DEF**innerproduct** on V,a scalar-valued map $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{F}$ that satisfies, for $\mathbf{x},\mathbf{y},\mathbf{z} \in V$, $a, b \in \mathbb{F}$: $(i)\langle x,x\rangle \geq 0, eq.iff \mathbf{x} = 0$ (ii) $\langle x, a\mathbf{y} + b\mathbf{z}\rangle = a\langle x,y\rangle + b\langle x,z\rangle$ (iii)$\langle x,y\rangle = \overline{\langle y,x\rangle}$ DEF A vector space together with an inner product is called an **inner product space** $(V,\langle\cdot,\cdot\rangle)$

$PROP3.1.3$ Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space. For any $\mathbf{x},\mathbf{y},\mathbf{z} \in V$ an any $a \in \mathbb{F}$, we have (i) $\langle\mathbf{x}+\mathbf{y},\mathbf{z}\rangle = \langle\mathbf{x},\mathbf{z}\rangle + \langle\mathbf{y},\mathbf{z}\rangle (iii) \langle a\mathbf{x},\mathbf{y}\rangle = \overline{a}\langle x,y\rangle$

DEF Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space. The length of a vector $\mathbf{x} \in V$ induced by the inner product is $||\mathbf{x}|| = \sqrt{\langle x,x\rangle}$. If $||\mathbf{x}|| = 1$, we say that x is a unit vector. The distance between two vectors $x, y \in V$ is the length of the difference, that is, $dist(x,y) = ||x-y||$

$PROPCauchyShwarz$ Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space. For all $\mathbf{x},\mathbf{y} \in V$, we have $|\langle x,y\rangle| \leq ||\mathbf{x}||||\mathbf{y}||$

DEF Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space. We define the angle between two nonzero vectors $\mathbf{x},\mathbf{y}$ be the unique angle $\theta \in [0,\pi]$ such that $\cos(\theta) = \langle x,y\rangle/||\mathbf{x}||||\mathbf{y}||$

$THMPythagoreanLaw$ If $\mathbf{x},\mathbf{y}$ are orthongonal vectors in the inner product space $(V,\langle\cdot,\cdot\rangle)$, then $||\mathbf{x}+\mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2$

DEF Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space. For any unit vector $\mathbf{u} \in V$ and any $\mathbf{x}inV$, define the orthogonal projection of $\mathbf{x}$ onto span($\{\mathbf{u}\}$) to be $\mathrm{proj}_{span(\{\mathbf{u}\})}(\mathbf{x}) = \langle u,x\rangle u$

$PROP3.1.23$ Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space. For any unit vector $\mathbf{u} \in V$ the map $\mathrm{proj}_u : V \to V$ is a linear operator. Moreover the following hold: (i) $\mathrm{proj}_u \circ \mathrm{proj}_u = \mathrm{proj}_u$ (ii) Residual vector $r = v - \mathrm{proj}_u(v)$ is orthogonal to vector in span(u), including $\mathrm{proj}_u(v)$. Thus r lies in $\mathscr{N}(\mathrm{proj}_u$ (iii) The vector $\mathrm{proj}u(v)$ is the unique vector in span(u) that is nearest to $\mathbf{v}$

REM for any $v \in V, u = v/||v||, \mathrm{proj}u(x) = \langle v/||v||, x\rangle v/||v|| = \langle v,x\rangle v/\langle v,v\rangle$

DEF Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space and $\{\mathbf{x}_i\}_{i=1}^m$ is a finite orthonormal set with span($\{\mathbf{x}_i\}_{i=1}^m$) $= X$, then for any $\mathbf{v} \in V$ we define orthogonal projection onto X as $\mathrm{proj}_X(\mathbf{v}) = \sum_{i=1}^m \langle\mathbf{x}_i,\mathbf{v}\rangle\mathbf{x}_i$.

$THM3.2.6$ If $\{\mathbf{x}_i\}_{i=1}^m$ is a finite orthonormal set with span $= X$, then the map $\mathrm{proj}_X : V \to V$ is a linear transformation, and (i) $proj_X \circ proj_X = proj_X$ (ii) For every $\mathbf{v} \in V$, r is orthogonal to every $x \in X$. Thus $\mathbf{r}$ lies in $\mathscr{N}proj_X$ (iii) The image $proj_X(\mathbf{v})$ is the unique vector in X that is nearest to $\mathbf{v}$. That is, $||v - \text{proj}_X(v)|| < ||\mathbf{v} - \mathbf{x}||$ for all $\mathbf{x} \in X$ where $\mathbf{x} \neq \text{proj}_X(v)$.

$THMPythagoreanTheorem$ LVbaips. If $\{\mathbf{x}_i\}_{i=1}^m$ is a finite orthonormal set with span $= X$, then every $\mathbf{v} \in V$ satisfies $||v||^2 = \sum_{i=1}^m |\langle x_i, v\rangle|^2 + ||v - \sum_{i=1}^m \langle x_i, v\rangle x_i||^2$

$Bessel'sInequality$ LVbaips. If $\{\mathbf{x}_i\}_{i=1}^m$ is a finite subset of an orthonormal set $\mathscr{C} \in V$, then every $v \in V$ satisfies $||v||^2 \geq \sum_{i=1}^m |\langle x_i, v\rangle|^2 = ||\text{proj}_X(v)||^2$

DEF A linear map L from an IPS V to an inner product space W is called an orthonormal transformation if for every $x, y \in V$ we have $\langle x, y\rangle_V = \langle Lx, Ly\rangle_W$ If $L : V \to V$, it is an orthonormal operator.

$PROP$ LVbaips. If L is an orthonormal operator, it is invertible.

DEF A square matrix Q is orthonormal if it is the matrix representation of an orthonormal operator on $\mathbb{F}^n$ with the standard bases and the standard inner products.

THM Let Q, Q_1, Q_2 be orthonormal square matrices and assuming the usual inner product. Then (i) $||Qx|| = ||x||$ (ii) Q_1Q_2 is an orthonormal matrix (iii) Q^{-1} is orthonormal matrix (iv) The matrix Q is an orthonormal matrix iff $Q^HQ = QQ^H = I$. (v) The columns of Q are orthonormal (vi) $|\det(Q)| = 1$

$THMGram - Schmidt$ Let $x_1, x_2, \ldots x_n$ be a linearly independent set in ipsV. Define $q_1 = x_1/||x_1||$ and $q_k = (x_k - p_{k-1})/||x_k - p_{k-1}||$ where $p_{k-1} = \text{proj}_{Q^{k-1}}(x_k) = \sum_{i=1}^{k-1}\langle q_i, x_k\rangle q_i$. Resulting set orthonormal with same span as $x_1, x_2, \cdots, x_n$.

$THMQRDecompostion$ Let A be an mxn matrix of rank n. Then A can be factored into a product QR, where Q is an mxn matrix with orthonormal columsn and R is a nonsingular nxn upper triangular matrix $R = Q^HA$

DEF A Hyperplane W in a vector space V is any subspace such that V/W is one dimensional.

DEF. Given a unit vector $v \in \mathbb{F}^n$, we define the hyperplane Y to be the subset of $\mathbb{F}^n$where every element of Y is orthogonal to v. More precisely $Y = \{y \in \mathbb{F}^n | \langle v, y\rangle = 0\}$. Reflection across hyperplane orthogonal to v given by $H_v = I - (2vv^H / v^H v)$

PROP reflection through the hyperplane orthogonal to v is an orthonormal transformation.