Math 320 Homework 5.7

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5.31

Suppose to the contrary that we have n nodes

$$\int_{-1}^{1} p_{2n}(x) = \sum_{i=0}^{n-1} f(x_i)w_i$$

Yielding the sytstem of equations

$$2 = \int_{-1}^{1} 1 = \sum_{i}^{n-1} w_i$$

$$0 = \int_{-1}^{1} x = \sum_{i}^{n-1} x_i^1 w_i$$

$$\frac{2}{3} = \int_{-1}^{1} x^2 = \sum_{i}^{n-1} x_i^2 w_i$$

$$0 = \int_{-1}^{1} x^3 = \sum_{i}^{n-1} x_i^3 w_i$$

$$\vdots$$

$$\frac{2}{2n+1} = \int_{-1}^{1} x^2 = \sum_{i}^{n-1} x_i^{2n} w_i$$

If the system has a solution, it will yield a gaussian quadrature for n nodes on $\mathbb{R}[x]_{2n}$

However, there is one more unknown than there are equations. Therefore, we cannot find a solution.

5.32

The third-degree Taylor series approximation around 0 is given by

$$
\begin{aligned}
p(x) &\approx f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \\
&= \sin(3) + \cos(3)x - \frac{sin(3)}{2}x^2 - \frac{\cos(3)}{6}x^3
\end{aligned}
$$

Now, by Example 5.7.2, we compute the integral and using a Taylor series approximation

$$
\begin{aligned}
&\int_{-1}^{1} \frac{-\cos(3)}{6}x^3 - \frac{\sin(3)}{2}x^2 + \cos(3)x + sin(3)dx \\
&= \frac{-cos(3)}{6}(\frac{-1}{\sqrt{3}})^3 - \frac{sin(3)}{2}(\frac{-1}{\sqrt{3}})^2 + \cos(3)(\frac{-1}{\sqrt{3}}) + \sin(3) \\
&- \frac{cos(3)}{6}(\frac{1}{\sqrt{3}})^3 - \frac{sin(3)}{2}(\frac{1}{\sqrt{3}})^2 + \cos(3)(\frac{1}{\sqrt{3}}) + \sin(3) \\
&= 0.2352
\end{aligned}
$$

Now, computing the integral of $\sin(x+3)$, we can compare it to the previous computation

$$
\int_{-1}^{1} sin(x+3)dx = -cos(x+3)|_{-1}^{1} = -cos(4) - (-cos(3)) = 0.2374
$$

Which are nearly the same.

5.33

Let $y_i = g(x_i) = a(1-x_i) + b(x_i)$. Then we have that if $\{y_i\}_{i=0}^{n} \subset [a,b]$ are the roots of the $n+1$st Legendre polynomial, then for all $q(x) = \mathbb{R}[x]_{2n+1}$ we have

$$
\int_a^b q(x)dx = \sum_{i=0}^{n} q(y_i)w_i
$$

where

$$
w_i = \int_a^b L_{i,n}(x)dx, \quad i = 0,1,2,\cdots,n
$$

are the integrals of the Lagrange basis polynomials. Since we have a map $g : [-1,1] \to [a,b]$ and a map $g^{-1} : [a,b] \to [-1,1]$, we can apply the proof of Theorem 5.7.4 without loss of generality by mapping back and forth between these intervals.

5.34

```
import numpy as np

def f(x):
    return np.abs(x)

def z(x):
    return np.cos(x)


def quadrature(f,n):
    a = np.linspace(-1,1,n+1)
    roots,weights = np.polynomial.legendre.leggauss(n+1)
    function_vals = f(roots)
    return np.sum(function_vals*weights)
print quadrature(z,4)
print np.sin(1)*2.

1.68294197041
1.68294196962
(Very close to one another)



print abs(x) yields the following values with for n = 10,20,30,...,100:
for i in xrange(10,110,10):
    print quadrature(f,i)

0.987523109474
0.996438310884
0.99834153543
0.999044665942
0.999379701294
0.999565044201
0.999678211086
0.999752337668
0.999803515872
0.999840326218
```

One's approximation is better since it is smooth and therefore more conducive to using polynomials.