

Math 344 Homework 3.2

Chris Rytting

October 7, 2015

3.7

Consider

$$\begin{aligned}\|\text{proj}_x(v)\|^2 &= \sum_{i=1}^m \left| \frac{\langle x_i, v \rangle}{\|x_i\|} \right|^2 \leq \|v\|^2 \\ \implies \sum_{i=1}^m \frac{|\langle x_i, v \rangle|^2}{\|x_i\|^2} &\leq \|v\|^2 \\ \implies \sum_{i=1}^m |\langle x_i, v \rangle|^2 &\leq \|v\|^2 \|x_i\|^2 \\ \implies \sum_{i=1}^m |\langle x_i, v \rangle| &\leq \|v\| \|x_i\| \\ \implies |\langle x_i, v \rangle| &\leq \|v\| \|x_i\|\end{aligned}$$

Which is the desired result.

3.8 (i)

We show that the set is normal by showing that for $s_i \in S$ $\|s_i\| = 1$, and that it is orthogonal by showing that $\langle s_i, s_j \rangle = 0 \ \forall s_i, s_j \in S$ where $i \neq j$.

$$\begin{aligned}\langle \cos(x), \cos(x) \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(x) dx = \frac{1}{\pi} [x + \sin(x)\cos(x)]_0^{\pi} = 1 \\ \langle \sin(x), \sin(x) \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2(x) dx = \frac{1}{\pi} [x - \sin(x)\cos(x)]_0^{\pi} = 1\end{aligned}$$

implying normality. The same result follows for the other two elements of the set as the only discrepancy between them and these elements of the set is the period. As

for orthogonality

$$\begin{aligned}
\langle \cos(x), \sin(x) \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(x) \sin(x) dx = \left(\frac{1}{\pi}\right) \left(\frac{-1}{2}\right) \cos^2(x) \Big|_{-\pi}^{\pi} = 0 \\
\langle \cos(x), \cos(2x) \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(x) \cos(2x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(x) [\cos^2(x) - \sin^2(x)] dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^3(x) dx - \int_{-\pi}^{\pi} \sin^2(x) \cos(x) dx = \frac{1}{\pi} (0) = 0 \\
\langle \cos(x), \sin(2x) \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(x) \sin(2x) dx = \frac{1}{\pi} \left[\frac{-2}{3} \cos^3(x) \right]_{-\pi}^{\pi} = 0 \\
\langle \sin(x), \cos(2x) \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(x) \sin(x) dx - \int_{-\pi}^{\pi} \sin^3(x) dx = -\frac{1}{3\pi} \cos^3(x) \Big|_{-\pi}^{\pi} = 0 \\
\langle \sin(x), \sin(2x) \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(x) \sin(2x) dx = \frac{2}{3\pi} \sin^3(x) \Big|_{-\pi}^{\pi} = 0
\end{aligned}$$

Again, the result follows for the other two elements of the set because the other two elements are only a change in period from $\langle \cos(x), \sin(x) \rangle$

3.8 (ii)

$$\|t\| = \langle t, t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 dt = \frac{1}{3\pi} [t^3]_{-\pi}^{\pi} = \frac{2\pi^2}{3}$$

3.8 (iii)

$$\begin{aligned}
proj_X(\cos(3t)) &= \langle \cos(t), \cos(3t) \rangle \cos(t) + \langle \sin(t), \cos(3t) \rangle \sin(t) + \\
&\quad \langle \cos(2t), \cos(3t) \rangle \cos(2t) + \langle \sin(2t), \cos(3t) \rangle \sin(2t) \\
&= \frac{1}{\pi} \left(\int_{-\pi}^{\pi} \cos(t) \cos(3t) dt \cos(t) + \int_{-\pi}^{\pi} \sin(t) \cos(3t) dt \sin(t) + \right. \\
&\quad \left. \int_{-\pi}^{\pi} \cos(2t) \cos(3t) dt \cos(2t) + \int_{-\pi}^{\pi} \sin(2t) \cos(3t) dt \sin(2t) \right) \\
&= \frac{1}{\pi} (0 \cdot \cos(t) + \frac{-3\pi}{2} \sin(t) + 0 + 4\pi \sin(2t)) = \frac{-3}{2} \sin(t) + 4\sin(2t)
\end{aligned}$$

3.8 (iv)

$$\begin{aligned}
proj_X(t) &= \langle \cos(t), t \rangle \cos(t) + \langle \sin(t), t \rangle \sin(t) + \langle \cos(2t), t \rangle \cos(2t) + \langle \sin(2t), t \rangle \sin(2t) \\
&= \frac{1}{\pi} \left(\int_{-\pi}^{\pi} \cos(t)t \, dt \cos(t) + \int_{-\pi}^{\pi} \sin(t)t \, dt \sin(t) + \right. \\
&\quad \left. \int_{-\pi}^{\pi} \cos(2t)t \, dt \cos(2t) + \int_{-\pi}^{\pi} \sin(2t)t \, dt \sin(2t) \right) \\
&= \frac{1}{\pi} (2\pi \sin(t) - \pi \sin(2t)) = 2\sin(t) - \sin(2t)
\end{aligned}$$

3.9

We know that an orthonormal transformation preserves vector length. Here, as a counterexample, let $x, y \in V$ be two unit vectors that are orthogonal to X . In this case, $proj_X(y) = 0 \neq 1$ $proj_X(x) = 0 \neq 1$ and we have the desired result.

3.10

If we rotate two vectors x, y by θ , we preserve the angle between two vectors since we rotate them both by θ . Furthermore, we preserve vector length since we are only rotating them and not performing other operations on them. Therefore, this is an orthonormal transformation.

3.11 (i)

By properties of orthogonality,

$$\|Qx\|_2 = \langle Qx, Qx \rangle = (Qx)^H Qx = x^H Q^H Qx = x^H Ix = x^H x = \langle x, x \rangle = \|x\|_2$$

3.11 (ii)

By (v)

$$(Q_1 Q_2)(Q_1 Q_2)^H = Q_1 Q_2 Q_2^H Q_1^H = Q_1 Q_1^H = I$$

which implies orthogonality.

3.11 (iii)

Because Q is orthonormal, when we take the Hermetian of the inverse this results in the original Q allowing us to perform the following proof:

$$\langle Q^{-1}, Q^{-1} \rangle = Q^{-1} (Q^{-1})^H = Q^{-1} Q = I$$

3.11 (iv)

\Rightarrow By (iii), if Q is orthonormal, the resulting statement is true.

$$Q^H Q = Q^{-1} Q = Q Q^{-1} = I$$

\Leftarrow If $Q^H Q = Q Q^H = Q^{-1} Q = Q Q^{-1} = I$ implying orthonormality yielding that $\|Q\| = 1$ and is orthonormal.

3.11 (v)

Because we know that if Q is orthonormal then either the rows or the columns must be orthonormal. Suppose that the rows are orthonormal. Because we know that Q^T is also orthonormal then the columns of Q are orthonormal.

3.11 (vi)

We know that $Q^{-1} = Q \Rightarrow Q^{-1} Q = I \Rightarrow \det(Q^{-1} Q) = \det(I) = \det(Q^{-1}) \det(Q) = \det(I) \Rightarrow \det(Q^H) \det(Q) = \det(I) \Rightarrow \det(Q) \det(Q) = 1 \Rightarrow \det(Q) = \pm 1 \Rightarrow |\det(Q)| = 1$

The converse is not true. Take the matrix A as an example where

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}$$

Whose determinant is 1 but whose columns are not orthonormal.