Math 344 Homework 4.1

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4.1

If λ is an eigenvalue of A, then we have that

$$\begin{aligned} Ax &= \lambda x \\ (Ax)^k &= (\lambda x)^k \\ A^k x^k &= \lambda^k x^k \end{aligned}$$

If A is nilpotent, though, we have that $A^k = 0$, which implies that

$$0x^k = \lambda^k x^k$$

Which only holds if and only if $\lambda = 0$, since x is nonzero by the definition 4.1.1.

4.2

Note that

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, we find eigenvalues by the following calculation

$$\begin{aligned} p_A(z) &= \det(zI - A) \\ &= \det\left(\begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \right) \\ &= \det \begin{bmatrix} z & -1 & 0 \\ 0 & z & -2 \\ 0 & 0 & z \end{bmatrix} \\ &= z^3 \\ \implies z &= 0 \end{aligned}$$

Now, we have there is one eigenvalue equal to zero with algebraic multiplicity of 3 and geometric multiplicity of one, the eigenvector being

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

4.3

Let A be as follows:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The characteristic polynomial is given by

$$\begin{aligned}
p_A(\lambda) &= \det(\lambda I - A) \\
&= \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \quad = \det \begin{bmatrix} \lambda - a & b \\ c & \lambda - d \end{bmatrix} = (\lambda - d)(\lambda - a) - bc \\
&= \lambda^2 - \lambda d - a\lambda + ad - bc \\
&= \lambda^2 - \lambda d - a\lambda + ad - bc \\
&= \lambda^2 - \lambda(d + a) + ad - bc \\
&= \lambda^2 - \lambda \mathrm{tr}(A) + \det(A)
\end{aligned}$$

as desired.

4.4 (i)

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Since A is Hermitian, $c = b$, and using the characteristic polynomial found in 4.3, we have that

$$\begin{aligned}
\lambda &= \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(1)(ad - b^2)}}{2} \\
&= \frac{a+d}{2} + \frac{1}{2}\sqrt{a^2 + 2ad + d^2 - 4ad + 4b^2} \\
&= \frac{a+d}{2} + \frac{1}{2}\sqrt{a^2 - 2ad + d^2 + 4b^2} \\
&= \frac{a+d}{2} + \frac{1}{2}\sqrt{(a-d)^2 + 4b^2}
\end{aligned}$$

Since all elements of A are real, and $(a-d)^2 + 4b^2$ will be non-negative, we know that no eigenvalue will be imaginary, implying that all will be real.

4.4 (ii)

As A is skew-Hermitian, we know that $-a = \overline{a}$ and that $-d = \overline{d}$, implying that A has strictly imaginary numbers on the diagonal. Moreover, $-b = \overline{c}$ and $-c = \overline{b}$, implying a matrix A of the form

$$A = \begin{bmatrix} wi & x + yi \\ -x + yi & vi \end{bmatrix}$$

where $x, y, v, w \in \mathbb{R}$ Using the characteristic polynomial and the quadratic formula, we have that

$$\begin{aligned} \lambda &= \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(1)(ad - b^2)}}{2} \\ &= \frac{a+d}{2} + \frac{1}{2}\sqrt{a^2 + 2ad + d^2 - 4ad + 4b^2} \\ &= \frac{a+d}{2} + \frac{1}{2}\sqrt{a^2 - 2ad + d^2 + 4b^2} \\ &= \frac{a+d}{2} + \frac{1}{2}\sqrt{(a-d)^2 + 4b^2} \end{aligned}$$

Since $(w - v)^2 + 4x^2 + 4y^2 \in \mathbb{R}_+$ and $\frac{w+v}{2}$ is real, the sum of these will be real, and a real number multiplied by an imaginary number will be strictly imaginary.

4.5

If $A(c - \lambda I)^{-1} B^H$ has an eigenvalue of 1, we have that

$$(A(c - \lambda I)^{-1} B^H)\mathbf{y} = \mathbf{y}$$

for some $\mathbf{y} \in \mathbb{F}^m$. If $\mathbf{x}$ is an eigenvector and λ is its eigenvalue $C - B^H A$ then we have that

$$(C - B^H A)\mathbf{x} = \lambda\mathbf{x}$$

Then we have the following:

$$\begin{aligned} (C - B^H A)\mathbf{x} &= \lambda\mathbf{x} \\ C\mathbf{x} - B^H A\mathbf{x} &= \lambda\mathbf{x} \\ B^H A\mathbf{x} &= C\mathbf{x} - \lambda\mathbf{x} \\ B^H A\mathbf{x} &= (C - \lambda I)\mathbf{x} \\ (C - \lambda I)^{-1} B^H A\mathbf{x} &= \mathbf{x} \\ A(C - \lambda I)^{-1} B^H A\mathbf{x} &= A\mathbf{x} \end{aligned}$$

Now, letting $\mathbf{y} = A\mathbf{x}$, we have that

$$(A(c - \lambda I)^{-1} B^H)\mathbf{y} = \mathbf{y}$$

Therefore, 1 is an eigenvalue of $A(c - \lambda I)^{-1} B^H$, as desired.

4.6

Let a matrix A be an upper-triangular matrix. To find its eigenvalues, we find the characteristic polynomial:

$$\begin{aligned}
p_A(z) &= \det(zi - A) \\
&= \det\left(\begin{bmatrix} z & 0 & 0 & \dots & 0 \\ 0 & z & 0 & \dots & 0 \\ 0 & 0 & z & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & z \end{bmatrix} - \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ 0 & b_1 & b_2 & \dots & b_{n-1} \\ 0 & 0 & c_1 & \dots & c_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n_1 \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} z - a_1 & -a_2 & -a_3 & \dots & -a_n \\ 0 & z - b_1 & -b_2 & \dots & -b_{n-1} \\ 0 & 0 & z - c_1 & \dots & -c_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & z - n_1 \end{bmatrix}\right) \\
&= (z - a_1)(z - b_1)(z - c_1)\dots(z - n_1)
\end{aligned}$$

which yields the desired result, that $a_1, b_1, c_1, \dots, n_1$ are eigenvalues of A since the characteristic polynomial equals zero when they do.