Math 344 Homework 4.2

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4.7 (i)

No element of this set is a linear combination of other elements of the set, so we have that the set is linearly independent, and we know by assumption that the set spans C^∞. Therefore, it forms a basis.

4.7 (ii)

$$D = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

4.7(iii)

Consider one space spanned by $sin(2x)$ and $cos(2x)$. Upon multiplying D by the vector representations of these spans, namely

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

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4.8

Consider a unidimensional subspace span$\{(0, 0, \ldots, a_n, 0, \ldots)\}$ of the vector space ℓ^∞ and the right shift operated span$\{(0, 0, \ldots, 0, a_n, \ldots)\}$. There exists no linear combination of elements from the first subspace that results in an element from the second, and therefore it is not invariant.

4.9

It suffices to show that $T'(a') = T'(b')$ if $a' = b'$. Now, let $a' = a + W$ and $b' = b + W$ where $a', b' \in V/W$. We will consider two cases, that $a, b \in W$ and that $a, b \notin W$.

Let $a, b \in W$. Then

$$T'(a') = T'(a + W) = T(a) + W$$

and

$$T'(b') = T'(b + W) = T(b) + W$$

and we have that, since by the definition of an invariant subspace, $T(a) \in W$ and $T(b) \in W$, that $T(a) + W = T(b) + W$ are in the same equivalence class and we have the desired result.

Now let $a, b \notin W$. We still have, though, that

$$a' = a + W = b' = b + W$$

Now, there exists a vector c such that, since a, b are in the same equivalence class, if we let

$$a'' = a - c + W$$

and

$$b'' = b - c + W$$

we still have that $a'' = b''$ and that

$$T'(a'') = T'(a - c + W) = T(a) - T(c) + W$$

and

$$T'(b'') = T'(b - c + W) = T(b) - T(c) + W$$

and, by the same logic as the other case, we have that

$$T(a) - T(c) + W = T(b) - T(c) + W$$

$$\implies T(a) + W = T(b) + W$$

Which is the desired result.

4.10

$$(i) \implies (ii)$$

For $w_1 \in W_1$ and $w_2 \in W_2$ note that

$$R^2(w_1 + w_2) = R(w_1 - w_2) = w_1 + w_2$$

Which is the desired result.

$$(ii) \implies (iii)$$

To find the nullspace fo $(R - I)$, we note that

$$(R - I)v = 0 \implies Rv - v = 0 \implies Rv = v$$

yielding

$$\mathscr{N}(R - I) = \{v \in V | Rv = v\}$$

and

$$(R + I)v = 0 \implies Rv + v = 0 \implies Rv = -v$$

yielding

$$\mathscr{N}(R + I) = \{v \in V | Rv = -v\}$$

It is clear that

$$\mathscr{N}(R + I) \cap \mathscr{N}(R + I) = \{0\}$$

Now, to show that these two spaces comprise all of V, we consider the vector

$$v \in V = \frac{1}{2}(I - R)v + \frac{1}{2}(I + R)v$$

and we have that

$$(R - I)(\frac{1}{2}(I + R)v) = \frac{1}{2}(R + R^2 - I - R)v = \frac{1}{2}(R + I - I - R)v = 0$$

and

$$(R + I)(\frac{1}{2}(I - R)v) = \frac{1}{2}(R - R^2 + I - R)v = \frac{1}{2}(R - I + I - R)v = 0$$

implying that the first and second terms of v are in the null space of $(R - I)$ and $(R + I)$, respectively. However, any $v \in V$ can be expressed as a linear combination of these nullspaces, implying that they span the whole space, and we have the desired result that they are complimentary.

$$(iii) \implies (i)$$

Let $W_1 = \mathscr{N}(R - I)$ and $W_2 = \mathscr{N}(R + I)$. We know by part (ii), that

$$R(w_1) = w_1 \quad w_1 \in W_1$$

and that

$$R(w_2) = -w_2 \quad w_2 \in W_2$$

Now, R is a linear operator, so we have that $R(w_1 + w_2) = R(w_1) + R(w_2) = w_1 - w_2$. Therefore, R is a reflection.

4.11 (i)

$$\text{span}\left\{\begin{bmatrix}1\\3\end{bmatrix}\right\} \quad \text{span}\left\{\begin{bmatrix}-3\\1\end{bmatrix}\right\}$$

4.11 (ii)

$$T = \left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}-3\\1\end{bmatrix}\right\} \quad L_{TT} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

4.11 (iii)

$$C_{ST} = \begin{bmatrix}1 & 3\\3 & -1\end{bmatrix} \quad C_{TS} = \begin{bmatrix}1/10 & 3/10\\3/10 & -1/10\end{bmatrix}$$

4.11 (iv)

$$L = C_{ST}L_{TT}C_{TS} = \begin{bmatrix}1 & 3\\3 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1/10 & 3/10\\3/10 & -1/10\end{bmatrix} = \begin{bmatrix}-4/5 & 3/5\\3/5 & 4/5\end{bmatrix}$$