Math 344 Homework 4.3

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October 28, 2015

4.12

Given the characteristic polynomial $\lambda^2 - 1.4\lambda + .4$, we have that our eigenvalues are $\lambda_1 = 1, \lambda_2 = .4$, yielding

$$\begin{bmatrix} .8-1 & .4 \\ .2 & .6-1 \end{bmatrix} = \begin{bmatrix} -.2 & .4 \\ .2 & -.4 \end{bmatrix} = \begin{bmatrix} -.2 & .4 \\ 0 & 0 \end{bmatrix}$$

Whose null space is

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

As for the second eigenvalue, we have that

$$\begin{bmatrix} .8-.4 & .4 \\ .2 & .6-.4 \end{bmatrix} = \begin{bmatrix} .4 & .4 \\ .2 & .2 \end{bmatrix} = \begin{bmatrix} .4 & .4 \\ 0 & 0 \end{bmatrix}$$

whose null space is

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

And we have that

$$P = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} D = \begin{bmatrix} 1 & 0 \\ 0 & .4 \end{bmatrix} P^{-1} = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix}$$

4.13

From Exercise 2 we know that

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus the Charcteristic equation is given by

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{vmatrix} \Rightarrow \lambda^3 = 0 \Rightarrow \lambda = 0$$

The eigen vectors are then given by the nullspace of

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Because the number of eigenvectors is less than the rank of D, we know that it is not semi-simple.

4.14 (i)

Let

$$\begin{aligned} B = \lim_{k\to\infty} P^{-1}(A)^k P &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-2}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .4 \end{bmatrix}^k \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-2}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \end{aligned}$$

and let

$$C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E = D - C = \begin{bmatrix} .4 & 0 \\ 0 & 0 \end{bmatrix}$$

Note,

$$PE^x P^{-1} = \begin{bmatrix} \frac{2}{3}(.4^x) & 0 \\ \frac{2}{3}(.4)^x & 0 \end{bmatrix}$$

Now, it should be clear that $\|A^k - B\|_1 = \frac{4}{3}(.4)^k < \epsilon \quad \epsilon > 0$, and we have that $\exists N > 0$ such that

$$\|A^k - B\|_1 < \epsilon \quad k > N$$

4.14 (ii)

As for the infinity norm, we have that

$$\begin{aligned} \|A^k - B\|_\infty &= \|P(D^k - C)P^{-1}\| \\ &= \|PE^k P^{-1}\|_\infty \\ &= \frac{2}{3}(.4)^k \end{aligned}$$

And we see that for $\epsilon > 0 \quad \exists N > 0$, such that

$$\frac{2}{3}(.4)^k < \epsilon \quad k > N$$

As for the Frobenius norm,

$$\begin{aligned}\|A^k - B\|_F &= \|P(D^k - C)P_F^{-1}\| \\ &= \|PE^kP^{-1}\|_F \\ &= \frac{2}{3}.4^k\end{aligned}$$

And we see that for $\epsilon > 0 \quad \exists N > 0$, such that

$$\frac{2}{3}(.4)^k < \epsilon \quad k > N$$

The choice of norm is irrelevant.

4.14 (iii)

By proposition 4.3.11, we have that the expression

$$p(A) = 3 + 5A + A^3$$

is equivalent to

$$p(\lambda) = 3 + 5\lambda + \lambda^3$$

yielding

$$3 + 5 \cdot 1 + 1 = 9 \quad 3 + .4 \cdot 5 + .4^3 = 2.064$$

4.15

We know that

$$Ax = \lambda_i x \forall i$$

Now, consider that

$$f(\lambda_i)x = a_0\lambda_i^0 + a_1\lambda_i^1 + \cdots + a_n\lambda_i^n$$

and that

$$f(A)x = a_0A^0 + a_1A^1 + \cdots + a_nA^n$$

Now, by proposition 4.3.9, we know that

$$\lambda_i x = Ax \implies \lambda_i^k x = A^k x$$

and we have that

$$f(\lambda_i)x = a_0\lambda_i^0 + a_1\lambda_i^1 + \cdots + a_n\lambda_i^n = a_0A^0 + a_1A^1 + \cdots + a_nA^n$$

which is the desired result, that $f(A)x = f(\lambda_i)x$.

4.16

Let $A \in M_n(\mathbb{F}$, where $p(x)$ be characteristic polynomial.

Note that

$$A = PDP^{-1}$$

where D is a diagonal matrix. Now we have that

$$\begin{aligned}
p(x) &= a_0 A^0 + a_1 A^1 + \cdots + a_n A^n \\
&= a_0 (PDP^{-1})^0 + a_1 (PDP^{-1})^1 + \cdots + a_n (PDP^{-1})^n \\
&= P(a_0 D^0 + a_1 D^1 + \cdots + a_n D^n) P^{-1} \\
&= P \left(\begin{bmatrix} a_0 & 0 & \cdots & 0 \\ 0 & a_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 \end{bmatrix} + \begin{bmatrix} a_1\lambda_1 & 0 & \cdots & 0 \\ 0 & a_1\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_1\lambda_n \end{bmatrix} + \begin{bmatrix} a_2\lambda_1^2 & 0 & \cdots & 0 \\ 0 & a_2\lambda_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_2\lambda_n^2 \end{bmatrix} + \cdots + \begin{bmatrix} a_n\lambda_1^n & 0 & \cdots & 0 \\ 0 & a_n\lambda_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n\lambda_n^n \end{bmatrix} \right) P^{-1} \\
&= P \left(\begin{bmatrix} a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \cdots + a_n\lambda_1^n & 0 & \cdots & 0 \\ 0 & a_0 + a_1\lambda_2 + a_2\lambda_2^2 + \cdots + a_n\lambda_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 + a_1\lambda_2 + a_2\lambda_2^2 + \cdots + a_n\lambda_2^n \end{bmatrix} \right) P^{-1} \\
&= P \left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right) P^{-1} \\
&= 0
\end{aligned}$$

Which is the desired result.

Exercise 4.17 (i)

Note that

$$\ell A = \lambda \ell$$

Therefore,

$$\ell A r = \lambda \ell r$$

$$\ell (Ar) = \lambda \ell r$$

$$\ell \rho r = \lambda \ell r$$

$$\rho \ell r = \lambda \ell r$$

Since $\rho \neq \lambda$, the only way for this equality to hold is if $\ell r = 0$.

Exercise 4.17 (ii)

By Remark 4.3.18, if a matrix is semisimple, it has a basis of right eigenvectors $\{\mathbf{p_1}, ..., \mathbf{p_n}\}$, and those right eigenvectors form the columns of a matrix P which diagonalizes A. Furthermore, for each i the ith row $\mathbf{q_i}$ of P^{-1} is a left eigenvector of

A with eigenvalue d_i. Therefore, if we diagonalize the matrix A and multiply P^{-1} by P, we get the I, and we see that the rows $\mathbf{q_i}$ of P^{-1} multiplied by the columns $\mathbf{p_i}$ of P equals one at every entry of the identity's diagonal.

$$P^{-1}P = I$$

Exercise 4.17 (iii)

Consider the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

$$\begin{aligned} \lambda &= \rho = 2 \neq 0 \\ \ell &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq 0 \\ \mathbf{r} &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \neq 0 \\ \ell\mathbf{r} &= 0 \end{aligned}$$

Note that the eigenvectors are distinct and non-zero, while the eigenvalues are the same and non-zero. Their product, however, is zero.

4.18

Consider the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 0 & -2 \end{bmatrix}$$

Since the matrix is uppertriangular, the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -2$, yielding eigenvectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

Upon performing the power method, we have

$$\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} 2 & 3 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

$$\begin{aligned} \mathbf{x}_2 = A\mathbf{x}_1 &= \begin{bmatrix} 2 & 3 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \mathbf{x}_3 = A\mathbf{x}_2 &= \begin{bmatrix} 2 & 3 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 20 \\ -8 \end{bmatrix} = 4 \begin{bmatrix} 5 \\ -2 \end{bmatrix} \\ \mathbf{x}_4 = A\mathbf{x}_3 &= \begin{bmatrix} 2 & 3 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 20 \\ -8 \end{bmatrix} = \begin{bmatrix} 16 \\ 16 \end{bmatrix} = 16 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

$$\vdots$$

Clearly, the method is oscillating between two distinct eigenvectors. We attribute this to separate dominant eigenvalues.

Now consider a matrix with two linearly independent eigenvectors but a dominant eigenvector

$$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

Which has only one eigenvalue 5, yields two eigenvectors:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Starting with the initial guess from before, we get:

$$\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$
$$\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 25 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vdots$$

which converges to the wrong eigenvector.

We find that the power method requires a dominant eigenvalue with a geometric multiplicity of just 1.