Math 320 Homework 1.8

Chris Rytting

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1.44 (i)

$$\binom{a}{0} = \frac{\Gamma(a+1)}{\Gamma(1)\Gamma(a+1)}\binom{a}{0} = \frac{\Gamma(a+1)}{0!\Gamma(a+1)}\binom{a}{0} = \frac{\Gamma(a+1)}{\Gamma(a+1)}$$

1.44 (ii)

$$\binom{a}{b+1} = \frac{\Gamma(a+1)}{\Gamma(a-b)\Gamma(b+2)}$$

Now, note that $\Gamma(n) = (n-1)\Gamma(n-1) \implies \Gamma(a-b) = \frac{\Gamma(a-b+1)}{a-b}$, and we have that

$$\begin{aligned}\binom{a}{b+1} = \frac{\Gamma(a+1)}{\Gamma(a-b)\Gamma(b+2)} &= \frac{\Gamma(a+1)(a-b)}{\Gamma(a-b+1)\Gamma(b+1)(b+1)} \\ &= \binom{a}{b+1}\frac{a-b}{b+1}\end{aligned}$$

1.44 (iii)

$$\begin{aligned}
\binom{a-1}{b-1} + \binom{a-1}{b} &= \frac{\Gamma(a)}{\Gamma(b)\Gamma(a-b+1)} + \frac{\Gamma(a)}{\Gamma(b+1)\Gamma(a-b)} \\
&= \frac{\Gamma(a)b}{\Gamma(b+1)\Gamma(a-b+1)} + \frac{(a-b)\Gamma(a)}{\Gamma(b+1)\Gamma(a-b+1)} \\
&= \frac{\Gamma(a)b + \Gamma(a)a - \Gamma(a)b}{\Gamma(b+1)\Gamma(a-b+1)} \\
&= \frac{\Gamma(a)a}{\Gamma(b+1)\Gamma(a-b+1)} \\
&= \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)} \\
&= \binom{a}{b}
\end{aligned}$$

1.44 (iv)

$$\begin{aligned}
\binom{a}{k} &= \frac{\Gamma(a+1)}{\Gamma(a-k+1)\Gamma(k+1)} \\
&= \frac{\Gamma(a+1)}{\Gamma(a-k+1)k!} \\
&= \frac{a\Gamma(a)}{(a-k)\Gamma(a-k)k!} \\
&= \frac{a(a-1)\Gamma(a-1)}{(a-k)(a-k-1)\Gamma(a-k-1)k!}
\end{aligned}$$

Proceeding inductively, since $k \in \mathbb{N}$, we know that eventually we will have

$$= \frac{a(a-1)(a-2)\ldots(a-k+1)(a-k)(a-k-1)\Gamma(a-k-1)}{(a-k)(a-k-1)\Gamma(a-k-1)k!}$$

It should be clear that the terms following and including $(a-k)$ in the numerator and denominator products will cancel out, leading to

$$= \frac{a(a-1)(a-2)\ldots(a-k+1)}{k!}$$

Which is the desired result.

1.45 (i)

$$
\begin{aligned}
(\int_{-\infty}^{\infty} e^{x^2/2})^2 dx &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dxdy \\
&= \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2} drd\theta \text{ where } r^2 = x^2+y^2 \\
&= \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2} drd\theta
\end{aligned}
$$

Now let $u = -\frac{r^2}{2}$, and we have

$$
\begin{aligned}
&= \int_0^{2\pi}\int_0^{\infty} e^{-u} dud\theta \\
&= \int_0^{2\pi}\int_0^{\infty} e^{-u} dud\theta \\
&= \int_0^{2\pi} \Big(- e^{-u}\Big|_0^{\infty}\Big) d\theta \\
&= \int_0^{2\pi} 1 d\theta \\
&= \theta\Big|_0^{2\pi} \\
&= 2\pi
\end{aligned}
$$

1.45 (ii)

$$
\begin{aligned}
\Gamma(x) &= \frac{1}{2x-1}\int_0^{\infty} e^{-\frac{u^2}{2}} u^{2x-1} du \\
&= \frac{1}{2^{x-1}}\int_0^{\infty} e^{-\frac{u^2}{2}} u^{2x-1} du \\
&= \frac{1}{2^{x-1}}\int_0^{\infty} e^{-\frac{u^2}{2}} u^{2x-1} du
\end{aligned}
$$

Now, letting $t = \frac{u^2}{2}$, we have

$$
\begin{aligned}
&= \frac{1}{2^{x-1}} \int_0^\infty e^{-t} u^{2x-2} dt \\
&= \frac{1}{2^{x-1}} \int_0^\infty e^{-t} u^{2x-2} dt \\
&= \frac{1}{2^{x-1}} \int_0^\infty e^{-t} 2^{x-1} (\frac{u^2}{2})^{x-1} dt \\
&= \frac{2^{x-1}}{2^{x-1}} \int_0^\infty e^{-t} t^{x-1} dt
\end{aligned}
$$

Which is equal to the definition of $\Gamma(x)$, yielding the desired result.

1.45 (iii)

By 1.45(ii)

$$
\begin{aligned}
\Gamma(\frac{1}{2}) &= \frac{1}{a^{2x-1}} \int_0^\infty e^{-\frac{u^2}{2}} u^{2x-1} du \\
&= \frac{1}{2^{-}1/2} \int_0^\infty e^{-\frac{u^2}{2}} u^0 du \\
&= \sqrt{2} \int_0^\infty e^{-\frac{u^2}{2}} du \\
&= \sqrt{2} \int_0^\infty e^{-\frac{u^2}{2}} du \\
&= \sqrt{2} \int_0^\infty \int_0^\infty e^{-\frac{x^2+y^2}{2}} du \\
&= \sqrt{2} \int_0^{\pi/2} \int_0^\infty e^{-\frac{r^2}{2}} r dr d\theta \\
&= \sqrt{2} \int_0^{\pi/2} \int_0^\infty e^{-\frac{r^2}{2}} r dr d\theta
\end{aligned}
$$

If we let $u = \frac{r^2}{2}$

$$
\begin{aligned}
&= \sqrt{2}\int_0^{\pi/2}\int_0^{\infty} e^{-u} r du d\theta \\
&= \sqrt{2}\int_0^{\pi/2}\int_0^{\infty} e^{-u} du d\theta \\
&= \sqrt{2}\int_0^{\pi/2} 1 d\theta \\
&= \sqrt{2}\int_0^{\pi/2} 1 d\theta \\
&= \sqrt{2}\frac{\pi}{2} 1 d\theta
\end{aligned}
$$

1.45 (iv)

$$
\begin{aligned}
\int_0^{\infty} e^{-xt^2} dt &= \Big(\int^{\pi} 0 e^{-xt^2} dt\Big)^2 \\
&= \Big(\int_0^{\infty} e^{-xt^2} dt\Big)^2 \\
&= \int_0^{\infty}\int^{\infty} 0 e^{-x(m^2+n^2)} dm dn
\end{aligned}
$$

Let $r^2 = m^2 + n^2$, we have

$$
\begin{aligned}
&= \int_0^{\pi/2}\int^{\infty} 0 e^{-xr^2} r dr d\theta \\
&= \int_0^{\pi/2}\int^{\infty} 0 e^{-xr^2} r dr d\theta
\end{aligned}
$$

Let $u = xr^2$, and we have that

$$\begin{aligned}
&= \int_0^{\pi/2} \int_0^{\infty} \frac{e^{-u}}{2x} dud\theta \\
&= \frac{1}{\sqrt{2x}} \int_0^{\pi/2} \int_0^{\infty} e^{-u} dud\theta \\
&= \frac{1}{\sqrt{2x}} \int_0^{\pi/2} 1 d\theta \\
&= \frac{1}{\sqrt{2x}} \sqrt{\frac{\pi}{2}} \\
&= \frac{1}{2} \sqrt{\frac{\pi}{x}}
\end{aligned}$$

1.46

The desired is equivalent to showing that

$$\begin{aligned}
&\lim_{x\to\infty} \frac{\text{Beta}(x,y)}{\Gamma(y)x^{-y}} = 1 \\
&= \lim_{x\to\infty} \frac{\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}}{\frac{\Gamma(y)x^{-y}}{1}} \\
&= \lim_{x\to\infty} \frac{\Gamma(x)\Gamma(y)}{\Gamma(y)\Gamma(x+y)x^{-y}} \\
&= \lim_{x\to\infty} \frac{\Gamma(x)}{\Gamma(x+y)x^{-y}} \\
&= \lim_{x\to\infty} \frac{\Gamma(x)x^{y}}{\Gamma(x+y)} \\
&= \frac{\int_0^x e^{-t}t^{x-1} dt x^y}{\int_0^x e^{-t}t^{x+y-1}dt}
\end{aligned}$$

Now, examining the denominator of this expression, we have

$$\int_0^x e^{-t}t^{x+y-1}dt = \left(-e^{-t}t^{x+y-1}\Big|_0^x \right) + (x+y-1)\int_0^x e^{-t}t^{x-1+y-1}$$

Resulting in

$$\frac{x \cdot x \cdot \dots \cdot \int_0^x e^{-t}t^{x-1}dt}{(x-1+y)(x-1+y-1)\dots(x-1)\int_0^x d^{-t}t^{x-1}dt}$$

Now, the right-most terms will cancel, and since we are taking the limit with respect to $x \to \infty$, we will need to differentiate with respect to x alone after encountering $\frac{\infty}{\infty}$. We will need to do so at least y times, meaning that all other x terms go to zero and we will have, effectively, $\frac{x^y}{x^y} = 1$.

1.47

$$\int_0^\infty e^{-xt}t^p dt$$

Now, letting

$$u = xt$$

$$du = x\ dt$$

$$t = \frac{u}{x}$$

We have

$$\begin{aligned}\int_0^\infty e^{-xt}t^p dt &= \int_0^\infty \frac{1}{x}e^{-u}(\frac{u}{x})^p du \frac{1}{x}\int_0^\infty e^{-u}(\frac{u}{x})^p du \\ &= \frac{1}{x}\int_0^\infty e^{-u}\frac{u^p}{x^p}du \\ &= \frac{1}{x^{1+p}}\int_0^\infty e^{-u}u^p du\end{aligned}$$

And since $\int_0^\infty e^{-u}u^p du = \Gamma(p+1)$, we have the desired result.

1.48

We know that log() strictly increasing on $[1, \infty)$

$$\implies \log(1) < \log(2) < \cdots < \log(n-1) < \log(n)$$

$$\begin{aligned}\sum_{k=1}^{n-1}\log(k) &= \log(1) + \log(2) + \cdots + \log(n-1) \\ &= \log(n-1)! \\ \sum_{k=1}^{n}\log(k) &= \log(1) + \log(2) + \cdots + \log(n) \\ &= \log(n)!\end{aligned}$$

Also, we have that

$$\begin{aligned}\int_1^n \log(k) &= k(\log(n-1))^n \\ &= n\log(k) - n + 1\end{aligned}$$

Where $\log(n)$ is a continually increasing funciton, we have that the first summation to $n-1$ is equivalent to the Riemann sums evaluated at the left endpoints of each interval, and the second sum corresponds to the right endpoints, meaning that the first is a lower bound on the true integral, and the second is an upper bound

$$\implies \sum_{k=1}^{n-1} \log(k) < \int_1^n \log(x)dx < \sum_{k=1}^{n} \log(k)$$
$$\implies \log(n-1)! < n\log(n) - n + 1 < \log(n!)$$

Now, notice that if we add $\log(n)$ to each term

$$\implies \log(n-1)! + \log(n) < n\log(n) - n + 1 + \log(n) < \log(n!) + \log(n)$$

which yields, combining the inequalities.

$$\implies n\log(n) - n + 1 < \log(n!) < n\log(n) - n + 1 + \log(n)$$

Which yields the second inequality. As for the third, let us raise every term to the exponent.

$$e^{n\log(n)-n+1} < e^{\log(n!)} < e^{n\log(n)-n+1+\log(n)}$$
$$\frac{e^{\log(n)^n}}{e^{n-1}} < e^{\log(n!)} < \frac{e^{(n+1)\log(n)}}{e^{n-1}}$$
$$\frac{(n)^n}{e^{n-1}} < (n!) < \frac{n^{(n+1)}}{e^{n-1}}$$

1.49

Consider

$$\int_{-1}^{1} e^{xcosh(t)}dt$$

Let

$$\alpha = x$$
$$f(t) = -cosh(t)$$
$$f'(t) = -sinh(t)$$
$$f''(t) = -cosh(t)$$

We can find x_0 by setting $f'(t)$ equal to 0.

$$-sinh(x_0) = 0$$

Differentiating, we have

$$-sinh^{-1}(sinh(x_0)) = 0$$
$$\implies x_0 = 0$$

We have that

$$\int_{-1}^{1} e^{xcosh(t)}dt = e^{\alpha f(x_0)}\sqrt{\frac{2\pi}{\alpha|f''(x_0)|}} = e^{-1x}\sqrt{\frac{2\pi}{x|1|}}$$
$$= e^{-x}\sqrt{\frac{2\pi}{x}}$$