Math 344 Homework 3.8

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3.44 (i)

We know that $Ax \in \mathscr{R}(A)$ by definition. Moreover, we know that

$$A^H Ax = 0 \implies Ax \in \mathscr{N}(A^H)$$

3.44 (ii)

We want to show that $x \in \mathscr{N}(A^H A)$ iff $\mathscr{N}(A)$.

$(\rightarrow)$ We know that $Ax \in \mathscr{R}(A)$ and $Ax \in \mathscr{N}(A^H)$. Since $\mathscr{R}(A) \cap \mathscr{R}(A) = 0$ by FST(Fundamental Subspaces Theorem), we know that $Ax = 0$ since it is in both.

$(\leftarrow)$

$$Ax = 0 \implies A^H Ax = 0$$

Which gives us the desired result.

3.44 (iii)

First, $\mathscr{N}(A^H A) = \mathscr{N}(A)$, implying that their dimensions are equal to some constant a. By rank nullity Theorem, we know that

$$\dim(\mathscr{R}(A^H A)) + \dim(\mathscr{N}(A^H A)) = n$$

and that

$$\dim(\mathscr{R}(A)) + \dim(\mathscr{N}(A)) = n$$

which yields

$$\dim(\mathscr{R}(A^H A)) + a = n$$
$$\dim(\mathscr{R}(A)) + a = n$$
$$\implies \dim(\mathscr{R}(A^H A)) = n - a = \dim(\mathscr{R}(A))$$
$$\implies \dim(\mathscr{R}(A^H A)) = \dim(\mathscr{R}(A))$$

Which is the desired result.

3.44 (iv)

We know that A being linearly independent implies that $\text{rank}(A) = n$. Since, by (iii),

$$n = \text{rank}(A) = \text{rank}(A^H A)$$

and $A^H A$ is $n \times n$, it must have rank n.

3.45 (i)

$$\begin{aligned} P^2 &= A(A^H A)^{-1} A^H A (A^H A)^{-1} A^H \\ &= A(A^H A)^{-1} I A^H = P \end{aligned}$$

3.45 (ii)

$$P^H = (A(A^H A)^{-1} A^H)^H = (A((A^H A)^{-1})^H A^H)^H = (A((A^H A)^H)^{-1} A^H) = (A(A^H A)^{-1} A^H) = P$$

3.45 (iii)

TODO

TODO

3.46

Because Q is orthonormal, we know that $Q^{-1} = Q^H$, and we have that

$$\begin{aligned} (QR)^H QR\hat{x} &= (QR)^H b \\ R^H Q^H QR\hat{x} &= (QR)^H b \\ R^H R\hat{x} &= R^H Q^H b \end{aligned}$$

Which holds iff $R\hat{x} = Q^H b$.

3.47

$$\begin{aligned} \begin{bmatrix} A & I \\ 0 & A^H \end{bmatrix} \begin{bmatrix} \hat{x} \\ r \end{bmatrix} &= \begin{bmatrix} b \\ 0 \end{bmatrix} \\ \begin{bmatrix} A\hat{x} + r \\ A^H r \end{bmatrix} &= \begin{bmatrix} b \\ 0 \end{bmatrix} \\ \begin{bmatrix} \text{proj}\mathscr{R}(A)(b) + b - \text{proj}\mathscr{R}(A)(b) \\ A^H r \end{bmatrix} &= \begin{bmatrix} b \\ 0 \end{bmatrix} \\ \begin{bmatrix} b \\ A^H r \end{bmatrix} &= \begin{bmatrix} b \\ 0 \end{bmatrix} \end{aligned}$$

Now, it is clear that $b = b$. As for the second equation, that $A^H r = 0$, we know that $r \in \mathscr{R}(A)^\perp = \mathscr{N}(A^H)$, meaning that $A^H r = 0$, as desired.

3.48

$$A^H Ax = \begin{bmatrix} \overline{x}_1^2 & \overline{x}_2^2 & \cdots & \overline{x}_n^2 \\ \overline{y}_1^2 & \overline{y}_2^2 & \cdots & \overline{y}_n^2 \end{bmatrix} \begin{bmatrix} x_1^2 & y_1^2 \\ x_2^2 & y_2^2 \\ \vdots & \vdots \\ x_n^2 & y_n^2 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \overline{x}_1^2 & \overline{x}_2^2 & \cdots & \overline{x}_n^2 \\ \overline{y}_1^2 & \overline{y}_2^2 & \cdots & \overline{y}_n^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = A^H x$$

3.49 (i)

Note that

$$P(\alpha A+\beta B) = \frac{(\alpha A+\beta B)+(\alpha A+\beta B)^2}{2} = \frac{\alpha(A+(A)^T)}{2}+\frac{\beta(A+(A)^T)}{2} = \alpha P(A)+\beta P(B)$$

Therefore, $P(A)$ is linear.

3.49 (ii)

$$P(A)^2 = P(P(A)) = \frac{\frac{A+A^T}{2}+\frac{(A+A^T)^T}{2}}{2} = \frac{\frac{A+A^T}{2}+\frac{A+A^T}{2}}{2} = \frac{A+A^T}{2} = P(A)$$

3.49 (iii)

By 3.41, we know that $A^* = A^H$ under the Frobenius inner product, and since we are in the reals, $A^* = A^H = A^T$. It suffices to show, then, that $P^T = P$. Well,

$$P(A)^T = \frac{(A+A^T)^T}{2} = \frac{(A^T+(A^T)^T)}{2} = \frac{A+A^T}{2} = P(A)$$

3.49 (iv)

Assume $A \in \text{Skew}_n(\mathbb{R})$. Then $A^T = -A$, and we have that

$$P(A) = \frac{A^T+A}{2} = 0$$

Then A is in the null space of A.

Now assume that A is in the null space of P. Then

$$P(A) = \frac{A^T+A}{2} = 0 \implies A^T = -A$$

And we have containment both in both directions, implying the equality at hand.

3.49 (v)

Assume $A \in \text{Sym}_n(\mathbb{R})$. Then $A^T = A$, and we have that

$$P(A) = \frac{A^T+A}{2} = A$$

which is in the range space of P since $P(A) = A$.

Now assume that A is in the range space of P. Then A is given by

$$P(B) = \frac{B^T+B}{2} = A$$

Now, we know that $A_{ij} = \frac{B_{ij}+B_{ji}}{2} = A_{ji}$, implying that A is symmetric, which is the desired result.

3.49 (vi)

$$
\begin{aligned}
||A - P(A)||_F &= ||\frac{2A - A - A^T}{2}||_F \\
&= ||\frac{A - A^T}{2}||_F \\
&= \sqrt{\operatorname{tr}\left(\frac{(A^T - A)}{2}\frac{(A - A^T)}{2}\right)} \\
&= \sqrt{\left(\frac{(\operatorname{tr}(A^TA) - \operatorname{tr}(A^2) - \operatorname{tr}((A^T)^2) + \operatorname{tr}(AA^T))}{4}\right)} \\
&= \sqrt{\left(\frac{(\operatorname{tr}(A^TA) - \operatorname{tr}(A^2) - \operatorname{tr}(A^2) + \operatorname{tr}(A^TA))}{4}\right)} \\
&= \sqrt{\left(\frac{(2\operatorname{tr}(A^TA) - 2\operatorname{tr}(A^2)}{4}\right)} \\
&= \sqrt{\left(\frac{(\operatorname{tr}(A^TA) - \operatorname{tr}(A^2)}{2}\right)}
\end{aligned}
$$