Mat 344 Homework 2.5

Chris Rytting

September 23, 2015

2.27

On the left side, we have

$$B_j^n(X) = \frac{n!}{j!(n-j)!}x^j(1-x)^{n-j}$$

On the right side, we have

$$\begin{aligned}
(1-x)B_j^{n-1}(x) + xB_{j-1}^{n-1}(x) &= \frac{(n-1)!}{(n-1-j)!j!}x^j(1-x)^{n-j} + \frac{(n-1)!}{(j-1)!(n-j)!}x^j(1-x)^{n-j} \\
&= x^j(1-x)^{n-j}\Big(\frac{(n-j)(n-1)!}{(n-j)!j!} + \frac{(n-1)!j}{(j)!(n-j)!}\Big) \\
&= x^j(1-x)^{n-j}\frac{(n-1)!}{j!(n-j)!}(n-j+j) \\
&= x^j(1-x)^{n-j}\frac{(n-1)!}{j!(n-j)!}(n) \\
&= x^j(1-x)^{n-j}\frac{n!}{j!(n-j)!}
\end{aligned}$$

2.28

On the left side, we have

$$B_j^n(x) = \frac{n!}{j!(n-j)!}x^j(1-x)^{n-j}$$

On the right side, we have

$$\begin{aligned}
\frac{j+1}{n+1}B_{j+1}^{n+1}(x) + \frac{n-j+1}{n+1}B_j^{n+1}(x) &= \frac{j+1}{n+1}\frac{(n+1)!}{(j+1)!(n-j)!}x^{j+1}(1-x)^{n-j} \\
&+ \frac{n-j+1}{n+1}\frac{(n+1)!}{(j)!(n-j+1)!}x^j(1-x)^{n-j+1} \\
&= \frac{n!}{j!(n-j)!}x^{j+1}(1-x)^{n-j} \\
&+ \frac{n!}{j!(n-j)!}x^j(1-x)^{n-j+1} \\
&= x\frac{n!}{j!(n-j)!}x^j(1-x)^{n-j} \\
&+ (1-x)\frac{n!}{j!(n-j)!}x^j(1-x)^{n-j} \\
&= (x+1-x)\frac{n!}{j!(n-j)!}x^j(1-x)^{n-j} \\
&= \frac{n!}{j!(n-j)!}x^j(1-x)^{n-j}
\end{aligned}$$

Now we have, by the recurrence relation, that

$$\begin{aligned}
B_0^n &= \frac{1}{n+1}B_1^{n+1} + \frac{n+1}{n+1}B_0^{n+1} \\
B_1^n &= \frac{2}{n+1}B_1^{n+1} + \frac{n}{n+1}B_0^{n+1} \\
B_2^n &= \frac{3}{n+1}B_1^{n+1} + \frac{n-1}{n+1}B_0^{n+1}
\end{aligned}$$

Resulting in the matrix

$$\begin{bmatrix}
\frac{n+1}{n+1} & \frac{1}{n+1} & 0 & 0 & \cdots & 0 \\
0 & \frac{n}{n+1} & \frac{2}{n+1} & 0 & \cdots & 0 \\
0 & 0 & \frac{n-1}{n+1} & \frac{3}{n+1} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \frac{2}{n+1} & \frac{n}{n+1} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{n+1}
\end{bmatrix}$$

2.29 (i)

On the left side, we have

$$
\begin{aligned}
DB_j^n(x) &= D\frac{n!}{j!(n-j)!}x^j(1-x)^{n-j} \\
DB_j^n(x) &= \frac{n!}{j!(n-j)!}Dx^j(1-x)^{n-j} \\
DB_j^n(x) &= \frac{n!}{j!(n-j)!}(jx^{j-1}(1-x)^{n-j} + (-1)(n-j)x^j(1-x)^{n-j-1}) \\
DB_j^n(x) &= \frac{n!}{j!(n-j)!}(jx^{j-1}(1-x)^{n-j} - (n-j)x^j(1-x)^{n-j-1})
\end{aligned}
$$

On the right side, we have

$$
\begin{aligned}
n(B_{j-1}^{n-1}(x) - B_j^{n-1}(x) &= n\Big(\frac{(n-1)!}{(j-1)!(n-j)!}x^{j-1}(1-x)^{n-j} - \frac{(n-1)!}{j!(n-1-j)!}x^j(1-x)^{n-j-1}\Big) \\
n(B_{j-1}^{n-1}(x) - B_j^{n-1}(x) &= \frac{n!}{j!(n-j)!}\Big(jx^{j-1}(1-x)^{n-j} - (n-j)x^j(1-x)^{n-j-1}\Big)
\end{aligned}
$$

Which is the desired result.

2.29 (ii)

We have, by the recurrence relation, that

$$
\begin{aligned}
DB_0^n &= nB_{-1}^{n-1}(x) - nB_0^{n-1}(x) \\
DB_1^n &= nB_0^{n-1}(x) - nB_1^{n-1}(x) \\
DB_2^n &= nB_1^{n-1}(x) - nB_2^{n-1}(x) \\
&\vdots
\end{aligned}
$$

Which yields the matrix

$$
\begin{bmatrix}
n & -n & 0 & \cdots & 0 \\
0 & n & -n & \cdots & 0 \\
0 & 0 & n & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & n & -n \\
0 & 0 & 0 & 0 & n
\end{bmatrix}
$$

2.29 (iii)

$$\begin{bmatrix} n & \frac{n}{n+1}-\frac{n^2}{n+1} & 0 & 0 & \cdots & 0 \\ 0 & n\frac{n}{n+1} & \frac{2n}{n+1}-n\frac{n-1}{n+1} & 0 & \cdots & 0 \\ 0 & 0 & n\frac{n-1}{n+1} & \frac{3n}{n+1}-n\frac{n-2}{n+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & n\frac{n-1}{n+1}+\frac{n}{n+1} \\ 0 & 0 & 0 & 0 & 0 & \frac{n}{n+1} \end{bmatrix}$$

2.30

$$\begin{aligned} DB_j^n(x) &= n\Big(B_{j-1}^{n-1}(x) - B_j^{n-1}(x)\Big) \\ DB_j^{n+1}(x) &= (n+1)\Big(B_{j-1}^{n}(x) - B_j^{n}(x)\Big) \\ B_j^{n+1}(x)\Big|_0^1 &= (n+1)\Big(\int_0^1 B_{j-1}^n(x)dx - (n+1)\int_0^1 B_j^n(x)dx\Big) \\ \Big((n+1)\int_0^1 B_j^n(x)dx\Big) &= (n+1)\Big(\int_0^1 B_{j-1}^n(x)dx\Big) - B_j^{n+1}(x)\Big|_0^1 \\ \Big(\int_0^1 B_j^n(x)dx\Big) &= \Big(\int_0^1 B_{j-1}^n(x)dx\Big) - \frac{1}{(n+1)}B_j^{n+1}(x)\Big|_0^1 \\ \Big(\int_0^1 B_j^n(x)dx\Big) &= \Big(\int_0^1 B_{j-1}^n(x)dx\Big) \\ \implies \Big(\int_0^1 B_j^n(x)dx\Big) &= \Big(\int_0^1 B_{j-1}^n(x)dx\Big) = \Big(\int_0^1 B_0^n(x)dx\Big) = \frac{1}{n+1} \end{aligned}$$

$$\implies I = [\frac{1}{n+1}, \cdots, \frac{1}{n+1}]$$

2.31 (i)

Let $L = \mathbf{B}_n$. We know the Bernstein basis of $n = 2$ to be as follows

$$\{(1-x)^2, 2(1-x)x, x^2\}$$

Now, by the definition given in the problem, let

$$\begin{aligned}
\mathbf{B}((1-x)^2) &= f(0)B_0^1(x) + f(1)B_1^1(x) \\
&= 1B_0^1(x) + 0B_1^1(x) \\
\mathbf{B}(2(1-x)x) &= f(0)B_0^1(x) + f(1)B_1^1(x) \\
&= 0B_0^1(x) + 0B_1^1(x) \\
\mathbf{B}(x^2) &= f(0)B_0^1(x) + f(1)B_1^1(x) \\
&= 0B_0^1(x) + 1B_1^1(x)
\end{aligned}$$

$$\mathbf{B_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.31 (ii)

Given the basis $\{\sin(x), \cos(x)\}$,

$$\begin{aligned}
\implies & \text{ for } n = 0 \\
\mathbf{B}_0(f(x)) &= [0, 0] \\
& \text{for } n = 1 \\
\mathbf{B}_1(f(x)) &= f(0)B_0^1(x) + f(1)B_1^1(x) \\
\mathbf{B}_1(\sin(x)) &= 0B_0^1(x) + \sin(1)B_1^1(x) \\
\mathbf{B}_1(\cos(x)) &= 1B_0^1(x) + \cos(1)B_1^1(x) \\
&\implies \begin{bmatrix} 0 & 1 \\ \sin(1) & \cos(1) \end{bmatrix} \\
& \text{for } n = 2 \\
\mathbf{B}_2(f(x)) &= f(0)B_0^1(x) + f(\frac{1}{2})B_1^1(x) + f(1)B_2^2(x) \\
\mathbf{B}_2(\sin(x)) &= 0B_0^2(x) + \sin(\frac{1}{2})B_1^2(x) + \sin(1)B_2^2(x) \\
\mathbf{B}_2(\cos(x)) &= 1B_0^2(x) + \cos(\frac{1}{2})B_1^2(x) + \cos(1)B_2^2(x) \\
&\implies \begin{bmatrix} 0 & 1 \\ \sin(\frac{1}{2}) & \cos(\frac{1}{2}) \\ \sin(1) & \cos(1) \end{bmatrix} \\
& \vdots
\end{aligned}$$

$$\begin{bmatrix} 0 & 1 \\ \sin(\frac{1}{n}) & \cos(\frac{1}{n}) \\ \vdots & \vdots \\ \sin(\frac{n}{n}) & \cos(\frac{n}{n}) \end{bmatrix}$$