Homework 1.5 344

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September 14, 2015

2.1 (i)

Note that

$$\begin{aligned}
L(a(x_1,x_2)+b(y_1,y_2)) &= L((ax_1,ax_2)+(by_1,by_2)) \\
&= ((ay_1,ay_2)+(bx_1,bx_2)) \\
&= aL(x_1,x_2)+bL(y_1,y_2)
\end{aligned}$$

$$\mathscr{N} = \mathbf{0}$$

$$\mathscr{R} = \mathbb{R}^2$$

Thus this is a linear transformation.

2.1 (ii)

$$\begin{aligned}
L(a(x_1,x_2)+b(y_1,y_2)) &= L((ax_1,ax_2)+(by_1,by_2)) \\
&= ((ax_1,0)+(by_1,0)) \\
&= aL(x_1,x_2)+bL(y_1,y_2)
\end{aligned}$$

$$\mathscr{N} = \{(0,y)|x \in \mathbb{R}^2\}$$

$$\mathscr{R} = \{(x,0)|x \in \mathbb{R}^2\}$$

Thus this is a linear transformation.

2.1 (iii)

$$\begin{aligned}
L(a(x_1,x_2)+b(y_1,y_2)) &= L((ax_1,ax_2)+(by_1,by_2)) \\
&= ((ax_1+1,ax_2+1)+(by_1+1,ax_2+1)) \\
&\neq aL(x_1,x_2)+bL(y_1,y_2) \\
&= (ax_1+a,ax_2+a)+(by_1+b,by_2+b)
\end{aligned}$$

Thus this is not a linear transformation.

2.1 (iv)

$$\begin{aligned} L(a(x_1,x_2)+b(y_1,y_2)) &= L((ax_1,ax_2)+(by_1,by_2)) \\ &= (a^2x_1^2,a^2x_2^2)+(b^2y_1^2,a^2x_2^2) \\ &\neq aL(x_1,x_2)+bL(y_1,y_2) \\ &= (ax_1^2,ax_2^2)+(by_1^2,by_2^2) \end{aligned}$$

Thus this is not a linear transformation.

2.2(i)

Let $p(x),q(x)\in\mathbb{F}_2$

$$\begin{aligned} L(a(p(x))+b(q(x))) &= x^2+y^2 \\ &\neq aL(p(x))+bL(q(x)) \\ &= ax^2+bx^2 \end{aligned}$$

2.2(ii)

Note that $xp(x)\in\mathbb{F}[x]_4 \quad \forall p(x)\in\mathbb{F}[x]_2$ Note that

$$\begin{aligned} L(a(p(x))+b(q(x))) &= axp(x)+bxq(x) \\ &= aL(p(x))+bL(q(x)) \end{aligned}$$

2.2(iii)

Note that $x^4+p(x)\in\mathbb{F}[x]_4 \quad \forall p(x)\in\mathbb{F}[x]_2$ Note that

$$\begin{aligned} L(a(p(x))+b(q(y))) &= x^4+ap(x)+y^4+bq(y) \\ &\neq aL(p(x))+bL(q(x)) \end{aligned}$$

Thus it is not a linear transformation

2.2(iv)

Note that $(4x^2-3x)p'(x)\in\mathbb{F}[x]_4 \quad \forall p(x)\in\mathbb{F}[x]_2$ Note that

$$\begin{aligned} L(a(p(x)))+L(b(q(x))) &= (4x^2-3x)ap'(x)+(4x^2-3x)bq'(x) \\ &= a((4x^2-3x)p'(x))+b((4x^2-3xq'(x)) \\ &= aL(p(x))+bL(q(x)) \end{aligned}$$

Thus it is a linear transformation

2.3

Let $f(x), g(x) \in C^1([0,1];\mathbb{F})$. Note also that $\forall f(x), h(x) = f(x) + f'(x)$ is continuous since $f(x)$ and $f'(x)$ are both continuous.

$$\begin{aligned} L(a(f(x))) + L(b(g(x))) &= af(x) + af'(x) + bg(x) + bg'(x) \\ &= a(f(x) + f'(x)) + b(g(x) + g'(x)) \\ &= aL(f(x)) + bL(g(x)) \end{aligned}$$

As for $L(f) = g$, note that

$$\begin{aligned} L(f) &= e^{-x}\int_0^x g(t)d^t dt + Ce^{-x} + (-e^{-x}\int_0^x g(t)e^t dt) + e^{-x}g(x)c^x - Ce^{-x} \\ &= g(x) + c^{-x} - e^{-x} \\ &= g(x) \end{aligned}$$

2.4

Let $L, K, M \in \mathscr{L}(V, W) \quad \mathbf{v} \in V, \quad a, b \in \mathbb{F}$.

2.4 (i)

By the properties of linear maps,

$$(L + K)(\mathbf{v}) = L(\mathbf{v}) + K(\mathbf{v}) = K(\mathbf{v}) + L(\mathbf{v}) = (K + L)(\mathbf{v})$$

2.4 (ii)

As with (i)

$$(L + K)(\mathbf{v}) + M(\mathbf{v}) = (L(\mathbf{v}) + K(\mathbf{v})) + M(\mathbf{v}) = L + (K + M)(\mathbf{v}) =$$

2.4 (iii)

$M(\mathbf{v}) = 0$ satisfies the additive identity

2.4 (iv)

Let $L'(\mathbf{v}) = -\mathbf{v}$. This linear transformation yields the additive inverse

2.4 (v)

As with (i),

$$a(L + K)(\mathbf{v}) = a(L(\mathbf{v}) + K(\mathbf{v})) = aL(\mathbf{v}) + aK(\mathbf{v}) = a(K + L)(\mathbf{v})$$

2.4 (vi)

$$(a+b)L(\mathbf{v}) = aL(\mathbf{v} + bL(\mathbf{v}) = bL(\mathbf{v} + aL(\mathbf{v}) = (b+a)L(\mathbf{v})$$

2.4 (vii)

$$\exists \mathbf{w} \in W \quad 1L(\mathbf{v}) = 1 * \mathbf{w} = \mathbf{w} = L(\mathbf{v})$$

2.4 (viii)

By properties of vector spaces, there are elements in W such that

$$(ab)L(\mathbf{v}) = ab(\mathbf{w}) = a(b\mathbf{w}) = a(bL(\mathbf{v}))$$

2.5

By induction, we see that for $n = 1$, we have $V_1, V_2, L_1 \quad L_1 : V_1 \to V_2 \quad (L_1)^{-1} = L_1^{-1}$ Suppose that $L_n L_{n-1} \cdots L_1)^{-1} = L_1^{-1} \cdots L_{n-1}^{-1} L_n^{-1}$. For $\{V_i\}_{i=1}^{n+1}\}$, and $\{L_i\}_{i=1}^{n}\}$, we have the expression

$$(L_n L_{n-1} \cdots L_1)^{-1} = (L_n(L_{n-1} \cdots L_1)^{-1}$$

And by remark 2.1.20, we can express it as follows

$$= ((L_{n-1} \cdots L_1)^{-1} L_n^{-1})$$

And inductively conclude

$$= L_1^{-1} \cdots L_n^{-1}$$

2.6

To show $\mathscr{N}(KL) = L^1 \mathscr{N}(K) = \mathbf{v} | L(\mathbf{v}) \in \mathscr{N}(K)$, we note by definiton:

$$\mathscr{N}(KL) = v \in V | KL(\mathbf{v}) = \mathbf{0}$$

$$\mathscr{N}(K) = w \in W | K(\mathbf{w}) = \mathbf{0}$$

We also know that $L^1 : W \to V$is a bijective map, because the two spaces are isomorphic. Let $v \in \mathscr{N}(KL)$. Thus $KL(\mathbf{v}) = 0$, and $KL(\mathbf{v}) \in W$. Thus, $vL1KL(\mathbf{v}) \in V$ To show the other direction, let $v \in L^{-1}\mathscr{N}(K)$. Because L inverse is bijective, there exists $v \in V$,forevery $w \in W$ that is in the null space of K, and $L^1\mathscr{N}(K) = v \in V | v = L^1(\mathscr{N}(K)$, and thus $v \in \mathscr{N}(KL)$. To show $\mathscr{R}(KL) \cong \mathscr{R}(K)$, we note by Definition:

$$\mathscr{R}(KL) = u \in U | \exists v \in V \quad \text{Where } KL(v) = u$$

$$\mathscr{R}(K) = u \in U | \exists w W \quad \text{Where } K(w) = u$$

Let $u \in \mathscr{R}(KL)$. Thus,$\exists v \in V$,where $KL(v) = w$. Note $L(v) \in W$, and $K(L(v)) = u$. Thus, $u \in \mathscr{R}(K)$. As for the other direction, let $u \in \mathscr{R}(K)$. Thus $\exists w \in W$, where $K(w) = u$. Because $L \cong W, \exists v \in V \quad$ s.t. $L(v) = w, KL(v) = u$. Thus $u \in \mathscr{R}(KL)$. Thus, $\mathscr{R}(KL) = \mathscr{R}(K)$.

2.7(i)

Let $\mathbf{x} \in V$, and $\mathbf{x} \in \mathscr{N}(L^k)$. Thus, $L^k\mathbf{x} = \mathbf{0}$. It follows that

$$L(L^k\mathbf{x}) = L(\mathbf{0}) = \mathbf{0})$$

And thus that

$$\mathbf{x} \in \mathscr{N}(L^{k+1})$$

2.7(ii)

Let $\mathbf{w} \in \mathscr{R}(L^{k+1})$. Thus, there exists $\mathbf{v} \in V$ such that $L^{k+1}(\mathbf{v}) = L(L(\mathbf{v}))$. Thus, $\exists \mathbf{v}' \in V \quad L(\mathbf{v}) = \mathbf{v}'$. Thus $L^k(\mathbf{v}') = \mathbf{w}$ and $\mathbf{w} \in \mathscr{R}(L^k)$.

$$\implies \mathscr{R}(L^{k+1}) \subset \mathscr{R}(L^k)$$