Math 344 Homework 2.8

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2.44

Since we know that $\det(A) = \det(A^T)$ and by Theorem 2.8.1, we have that

$$\det(V_n) = \det \begin{bmatrix} 1 & 1 & 1 \cdots 1 \\ x_0 & x_1 & x_2 \cdots x_n \\ x_0^2 & x_1^2 & x_2^2 \cdots x_n^2 \\ \cdots & \cdots & \cdots\cdots\cdots \\ x_0^n & x_1^n & x_2^n \cdots x_n^n \end{bmatrix} \sim \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & x_1 - x_0 & x_2 - x_0 & \cdots & x_n - x_0 \\ 0 & x_1^2 - x_1x_0 & x_2^2 - x_2x_0 & \cdots & x_n^2 - x_nx_0 \\ 0 & x_1^3 - x_1^2x_0 & x_2^3 - x_2^2x_0 & \cdots & x_n^3 - x_n^2x_0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & x_1^n - x_1^{n-1}x_0 & x_2^n - x_2^{n-1}x_0 & \cdots & x_n^3 - x_n^2x_0 \end{bmatrix}$$

$$\sim 1 \cdot \det \begin{bmatrix} x_1 - x_0 & x_2 - x_0 & \cdots & x_n - x_0 & \\ x_1^2 - x_1x_0 & x_2^2 - x_2x_0 & \cdots & x_n^2 - x_nx_0 & \\ x_1^3 - x_1^2x_0 & x_2^3 - x_2^2x_0 & \cdots & x_n^3 - x_n^2x_0 & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_1^n - x_1^{n-1}x_0 & x_2^n - x_2^{n-1}x_0 & \cdots & x_n^3 - x_n^2x_0 & \end{bmatrix}$$

$$\sim 1 \cdot \det \begin{bmatrix} x_1 - x_0 & x_1^2 - x_1x_0 & x_1^3 - x_1^2x_0 & \cdots & x_1^n - x_1^{n-1}x_0 \\ x_2 - x_0 & x_2^2 - x_2x_0 & x_2^3 - x_2^2x_0 & \cdots & x_2^n - x_2^{n-1}x_0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_n - x_0 & x_n^2 - x_nx_0 & x_n^3 - x_n^2x_0 & \cdots & x_n^3 - x_n^2x_0 \end{bmatrix}$$

$$\sim 1(x_1 - x_0)(x_2 - x_0)\cdots(x_n - x_0)\det \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ 1 & x_3 & x_3^2 & \cdots & x_3^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_3 & x_3^2 & \cdots & x_3^n \end{bmatrix}$$

$$\sim 1(x_1 - x_0)(x_2 - x_0)\cdots(x_n - x_0)\det \begin{bmatrix} 1 & 1 & 1 \cdots 1 \\ x_1 & x_2 & x_2 \cdots x_n \\ x_1^2 & x_2^2 & x_2^2 \cdots x_n^2 \\ \cdots & \cdots & \cdots\cdots\cdots \\ x_1^n & x_2^n & x_2^n \cdots x_n^n \end{bmatrix}$$

$$\sim (x_1-x_0)(x_2-x_0)\cdots(x_n-x_0)\det\begin{bmatrix}1 & 1 & 1 & \cdots & 1\\ 0 & x_2-x_1 & x_3-x_1 & \cdots & x_n-x_1\\ 0 & x_2^2-x_2x_1 & x_3^2-x_3x_1 & \cdots & x_n^2-x_nx_1\\ 0 & x_2^3-x_2^2x_1 & x_3^3-x_3^2x_1 & \cdots & x_n^3-x_n^2x_1\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & x_2^n-x_2^{n-1}x_1 & x_3^n-x_3^{n-1}x_1 & \cdots & x_n^3-x_n^2x_1\end{bmatrix}$$

$$\sim 1(x_1-x_0)(x_2-x_0)\cdots(x_n-x_0)\det\begin{bmatrix}x_2-x_1 & x_3-x_1 & \cdots & x_n-x_1\\ x_2^2-x_2x_1 & x_3^2-x_3x_1 & \cdots & x_n^2-x_nx_1\\ x_2^3-x_2^2x_1 & x_3^3-x_3^2x_1 & \cdots & x_n^3-x_n^2x_1\\ \cdots & \cdots & \cdots & \cdots\\ x_2^n-x_2^{n-1}x_1 & x_3^n-x_3^{n-1}x_1 & \cdots & x_n^3-x_n^2x_1\end{bmatrix}$$

$$\sim 1(x_1-x_0)(x_2-x_0)\cdots(x_n-x_0)\det\begin{bmatrix}x_2-x_1 & x_2^2-x_2x_1 & x_2^3-x_2^2x_1 & \cdots & x_2^n-x_2^{n-1}x_1\\ x_3-x_1 & x_3^2-x_3x_1 & x_3^3-x_3^2x_1 & \cdots & x_3^n-x_3^{n-1}x_1\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ x_n-x_1 & x_n^2-x_nx_1 & x_n^3-x_n^2x_1 & \cdots & x_n^3-x_n^2x_1\end{bmatrix}$$

$$\sim 1(x_1-x_0)(x_2-x_0)\cdots(x_n-x_0)(x_2-x_1)(x_3-x_1)\cdots(x_n-x_1)\det\begin{bmatrix}1 & x_2 & x_2^2 & \cdots & x_2^n\\ 1 & x_3 & x_3^2 & \cdots & x_3^n\\ 1 & x_4 & x_4^2 & \cdots & x_4^n\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 1 & x_n & x_n^2 & \cdots & x_n^n\end{bmatrix}$$

Proceeding recursively, performing the same operations on this matrix, it should be clear that eventually we will have all ones on the diagonal of this matrix, yielding a determinant of 1 times all $(x_j - x_i)$ where $i < j$, and we have that

$$\det(V_n) = \Pi_{i<j}(x_j - x_i)$$

2.45

Let

$$A = \begin{bmatrix}x_{11} & x_{12} & x_{13} & \cdots & x_{1n}\\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n}\\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn}\end{bmatrix}$$

$$\implies \det(A) = \det\begin{bmatrix}x_{11} & x_{12} & x_{13} & \cdots & x_{1n}\\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n}\\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn}\end{bmatrix}$$

Now we also have that

$$\alpha A = \begin{bmatrix} \alpha x_{11} & \alpha x_{12} & \alpha x_{13} & \cdots & \alpha x_{1n} \\ \alpha x_{21} & \alpha x_{22} & \alpha x_{23} & \cdots & \alpha x_{2n} \\ \alpha x_{31} & \alpha x_{32} & \alpha x_{33} & \cdots & \alpha x_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha x_{n1} & \alpha x_{n2} & \alpha x_{n3} & \cdots & \alpha x_{nn} \end{bmatrix}$$

$$\implies \det(\alpha A) =\sim \det \begin{bmatrix} \alpha x_{11} & \alpha x_{12} & \alpha x_{13} & \cdots & \alpha x_{1n} \\ \alpha x_{21} & \alpha x_{22} & \alpha x_{23} & \cdots & \alpha x_{2n} \\ \alpha x_{31} & \alpha x_{32} & \alpha x_{33} & \cdots & \alpha x_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha x_{n1} & \alpha x_{n2} & \alpha x_{n3} & \cdots & \alpha x_{nn} \end{bmatrix}$$

$$\sim \alpha \det \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ \alpha x_{21} & \alpha x_{22} & \alpha x_{23} & \cdots & \alpha x_{2n} \\ \alpha x_{31} & \alpha x_{32} & \alpha x_{33} & \cdots & \alpha x_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha x_{n1} & \alpha x_{n2} & \alpha x_{n3} & \cdots & \alpha x_{nn} \end{bmatrix}$$

$$\sim \alpha^2 \det \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ \alpha x_{31} & \alpha x_{32} & \alpha x_{33} & \cdots & \alpha x_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha x_{n1} & \alpha x_{n2} & \alpha x_{n3} & \cdots & \alpha x_{nn} \end{bmatrix}$$

Proceeding recursively, we have that

$$\implies \det(\alpha A) = \alpha^n \det \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} \end{bmatrix} = \alpha^n \det(A)$$

Which is the desired result.

2.46

We know that for an upper triangular matrix B,

$$\det(B) = b_{11} b_{22} \cdots b_{nn}$$

where $b_{ii} \quad i = 0, 1, \cdots, n$ are the diagonal entries of B. Now, since A is in block-triangular form, we know that A_{11} and A_{22} are square matrices that can be row reduced so that they are upper triangular matrices A'_{11} and A'_{22}, so that $\det(A'_{11}) = a_{11,11} a_{11,22} \cdots a_{11,nn} \quad \det(A'_{22}) = a_{22,11} a_{22,22} \cdots a_{22,nn}$ where $a_{11,ii}$ are the diagonal entries of A'_{11} and $a_{22,ii}$ are the diagonal entries of A'_{22}. However, these are also the diagonal entries of A, and we have the following

$$\det(A) = a_{11,11} a_{11,22} \cdots a_{11,nn} a_{22,11} a_{22,22} \cdots a_{22,nn} = \det(A'_{11}) \det(A'_{22}) = \det(A_{11}) \det(A_{22})$$

2.47

Knowing that

$$\begin{bmatrix} I & \mathbf{0} \\ -\mathbf{y^H} & 1 \end{bmatrix} \begin{bmatrix} I - \mathbf{x}\mathbf{y^H} & \mathbf{x} \\ \mathbf{0^H} & 1 \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ \mathbf{y^H} & 1 \end{bmatrix} = \begin{bmatrix} I & \mathbf{x} \\ \mathbf{0^H} & 1 - \mathbf{y^H}\mathbf{x} \end{bmatrix}$$

$$\implies \det\left(\begin{bmatrix} I & \mathbf{0} \\ -\mathbf{y^H} & 1 \end{bmatrix} \begin{bmatrix} I - \mathbf{x}\mathbf{y^H} & \mathbf{x} \\ \mathbf{0^H} & 1 \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ \mathbf{y^H} & 1 \end{bmatrix} \right) = \det\left(\begin{bmatrix} I & \mathbf{x} \\ \mathbf{0^H} & 1 - \mathbf{y^H}\mathbf{x} \end{bmatrix} \right)$$

By theorem 2.8.7, we have that

$$\det\left(\begin{bmatrix} I & \mathbf{0} \\ -\mathbf{y^H} & 1 \end{bmatrix} \right) \det\left(\begin{bmatrix} I - \mathbf{x}\mathbf{y^H} & \mathbf{x} \\ \mathbf{0^H} & 1 \end{bmatrix} \right) \det\left(\begin{bmatrix} I & \mathbf{0} \\ \mathbf{y^H} & 1 \end{bmatrix} \right) = \det\left(\begin{bmatrix} I & \mathbf{x} \\ \mathbf{0^H} & 1 - \mathbf{y^H}\mathbf{x} \end{bmatrix} \right)$$

Now, since the first and the third matrices are upper-triangular once transposed, we know that their determinant is equal to the product of their diagonal entries, which in this case are all ones. We also know that

$$\det\left(\begin{bmatrix} I & \mathbf{x} \\ \mathbf{0^H} & 1 - \mathbf{y^H}\mathbf{x} \end{bmatrix} \right) = 1 \cdot 1 \cdot \cdots \cdot (1 - \mathbf{y^H}\mathbf{x}) = (1 - \mathbf{y^H}\mathbf{x})$$

for the same reason, that it is upper triangular, and the only non-one entry on the diagonal is $(1 - \mathbf{y^H}\mathbf{x})$. Finally, we know by the previous exercise (2.46) that

$$\det\left(\begin{bmatrix} I - \mathbf{x}\mathbf{y^H} & \mathbf{x} \\ \mathbf{0^H} & 1 \end{bmatrix} \right) = \det(I - \mathbf{x}\mathbf{y^H})\det(1) = \det(I - \mathbf{x}\mathbf{y^H})$$

And so we have the following

$$\det(I - \mathbf{x}\mathbf{y^H}) = (1 - \mathbf{y^H}\mathbf{x})$$

Which is the desired result.

2.48

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$

We have that $A^{-1} = \frac{\text{adj}(A)}{\det(A)}$. Now, we know that the adjoint of A is given by

$$\begin{bmatrix} 14 - 24 & -(14 - 18) & 8 - 6 \\ -(7 - 20) & 7 - 15 & -(4 - 3) \\ 6 - 10 & -(6 - 10) & 2 - 2 \end{bmatrix} = \begin{bmatrix} -10 & 4 & 2 \\ 13 & -8 & -1 \\ -4 & 4 & 0 \end{bmatrix}$$

And we have that $\det(A) = 4$

$$\implies \frac{1}{4} \begin{bmatrix} -10 & 4 & 2 \\ 13 & -8 & -1 \\ -4 & 4 & 0 \end{bmatrix} = \begin{bmatrix} -2.5 & 1 & .5 \\ 3.25 & -2 & -.25 \\ -1 & 1 & 0 \end{bmatrix}$$

2.49

The matrix is singular, and so Cramer's Rule does not apply.

2.50 (i)

It suffices to show that if $\mathscr{C}$ is not linearly independent, then there exists no $x \in [a, b] \quad W(x) \neq 0$. So if $\mathscr{C}$ is linearly dependent, then we know that some $y_i \in \mathscr{C}$ will be a linear combination of other elements of $\mathscr{C}$ such that $\mathbf{y}_i = a_1\mathbf{y}_1 + \cdots + a_{i-1}\mathbf{y}_{i-1} + a_{i+1}\mathbf{y}_{i+1} + \cdots + a_{n-1}\mathbf{y}_{n-1} \quad a_j \in \mathbb{R} y_j \in \mathscr{C}$. Now, this implies that, using row operations, we can make this $y_i = 0$ in the transpose matrix. This will imply that all the derivatives $y_i', y_i'', \cdots y_i^{n-1}$ are zero as well, resulting in a whole row of zeros in $W(x)$. We have by exercise 2.43 that this will result in $W(x) = 0$, regardless of what x is, implying that there exists no x such that $W(x) \neq 0$.

2.50 (ii)

We have that the Wronksian of S is given by

$$\begin{aligned} W(x) &= \det \begin{bmatrix} e^{\alpha x} & xe^{\alpha x} & x^2e^{\alpha x} \\ \alpha e^{\alpha x} & e^{\alpha x} + \alpha x e^{\alpha x} & 2xe^{\alpha x} + \alpha x^2 e^{\alpha x} \\ \alpha^2 e^{\alpha x} & 2\alpha e^{\alpha x} + \alpha^2 x e^{\alpha x} & 2e^{\alpha x} + 4xe^{\alpha x} + \alpha^2 x^2 e^{\alpha x} \end{bmatrix} \\ &= \det \begin{bmatrix} e^{\alpha x} & xe^{\alpha x} & x^2 e^{\alpha x} \\ 0 & e^{\alpha x} & 2xe^{\alpha x} \\ 0 & 2\alpha e^{\alpha x} & 2e^{\alpha x} + 4xe^{\alpha x} \end{bmatrix} \\ &= \det \begin{bmatrix} e^{\alpha x} & xe^{\alpha x} & x^2 e^{\alpha x} \\ 0 & e^{\alpha x} & 2xe^{\alpha x} \\ 0 & 0 & 2e^{\alpha x} \end{bmatrix} \\ &= 2e^{3(\alpha x)} \neq 0 \end{aligned}$$

for any x, implying the desired result.