Math 320 Homework 5.6

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December 7, 2015

5.24

$$\begin{aligned}
w_0 &= \int_{x_0}^{x_2} \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}dx \\
&= \frac{1}{(x_0-x_1)(x_0-x_2)} \int_{x_0}^{x_2} (x-x_1)(x-x_2)dx \\
&= \frac{1}{(x_0-x_1)(x_0-x_2)} \int_{x_0}^{x_2} x^2 - x_1x - x_2x + x_1x_2dx \\
&= \frac{1}{(x_0-x_1)(x_0-x_2)} \left[x^2 - x_1x - x_2x + x_1x_2\right]_{x_0}^{x_2} \\
&= \frac{1}{(x_0-x_1)(x_0-x_2)} \left[\frac{{x_2}^3}{3} - \frac{x_1x_2^2}{2} - \frac{x_2^3}{2} + x_1x_2^2 - \frac{x_0^3}{3} + \frac{x_1x_0^2}{2} + \frac{x_2x_0^2}{2} - x_1x_2x_0\right] \\
&= \frac{1}{h^2}\left[\frac{2{x_2}^3}{6} - \frac{3x_1x_2^2}{6} - \frac{3x_2^3}{6} + \frac{6x_1x_2^2}{6} - \frac{2x_0^3}{6} + \frac{3x_1x_0^2}{6} + \frac{3x_2x_0^2}{6} - \frac{6x_1x_2x_0}{6}\right] \\
&= \frac{1}{h^2}\left[-\frac{{x_2}^3}{6} - \frac{3x_1x_2^2}{6} + \frac{6x_1x_2^2}{6} - \frac{2x_0^3}{6} + \frac{3x_1x_0^2}{6} + \frac{3x_2x_0^2}{6} - \frac{6x_1x_2x_0}{6}\right] \\
&= \frac{1}{h^2}\left[\frac{-x_2^3 + 3x_1x_2^2 - 2x_0^3 + 3x_1x_0^2 + 3x_2x_0^2 - 6x_1x_2x_0}{6}\right] \\
&= \frac{1}{h^2}\left[\frac{-x_2^2(x_2 - x_1 - 2x_1) - x_0^2(2x_0 - 3x_1) + x_2x_0(4x_0 - 6x_1)}{6}\right] \\
&= \frac{1}{h^2}\left[\frac{-x_2^2(h - 2x_1) - x_0^2(2x_0 - 2x_1 - x_1) + x_2x_0(3x_0 - 3x_1 - 3x_1)}{6}\right] \\
&= \frac{1}{h^2}\left[\frac{-x_2^2(h - 2x_1) - x_0^2(-2h - x_1) + x_2x_0(-3h - 3x_1)}{6}\right] \\
&= \frac{1}{6}\frac{1}{h^2}\left[-x_2^2(h - 2x_1) - x_0^2(-2h - x_1) + x_2x_0(-3h - 3x_1)\right] \\
&= \frac{1}{6}\left[\frac{-x_2^2 + x_0^2 2 - x_2x_0}{h} + \frac{2x_2^2x_1 + x_0^2x_1 - 3x_1x_2x_0}{h^2}\right] \\
&= \frac{1}{6}\left[\frac{(x_0 - x_2)(2x_0 + x_2)}{h} + \frac{2x_2^2x_1 + x_0x_1(x_0 - x_2 - 2x_2}{h^2} + \right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{6}\left[-2(2x_0 + x_2) + \frac{2x_2^2 x_1 + x_0 x_1(x_0 - x_2 - 2x_2)}{h^2}\right] \\
&= \frac{x_2 - x_0}{6} \\
&= \frac{h}{3}
\end{aligned}
$$

5.26

For the case where k = 0, we have

$$\int_a^b 1 \, dx = b - a$$

Which implies

$$
\begin{aligned}
\frac{b-a}{3n}\big(1 + 4 + 2 + \cdots + 2 + 4 + 1\big) &= \frac{b-a}{3n}\big((1+4+1) + (1+4+1) + \cdots + (1+4+1)\big) \\
&= \frac{b-a}{3n} \cdot (3n) \\
&= b - a
\end{aligned}
$$

There are $\frac{n}{2}$ instances of 4 and summation is equal to $\frac{n}{2} \cdot (6) = 3n$.

For the case where k = 1, we have:

$$\int_a^b x \, dx = \frac{1}{2}(b^2 - a^2) = \frac{1}{2}(b-a)(b+a)$$

which yields

$$
\begin{aligned}
&\frac{b-a}{3n}\big(x_0 + 4x_1 + 2x_2 + \cdots + 2x_{n-2} + 4x_{n-1} + x_n\big) \\
&\quad = \frac{b-a}{3n}\big(x_0 + 4x_1 + 2x_2 + \cdots + 2x_{n-2} + 4x_{n-1} + x_n\big) \\
&\quad = \frac{b-a}{3n} \cdot (b+a)\big(1 + 4 + 2 + \cdots + 2 + 4 + 1\big) \\
&\quad = \frac{b-a}{3n} \cdot (b+a)\big((1+4+1) + (1+4+1) + \cdots + (1+4+1)\big) \\
&\quad = \frac{b-a}{3n} \cdot (b+a)(3n) \\
&\quad = b^2 + a^2
\end{aligned}
$$

implying

$$x_0 + x_n = x_1 + x_{n-1} = x_2 + x_{n-2} = \cdots = b + a$$

For the case where k = 2, we have:

$$\int_a^b x^2 \; dx = \frac{1}{3}(b^3 - a^3)$$

which yields

$$\begin{aligned}
&\frac{b-a}{3n}\big(x_0^2 + 4x_1^2 + 2x_2^2 + \cdots + 2x_{n-2}^2 + 4x_{n-1}^2 + x_n^2\big) \\
&\qquad = \frac{b-a}{3n}\big(x_0^2 + x_n^2 + 4x_1^2 + 4x_{n-1}^2 + 2x_2^2 + 2x_{n-2}^2 + \dots\big) \\
&\qquad = \frac{b-a}{3n}\big(x_0^2 + x_n^2 + 4(x_1^2 + x_{n-1}^2) + 2(x_2^2 + x_{n-2}^2) + \dots\big) \\
&\qquad = \frac{b-a}{3n} \cdot \frac{1}{3(b-a)}(b^3 - a^3)\big(1 + 4 + 2 + \cdots + 2 + 4 + 1\big) \\
&\qquad = \frac{b-a}{3n} \cdot \frac{1}{3(b-a)}(b^3 - a^3)\big((1+4+1) + (1+4+1) + \cdots + (1+4+1)\big) \\
&\qquad = \frac{b-a}{3n} \cdot \frac{1}{3(b-a)}(b^3 - a^3)(3n) \\
&\qquad = \frac{1}{3}(b^3 - a^3)
\end{aligned}$$

By the integral MVT,

$$\frac{1}{b-a}\int_a^b f(x)dx$$

For the case where k = 3, we have

$$\int_a^b x^3 \; dx = \frac{1}{4}(b^4 - a^4)$$

yielding

$$\begin{aligned}
&\frac{b-a}{3n}\big(x_0^2 + 4x_1^3 + 2x_2^3 + \cdots + 2x_{n-2}^3 + 4x_{n-1}^3 + x_n^3\big) \\
&\qquad = \frac{b-a}{3n}\big(x_0^3 + x_n^3 + 4x_1^3 + 4x_{n-1}^3 + 2x_2^3 + 2x_{n-2}^3 + \dots\big) \\
&\qquad = \frac{b-a}{3n}\big(x_0^3 + x_n^3 + 4(x_1^3 + x_{n-1}^3) + 2(x_2^3 + x_{n-2}^3) + \dots\big) \\
&\qquad = \frac{b-a}{3n} \cdot \frac{1}{4(b-a)}(b^4 - a^4)\big(1 + 4 + 2 + \cdots + 2 + 4 + 1\big) \\
&\qquad = \frac{b-a}{3n} \cdot \frac{1}{4(b-a)}(b^4 - a^4)\big((1+4+1) + (1+4+1) + \cdots + (1+4+1)\big) \\
&\qquad = \frac{b-a}{3n} \cdot \frac{1}{4(b-a)}(b^4 - a^4)(3n) \\
&\qquad = \frac{1}{4}(b^4 - a^4)
\end{aligned}$$

By the integral MVT,

$$\frac{1}{b-a}\int_a^b f(x)\ dx$$

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We know that:

$$t = kh^a$$

Logarithimically scaling, and evaluating the equation at two different values of t_1, t_2, we have

$$\log(t_1) = \log(k) + a\log(h_1)$$
$$\log(t_2) = \log(k) + a\log(h_2)$$

yielding

$$\frac{\log(t_1) - \log(t_2)}{\log(h_1) - log(h_2)} = a$$

By values in table 5.2, yields

$$\frac{\log(2.7 * 10^{-5}) - \log(1.7 * 10^{-6})}{\log(1024) - \log(4096)} \approx 2.00003$$

as for Simpson's method

$$\frac{\log(1.3162e-9) - \log(5.1585e-12)}{\log(1024) - \log(4096)} \approx 3.99997$$

yielding the desired result.