Math 320 Homework 5.1

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5.1

We know that

$$B_n^k = \binom{n}{k}(1-x)^{n-k}x^k$$

Differentiating, we get

$$\binom{n}{k}(kx^{k-1}(1-x)^{n-k} - (n-k)x^k(1-x)^{n-k-1}) = 0$$

$$\implies k(1-x) - (n-k)x = 0$$

$$\implies k - kx - nx + kx = 0$$

and we have that $x = n/k$.

5.2

$$B_n[f](0) = \sum_{k=0}^{n} f(k/n)B_k^n(0)$$

$$= \sum_{k=1}^{n} f(k/n) * 0 + f(0) = f(0)$$

$$B_n[f](1) = \sum_{k=0}^{n-1} f(k/n)B_k^n(1)$$

$$= \sum_{k=0}^{n-1} f(k/n) * 0 + f(1) * 1$$

and by lemma 5.1.2

$$= f(1)$$

as desired.

5.3 (i)

$$\begin{aligned} B_n[1] &= \sum_{k=0}^{n} 1 \binom{n}{k} x^k (1-x)^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \end{aligned}$$

We know, though, that the Bernstein polynomials sum to 1 and we have the desired result.

5.3 (ii)

$$\begin{aligned} B_n[x] &= \sum_{k=0}^{n} \frac{k}{n} \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \\ &= \sum_{k=0}^{n} \frac{(n-1)!}{n(k-1)!(n-k)!} x^k (1-x)^{n-k} \\ &= \sum_{k=0}^{n} x \binom{n-1}{k-1} x^k (1-x)^{n-k} \\ &= x \sum_{k=0}^{n} \mathrm{B}_{k-1}^{n-1}(x) \\ &= x \end{aligned}$$

since the Bernstein polynomials sum to one.

5.3 (iii)

$$B_n[x^2] = \sum_{k=0}^{n} f(\frac{k}{n}) B_k^n(x)$$

$$WTS$$

$$\sum_{k=0}^{n} \frac{k^2}{n^2} \frac{n!}{(k!(n-k)!} x^k (1-x)^{n-k} = x^2 + \frac{x - x^2}{n}$$

$$\sum_{k=0}^{n} \frac{k}{n} \frac{(n-1)!}{(k-1)!(n-k)!} x^k (1-x)^{n-k} = x^2 + \frac{x - x^2}{n}$$

$$\sum_{k=0}^{n} k \frac{(n-1)!}{(k-1)!(n-k)!} x^k (1-x)^{n-k} = nx^2 + x - x^2$$

$$\sum_{k=0}^{n} k \frac{(n-1)!}{(k-1)!(n-k)!} x^k (1-x)^{n-k} = (n-1)x^2 + x$$

$$x \sum_{k=1}^{n} k \binom{n-1}{k-1} x^{k-1} (1-x)^{n-k} = (n-1)x^2 + x$$

Now, letting $j = k - 1$, we have

$$x \sum_{j=0}^{n-1} (j+1) \binom{n-1}{j} x^j (1-x)^{n-1-j} = (n-1)x^2 + x$$

which yields, by the binomial theorem and distributing through,

$$\begin{aligned} &= x \sum_{j=0}^{n-1} (j) \binom{n-1}{j} x^j (1-x)^{n-1-j} + x \\ &= (n-1)x^2 + x \end{aligned}$$

which is the desired result.

5.4

We have the system of equations:

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ -6 \\ 16 \end{bmatrix}$$

And by applying the inverse to both sides we get

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1/3 & -1/2 & 1 & -1/6 \\ 1/2 & -1 & 1/2 & 0 \\ -1/6 & 1/2 & -1/2 & 1/6 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ -6 \\ 16 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 2 \\ -2 \end{bmatrix}$$

Yielding the polynomial

$$p_3(x) = -2x^3 + 2x^2 - 2x - 4$$

5.5

Given that $p_3(x) = \sum_{j=0}^{3} f(x_j) L_{3,j}$.

We have $x = (-1, 0, 1, 2)$ and $f(x) = (2, -4, -6, -16)$ we find $L_{3,j}$:

$$L_{3,0} = \left(\frac{x}{-1}\right)\left(\frac{x-1}{-2}\right)\left(\frac{x-2}{-3}\right)$$
$$L_{3,1} = \left(\frac{x+1}{1}\right)\left(\frac{x-1}{-1}\right)\left(\frac{x-2}{-2}\right)$$
$$L_{3,2} = \left(\frac{x+1}{2}\right)\left(\frac{x}{1}\right)\left(\frac{x-2}{-1}\right)$$
$$L_{3,3} = \left(\frac{x+1}{3}\right)\left(\frac{x}{2}\right)\left(\frac{x-1}{1}\right)$$

Yielding

$$p_3(x) = \frac{-2}{6}\big(x(x-1)(x-2)\big) - \frac{4}{2}\big((x+1)(x-1)(x-2)\big) + \frac{6}{2}\big((x+1)(x)(x-2)\big) - \frac{16}{6}\big((x+1)(x)(x-1)\big)$$

Simplifying, we get

$$p_3(x) = -2x^3 + 2x^2 - 2x - 4$$