Math 344 Homework

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6.6

Given f, note that $|x| \leq (x^2+y^2)^{\frac{1}{2}}$, and $|y| \leq (x^2+y^2)^{\frac{1}{2}}$. Now, given $\epsilon > 0 \quad |x^2+y^2| < \delta$

$$\implies |f(x,y) - f(0,0)| = \frac{xy^2}{X^2+y^2} \leq \frac{(x^2+y^2)^{3/2}}{x^2+y^2} = (x^2+y^2)^{1/2} < \delta = \epsilon$$

Now, differentiating, we get

$$f_y = \frac{2xy(x^2+y^2) - 2xy^3}{(x^2+y^2)^2}$$

Now consider the sequence $\{x_n\}_{n=2}^{\infty} = (1/n, 1/n)$. Plugging this into f_y, we have

$$f_y = \frac{2xy(x^2+y^2) - 2xy^3}{(x^2+y^2)^2} = \frac{\frac{4}{n^4} - \frac{2}{n^4}}{\frac{4}{n^4}} = \frac{1}{2}$$

Next, consider the sequence $\{y_n\}_{n=2}^{\infty} = (1/n, -1/n)$, and we have

$$= \frac{\frac{-4}{n^4} + \frac{2}{n^4}}{\frac{4}{n^4}} = \frac{-1}{2}$$

implying that the limit is not the same coming from both sides, implying that f is not differentiable at $(0,0)$, the desired result.

6.7

Directional derivatives are as follows:

$$D_1 f(0,0) = \lim_{h \to 0} \frac{f(h,0) - 0}{h} = \frac{0}{h^2} = 0$$

$$D_2 f(0,0) = \lim_{h \to 0} \frac{f(0,h) - 0}{h} = \frac{0}{h} = 0$$

Therefore, the partial derivatives exist. Converting x and y into polar coordinates, where

$$x = r\cos(\theta) \quad y = r\sin(\theta)$$

yielding

$$\begin{aligned}
\lim_{r\to 0} \frac{r^2\cos(\theta)\sin(\theta)}{r^2\cos^2(\theta)+r\sin(\theta)} &= \lim_{r\to 0} \frac{r\cos(\theta)\sin(\theta)}{r\cos^2(\theta)+\sin(\theta)} \\
&= \lim_{r\to 0} \frac{\cos(\theta)\sin(\theta)}{\cos^2(\theta)} \\
&= \lim_{r\to 0} \frac{\sin(\theta)}{\cos(\theta)} \\
&\neq 0
\end{aligned}$$

Consider the sequence $\{x_n\}_{i=n}^{\infty} = (1/n, 1/n)$ which yields

$$\lim_{n\to 0} \frac{1}{n} = 0$$

and the derivatives do not converge to same value, so we have that f is not differentiable.

6.8

Directional derivatives are as follows:

$$D_1 f(x,y) = \lim_{h\to 0} \frac{0\cdot h}{0^2+h^2} = \lim_{h\to 0} \frac{0}{h^2} = 0$$

$$D_2 f(x,y) = \lim_{h\to 0} \frac{h\cdot o}{h^2+0^2} = \lim_{h\to 0} \frac{0}{h^2} = 0$$

Therefore, partial derivatives exist, and if f is differentiable, total derivative is 0.

Now, if f is differentiable, we have the following:

$$\begin{aligned}
\lim_{h\to 0} \frac{\|f(0+h)-f(0,0)-0\|}{\|h\|} &= 0 \\
&= \lim_{h\to 0} \frac{1}{\sqrt{h_x^2+h_y^2}} \cdot \frac{\|h_x\|\|h_y\|}{\sqrt{h_x^2+h_y^2}} \\
&= \lim_{h\to 0} \frac{\|h_x\|\|h_y\|}{h_x^2+h_y^2}
\end{aligned}$$

Consider the sequence $\{(\frac{1}{n}, \frac{1}{n})\}_{n=1}^{\infty}$. The limit $\frac{1/n^2}{2/n^2} = \frac{1}{2} \neq 0$ yielding a contradiction and we have the desired result.

6.9

Given

$$Df(0,0)(h) = 0\cdot h = 0 \;\forall\; h$$

Let $h = (x, y)$. Given $\varepsilon > 0$ let $\delta = \varepsilon$. Then $\|h - (0,0)\| = (x^2 + y^2)^{\frac{1}{2}} < \delta$. and we have the result.

$$\begin{aligned}
\lim_{h \to 0} \frac{\|f(h+0) + f(0,0) + Df(0,0)(h\|}{\|h\|} &= \lim_{h \to 0} \frac{|(x^2+y^2) sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)}{\sqrt{x^2+y^2}} \\
&\leq \frac{x^2+y^2}{\sqrt{x^2+y^2}} \\
&< \varepsilon
\end{aligned}$$

And we have that f is differentiable at $(0,0)$.

For $D_1 f(x, y)$ and $\varepsilon_1, \varepsilon_2 > 0$,

$$\begin{aligned}
&\frac{1}{h} \cdot f(h + \varepsilon_1, \varepsilon_2) - f(\varepsilon_1, \varepsilon_2) \\
&\quad = \frac{1}{h} \cdot \left(((h+\varepsilon_1)^2 + \varepsilon_2^2) \cdot \sin\left(\frac{1}{\sqrt{(h+\varepsilon_1)^2 + \varepsilon_2^2}}\right) - (\varepsilon_1^2 + \varepsilon_2^2) \cdot \sin\left(\frac{1}{\sqrt{\varepsilon_1^2 + \varepsilon_2^2}}\right) \right) \\
&\quad \leq \frac{1}{h} \cdot \left((h+\varepsilon_1)^2 + \varepsilon_2^2 - \varepsilon_1^2 - \varepsilon_2^2\right) \\
&\quad = \frac{1}{h}(h^2 + 2\varepsilon_1) = h + 2\varepsilon_1 < M_1
\end{aligned}$$

For $D_2 f(x, y)$ and $\varepsilon_1, \varepsilon_2 > 0$,

$$\begin{aligned}
&\frac{1}{h} \cdot f(\varepsilon_1, h + \varepsilon_2) - f(\varepsilon_1, \varepsilon_2) \\
&\quad = \frac{1}{h} \cdot \left((\varepsilon_1^2 + (h+\varepsilon_2)^2) \cdot \sin\left(\frac{1}{\sqrt{(\varepsilon_1^2 + (h+\varepsilon_2)^2}}\right) - (\varepsilon_1^2 + \varepsilon_2^2) \cdot \sin\left(\frac{1}{\sqrt{\varepsilon_1^2 + \varepsilon_2^2}}\right) \right) \\
&\quad \leq \frac{1}{h} \cdot \left(\varepsilon_1^2 + (h+\varepsilon_2)^2 - \varepsilon_1^2 - \varepsilon_2^2\right) \\
&\quad = \frac{1}{h}(h^2 + 2\varepsilon_2) = h + 2\varepsilon_2 < M_2
\end{aligned}$$

So the partial derivatives are bounded at $(0,0)$. Discontinuous as f doesn't converge to zero for all $\varepsilon < 0$, while $f(0,0) = 0$.

6.10

Given

$$f(x) = (f_1(x), ..., f_n(x))$$

Since each derivative exists, given that $\epsilon/n > 0$,$\exists\delta$

$$\implies \lim_{n \to 0} \frac{\|f_i(x+h) - f_i(x) - Df_i(x)h\|_{y_i}}{\|h\|_x} < \epsilon_i$$

when $\|h\|_x < \delta_i$. Now we let $\delta = min\{\delta_i\}_{i=1}^n$. Note that

$$\epsilon > n{\cdot}\sup_i \frac{\|f_i(x+h) - f_i(x) - Df_i(x)h\|_{y_i}}{\|h\|_x} \geq \frac{\|\vec{f}(x+h) - \vec{f}(x) - D(f_1(x)h, ..., Df_n(x)h)\|_y}{\|h\|_x}$$

Therefore the total derivative exists and is

$$D(f_1(x)h, ..., Df_n(x)h)$$

6.11

Note that

$$\begin{aligned}
\lim_{h\to\infty} \frac{\|E(t+h) - E(t) - Ae^{Ath}\|_x}{\|h\|} &= \lim_{h\to\infty} \frac{\|E(t+h) - E(t) - Ae^{Ath}\|_x}{\|h\|} \\
&= \lim_{h\to\infty} \frac{\|e^{At}e^{Ah} - e^{At} - Ae^{Ath}\|_x}{\|h\|} \\
&= \|e^{At}\| \lim_{h\to\infty} \frac{\|e^{Ah} - I - Ae^{Ah}\|_x}{\|h\|} \\
&\text{Since } \|e^{At}\| \text{ is bounded by a constant } M_1\text{, we have} \\
&\leq M_1 \lim_{h\to\infty} \frac{\|e^{Ah} - I - Ae^{Ah}\|_x}{\|h\|} \\
&\text{and by example 5.1.19,} \\
&\leq M_1 \lim_{h\to\infty} \frac{\sum_{k=0}^\infty \frac{\|Ah\|^k}{k!} + \|I\| + A\sum_{k=0}^\infty \frac{\|h\|^k}{k!}}{\|h\|} \\
&= M_1 \lim_{h\to\infty} \frac{\sum_{k=0}^\infty \frac{\|h\|\|A\|^k}{k!} + \|I\| + A\sum_{k=0}^\infty \frac{\|h\|\|1\|^k}{k!}}{\|h\|} \\
&= M_1 \lim_{h\to\infty} \sum_{k=0}^\infty \frac{\|A\|^k}{k!} + \|I\| + A\sum_{k=0}^\infty \frac{\|1\|^k}{k!} \\
&\text{and since each of these are bounded above,} \\
&\text{say by a constant } M_2\text{, we have} \\
&\leq M_1M_2
\end{aligned}$$