Math 344 Homework 6.4

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6.19

We have that

$$\begin{aligned}
\|f(\mathbf{y}) - f(\mathbf{x})\|_Y &= \|\int_0^1 Df(t\mathbf{y} + (1-t)\mathbf{x}))(\mathbf{y}-\mathbf{x})dt\|_Y \\
&\leq \|\int_0^1 Df\big(t\mathbf{y} + (1-t)\mathbf{x}\big)(\mathbf{y}-\mathbf{x})dt\|_Y \\
&\leq \int_0^1 \|Df\big(t\mathbf{y} + (1-t)\mathbf{x}\big)(\mathbf{y}-\mathbf{x})dt\|_X \\
&\leq \int_0^1 \|Df\big(t\mathbf{y} + (1-t)\mathbf{x}\big)\|_X, Y\|(\mathbf{y}-\mathbf{x})dt\|_X \\
&\leq \int_0^1 \sup_{\mathbf{c}\in\ell(\mathbf{x},\mathbf{y})} \|Df\big(\mathbf{c}\big)\|_{X,Y}\|(\mathbf{y}-\mathbf{x})dt\|_X \\
&\leq \sup_{\mathbf{c}\in\ell(\mathbf{x},\mathbf{y})} \|Df\big(\mathbf{c}\big)\|_{X,Y}\|(\mathbf{y}-\mathbf{x})\|_X
\end{aligned}$$

6.20

Consider

$$F(t) = \int_{g(c)}^{t} f(\tau)d\tau$$

By FTC, $F'(t) = f(t)$. By this and by chain rule,

$$\int_c^d f(g(s))g'(s)ds = \int_c^d F'(g(s))g'(s)ds = \int_{g(c)}^{g(d)} DF(g(s))ds$$

Now by FTC

$$= F(g(d)) - F(g(c)) = \int_{g(c)}^{g(d)} f(\tau)d\tau - \int_{g(c)}^{g(c)} f(\tau)d\tau = \int_{g(c)}^{g(d)} f(\tau)d\tau$$

which is the desired result.

6.21

If we know that a sequence $(f_n)_{n=0}^\infty \in C(U;Y)$ is Cauchy in $C(U;Y)$, then the restriction $(f_n|_K)_{n=0}^\infty \in (C(K;Y), \|\cdot\|_{L^\infty})$ is Cauchy for every compact subset $K \subset U$. We also know that $(f_n)_{n=0}^\infty \in C(U;Y)$ is uniformly convergent, meaning that $(f_n|_K)_{n=0}^\infty$ converges to $f|_K$ in $(C(K;Y), \|\cdot\|_{L^\infty})$ for every compact subset $K \subset U$. Now, if this is true for an arbitrary sequence $f_n \in C(U;Y)$, then it will be true for all sequences in an open set contained in the closed set, which we know it is because the closure of a set contains the interior and exterior points of the set.

6.22 (i)

As derivative exists and is given by

$$f'(x) = \frac{x}{\sqrt{\frac{1}{n^2} + x^2}}$$

for any $x \in [-1, 1]$.

6.22 (ii)

We have that

$$\sup_{(0,1)} f_n(x) = \sqrt{\frac{n^2+1}{n^2}}$$

Furthermore, any compact set lies in the interval $[a, b]$ where $0 < a < b < \sqrt{\frac{n^2+1}{n^2}}$ Therefore, we have that

$$\|f_n(x)|_{[a,b]}\|_{L^\infty} = \sqrt{\frac{n^2+1}{n^2}} \to |x| \text{ as } n \to \infty$$

So $f_n(x)$ converges uniformly to $|x|$ on $[-1, 1]$.

6.22 (iii)

Note $f(x) = |x|$. Then

$$f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Which is not discontinuous at $x = 0$, implying that it is not differentiable there.

6.22 (iv)

The criterion that $f_n(\mathbf{x}_*))_{n=0}^\infty \subset C^1(U;Y)$ does not converge in Y is not fulfilled, and we have that the theorem holds.

6.23

Let

$$S_k = \sum_{n=0}^{k} Df_n = D \sum_{n=0}^{k} f_n$$

so $\{s_k\}_{k=0}^{\infty}$ converges ass. on U. Now

$$t_k = \sum_{n=0}^{k} f_n(x_0)$$

so $\{t_k\}_{k=0}^{\infty}$ converges on Y.