Math 320 Homework 2.1

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2.1

Since Abe said we don't actually have to list out the subgraphs, as the question indicates, I will show how to derive the number of subgraphs. Let $G_i = (V_i, E_i)$ be a sequence of subgraphs of G for $i = 1, 2, \cdots, 48$, as we know the following:
For subgraphs with one vertex, there will be 4 subgraphs, as there are no edges between a vertex and itself.
For subgraphs with two vertices, there will be 10 subgraphs, since between a and d, there is a subgraph with no edge but that includes both vertices and a subgraph with an edge that includes both vertices. On the other hand, there exists a subgraph between b and d, there is only one subgraph since there is no edge in G between b and d.
For subgraphs with three vertices, there will be 18 total subgraphs.
For subgraphs with four vertices, there will be 16 total subgraphs.
Therefore, we have that there are

$$4 + 10 + 16 + 18 = 48$$

total subgraphs.

2.2

For undirected graphs, we know there are $\binom{7}{2} = 21$ edges between the seven vertices. Therefore, the number of distinct graphs is given by $\binom{21}{13} = 203490$.

For directed graphs, we know there are $2\binom{7}{2} = 42$ edges between the seven vertices, (since now there are simply twice as many). Therefore, the number of distinct graphs is given by $\binom{42}{13} = 25518731280$.

2.3

Let A be our adjacency matrix. Then we have that

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 0 & 0 & 0 & 3 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 & 1 \\ 4 & 1 & 1 & 0 & 2 & 2 \\ 0 & 3 & 3 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 & 0 & 1 \end{bmatrix}$$

$\implies$ by Proposition 2.1.13, we have that there are 3 length-4 paths from node 1 to 4.

2.4

Let A be our adjacency matrix. Then we have that

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2 & 5 & 6 & 0 & 3 & 3 \\ 5 & 4 & 7 & 0 & 7 & 3 \\ 6 & 7 & 6 & 0 & 7 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 7 & 7 & 0 & 4 & 5 \\ 3 & 3 & 6 & 0 & 5 & 2 \end{bmatrix}$$

$\implies$ by Proposition 2.1.13, we have that there are 6 length-3 paths from node 3 to 3.

2.5

As our base case, for $k = 1$, A^1 is clearly the number of length 1 paths from i to j. Suppose, as our inductive hypothesis, then, that A^{k-1} is the number of length $k - 1$

paths from i to j. Note that

$$AA^{k-1} = \sum_{b=1}^{n} A_{ib} A_{bj}^{k-1}$$

Note that A_{ib} will either be 0 or A_{bj}^{k-1} for each b, j since A_{ib} is equal to either 0 or 1. Thus $\sum_{b=1}^{n} A_{ib} A_{bj}^{k-1}$ is the sum of the number of length $k-1$ paths that ends at a node that is connected to i. If we extend the path length by one, each of these paths will end at i. Therefore, we have that $\sum_{b=1}^{n} A_{ib} A_{bj}^{k-1}$ is the number of length k paths connecting i and j, and we have the desired result.

2.6

It is enough to show that $1 \implies 2 \implies 3 \implies 4 \implies 1$.

$1 \implies 2$

Let G be a tree. By definition of a tree, there is a unique path between v_i and v_j

$2 \implies 3$

Let $(v_1, v_2, \ldots, v_j)$ be the unique path from v_i to v_j. Now remove v_n from V where $1 < n < j$. Note v_i is no longer connected to v_j. Since the two nodes are arbitrary, removing any edge makes a disconnected graph.

$3 \implies 4$

Suppose G has a cycle. By removing one of the edges of a cycle, G is still connected, which is a contradiction. As G is connected, there exists a path $(v_1, v_2, \ldots, v_j)$ between v_i and v_j of path length 2 or greater, we add an edge between v_i and v_j, which yields a cycle from v_i to v_j through our path, and v_j to v_i through our edge.

$4 \implies 1$

It is sufficient to show here that G is connected and undirected. That G is undirected is assumed. Note that upon placing an edge between v_i and v_j, we get a cycle $(v_i, v_1, \ldots, v_j, v_i)$. We have then, that there exists a path from v_1 to v_j, namely $(v_i, v_1, \ldots, v_j)$. Since v_i, v_j are arbitrary, we have that G is connected, implying that it is a tree.