Math 344 Homework 5.5

Chris Rytting

November 18, 2015

5.30

If we have that
$$(f_n)_{n=0}^{\infty}$$
is converges uniformly, then
$$\exists N \in \mathbb{N} \text{ s.t. } |f_n(x) - f(x)| < \epsilon \quad \forall \mathbf{x} \in X$$
And by Corollary 5.2.35.
$$f(x) = lim_{n\to\infty} f_n(x)$$

5.31 (i)

Let
$$f(x) = \begin{cases} 1 & \text{if } x = \frac{\pi}{2} \\ 0 & \text{else} \end{cases}$$
for $x \in [0, \frac{\pi}{2}]$. We note that $\sin(x) < 1 \quad \forall\ x\ \in [0, \pi] \setminus \{\frac{\pi}{2}\}$
Therefore $\sin^n(x) \to 0$ and $\sin^n(\frac{\pi}{2}) = 1 \quad \forall n$.
Therefore $\sin^n(x) \to \begin{cases} 1 & \text{if } x = \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$

5.31 (iii)

Let
$$\epsilon = \frac{1}{2}, \nexists \delta \text{ s.t. } \forall x \in B(\frac{\pi}{2}, \delta) \implies \|sin^n(x) - f(x)\|_\infty < \frac{1}{2}$$
Since every epsilon potential ball contains the point $\sin^n(\frac{\pi}{2}) = 1$, it is not uniformly convergent.

5.31 (iv)

Sequence not Cauchy.

5.32 (i)

Note, by ratio test, we have

$$\begin{aligned}
\left\| (-1)^{k+1} \frac{A^{2(k+1)}}{(2(k+1))!} \right\| \left\| (-1)^k \frac{A^{2k}}{(2k)!} \right\|^{-1} &= \left\| \frac{A^{2k+2}}{(2k+2)!} \right\| \left\| \frac{A^{2k}}{(2k)!} \right\|^{-1} \\
&= \left\| \frac{A^2}{(2k+2)(2k+1)} \right\|
\end{aligned}$$

which approaches 0 as $k \to \infty$.

5.32 (ii)

Note, by ratio test, we have

$$\begin{aligned}
\left\| (-1)^{k+1} \frac{A^{2(k+1)+1}}{(2(k+1)+1)!} \right\| \left\| (-1)^k \frac{A^{2k+1}}{(2k+1)!} \right\|^{-1} &= \left\| \frac{A^{2k+3}}{(2k+3)!} \right\| \left\| \frac{A^{2k+1}}{(2k+1)!} \right\|^{-1} \\
&= \left\| \frac{A^2}{(2k+3)(2k+2)} \right\|
\end{aligned}$$

which approaches 0 as $k \to \infty$.

5.32 (iii)

Note, by ratio test, we have

$$\begin{aligned}
\left\| (-1)^k \frac{A^{k+1}}{k+1} \right\| \left\| (-1)^{k-1} \frac{A^k}{k} \right\|^{-1} &= \left\| \frac{A^{k+1}}{k+1} \right\| \left\| \frac{A^k}{k} \right\|^{-1} \\
&= \left\| \frac{kA}{k+1} \right\| \\
&= \left\| \frac{k}{k+1} \right\| \|A\| \\
&\text{and as } k \to \infty \\
&= 1 \|A\|
\end{aligned}$$

Since $\|A\| < 1$ as $k \to \infty$ we know that this value is less than one and by the ratio test converges absolutely.

5.33

Absolute value of sum is going to zero, and can can always be expressed as less than ε, however, absolute value of summation diverges.

$$\left(-1+\frac{1}{2}\right)+\left(\frac{-1}{3}+\frac{1}{4}+\frac{1}{6}\right)\dots$$

adding sufficient terms so that each term is greater than $\frac{1}{10}$, justified by the fact that

$$\sum_{n=1}^{\infty}\frac{1}{2n}$$

diverges. Then each term will be greater than $\frac{1}{10}$, and we have that the sum is greater than

$$\left(\frac{1}{10}\right)+\left(\frac{1}{10}\right)+\left(\frac{1}{10}\right)+\left(\frac{1}{10}\right)\dots=\infty$$

5.34

Assume to the contrary that $(I-A)$ is singular. Therefore, $\exists\mathbf{x}\neq 0$ such that

$$(I-A)\mathbf{x}=0$$

so $A\mathbf{x}=\mathbf{x}$. Now, scale $\mathbf{x}$ such that $\|\mathbf{x}\|=1$. Therefore, $\|A\mathbf{x}\|=1$. Thus

$$\sup_{\|x\|=1}\|A\mathbf{x}\|\geq 1$$

which is a contradiction. Now by definition, let

$$\|(I-A)^{-1}\|=\sup_{\|y\|=1}\|(I-A)^{-1}\mathbf{y}\|$$

Now, let $x_y=(I-A)^{-1}\mathbf{y}$ or $y=(I-A)x_y$. Then

$$\sup_{\|y\|=1}\|\mathbf{x}_y\|$$

Now we get

$$\begin{aligned}
1&=\|\mathbf{y}\|\\
&=\|(I-A)\mathbf{x}_y\|\\
&=\|\mathbf{x}_y-A\mathbf{x}_y\|\\
&\geq\|\mathbf{x}_y\|-\|A\mathbf{x}_y\|\\
&\geq\|\mathbf{x}_y\|-\|A\|\|\mathbf{x}_y\|\\
&=(1-\|A\|)(\mathbf{x}_y)
\end{aligned}$$

Therefore, we can say

$$\begin{aligned}
\mathbf{x}_y\leq\frac{1}{1-\|A\|}&=(1-\|A\|)^{-1}\\
\implies\|(I-A)^{-1}\mathbf{y}\|&<(1-\|A\|)^{-1}\\
\implies\|(I-A)^{-1}\|&\leq(1-\|A\|)^{-1}
\end{aligned}$$

which is the desired result.

5.35

We know that $\|A\|^{-1} < M$, and $\|E\| < \frac{1}{\|A\|^{-1}}\|$. Suppose to the contrary

$$\begin{aligned}
\|A+E\|^{-1} &\leq \frac{\|A\|^{-1}}{1-\|E\|\|A^{-1}\|} \\
\frac{\|A+E\|^{-1}}{\|A\|^{-1}} &\leq 1 - \frac{1}{1-\|E\|M} \\
\frac{\|A\|^{-1}}{\|A+E\|^{-1}} &\geq 1 - \frac{1-\|E\|M}{1} \\
\frac{\|A\|^{-1}}{\|A+E\|^{-1}} + M\|E\| &\leq 1 \\
\implies \frac{1}{M} &< \|A+E\| + \frac{1}{\|A^{-1}\|} \\
\frac{1}{M} - \frac{1}{\|A^{-1}\|} &< \|A+E\| \\
\implies \|A+E\| &< 0
\end{aligned}$$

a contradiction.