Problem Set 1

Chapter 5

5.1

The following condition characterizes the optimal amount of cake to eat in period 1:

$$u(w_1 - \psi(w_1)) = u(w_1)$$

For period 2

$$w_{t+1} = \psi(w_1) = 0 \quad \forall w_1$$

5.2

Optimal amount of cake to leave for period 3 from period 2:

$$w_3 = \psi_2(w_2) = 0$$

$$\beta u'(w_2) = u'(w_1 - w_2)$$

5.3

If the individual lives for three periods $T = 3$ what are the conditions that characterize the optimal amount of cake to leave for the next period in each period $\{W_2, W_3, W_4\}$? Now assume that the initial cake size is $W_1 = 1$, the discount factor is $\beta = 0.9$, and the period utility function is $\ln(c_t)$. Show how $\{c_t\}_{t=1}^3$ and $\{W_t\}_{t=1}^4$ evolve over the three periods.

$$\beta u'(w_3) = u'(w_2 - w_3)$$

$$\beta u'(w_2 - \psi_2(w_2)) = u'(w_1 - w_2)$$

$w_1 = 1$
$w_2 = .631$
$w_3 = .299$
$w_4 = 0$
$c_1 = .369$
$c_2 = .332$
$c_3 = .299$

5.4

$$\beta v_T'(w_T) = u'(W_{T-1} - W_T)$$

$$W_T = \psi(W_{T-1}) : u(W_{T-1} - \psi(W_{T-1})) + \beta V_T(\psi_{T-1}(W_{T-1}) = 0$$

5.5

$$V_T = ln(\bar{W}) \neq V_{T-1} = (ln\bar{W} - \psi_{T-1}(\bar{W}))) + \beta ln(\psi_{T-1} barW))$$

$$\begin{gathered} W_{T+1} = \psi_T(\bar{W}) = 0 \\ W_T = \psi_{T-1}(\bar{W}) = ln(\bar{W} - \psi_{T-1}(\bar{W})) + \beta ln(\psi_{T-1}(\bar{W})) \neq 0 \\ \implies \psi_{T-1}(\bar{W}) \neq \psi_T(\bar{W}) \end{gathered}$$

5.6

Solution for the period $T-2$ policy function for how much cake to save for the next period W_{T-1}:

$$W_{T-1} = \beta \frac{(1+\beta)}{(1+\beta+\beta^2)W} T_{T-2}$$

Analytical Solution for V_{T-2}:

$$V_{T-2} = ln(\frac{1}{1+\beta+\beta^2} W_{T-2}) + \beta ln(\frac{\beta}{1+\beta+\beta^2} W_{T-2}) + \beta^2 ln(\frac{\beta^2}{1+\beta+\beta^2} W_{T-2})$$

5.7

Analytical solution for $\psi_{T-s}(W_{T-s})$:

$$\psi_{T-s}(W_{T-s}) = \frac{\sum_{i=0}^{s} \beta^i - 1}{\sum_{i=0}^{s} \beta^i}$$

Analytical solution for $V_{T-s}(W_{T-s})$:

$$V_{T-s}(W_{T-s}) = \sum_{i=0}^{s} \beta^i ln(\frac{\beta^i}{\sum_{j=0}^{s} \beta^j} W_{T-s})$$

Proofs:

$$\lim_{s\to\infty} \psi_{T-s}(W_{T-s}) = \lim_{s\to\infty} \frac{\frac{1}{1-\beta} - 1}{\frac{1}{1-\beta}} W_{T-s} = \lim_{s\to\infty} \frac{\frac{\beta}{1-\beta}}{\frac{1}{1-\beta}} W_{T-s} = \beta(W_{T-s}) = \psi(W_{T-s})$$

And as $s \to \infty$:

$$V_{T-s}(W_{T-s}) = \sum_{i=0}^{s} \beta^i ln(\frac{\beta^i}{\sum_{j=0}^{s} \beta^j} W_{T-s}) = \sum_{i=0}^{\infty} \beta^i ln(\frac{\beta^i}{\frac{1}{1-\beta}} W_{T-s}) - \sum_{i=0}^{\infty} \beta^i ln(\beta^i(1-\beta)W_{T-s})$$

$$= \lim_{s \to \infty} \sum_{i=0}^{s} i\beta^i ln(\beta) + \frac{ln((1-\beta)W_{T-s})}{1-\beta} = ln(\beta) \lim_{s \to \infty} \sum_{i=0}^{s} i\beta^i + \frac{ln((1-\beta)W_{T-s})}{1-\beta}$$

$\implies$ $V_{T-s}(W_{T-s})$ converges to a function $V(W_{T-s})$, which is independent of time.

5.8

Bellman equation with infinite horizon:

$$V(W) = \max_{W' \in [0,W]} u(W - W') + \beta V(W')$$

5.11

$$\delta_T = 709.1159$$

5.12

$$\delta_{T-1} = 855.9830$$

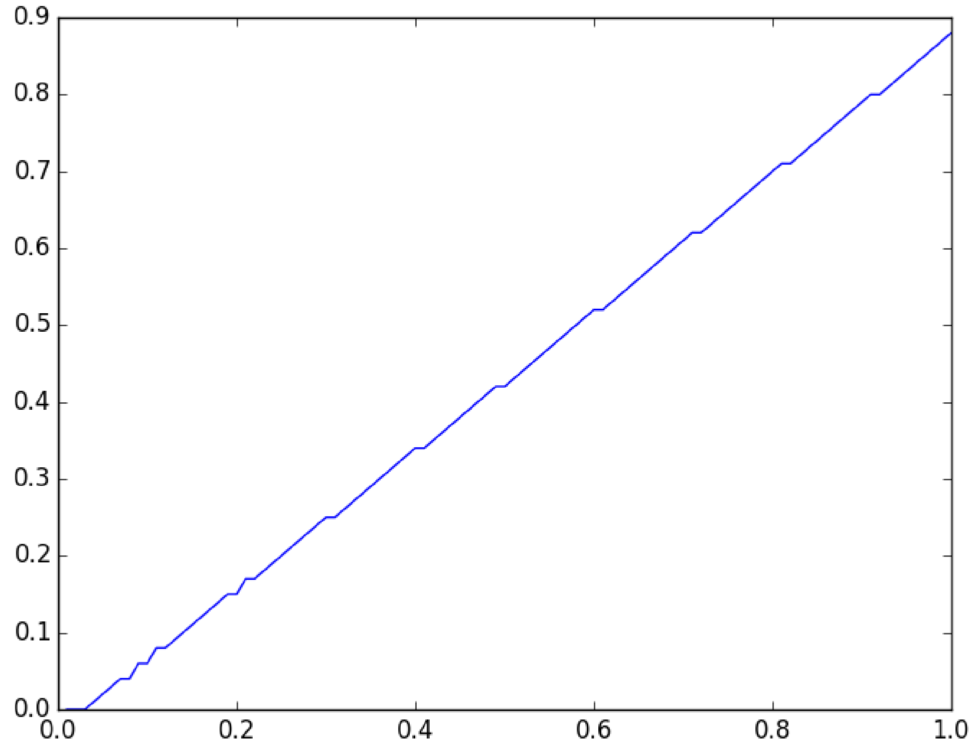
δ_{T-1} is higher than δ_T

5.13

$$\delta_{T-2} = 838.5298$$

δ_{T-2} is higher than δ_T, but lower than δ_{T-1}.

5.15



5.18

$$\delta_T = 7,006,262.4138$$

5.19

$$\delta_{T-1} = 5,684,014.4287$$

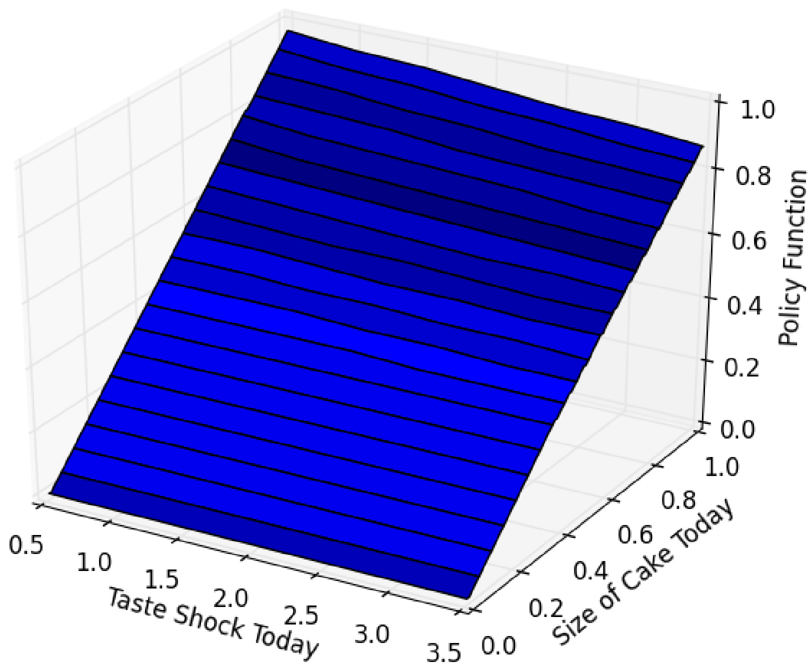
δ_{T-1} is lower than δ_T

5.20

$$\delta_{T-2} = 4,609,092.4524$$

δ_{T-2} is lower than both δ_{T-1} and δ_T

5.22



5.25

$$\delta_T = 24,819.0573$$

5.26

$$\delta_{T-1} = 85,327.4341$$

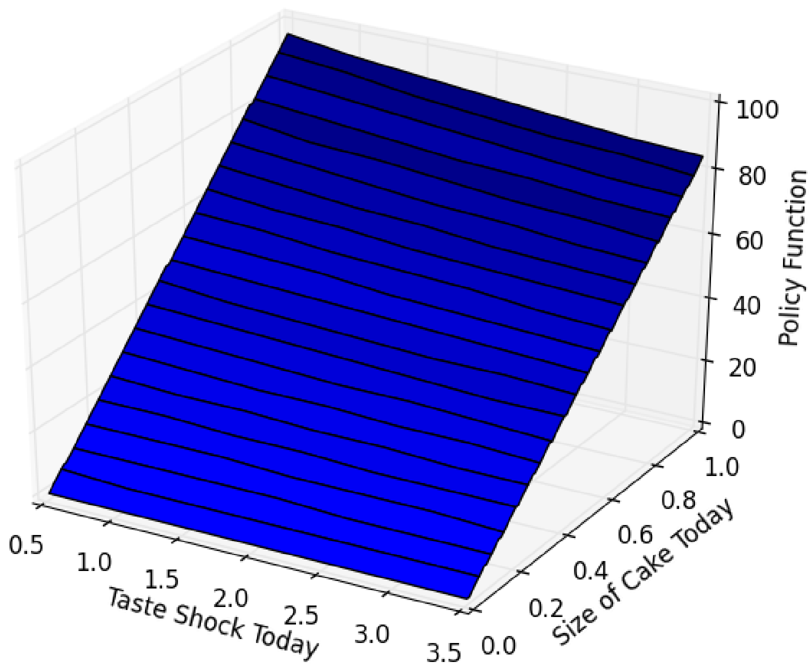
δ_{T-1} is higher than δ_T

5.27

$$\delta_{T-2} = 71,280.8333$$

δ_{T-2} is lower than δ_{T-1} but not δ_T

5.29



Chapter 4

4.1

$$\bar{k2} = .028$$

$$\bar{k3} = .091$$

$$\bar{c1} = .214$$

$$\bar{c2} = .223$$

$$\bar{c3} = .232$$

$$\bar{w} = .242$$

$$\bar{r} = 2.192$$

4.2

$$\bar{k}2 = .041$$

$$\bar{k}3 = .117$$

$$\bar{c}1 = .226$$

$$\bar{c}2 = .240$$

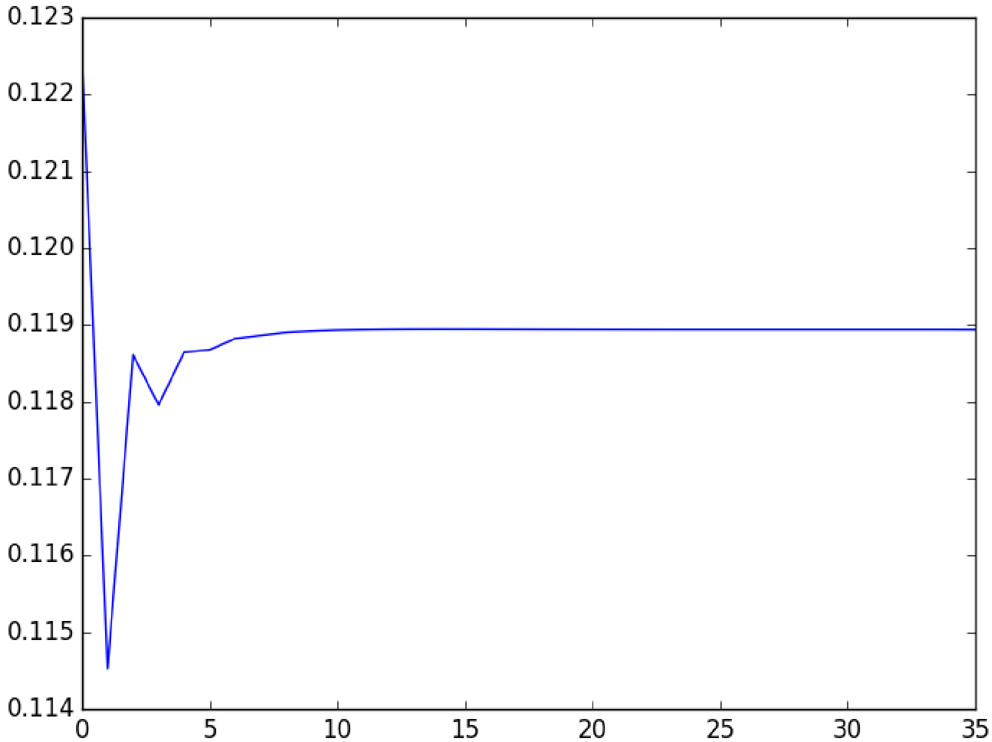
$$\bar{c}3 = .255$$

$$\bar{w} = .268$$

$$\bar{r} = 1.818$$

Consumption increases, kbar increases. This is due to the increase in patience. People will save more and therefore in the long-run will be able to consume more. The only variable which decreases is the interest rate, which also makes sense be cause more people are saving more.

4.4



$T = 7$ before within .0001 of steady state.