Homework 1.5

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1.26

$$\begin{aligned}
\sum_{k=5}^{n}(k-5)^2 &= \sum_{j=0}^{n-5} j^2 \\
&= \sum_{j=1}^{n-5} j^2 \\
&= \frac{(2n-9)(n-4)(n-5)}{6}
\end{aligned}$$

1.27

$$\sum_{k=4}^{n}\sum_{j=-3}^{k-8}(k-4) \quad \text{Now if we let } i = k-4 \text{, we have}$$

$$\begin{aligned}
&= \sum_{k=0}^{n}\sum_{j=-3}^{i-4} i \\
&= \sum_{k=0}^{n} i^2 \\
&= \frac{(2n+1)(n+1)n}{6}
\end{aligned}$$

1.28

$$\sum_{j=-3}^{n-3}\sum_{k=j+3}^{n+3}(k-3) \quad \text{Now if we let } l=k-3\text{, we have}$$
$$=\sum_{j=-3}^{n-3}\sum_{l=j}^{n} l$$
$$=\sum_{j=-3}^{n-3}(\sum_{l=0}^{n} l-\sum_{l=0}^{j-1} l)$$
$$=\sum_{j=-3}^{n-3}\frac{(n)(n+1)}{2}-\frac{(j-1)(j)}{2})$$
$$=\sum_{j=-3}^{n-3}\frac{(n)(n+1)}{2}-\sum_{j=-3}^{n-3}\frac{(j-1)(j)}{2})$$
$$=\frac{(n+1)^2n}{2}-\frac{1}{2}(\sum_{j=-3}^{n-3} j^2-\sum_{j=-3}^{n-3} j)$$
$$=\frac{(n+1)^2n}{2}-\frac{1}{2}(\sum_{j=0}^{n-3} j^2-\sum_{j=-3}^{0} j^2-(\sum_{j=0}^{n-3} j-\sum_{j=-3}^{0} j))$$
$$=\frac{(n+1)^2n}{2}-\frac{1}{2}(\frac{(n-3)(n-2)(2n-5)}{6}-14-\frac{(n-3)(n-2)}{2}-6))$$
$$=\frac{1}{2}((n+1)^2n-(\frac{(n-3)(n-2)(2n-5)}{6}-\frac{(n-3)(n-2)}{2}-20))$$

1.29

$$\sum_{k=1}^{n} k^3=(\frac{n(n+1)}{2})^2$$

If $f(k)=k^4$, then we know by the FTFC that

$$(\Delta f)(k)=(k+1)^4-k^4=k^3+6k^2+4k+1$$

$$
\begin{aligned}
\sum_{k=a}^{b-1}(4k^3+6k^2+4k+1) &= \sum_{k=a}^{b-1}(\Delta f) \\
&= f(b)-f(a) \\
&= b^4-a^4 \\
&= \sum_{k=a}^{b-1}(4k^3+6k^2+4k+1) \\
&= 4\sum_{k=a}^{b-1}k^3+6\sum_{k=a}^{b-1}k^2+4\sum_{k=a}^{b-1}k+\sum_{k=a}^{b-1}1 \\
&= 4\sum_{k=a}^{b-1}k^3+6\sum_{k=a}^{b-1}k^2+4\sum_{k=a}^{b-1}k+(b-a) \\
\implies b^4-a^4-(b-a) &= b^4-b-(a^4-a) \\
&= b(b^3-1)-a(a^3-1) \\
&= 4\sum_{k=a}^{b-1}k^3+6\sum_{k=a}^{b-1}k^2+4\sum_{k=a}^{b-1}k
\end{aligned}
$$

Now let $a=1, b=n+1$, then we have that

$$
\begin{aligned}
(n+1)((n+1)^3-1) &= 4\sum_{k=1}^{n}k^3+(2n+1)(n+1)n+2n(n+1) \\
\implies \sum_{k=1}^{n}k^3 &= \frac{(n(n+1))^2}{2}
\end{aligned}
$$

1.30

Note that if $f(i)=\frac{-1}{i}$, then

$$
(\Delta f)(i) = \frac{-1}{i+1}+\frac{1}{i}
$$

$$
\implies \frac{-i}{i(i+1)}+\frac{i+1}{i(i+1)} = \frac{1}{i(i+1)}
$$

so we have that

$$
\sum_{i=1}^{n}(\Delta f)(i) = \frac{-1}{n+1}-\frac{-1}{1} = 1-\frac{1}{n+1}
$$

1.31 (i)

$$\begin{aligned}
\sum_{k=0}^{n}\sum_{j=k}^{n} &= \sum_{k=0}^{n}(\sum_{j=0}^{n} j - \sum_{j=0}^{k-1} j) \\
&= \sum_{k=0}^{n}(\frac{n(n+1)}{2} - \frac{(k-1)k}{2}) \\
&= (n+1)^2 n - \sum_{k=0}^{n} k^2 + \sum_{k=0}^{n} k \\
&= \frac{1}{2}[(n+1)^2 n - \frac{n(n+1)(2n+1}{6} + \frac{n(n+1)}{2}] \\
&= \frac{1}{3}n^3 + \frac{3}{2}n^2 + \frac{2}{3}n
\end{aligned}$$

Going the other way we have $\sum_{j=0}^{n}\sum_{k=0}^{j} j = \sum_{j=1}^{n} j^2 + \sum_{j=1}^{n}$

$$\begin{aligned}
&= \frac{n(n+1)(2n+1}{6} + \frac{n(n+1)}{2} \\
&= \frac{1}{3}n^3 + \frac{3}{2}n^2 + \frac{2}{3}n
\end{aligned}$$

Which is equal to the first part so we have the desired result.

1.31 (ii)

$$
\begin{aligned}
\sum_{k=0}^{n}\sum_{j=0}^{k} j &= \sum_{k=0}^{n}(\frac{(k+1)k}{2} \\
&= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k \\
&= \frac{1}{2}[\frac{(2n+1)(n+1)n}{6} + \frac{n(n+1)}{2}] \\
&= \frac{n^3+3n^2+2n}{6}
\end{aligned}
$$

Going the other way, we have

$$
\begin{aligned}
\sum_{j=0}^{k}\sum_{k=0}^{n} j &= \sum_{j=0}^{n}\sum_{k=j}^{n} j \\
&= \sum_{j=0}^{n}(n-j+1)j \\
&= \sum_{j=0}^{n} jn - j^2 + j \\
&= \sum_{j=0}^{n} jn - \sum_{j=0}^{n} + \sum_{j=0}^{n} \\
&= \frac{n^2(n+1)}{2} - \frac{(2n+1)(n+1)n}{6} + \frac{n(n+1)}{2} \\
&= \frac{n^3+3n^2+2n}{6}
\end{aligned}
$$

Which is equal to the first part so we have the desired result.

1.32

$$\sum_{t=0}^{N} \beta^t = \frac{\beta^{N+1} - 1}{\beta - 1}$$

$$\implies \sum_{t=0}^{N} t\beta^{t-1} = \frac{(\beta - 1)[(N+1)\beta^N] - (\beta^{N+1} - 1}{(\beta - 1)^2}$$

Note that $|\beta| < 1$, so taking the limit as $N \to \infty$ yields

$$\lim_N \to \infty \frac{(\beta - 1)[(N+1)\beta^N] - (\beta^{N+1} - 1}{(\beta - 1)^2}$$

$$= \frac{1}{1 - \beta^2}$$Now, note that

$$= \sum_{t=0}^{\infty} t\beta^{t-1}$$

$$= \sum_{t=0}^{\infty}$$

$$= \frac{1}{(1 - \beta^2)}$$

$$\implies \frac{1}{\beta} \sum_{t=0}^{\infty} t\beta^t = \frac{1}{(1 - \beta^2)}$$

$$\implies \sum_{t=0}^{\infty} t\beta^t = \frac{\beta}{(1 - \beta^2)}$$