Math 344 Homework 3.7

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3.37

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} D^* = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

$$\mathscr{N}(D) = \text{span}\{(1,0,0)^T\}$$

$$\mathscr{R}(D) = \text{span}\{(0,1,0)^T, (0,0,1)^T\}$$

$$\mathscr{N}(D^*) = \text{span}\{(0,0,1)^T\}$$

$$\mathscr{R}(D^*) = \text{span}\{(0,0,1)^T, (0,1,0)^T\}$$

3.38

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 \end{bmatrix}$$

$$A^* = A^H = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathscr{N}(A) = \text{span}\{(0,0,0,1)^T, (1,0,-1,0)^T, (-1,1,0,0)^T\}$$

$$\mathscr{R}(A) = \text{span}\{(1,0,0,0)^T\}$$

$$\mathscr{R}(A^*) = \text{span}\{(1,0,0)^T\}$$

$$\mathscr{N}(A^*) = \text{span}\{(0,1,0)^T, (2,0,-1)^T\}$$

3.39 (i)

$$\begin{aligned}
\langle \mathbf{w}, (S+T)\mathbf{n}\rangle_w &= \langle \mathbf{w}, S\mathbf{v} + T\mathbf{v}\rangle_w \\
&= \langle \mathbf{w}, S\mathbf{v}\rangle_w + \langle \mathbf{w}, T\mathbf{v}\rangle_w \\
&= \langle S^*\mathbf{w}, \mathbf{v}\rangle_v + \langle T^*\mathbf{w}, \mathbf{v}\rangle_v \\
&= \langle (S^* + T^*)\mathbf{w}, \mathbf{v}\rangle_v
\end{aligned}$$

And we have that $(S+T)^* = S^* + T^*$

$$\begin{aligned}
\langle \mathbf{w}, \alpha T\mathbf{v}\rangle_w &= \langle \bar{\alpha}\mathbf{w}, T\mathbf{v}\rangle_w \\
&= \langle \bar{\alpha}\mathbf{w}, T^*\mathbf{v}\rangle_v
\end{aligned}$$

And we have that $(\alpha T)^* = \bar{\alpha} T^*$

3.39 (ii)

$$\langle \mathbf{w}, S\mathbf{v}\rangle_w = \langle S^*\mathbf{w}, \mathbf{v}\rangle_v = \langle \mathbf{w}, (S^*)^*\mathbf{v}\rangle_w$$

And we have that $(S^*)^* = S$

3.39 (iii)

$$\langle \mathbf{v}, ST\mathbf{v} \rangle = \langle S^*\mathbf{v}, T\mathbf{v} \rangle = \langle T^*S^*\mathbf{v}, \mathbf{v} \rangle$$

And we have that $(ST)^* = T^*S^*$

3.39 (iv)

$$\begin{aligned} \langle \mathbf{v}, \mathbf{v} \rangle &= \langle TT^{-1}\mathbf{v}, \mathbf{v} \rangle \\ &= \langle T^{-1}\mathbf{v}, T^*\mathbf{v} \rangle \\ &= \langle \mathbf{v}, (T^{-1})^*T^*\mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{v} \rangle \end{aligned}$$

And we have that $(T^{-1})^*T^* = I$, implying $(T^{-1})^* = (T^*)^{-1}$.

3.40

We know that $L^* : W \to V$ and $\mathbf{v} \in \mathscr{R}(L^*)^\perp$ iff

$$\langle \mathbf{v}, L^*\mathbf{w} \rangle = 0 \forall \mathbf{w} \in W$$

This is true iff

$$\langle L\mathbf{v}, \mathbf{w} \rangle = 0 \quad \forall \mathbf{v} \in V$$

which happens iff

$$\mathbf{v} \in \mathscr{N}(L)$$

Therefore, $\mathscr{N}(L) = \mathscr{R}(L^*)^\perp$, and by Lemma 3.7.18, $\mathscr{N}(L)^\perp = \mathscr{R}(L^*)$

3.41 (i)

It is sufficient to show $\langle Y, AX \rangle = \langle A^H Y, X \rangle$

$$\langle Y, AX \rangle = tr(Y^H AX) = tr(A^H Y)^H X) = \langle A^H Y, X \rangle$$

3.41 (ii)

$$\langle V, WA \rangle = tr(V^H WA) = tr(AV^H W) = tr((VA^H)^H W) = \langle VA^*, W \rangle$$

3.41 (iii)

$$\begin{aligned}
\langle (T_A(B))^*, C\rangle &= \langle B, T_A(C)\rangle \\
&= \langle B, AC - CA\rangle = tr(B^H(AC_CA)) \\
&= tr(B^HAC) - tr(B^HCA) = \langle B, AC\rangle - \langle B, CA\rangle \\
&= \langle A^*B, C\rangle - \langle BA^*, C\rangle \\
&= tr(B^H(A^*)^HC) - tr((A^*)^HB^HC) = tr(B^H(A^*)^HC - (A^*)^HB^HC) \\
&= \langle A^*B - BA^*, C\rangle \\
&= \langle T_{A^*}(B), C\rangle
\end{aligned}$$

And we have that $(T_A)^* = T_{A^*}$

3.42

Note that $A^* = A^H$, and if there exists a solution to

$$A\mathbf{x} = \mathbf{b} \implies \mathbf{x} \in \mathscr{R}(A)$$

Otherwise, it would be the case that

$$\mathbf{x} \in \mathscr{R}(A)^\perp$$

because they are complement subspaces. By Theorem 3.7.21, we know

$$\mathscr{R}(A)^\perp = \mathscr{N}(A^H)$$

Therefore, we have that

$$\mathbf{x} \in \mathscr{R}(A) \quad \text{or} \quad \mathbf{x} \in \mathscr{N}(A^H)$$

3.43

If A and B are arbitrary matrices in the spaces $\text{Sym}_n(\mathbb{R})$ and $\text{Skew}_n(\mathbb{R})$, respectively, then So $A^T = A$ and $B^T = -B$. We also know that the following is true:

$$tr(CD) = tr(DC) \quad \text{and} \quad tr(C) = tr(C^T)$$

$$\implies \langle A, B\rangle = \text{tr}(A^TB) = \text{tr}((-1)A^TB^T) = -\text{tr}(A^TB^T) = -\text{tr}((BA)^T) = -\text{tr}(AB)$$

but $\text{tr}(AB) = -\text{tr}(AB)$ only holds iff $\text{tr}(AB) = 0$.

Therefore, the set of symmetric matrices in M is orthogonal to the set of skew matrices.