10.1

Note that the gradient of g is as follows:

$$Dg(x, y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

Note, further that only at $(0, 0)$ do we have that

$$Dg(x, y) = \mathbf{0}$$

As this point is infeasible, no point on the circle is singular so all must be feasible. The parametrization is as follows:

$$g(\theta) = (cos(\theta), sin(\theta)) \qquad \theta \in (0, 2\pi)$$

10.2

Note that the gradient of g is given by

$$Dg(\mathbf{p}) = \begin{bmatrix} 2(c - \sqrt{x^2 + y^2})(-x)(x^2 + y^2)^{-1/2} \\ 2(c - \sqrt{x^2 + y^2})(-y)(x^2 + y^2)^{-1/2} \\ 2z \end{bmatrix}$$

if $z \neq 0$, then $Dg(\mathbf{p}) \neq \mathbf{0}$. On the other hand, it could be that $z = 0$. Note that

$$\mathbf{p} = [x, y, z] \in T^2 \implies x^2 + y^2 \neq 0 \implies x, y \neq 0$$

It only remains to show that

$$c - \sqrt{x^2 + y^2} \neq 0$$

Assume to the contrary that

$$c - \sqrt{x^2 + y^2} = 0 \implies g(x, y, z) = (c - \sqrt{x^2 + y^2})^2 + z^2 - a^2 = z^2 - a^2 = 0 \implies z = \pm a \neq 0$$

since $a \neq 0$, which is a contradiction, as $z = 0$.

10.3 (i)

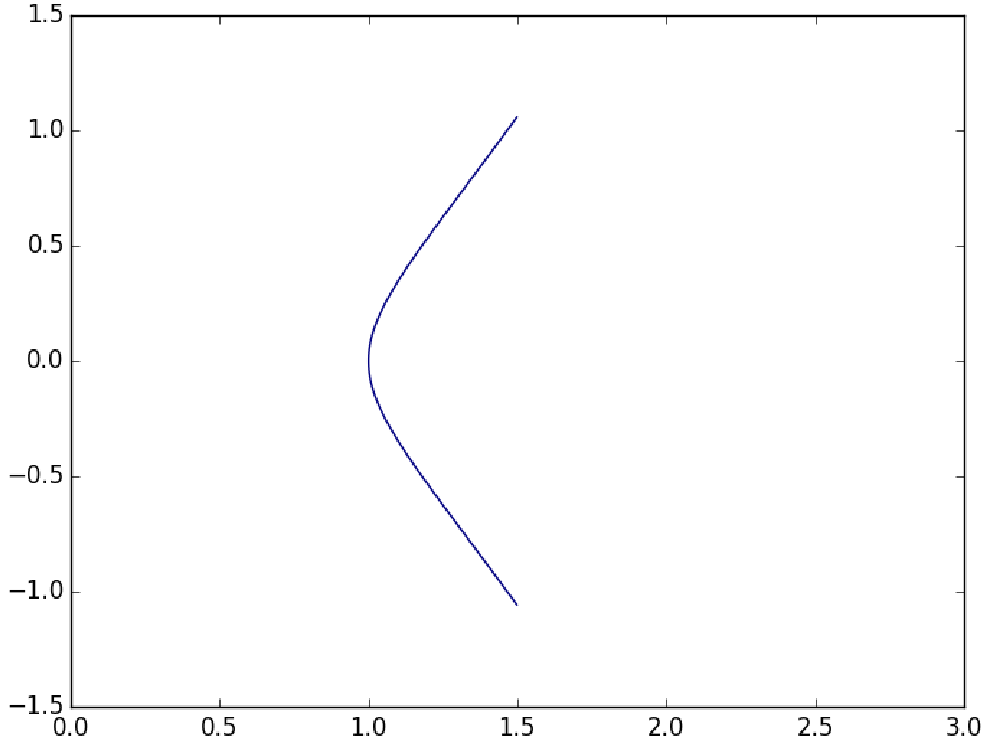
To find singular points, in general, we want to find the points where

$$Dg(x) = mathbb0$$

$$Dg(x, y) = \begin{bmatrix} 3x^2 - 2x \\ -2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies x = 0, \frac{2}{3}, y = 0$$

$$\text{singular points} : (0,0), (\frac{2}{3}, 0)$$



Since both entries of $Dg(x,y) = 0$ at the singular points, we have that

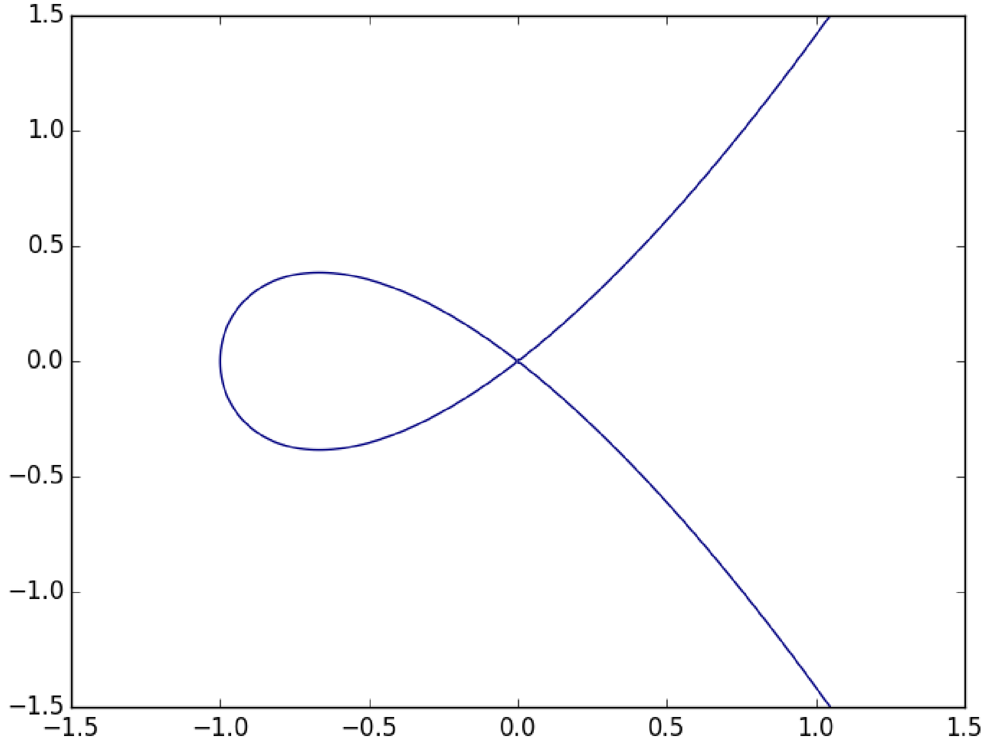
$$N_{\mathbf{x}}S = \mathbb{R}^2 \qquad N_{\mathbf{x}}S^{\perp} = \{\mathbf{0}\}$$

10.3 (ii)

$$Dg(x,y) = \begin{bmatrix} 3x^2 + 2x \\ -2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies x = 0, -\frac{2}{3}, y = 0$$

$$\text{singular points} : (0,0), (-\frac{2}{3}, 0)$$

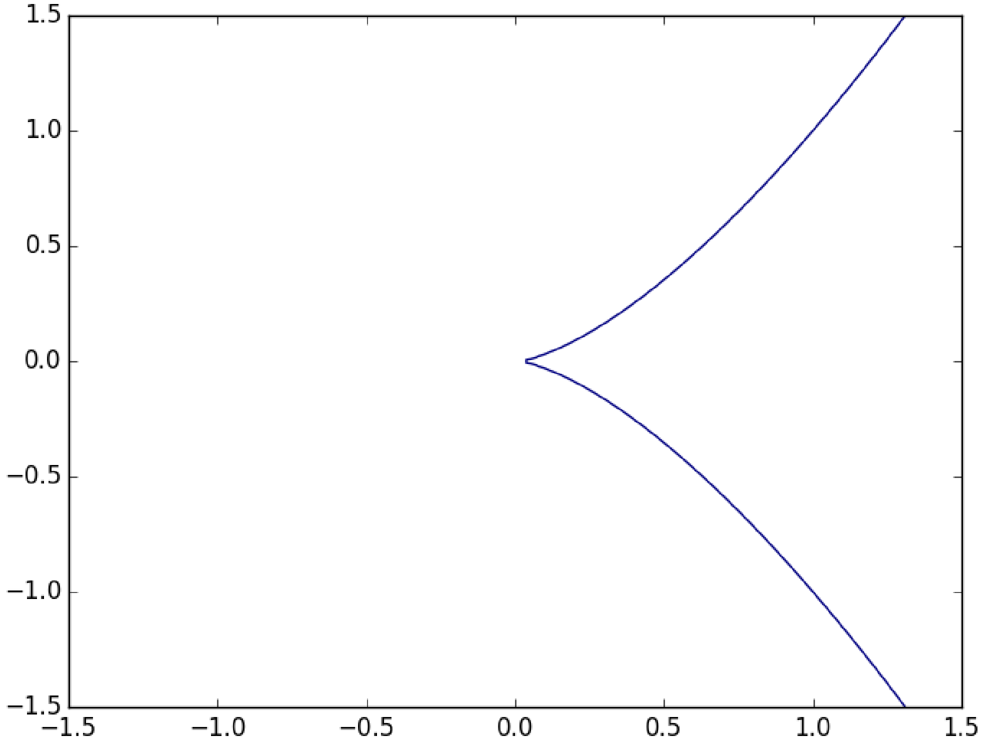


Since both entries of $Dg(x, y) = 0$ at the singular points, we have that

$$N_{\mathbf{x}}S = \mathbb{R}^2 \qquad N_{\mathbf{x}}S^{\perp} = \{\mathbf{0}\}$$

10.3 (iii)

$$Dg(x, y) = \begin{bmatrix} 3x^2 \\ -2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\implies x = 0, y = 0$$
$$\implies \text{Singular point} : (0, 0)$$



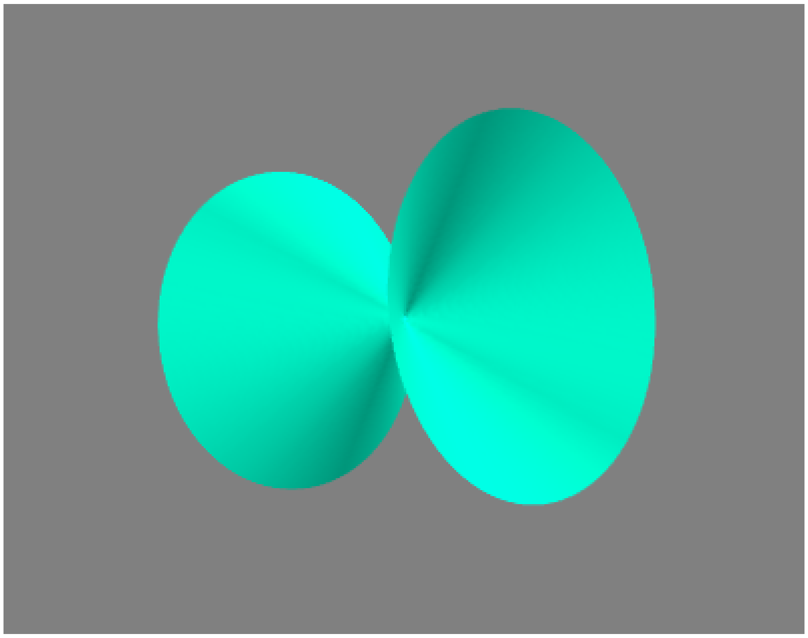
Since both entries of $Dg(x, y) = 0$ at the singular points, we have that

$$N_{\mathbf{x}}S = \mathbb{R}^2 \qquad N_{\mathbf{x}}S^{\perp} = \{\mathbf{0}\}$$

10.3 (iv)

$$Dg(x, y, z) = \begin{bmatrix} -2x \\ 2y \\ 2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\implies x = 0, y = 0, z = 0$$

$$\implies \text{Singular point} : (0, 0, 0)$$



Since all entries of $Dg(x, y, z) = 0$ at the singular points, we have that

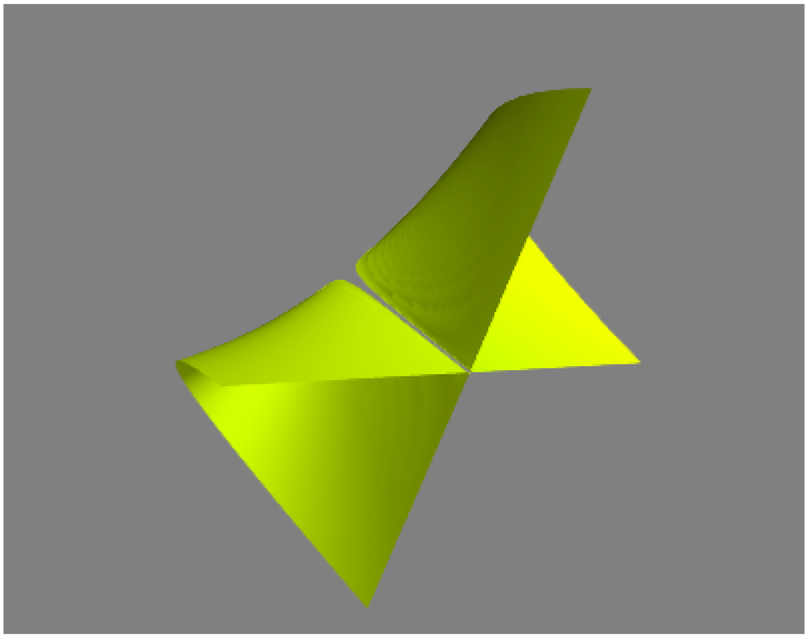
$$N_{\mathbf{x}}S = \mathbb{R}^3 \qquad N_{\mathbf{x}}S^{\perp} = \{\mathbf{0}\}$$

10.3 (v)

$$Dg(x, y, z) = \begin{bmatrix} 2xy \\ x^2 \\ -2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies x = 0, z = 0, y \in \mathbb{R}$$

As y can be any real number, granted that $x = 0, z = 0$, we have an infinite amount of singular points.



Since all entries of $Dg(x, y, z) = 0$ at the singular points, we have that

$$N_{\mathbf{x}}S = \mathbb{R}^3 \qquad N_{\mathbf{x}}S^{\perp} = \{\mathbf{0}\}$$

10.4

From 3.44(ii), we know that $A^H A$ has the same rank as A. So $Rank(A^H A) = m$ and is invertible so $(A^H A)^H = AA^H$ must likewise be invertible.

10.6

Lagrange is as follows:

$$\mathscr{L} = x^2 + 2xy + 3y^2 + 4x + 5y + 6z + \lambda(3 - 2y - x) + \mu(6 - 4x - 5z)$$

Our first-order conditions being as follows:

$$\frac{d\mathscr{L}}{dx} = 2x + 2y + 4 - \lambda - 4\mu = 0$$

$$\frac{d\mathscr{L}}{dy} = 2x + 6y + 5 - 2\lambda = 0$$

$$\frac{d\mathscr{L}}{dz} = 6 - 5\mu = 0$$

$$\frac{d\mathscr{L}}{d\lambda} = 3 - 2y - x = 0$$

$$\frac{d\mathscr{L}}{d\mu} = 6 - 4x - 5z = 0$$

Which yields the system of equations

$$2x - 2y - \frac{33}{5} = 0$$

$$3 - 2y - x = 0$$

$$6 - 4x - 5z = 0$$

And we get that

$$x = \frac{16}{5}$$

$$y = -\frac{1}{10}$$

$$z = -\frac{34}{25}$$

10.8

We will attempt to do find the second order conditions using the following equation:

$$D^2\mathscr{L} = \begin{bmatrix} D^2 f + \lambda D^2 G & DG^T \\ DG & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 & 1 & 4 \\ 2 & 6 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 1 & 2 & 0 & 0 & 0 \\ 4 & 0 & 5 & 0 & 0 \end{bmatrix}$$

The eigenvalues are as follows:

$$\lambda_1 = -6.14 \qquad \lambda_2 = 8.19 \qquad \lambda_3 = 5.85 \qquad \lambda_4 = 0.77 \qquad \lambda_5 = 0.66$$

this would indicate neither positive or negative definites conclusively, so the point must be a saddle point.

10.14

The volume of a box with consisting of sides with lengths x, y, z can be expressed as $V = xyz$. Therefore, the lagrangian can be written as follows:

$$\mathscr{L}(x, y, z, \lambda) = xyz - \lambda(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1)$$

With first-order conditions as follow:

$$\frac{\partial \mathscr{L}}{\partial x} = yz - 2\frac{\lambda}{a^2}x = 0$$

$$\frac{\partial \mathscr{L}}{\partial y} = xz - 2\frac{\lambda}{b^2}y = 0$$

$$\frac{\partial \mathscr{L}}{\partial z} = xy - 2\frac{\lambda}{c^2}z = 0$$

$$\frac{\partial \mathscr{L}}{\partial \lambda} = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

We get

$$x = \pm\sqrt{\frac{4\lambda^2}{b^2c^2}} \qquad y = \pm\sqrt{\frac{4\lambda^2}{a^2c^2}}, \qquad \pm\sqrt{\frac{4\lambda^2}{a^2b^2}}$$

Plugging this into the constraint we get the following:

$$\lambda = \pm\frac{abc}{\sqrt{12}}$$

Plugging in for λ, we get

$$x^* = \pm\frac{a}{\sqrt{3}} \qquad y^* = \pm\frac{b}{\sqrt{3}} \qquad z^* = \pm\frac{c}{\sqrt{3}}$$

Since x^*, y^*, z^* are half the lengths of the box's sides, we have that the maximum volume is

$$8x^*y^*z^* = \frac{8abc}{3\sqrt{3}}$$

10.17

We have that

$$\mathscr{L}(x_1, x_2, \lambda) = x_1x_2 + \lambda(x_1 + x_2^2 - 2)$$

Therefore,

$$\frac{\partial \mathscr{L}}{\partial x_1} = x_2 + \lambda = 0$$

$$\frac{\partial \mathscr{L}}{\partial x_2} = x_1 + 2x_2\lambda = 0$$

$$\frac{\partial \mathscr{L}}{\partial \lambda} = x_1 + x_2^2 - 2 = 0$$

$$\implies x_1^* = \frac{4}{3}, \qquad x_2^* = \sqrt{\frac{2}{3}}$$

so we have that x_2^* is the maximum.

10.20

The Lagrangian is given as follows:

$$\mathscr{L} = \mathbf{x}^T\mathbf{x} - \lambda(\mathbf{c}^T\mathbf{x} - b)$$

The first-order conditions are as follows:

$$\frac{\partial \mathscr{L}}{\partial \mathbf{x}} = 2\mathbf{x} - \lambda\mathbf{c} = 0$$

$$\frac{\partial \mathscr{L}}{\partial \lambda} = \mathbf{c}^T \mathbf{x} - b = 0$$

And we get the following system of equations:

$$\mathbf{x} = \frac{\lambda \mathbf{c}}{2}$$

$$\mathbf{c}^T \mathbf{x} = b$$

$$\implies \lambda \mathbf{c}^T \mathbf{c} = 2b$$

$$\implies \lambda = \frac{2b}{||c||^2}$$

And finally

$$\implies \mathbf{x} = \frac{bc}{||c||^2}$$

Measure Theory (MT)

MT.1 (i)

We must show that the three properties of σ-algebras hold for $(i) - (iii)$. First, that $X \in \cap_{\alpha \in I} \mathscr{F}_\alpha$. Each $\mathscr{F}_\alpha$ is a σ-algebra, so it follows that

$$X \in \mathscr{F}_\alpha \qquad \forall \alpha \in I$$

$$\implies X \in \cap_{\alpha \in I} \mathscr{F}_\alpha$$

MT.1 (ii)

Next, that if

$$A \in \cap_{\alpha \in I} \mathscr{F}_\alpha \implies A^C \in \cap_{\alpha \in I} \mathscr{F}_\alpha$$

As

$$A \in \cap_{\alpha \in I} \mathscr{F}_\alpha \implies \forall \alpha \in I, A \in \mathscr{F}_\alpha$$

. But we know that $\mathscr{F}_\alpha$ is a σ-algebra, so we have that

$$A^C \in \mathscr{F}_\alpha \qquad \alpha \in I \implies A^C \cap_{\alpha \in I} \mathscr{F}_\alpha$$

MT.1 (iii)

Next, if we have a set $\{A_i\}$ such that

$$\{A_i\}_{i=1}^{\infty} \subset \cap_{\alpha \in I} \mathscr{F}_\alpha \implies \cup_{i=1}^{\infty} A_i \in \cap_{\alpha \in I} \mathscr{F}_\alpha$$

Note that

$$\{A_i\}_{i=1}^{\infty} \subset \cap_{\alpha \in I} \mathscr{F}_\alpha \implies \{A_i\}_{i=1}^{\infty} \subset \mathscr{F}_\alpha \qquad \alpha \in I$$

Also, as each $\mathscr{F}_\alpha$ is a σ-algebra, we have that

$$\cup_{i=1}^{\infty} A_i \in \mathscr{F}_\alpha \quad \alpha \in I$$

$$\implies \cup_{i=1}^{\infty} A_i \subset \cap_{\alpha \in I} \mathscr{F}_\alpha$$

With all these facts, we have that $\cap_{\alpha \in I} \mathscr{F}_\alpha$ is a σ-algebra.

MT.2 (i)

As $\mathscr{B}$ is the Borel σ-algebra of $\mathbb{R}$, we have that

$$\mathscr{B} = \{(a, b) | a < b, \forall a, b \in \mathbb{R}\}$$

We need to show that all intervals of the forms $(-\infty, x)$, $(-\infty, x]$, $(x, y]$, and $[x, y]$, as well as the singleton set $\{x\}$, are all in $\mathscr{B}$.
Let us express $(-\infty, x)$ as

$$\cup_{i=1}^{\infty} (x - i, x)$$

Note that this is the countable union of open intervals.

$$\implies (-\infty, x) \in \mathscr{B}$$

MT.2 (ii)

Let us express $(-\infty, x]$ as follows:

$$(x, \infty)^C = (\ \cup_{i=1}^{\infty} (x, x + i)\ )^C$$

This is the complement of the countable union of open intervals, which are in $\mathscr{B}$.

$$\implies (-\infty, x] \in \mathscr{B}$$

MT.2 (iii)

Let us express $(x, y]$ as follows:

$$(\ (-\infty, x] \cup (y, \infty) \ )^C$$

By $(i), (ii)$, we have that

$$\begin{aligned} (-\infty, x] \in \mathscr{B} \text{and} (y, \infty)^C \in \mathscr{B} &\implies (y, \infty) \in \mathscr{B} \\ &\implies (\ (-\infty, x] \cup (y, \infty) \ ) \in \mathscr{B} \\ &\implies (\ (-\infty, x] \cup (y, \infty) \ )^C = (x, y] \in \mathscr{B} \end{aligned}$$

MT.2 (iv)

Let us express $[x, y]$ as follows:

$$(\ (-\infty, x) \cup (y, \infty) \ )^C$$

By parts $(i) - (iii)$, we see that

$$(\ (-\infty, x) \cup (y, \infty) \ )^C \in \mathscr{B} \implies [x, y] \in \mathscr{B}$$

MT.2 (v)

Let us express $\{x\}$ as follows:

$$(\ (-\infty, x) \cup (x, \infty) \ )^C \in \mathscr{B}$$

MT.3

Let us first re-index the α_i's so that they are monotonically increasing. Then we have the following:

$$x \in A_i, s(x) = \alpha_i \geq \alpha_{i-1} \geq \ldots \geq \alpha_1$$

We can show with this fact that $s^{-1}((-\infty, a))$ is measurable where $a \in \mathbb{R}$.
As the sum approaches some finite n, the range of A is finite. Moreover, because we have re-indexed the α_i's, granted that$a \in A_1$

$$\implies s^{-1}((-\infty, a)) = A_1$$

Granted that $a \in A_2$

$$\implies s^{-1}((-\infty, a)) = A_1 \bigcup A_2$$

Continuing inductively, we get, for large values of a that

$$s^{-1}((-\infty, a)) = A_1 \bigcup A_2 \ldots \bigcup A_n$$

This implies that at worst, $s^{-1}((-\infty, a))$ is the union of n measurable sets, which is measurable. s, therefore, must be measurable.

MT.4

We want to show that

$$v(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} v(A_i)$$

Note that

$$\begin{aligned} \implies v(\cup_{i=1}^{\infty} A_i) &= \mu((\cup_{i=1}^{\infty} A_i) \cap D) \\ &= \mu(\cup_{i=1}^{\infty} (A_i \cap D)) \end{aligned}$$

by DeMorgan's Laws. Since $D \in F$ we have that

$$A \cap D \in F \quad \forall A \in F$$

As μ is a measure on F we have

$$\mu(\cup_{i=1}^{\infty} (A_i \cap D)) = \sum_{i=1}^{\infty} \mu(A_i \cap D) = \sum_{i=1}^{\infty} v(A_i \cap D)$$

$$\implies v(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} v(A_i)$$

$\implies v$ is a measure on F.

MT.4

Note that

$$f(x) = x^2 + 1 \quad \forall x \in \Omega = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$$

In order to express f as a simple function, let $\alpha_i \in \alpha$ be expressed as follows:

$$\alpha = \{26, 17, 10, 5, 2, 1, 2, 5, 10, 17, 26\}$$

So $f : \Omega \to \mathbb{R}$ can be expressed as a simple function s according to the following:

$$s(x) = \sum_{i=1}^{11} \alpha_i \chi_{A_i}(x)$$

Where

$$A_1 = \{-5\}, A_2 = \{-4\}, \ldots A_{10} = \{4\}, A_{11} = \{5\}$$

Now, f is simple and therefore measurable by MT.3.

$$\implies \int_{\Omega} f d\mu = \sum_{i=1}^{11} \alpha_i \mu(A_i \cap \Omega)$$

Our measure is given by $\mu(\{x\}) = \frac{1}{|x|}$ where $x \neq 0$, and $\mu(\{x\}) = 0$ where $x = 0$. And since $A_i \cap \Omega = A_i$

$$\implies \int_\Omega f d\mu = \sum_{i=1}^{11} \alpha_i \mu(A_i)$$

$$= 26(\frac{1}{5}) + 17(\frac{1}{4}) + 10(\frac{1}{3}) + 5(\frac{1}{2}) + 2(1) + 1(0) + 2(1) + 5(\frac{1}{2}) + 10(\frac{1}{3}) + 17(\frac{1}{4}) + 26(\frac{1}{5})$$

$$= 1037/30 \approx 34.566$$