Chapter 6 Section 4

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6.19

The proof is as follows:

$$\begin{aligned}
\|f(\mathbf{y}) - f(\mathbf{x})\|_Y &= \| \int_0^1 Df(t\mathbf{y} + (1-t)\mathbf{x}))(\mathbf{y} - \mathbf{x}) dt\|_Y \\
&\leq \| \int_0^1 Df\big(t\mathbf{y} + (1-t)\mathbf{x}\big)(\mathbf{y} - \mathbf{x}) dt\|_Y \\
&\leq \int_0^1 \|Df\big(t\mathbf{y} + (1-t)\mathbf{x}\big)(\mathbf{y} - \mathbf{x}) dt\|_X \\
&\leq \int_0^1 \|Df\big(t\mathbf{y} + (1-t)\mathbf{x}\big)\|_X, Y\|(\mathbf{y} - \mathbf{x}) dt\|_X \\
&\leq \int_0^1 \sup_{\mathbf{c}\in\ell(\mathbf{x},\mathbf{y})} \|Df\big(\mathbf{c}\big)\|_{X,Y}\|(\mathbf{y} - \mathbf{x}) dt\|_X \\
&\leq \sup_{\mathbf{c}\in\ell(\mathbf{x},\mathbf{y})} \|Df\big(\mathbf{c}\big)\|_{X,Y}\|(\mathbf{y} - \mathbf{x})\|_X
\end{aligned}$$

6.20

If we consider the function:

$$F(t) = \int_{g(c)}^t f(\tau) d\tau$$

Using the fundamental theorem of calculus $F'(t) = f(t)$, and then we use the chain rule for the following:

$$\int_c^d f(g(s))g'(s) ds = \int_c^d F'(g(s))g'(s) ds = \int_{g(c)}^{g(d)} DF(g(s)) ds$$

By the fundamental theorem of calculus:

$$= F(g(d)) - F(g(c)) = \int_{g(c)}^{g(d)} f(\tau) d\tau - \int_{g(c)}^{g(c)} f(\tau) d\tau = \int_{g(c)}^{g(d)} f(\tau) d\tau$$

6.21

If a sequence, $(f_n)_{n=0}^{\infty} \in C(U;Y)$, is cauchy then we know that $(f_n|_K)_{n=0}^{\infty} \in (C(K;Y), \|\cdot\|_{L^\infty})$ is also cauchy for all compact subsets $K \subset U$, and that $(f_n)_{n=0}^{\infty} \in C(U;Y)$ is uniformly convergent. Thus $(f_n|_K)_{n=0}^{\infty}$ converges to $f|_K$ in $(C(K;Y), \|\cdot\|_{L^\infty})$ for all described compact subsets.

Let these assumptions be true of the sequence $f_n \in C(U;Y)$. Thus we know that it will be true for all sequences that are in any open set that is a subset of the closed set, as this is.

6.22

(i)
There exists a derivative and is as follows for all $x \in [-1, 1]$:

$$f'(x) = \frac{x}{\sqrt{\frac{1}{n^2} + x^2}}$$

(ii)
We now that:

$$\sup_{(0,1)} f_n(x) = \sqrt{\frac{n^2+1}{n^2}}$$

We also know that any compact set is in $[a, b]$ where $0 < a < b < \sqrt{\frac{n^2+1}{n^2}}$, giving us that:

$$\|f_n(x)|_{[a,b]}\|_{L^\infty} = \sqrt{\frac{n^2+1}{n^2}} \to |x| \text{ as } n \to \infty$$

Thus proving that $f_n(x)$ converges uniformly to $|x|$ on $[-1, 1]$.

(iii)
If $f(x) = |x|$, then

$$f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

And this is discontinuous at $x = 0$, implying that it is not differentiable.

(iv)
The assumption thst $f_n(\mathbf{x}_*))_{n=0}^{\infty} \subset C^1(U;Y)$ does not converge in Y does not hold and thus the theorem holds.

6.23

If we have that:

$$S_k = \sum_{n=0}^{k} Df_n = D\sum_{n=0}^{k} f_n$$

Then $\{s_k\}_{k=0}^{\infty}$ converges assymptotically on U. We also know that:

$$t_k = \sum_{n=0}^{k} f_n(x_0)$$

And thus $\{t_k\}_{k=0}^{\infty}$ converges on Y.