# Math 344 Homework 3.1

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## 3.1 (i)

$$\begin{aligned}
\langle x, y\rangle &= \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) \\
&= \frac{1}{4}[(\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle) - (\langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle)] \\
&= \frac{1}{4}(2\langle x,y\rangle + 2\langle y,x\rangle) \\
&= \frac{1}{4}(4\langle x,y\rangle) \\
&= \langle x,y\rangle
\end{aligned}$$

## 3.1 (ii)

$$\begin{aligned}
\|x\|^2\|y\|^2 &= \frac{1}{2}(\|x+y\|^2 + \|x-y\|^2) \\
&= \frac{1}{2}(\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle + \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle) \\
&= \frac{1}{2}(2\langle x,x\rangle + 2\langle y.y\rangle) \\
&= \langle x,x\rangle + \langle y,y\rangle
\end{aligned}$$

## 3.2

$$\begin{aligned}
\langle x, y\rangle &= \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x-iy\|^2 - i\|x+iy\|^2) \\
&= \frac{1}{4}(4\langle x, y\rangle + i\langle x-iy, x-iy\rangle - i\langle x+iy, x+iy\rangle) \\
&= \frac{1}{4}[4\langle x, y\rangle + i(\langle x, x\rangle + \langle x, -iy\rangle + \langle -iy, x\rangle + \langle -iy, -iy\rangle) \\
&\quad - i(\langle x, x\rangle + \langle x, iy\rangle + \langle iy, x\rangle + \langle iy, iy\rangle)] \\
&= \frac{1}{4}(4\langle x, y\rangle + i(\langle iy, x\rangle + \langle -iy, x\rangle) - i(\langle -iy, x\rangle + \langle iy, x\rangle) \\
&= \frac{1}{4}(4\langle x, y\rangle)
\end{aligned}$$

## 3.3 (i)

$$\begin{aligned}
cos(\theta) &= \frac{\langle x, x^5\rangle}{\|x\|\|x^5\|} \\
\langle x, x^5\rangle &= \int_0^1 x^6 dx = \frac{1}{7}x^7|_0^1 = \frac{1}{7} \\
\|x\| = \sqrt{\langle x, x\rangle} &= \int_0^1 x^2 dx = \frac{1}{3}x^3|_0^1 = \frac{1}{\sqrt{3}} \\
\|x^5\| = \sqrt{\langle x, x\rangle} &= \int_0^1 x^{10} dx = \frac{1}{11}x^{11}|_0^1 = \frac{1}{\sqrt{11}} \\
\theta &= cos^{-1}(\frac{\sqrt{33}}{7}) \approx .60824
\end{aligned}$$

## 3.3 (ii)

$$\begin{aligned}
cos(\theta) &= \frac{\langle x^2, x^4\rangle}{\|x^2\|\|x^4\|} \\
\langle x^2, x^4\rangle &= \int_0^1 x^6 dx = \frac{1}{7}x^7|_0^1 = \frac{1}{7} \\
\|x^2\| = \sqrt{\langle x, x\rangle} &= \int_0^1 x^4 dx = \frac{1}{5}x^5|_0^1 = \frac{1}{\sqrt{5}} \\
\|x^4\| = \sqrt{\langle x, x\rangle} &= \int_0^1 x^8 dx = \frac{1}{9}x^9|_0^1 = \frac{1}{\sqrt{9}} \\
\theta &= cos^{-1}(\frac{\sqrt{45}}{7}) \approx .289
\end{aligned}$$

## 3.4

Suppose $\|T\mathbf{x}\| = a\|\mathbf{x}\|$, Thus,

$$\begin{aligned}
\frac{\langle \mathbf{Tx}, \mathbf{Ty} \rangle}{\|x\|\|y\|} &= \frac{1}{4}\frac{(||Tx+Ty||)^2 - (||Tx-Ty||^2)}{(\|Tx\|\|Ty\|)} \\
&= \frac{1}{4}\frac{a^2\|x+y\|^2 - \|x-y\|^2}{a^2\|x\|\|y\|} \\
&= \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|x\|\|y\|}
\end{aligned}$$

Now suppose, $\frac{\langle \mathbf{Tx}, \mathbf{Ty} \rangle}{\|Tx\|\|Ty\|} = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|x\|\|y\|}$. Thus, $\langle \mathbf{Tx}, \mathbf{Ty} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle = 0$ if and only if $\mathbf{x}, \mathbf{y}$ are orthogonal (since $T$ is linear), thus it preserves angle. Note that $\mathbf{y} = \text{proj}_x \mathbf{y} + \mathbf{r}$,

$$\frac{\langle \mathbf{Tx}, \mathbf{Ty} \rangle}{\|T\mathbf{x}\|\|T\mathbf{y}\|} = \frac{\langle \mathbf{Tx}, \mathbf{T}(\alpha\mathbf{x} + \mathbf{r}) \rangle}{\|Tx\|\|Ty\|}$$

Thus,

$$\begin{aligned}
\frac{\langle \mathbf{Tx}, \mathbf{T}(\alpha\mathbf{x} + \mathbf{r}) \rangle}{\|Tx\|\|Ty\|} &= \frac{\langle \mathbf{x}, (\alpha\mathbf{x} + \mathbf{r}) \rangle}{\|x\|\|y\|} \\
\frac{\alpha\|Tx\|^2}{\|Tx\|\|Ty\|} &= \frac{\alpha\|x\|^2}{\|x\|\|y\|} \\
\frac{\alpha\|Tx\|}{\|Ty\|} &= \frac{\alpha\|x\|}{\|y\|} \\
\frac{\alpha\|Tx\|}{\|x\|} &= \frac{\alpha\|Ty\|}{\|y\|}
\end{aligned}$$

Which is equal to some constant for both, and thus,

$$(\|T\mathbf{x}\| = a\|\mathbf{x}\|)$$

and,

$$(\|T\mathbf{y}\| = a\|\mathbf{y}\|)$$

## 3.5

$$f = e^x \qquad g = x - 1$$

$$proj_u(f) = \frac{\langle g, f \rangle}{\langle g, g \rangle} \cdot g$$

$$\langle g, f \rangle = \langle e^x, x-1 \rangle = \int_0^1 xe^x - e^x dx = xe^x - 2e^x|_0^1 = (e - 2e) + (2) = -e + 2$$

$$\langle g, g \rangle = \int_0^1 (x-1)^2 = \int_0^1 x^2 - 2x + 1 dx = \frac{1}{3}x^3 - x^2 + x|_0^1 = \frac{1}{3}$$

$$\frac{-e+2}{\frac{1}{3}} \cdot (x-1) = (-3e+6)(x-1)$$

## 3.6

Starting with $0 \le \|x - \lambda y\|^2$ we define $\lambda = \frac{\langle x,y \rangle}{\langle y,y \rangle}$ and we continue as follows:

$$\|x - \lambda y\|^2 = \langle x - \lambda y, x - \lambda y \rangle$$

Note that with $\lambda = \frac{\langle x,y \rangle}{||y||^2}$, we have that $\langle x - \lambda y, -\lambda y \rangle = \langle x - \frac{\langle x,y \rangle}{||y||^2} y, -\frac{\langle x,y \rangle}{||y||^2} y \rangle = \langle x - \text{proj}_y(x), -\text{proj}_y(x) \rangle$, implying that $-\lambda y$ is orthogonal to $x - \lambda y$. This implies that that $\langle x - \lambda y, -\lambda y \rangle = 0$ and $\langle -\lambda y, x - \lambda y \rangle = 0$.

$$\begin{aligned}
0 \le \|x - \lambda y\|^2 &= \langle x - \lambda y, x - \lambda y \rangle \\
&= \langle x, x \rangle - \langle \lambda y, x \rangle \\
&= \|x^2\| - \frac{\langle x, y \rangle^2}{\langle y, y \rangle} \\
&= \|x^2\| - \frac{\langle x, y \rangle^2}{\|y\|^2} \\
\implies 0 &\le \|x^2\| - \frac{\langle x, y \rangle^2}{\|y\|^2} \\
&\Rightarrow \frac{\langle x, y \rangle^2}{\|y\|^2} \le \|x^2\| \\
&\Rightarrow \frac{\langle x, y \rangle}{\|y\|} \le \|x\| \\
&\Rightarrow \langle x, y \rangle \le \|x\| \ \|y\|
\end{aligned}$$