Homework 1.3

Chris Rytting

September 9, 2015

1.15 (i)

We want to show that V is a vector space. Take any three vectors $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$, $\mathbf{y} = \{y_1, y_2, \ldots, y_n\}$, and $\mathbf{z} = \{z_1, z_2, \ldots, z_n\}$ where $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V \quad a, b \in \mathbb{R}$
(i)

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= \{x_1, x_2, \ldots, x_n\} + \{y_1, y_2, \ldots, y_n\} \\
&= \{x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n\} \\
&= \{y_1 + x_1, y_2 + x_2, \ldots, y_n + x_n\} \\
&= \mathbf{y} + \mathbf{x}
\end{aligned}$$

(ii)

$$\begin{aligned}
(\mathbf{x} + \mathbf{y}) + \mathbf{z} &= \{x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n\} + \{z_1, z_2, \ldots, z_n\} \\
&= \{x_1 + y_1 + z_1, x_2 + y_2 + z_2, \ldots, x_n + y_n + z_n\} \\
&= \{x_1, x_2, \ldots, x_n\} + \{y_1 + z_1, y_2 + z_2, \ldots, y_n + z_n\} \\
&= \mathbf{x} + (\mathbf{y} + \mathbf{z})
\end{aligned}$$

(iii)

$$\begin{aligned}
\mathbf{x} + \mathbf{0} &= \{x_1, x_2, \ldots, x_n\} + \{0, 0, \ldots, 0\} \\
&= \{x_1 + 0, x_2 + 0, \ldots, x_n + 0\} \\
&= \{x_1, x_2, \ldots, x_n\} \\
&= \mathbf{x}
\end{aligned}$$

(iv)

$$\begin{aligned}
\mathbf{x} + -\mathbf{x} &= \{x_1, x_2, \ldots, x_n\} - \{x_1, x_2, \ldots, x_n\} \\
&= \{x_1 - x_1, x_2 - x_2, \ldots, x_n - x_n\} \\
&= \{0, 0, \ldots, 0\} \\
&= \mathbf{0}
\end{aligned}$$

(v)

$$\begin{aligned}a(\mathbf{x}+\mathbf{y}) &= a(\{x_1, x_2, \dots, x_n\} + \{y_1, y_2, \dots y_n\})\\ &= a(\{x_1+y_1, x_2+y_2, \dots, x_n+y_n\})\\ &= \{ax_1+ay_1, ax_2+ay_2, \dots, ax_n+ay_n\}\\ &= a\{x_1, x_2, \dots, x_n\} + a\{y_1, y_2, \dots, y_n\}\\ &= a\mathbf{x} + a\mathbf{y}\end{aligned}$$

(vi)

$$\begin{aligned}(a+b)\mathbf{x} &= (a+b)\{x_1, x_2, \dots x_n\}\\ &= \{(a+b)x_1, (a+b)x_2, \dots (a+b)x_n\}\\ &= \{ax_1+bx_1, ax_2+bx_2, \dots ax_n+bx_n\}\\ &= a\{x_1, x_2, \dots x_n\} + b\{x_1, x_2, \dots x_n\}\\ &= a\mathbf{x} + b\mathbf{x}\end{aligned}$$

(vii)

$$\begin{aligned}1\mathbf{x} &= 1\{x_1, x_2, \dots x_n\}\\ &= \{1x_1, 1x_2, \dots 1x_n\}\\ &= \{x_1, x_2, \dots x_n\}\\ &= \mathbf{x}\end{aligned}$$

$(viii)$

$$\begin{aligned}(ab)\mathbf{x} &= (ab)\{x_1, x_2, \dots x_n\}\\ &= (a)\{bx_1, bx_2, \dots bx_n\}\\ &= a(b\mathbf{x})\end{aligned}$$

$\implies$ V is a vector space.

1.15 (ii)

By Theorem 1.3.21, for $V_i \quad \forall i$, there exists a basis $S_i = \{s_1, s_2, \dots, s_{m_i}\}$ where $m_i = \dim(V_i)$. Now, we know that $v \in V$ can be expressed as follows $\alpha_{(i)j} \in \mathbb{R}$:

$$v = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) = (\sum_{j=1}^{m_1} \alpha_{(1)j} S_{(1)j}, \sum_{j=m_1+1}^{m_2-m_1} \alpha_{(2)j} S_{(2)j}, \dots, \sum_{j=m_{n-1}+1}^{m_n-m_{n-1}} \alpha_{(n)j} S_{(n)j})$$

$$= (\sum_{j=1}^{m_1} \alpha_{(1)j} S_{(1)j}, 0, \dots, 0) + (0, \sum_{j=m_1+1}^{m_2-m_1} \alpha_{(2)j} S_{(2)j}, \dots, 0) + (0, 0, \dots, \sum_{j=m_{n-1}+1}^{m_n-m_{n-1}} \alpha_{(n)j} S_{(n)j})$$

With $\dim(\mathbf{v}_i) = m_i$, we have that

$$\implies \dim(V_1 \times V_2 \times \cdots \times V_n) = \sum_{i=1}^{n} \dim(V_i)$$

1.16

Let $W \in V$. by 1.3.16, we know that if there are bases $T = \{t_i\}_{i=1}^{n}$ and $S = \{x_i\}_{i=1}^{m}$ for V and W, respectively, then there exists $S' \in S$ having $m - n$ elements such that $T \cup S'$ is a basis for V. This suggests that t and s' consist of linearly independent vectors, $\implies T \cap S' = \{\mathbf{0}\}$, and since S' spans the rest of V, S' is a basis, implying the existence of a subspace X.

1.17

Consider the subspaces

$$\begin{aligned}
A_1 &= \{x^{ni}\}_{i=1}^{\infty} \\
A_2 &= \{x^{ni-1}\}_{i=1}^{\infty} \\
A_3 &= \{x^{ni-2}\}_{i=1}^{\infty} \\
&\vdots \\
A_{n-1} &= \{x^{ni-(n-1)}\}_{i=1}^{\infty}
\end{aligned}$$

1.18

Let $B = \mathrm{Sym}_n(\mathbb{F}), C = \mathrm{Skew}_n(\mathbb{F}), D = \mathrm{M}_n(\mathbb{F})$

1.18 (i)

Let $X, Y \in B \quad a, b \in \mathbb{R}$
We have, then, that $X^T = X, Y^T = Y$. Now, note that

$$\begin{aligned}
(aX + bY)^T &= aX^T + bY^T && (1) \\
&= aX + bY && (2)
\end{aligned}$$

and we have that

$$(aX + bY)^T = aX + bY$$

1.18 (ii)

Let $X, Y \in C \quad a, b \in \mathbb{R}$
We have, then, that $X^T = -X, Y^T = -Y$. Now, note that

$$\begin{aligned}(aX + bY)^T &= aX^T + bY^T && (3)\\ &= a(-X) + b(-Y) && (4)\\ &= -aX - bY && (5)\end{aligned}$$

and we have that

$$(aX + bY)^T = -(aX + bY)$$

1.18 (iii)

Let any square matrix $A = B_1 + C_1$ Let

$$B_1 = \frac{1}{2}(A + A^T) \quad C_1 = \frac{1}{2}(A - A^T)$$

We see that

$$B_1^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A^T + A) = \frac{1}{2}(A + A^T) = B_1$$

and that

$$C_1^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - A) = -C_1$$

$$\implies B_1 \in B, C_1 \in C$$

Now, by theorem 1.3.7, if A is a unique combination of $B' \in B$ and $C' \in C$, then $B + C$ is an internal direct sum. Suppose to the contrary, that there exist some $B_2 \in B \quad C_2 \in C$

$$s.t. \quad A = B_1 + C_1 = B_2 + C_2 \text{ where } B_2 \neq B_1 \quad C_2 \neq C_1$$

Now, since we proved in (i) and (ii) that B and C are subspaces, we know that

$$B_1 - B_2 \in B \quad C_1 - C_2 \in C$$

and by the definition of symmmetric and skew matrices,

$$B \cap C = \{\mathbf{0}\}$$

$$\implies B_1 - B_2 = C_1 - C_2 = \{\mathbf{0}\}$$

$$\implies B_1 = B_2 \text{ and } C_1 = C_2$$

$$\Rightarrow\Leftarrow$$

$$\implies M_n(\mathbb{F}) = B \oplus C$$

1.19

Let

$$f(x) = g(x) + h(x)$$
$$g(x) = \frac{1}{2}(f(x) + f(-x))$$

Which is even because $g(-x) = g(x)$

$$h(x) = \frac{1}{2}(f(x) - f(-x))$$

Which is odd because $g(-x) = -g(x)$ Now, to show uniqueness, assume to the contrary that there exists some even function $g(x)'$ and some odd function $h(x)'$ such that $g(x)' + h(x)' = f(x) = g(x) + h(x)$

$$\implies g(x) - g(x)' = h(x) - h(x)' = \mathbf{0}$$

since the zero function is the only odd and even function. However,

$$\implies g(x) = g(x)' \text{ and } h(x) = h(x)'$$
$$\Rightarrow\Leftarrow$$

Showing uniqueness.

To show that even functions are subspaces, let $f(x)$ and $g(x)$ be even functions. Now let

$$h(x) = af(x) + bg(x)$$

and note that

$$h(-x) = af(-x) + bg(-x) = af(x) + bg(x) = h(x)$$

$\implies h(x)$ is even.

To show that odd functions are subspaces, let $f(x)$ and $g(x)$ be odd functions. Now let

$$h(x) = af(x) + bg(x)$$

and note that

$$h(-x) = af(-x) + bg(-x) = -af(x) - bg(x) = -(af(x) + bg(x)) = -h(x)$$

$\implies h(x)$ is odd. Therefore, both the spaces of even and odd functions form subspaces.