Math 320 3.3

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3.14 (i)

$$f_X(\alpha) = \begin{cases} \frac{1}{2} & \text{if } \alpha = a, b, c \\ \frac{1}{6} & \text{if } \alpha = d \\ \frac{1}{3} & \text{if } \alpha = e, f \\ 0 & \text{if } \alpha \text{ equals anything else.} \end{cases}$$

3.14 (ii)

Mean given by

$$\frac{1}{2} + \frac{1}{6} \cdot 2 + \frac{1}{3} \cdot 3.5 = 2$$

3.14 (iii)

Variance given by

$$\mathrm{V}(X) = (1-2)^2 \cdot \frac{1}{2} + (0)^2 \cdot \frac{1}{6} + (1.5)^2 \cdot \frac{1}{3} = \frac{5}{4}$$

3.15

Let X be an arbitary discrete set. Let the probability distribution be given by $p(X = i) = g_X$

Suppose $h : \mathbb{R} \to \mathbb{R}$ is a continuous function such that there exists an $\varepsilon > 0$ between each $x \in X$ where there is a countable number of points.

Since h is continuous, we have that there exists an $\epsilon' > 0$ between each x. Let $A = \{\epsilon'\}_{i \in I}$.

$inf(A)$ will yield an ϵ', such that for every point p, there exists an $E(p, \frac{\epsilon'}{2})$ where the neighborhood E doesn't contain other points $x \in X$. Therefore, we have that $h(X)$ is a random variable, and by the Law of the Unconscious Statistitian, we have

$$E[h(X)] = \sum_{i \in I} h(i)p(x = i) = \sum_{i \in I} h(i)g_X(i)$$

3.16

$$\begin{aligned} \mathrm{V}(X) &= E[(X - \mu)^2] = E[X^2 - 2X\mu + \mu^2] \\ &= E[X^2] - 2\mu E[X] + E[\mu^2] \\ &= E[X^2] - \mu^2 \end{aligned}$$

3.17

$$\begin{aligned} \mathrm{V}(\alpha x + \beta y) &= E[\alpha x + \beta y]^2 - (E[\alpha x + \beta y])^2 \\ &= E[\alpha^2 x^2 + \alpha x \beta y + \alpha \beta xy + \beta^2 y^2] - (\alpha E[x] + \beta E[Y])^2 \\ &= \alpha^2 E[x^2] + 2\alpha\beta E[xy] + \beta^2 E[y^2] - \alpha E[x]^2 - 2\alpha\beta E[x]E[y] - \beta^2 E[y]^2 \\ &= \alpha^2 (E[x^2] - E[x]^2) + 2\alpha\beta(E[xy] - E[x]E[y]) + \beta^2(E[y^2] - E[y]^2) \\ &= \alpha^2 \mathrm{var}[x] + 2\alpha\beta(E[xy] - E[x]E[y]) + \beta^2 \mathrm{V}(y) \end{aligned}$$

Which is the desired result. Moreover, if x, y are independent, we have $E[xy] - E[x]E[y] = 0$, yielding

$$\mathrm{V}(\alpha x + \beta y) = \alpha^2 \mathrm{V}[x] + \beta^2 \mathrm{V}[y]$$

3.18

Note, by the Law of the Unconscious Statistician, we have that $E(\frac{1}{X+1})$ is a random variable.

$$\begin{aligned}
E\left[\frac{1}{X+1}\right] &= \sum_{i=0}^{n} \frac{1}{i+1}\binom{n}{i} p^i (1-p)^{n-i} \\
&= \sum_{i=0}^{n} \frac{n!}{(i+1)!(n-i)!} p^i (1-p)^{n-i} \\
&= \sum_{i=1}^{n+1} \frac{n!}{i!(n-i+1)!} p^{i-1} (1-p)^{n-i+1} \\
&= \frac{1}{p(n+1)} \sum_{i=1}^{n+1} \frac{(n+1)!}{i!(n+1-i)!} p^i (1-p)^{n+1-i} \\
&= \frac{1}{p(n+1)} \sum_{i=1}^{n+1} \binom{n+1}{i} p^i (1-p)^{n+1-i} \\
&= \frac{1}{p(n+1)} \Big(\sum_{i=0}^{n+1} \binom{n+1}{i} p^i (1-p)^{n+1-i} - (1-p)^{n+1} \Big) \\
&= \frac{1}{p(n+1)} \cdot (1 - (1-p)^{n+1}) \\
&= \frac{1-(1-p)^{n+1}}{p(n+1)}
\end{aligned}$$

Which is the desired result.

3.19

$$\begin{aligned}
E[X] &= \sum_{x \in X} x \cdot P(X = x) \\
&= \sum_{i \in I} x \sum x \in X \frac{xP(X = x) \bigcap B_i) P(B_i)}{P(B_i)} \\
&= \sum_{i} \sum_{x} x P(X = x | B_i) P(B_i) \\
&= \sum_{i} E(X|B_i) P(B_i)
\end{aligned}$$