Homework 1.7 Math 320

Chris Rytting

September 15, 2015

1.37

For $n = 1$, it is obvious that there are $1! = 1$ permutations. Assume that it is true that for $n-1$ there are $(n-1)!$ permutations such that for

$$(k_1, k_2, k_3, \cdots, k_{n-1}) \text{ there are } (n-1)! \text{ permutations.}$$

Now, let us add an nth element. We know that for

$$(k_n, k_1, k_2, k_3, \cdots, k_{n-1}) \text{ there are } (n-1)! \text{ permutations, and that for}$$

$$(k_1, k_n, k_2, k_3, \cdots, k_{n-1}) \text{ there are } (n-1)! \text{ permutations.}$$

Proceeding inductively, there are n spots into which we could insert k_n, each resulting in a set with $(n-1)!$, permutations, leading to the conclusion that there are $n(n-1)! = n!$ permutations in a set with n elements.

1.38 (i)

$$6!$$

1.38 (ii)

$$5!2!$$

1.38 (iii)

$$4!3!$$

1.38 (iv)

$$3!3!2!$$

1.39

$$C(13,2)C(4,2)C(4,2)C(11,1)4 = 78 \cdot 6 \cdot 6 \cdot 11 \cdot 4 = 123,552$$

1.40

$$\text{Total number of ways to win \$100: } C(1,1)C(5,3)C(54,2) = 1 \cdot 10 \cdot 1431 = 14,310$$
$$\text{Total possible combinations: } C(35,1)C(59,5) = 35 \cdot 5,006,386 = 175,223,510$$
$$\text{Probability of winning \$100: } 14,310/175,223,510 = .000081667$$

1.41 (i)

Note that

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Differentiating both sides with respect to x, we have

$$n(1+x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1}$$

Letting $x = 1$, we have

$$n2^{n-1} = \sum_{k=0}^{n} \binom{n}{k} kn2^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k + 0$$

1.41 (ii)

Note that

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Differentiating both sides with respect to x, we have

$$n(1+x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1}$$
$$xn(1+x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^k$$

Differentiating both sides again (with respect to x), we have

$$n(1+x)^{n-1} + xn(n-1)(1+x)^{n-2} = \sum_{k=0}^{n} \binom{n}{k} k^2 x^{k-1}$$

Letting $x = 1$, we have

$$\begin{aligned} n(2)^{n-1} + n(n-1)(2)^{n-2} &= \sum_{k=0}^{n} \binom{n}{k} k^2 \\ n2^{n-2}(1+n) &= \sum_{k=0}^{n} \binom{n}{k} k^2 \\ n2^{n-2}(1+n) &= \sum_{k=1}^{n} \binom{n}{k} k^2 + 0 \end{aligned}$$

1.42

$$\begin{aligned} (1+x)^n(1+x)^m &= \sum_{k=0}^{n} \binom{n}{k} x^k \cdot \sum_{j=0}^{m} \binom{m}{j} x^j \\ (1+x)^{n+m} &= \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{n}{k} \binom{m}{j} x^j x^k \end{aligned}$$

Now, let $r = k+j$, implying $j = r-k$. We also know that the monomials $x_1, x_2, \ldots, x_n$ are linearly independent, implying that

$$\begin{aligned} &\implies \sum_{r=0}^{m+n} \sum_{k=0}^{r} \binom{n}{k} \binom{m}{r-k} x^r = \sum_{r=0}^{n+m} x^r \\ &\implies \sum_{k=0}^{r} \binom{n}{k} \binom{m}{r-k} = \binom{m+n}{r} \end{aligned}$$