\boxdot(\mathbf{y}+W)$$

(vi)

$$(a+b)\boxdot(\mathbf{x}+\mathbf{y})=(a+b)\mathbf{x}+W=(a\mathbf{x}+b\mathbf{y})=a\boxdot(\mathbf{x}+W)+b\boxdot(\mathbf{x}+W)$$

(vii)

$$1\boxdot(\mathbf{x}+W)=(1\mathbf{x})\boxplus W=\mathbf{x}+W$$

$(viii)$

$(ab)\boxdot(\mathbf{x}+W)=a\boxdot(b\mathbf{x}+w)=(ab\mathbf{x}+w=ba\mathbf{x}+W=b\boxdot(a\mathbf{x}+w)=(ba)\boxdot(\mathbf{x}+W)$

1.21

$$(a\boxdot(\mathbf{x}+W))\boxplus(b\boxdot(\mathbf{y}+W))=(a\mathbf{x}+W)\boxplus(b\mathbf{y}+W)=(a\mathbf{x}+b\mathbf{y})+W$$

1.22

We know from the definition of translates that, given V/W, the translates of W are sets of the form

$$\mathbf{x}+W=\{\mathbf{x}+\mathbf{w}|\mathbf{w}\in W\}$$

where $\mathbf{x}$ is any element of V. Given V/V, then, the translates of V are sets of the form

$$\mathbf{x}+V=\{\mathbf{x}+\mathbf{v}|\mathbf{v}\in V\}$$

where $\mathbf{x}$ is any element of V. However, x is necessarily in V, so only one quotient exists in the set, for no translate is necessary.

1.23

Let $\varphi : V/\{\mathbf{0}\} \to V$ be such that

$$\varphi(\mathbf{x} + \{\mathbf{0}\}) = \mathbf{x} \quad \text{for } \mathbf{x} \in V$$

Then we have that

$$\varphi(\mathbf{x} + \{\mathbf{0}\} \boxplus \mathbf{y} + \{\mathbf{0}\}) = \varphi(\mathbf{x} + \mathbf{y}\{\mathbf{0}\}) = \mathbf{x} + \mathbf{y}$$

Furthermore

$$\varphi(\mathbf{x} + \{\mathbf{0}\} + \varphi\mathbf{y} + \{\mathbf{0}\}) = \mathbf{x} + \mathbf{y}$$

$$\implies \varphi(\mathbf{x} + \{\mathbf{0}\} \boxplus \mathbf{y} + \{\mathbf{0}\}) = \varphi(\mathbf{x} + \{\mathbf{0}\}) + \varphi(\mathbf{y} + \{\mathbf{0}\})$$

Next, we have that for $\mathbf{x}, \mathbf{y} \in V, \quad c \in \mathbb{F}$,

$$\varphi(c \boxdot (\mathbf{x} + \{\mathbf{0}\})) = \varphi(c\mathbf{x} + \{\not{\mathbf{0}}\}) = c\mathbf{x}$$

and that

$$c\varphi(\mathbf{x} + \{\mathbf{0}\}) = c\mathbf{x}$$

so we have that

$$\varphi(c \boxdot (\mathbf{x} + \{\mathbf{0}\})) = c\varphi(\mathbf{x} + \{\mathbf{0}\})$$

1.24

Consider the map ψ with $c_i = i \in \mathbb{N}$

$$\psi : V/W \to \mathbb{F}[y] \quad \text{s.t.} \quad \psi(a_1x^{c_1} + a_2x^{c_2} + \cdots + a_nx^{c_n} + W) = (a_1y^{c_1/2} + a_2y^{c_2/2} + \cdots + a_ny^{c_n/2})$$

$$\text{where } a_i = \begin{cases} 0 & \text{if } c_i \text{ is even} \\ a_i & \text{if } c_i \text{ is odd} \end{cases}$$

Bijectivity satisfied because every monomial exists in V/W (odds covered by W and evens covered by rest of function), and it is mapped to every monomial in $\mathbb{F}[y]$ since every number is an even number divided by two, which is what characterizes the second space. As for addition and multiplication closure, for $p, q \in V \quad d \in \mathbb{F}$,

$$\psi(p + W) \boxplus (q + W)$$

$$= \psi((a_1p^{c_1} + a_2p^{c_2} + \cdots + p_nx^{c_n} + W)) \boxplus ((a_1q^{c_1} + a_2q^{c_2} + \cdots + q_nx^{c_n} + W)) = \psi(p + W) + \psi(q + W)$$

$$\psi(d \boxdot (p + W)) = \psi((da_1p^{c_1} + da_2p^{c_2} + \cdots + dp_nx^{c_n} + W)) = d\psi(p + W)$$