Math 344 Homework 5.2

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5.10

Consider a point $x \in (\overline{E})^c$. As $(\overline{E}^c)$ is open, x is an interior point of $(\overline{E})^c$, and $(\overline{E})^c \subset E^c$, thus x is an interior point of E^c.

$$\implies x \in (E^c)^\circ$$

Now consider a point $x \in (E^c)^\circ$. Thus x is an interior point of (E^c) and we have that there exists a $B(x, \delta)$ such that

$$B \subset E^c, \quad (\overline{E})^c \subset E^c$$

5.11

$(\rightarrow)$ If $\mathbf{x}_0 \in \overline{E}$, then $\inf_{x \in E} d(\mathbf{x}_0, \mathbf{x}) = 0$ since $d(\mathbf{x}_0, \mathbf{x}_0) = 0$ and we know that metrics have to be nonnegative, so we couldn't find $d(\mathbf{x}_0, \mathbf{x}) < 0$, and we have the desired result.

$(\leftarrow)$ If $\inf_{x \in E} d(\mathbf{x}_0, \mathbf{x}) = 0$, then we know by the criteria of metrics that $d(x, y) = 0 iff x = y$. Therefore, we have that $\mathbf{x}_0 = \mathbf{x}$, implying that $\mathbf{x}_0 \in \overline{E}$ since $m \in \overline{E}$

5.12

We know that since T is unbounded, that for any sequence of normalized vectors, that

$$\frac{\|T\mathbf{x}_k\|}{\|\mathbf{x}_k\|} = \|T\mathbf{x}_k\| > k \quad \forall k$$

Now, if we let $z_k = \frac{x_k}{k}$, then we have that

$$\|T\mathbf{z}_k\| = \frac{1}{k}\|T\mathbf{x}_k\| > \frac{k}{k} = 1$$

and we have that at the origin, if $\mathbf{z} \rightarrow 0$, then T is continuous at zero iff $T\mathbf{x}_k \rightarrow T\mathbf{0}$, but we know this isn't true since $T\mathbf{z}_k \rightarrow$ something that is not zero.

This proves discontinuity at zero. We proceed to prove that T is discontinuous at $j \neq 0, j \in \mathbb{R}$. We know that

$$j \to j \text{ as } k \to \infty \quad \lim_{k\to\infty} T(j) = \lim_{k\to\infty} T(j + z_k) = \lim_{k\to\infty} T(j) + \lim_{k\to\infty} T(z_k) \neq T(j)$$

and we have a contradiction, implying that T is not continuous at $\mathbf{0}, \mathbf{j}$, encompassing all of $\mathbb{R}$.

5.13 (i)

Where

$$f(x,y) = \frac{\sqrt{xy}}{x^2 + y^2}$$

Let $y = x$. Therefore, we have that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x}{2x^2} = \infty$$

Hence the limit doesn't exist.

5.13 (ii)

Where

$$f(x,y) = \frac{xy}{x^2 + y^2}$$

Let $y = x$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

Let $y = \frac{1}{x}$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2 + \frac{1}{x^2}} = \lim_{x\to 0} \frac{1}{x^2 + 1} = 1$$

Hence the limit doesn't exist.

5.13 (iii)

Given

$$c = \frac{xy}{\sqrt{x^2 + y^2}}$$

we have

$$\frac{1}{c^2} = \frac{1}{x^2} + \frac{1}{y^2}$$

$$\implies \lim_{(x,y)\to(0,0)} \frac{1}{c^2} = \infty$$

implying

$$\lim_{(x,y)\to(0,0)} c = 0$$

5.14

Consider

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

at (0,0). Under the metric $d(a,b) = |b-a|$ we have

$$\begin{aligned} |x^2 - y^2| &\leq |x^2| + |-y^2| \\ &= |x^2| + |y^2| \\ &= x^2 + y^2 \end{aligned}$$

$$\implies \left| \frac{x^2 - y^2}{\sqrt{x^2+y^2}} \right| \leq \frac{x^2 + y^2}{|\sqrt{x^2+y^2}|} = \sqrt{x^2+y^2}$$

$$\implies |f(x,y) - 0| \leq \sqrt{x^2+y^2}$$

and we have that $f(x,y)$ is continuous at $(0,0)$.

5.15 (i)

Where $x, y \neq 0$,

$$\frac{2at^3}{t^4 + a^2t^2} = \frac{2at}{t^2 + a^2} = 0$$

5.15 (ii)

Where $x, y \neq 0$,

$$\frac{2t^4}{t^4 + t^4} = \frac{2t^4}{2t^4} = 1$$

This implies that the limit doesn't exist.