Math 320 Homework 4.1

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4.1

Note that

$$\begin{aligned} f(x) + c &= a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) \\ &= (a_0 - c) + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) \\ &= \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) + c' \end{aligned}$$

Since $f(x) + c$ is an odd function, a_k must necessarily vanish since $\cos(kx)$ is an even function and $\sin(kx)$ is an odd function and an odd function cannot be the sum of both odd and even functions.

For the second case, note that

$$\begin{aligned} f(x) + c &= a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) \\ &= (a_0 - c) + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) \\ &= \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) + c' \end{aligned}$$

Since $f(x) + c$ is an even function, b_k must necessarily vanish since $\cos(kx)$ is an even function and $\sin(kx)$ is an odd function and an even function cannot be the sum of both odd and even functions.

4.2

We have that

$$\begin{aligned}
c_k &= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(x)e^{-ikx}dx\right) \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{0} f(x)e^{-ikx}dx + \int_{0}^{\pi} f(x)e^{-ikx}dx\right) \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{0} -xe^{-ikx}dx + \int_{0}^{\pi} xe^{-ikx}dx\right) \\
&= \frac{\frac{-1+e^{i\pi k}(1-i\pi k)}{k^2} + \frac{-1+e^{-i\pi k}(1+i\pi k)}{k^2}}{2\pi} \\
&= \frac{-(1-e^{i\pi k})}{\pi k^2}
\end{aligned}$$

and this gives us

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{-(1-e^{i\pi k})}{\pi k^2} e^{inx}$$

4.3

We have that

$$\begin{aligned}
c_k &= \frac{1}{2\pi}\left(\int_{0}^{2\pi} f(x)e^{-ikx}dx\right) \\
&= \frac{1}{2\pi}\left(\int_{0}^{\pi} f(x)e^{-ikx}dx + \int_{\pi}^{2\pi} f(x)e^{-ikx}dx\right) \\
&= \frac{1}{2\pi}\left(\int_{0}^{\pi} -e^{-ikx}dx + \int_{\pi}^{2\pi} e^{-ikx}dx\right) \\
&= \frac{-1}{2\pi k}(2ie^{-ik\pi} - ie^{-2ik\pi} - i)
\end{aligned}$$

and that

$$\begin{aligned}
||f||^2 &= \langle f, f\rangle \\
&= \left(\frac{1}{T}\int_{0}^{T} \overline{f(x)}f(x)dx\right) \\
&= \left(\frac{1}{2\pi}\int_{0}^{\pi} (-1)^2 dx + \int_{\pi}^{2\pi} (1)^2 dx\right) \\
&= \left(\frac{1}{2\pi}\int_{0}^{2\pi} dx\right) \\
&= \left(\frac{1}{2\pi}2\pi\right) \\
&= 1
\end{aligned}$$

and this gives us

$$f(x) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi} 2\pi \right) e^{inx}$$

4.4

We have that

$$\begin{aligned}
b_k &= \frac{1}{\pi} \int_0^{2\pi} (\pi - x)\sin(kx)dx \\
&= \frac{1}{\pi} \left(\int_0^{2\pi} \pi\sin(kx)dx - \int_0^{2\pi} x\sin(kx)dx \right) \\
&= \frac{1}{\pi} \left(0 - \int_0^{2\pi} x\sin(kx)dx \right) \\
&= -\frac{1}{\pi} \left(\int_0^{2\pi} x\sin(kx)dx \right) \\
&= -\frac{1}{\pi} \left(-\frac{2\pi}{k} \right) \\
&= \frac{2\pi}{k\pi} \\
&= \frac{2}{k}
\end{aligned}$$

As in the example, we find that the a_k's go to zero.
Furthermore, a_0 is given by

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{\pi - x}{\sqrt{2}} dx = 0$$

4.5 (i)

```
import numpy as np
from matplotlib import pyplot as plt


def g_n(x,n):
    total = 0
    for k in xrange(1,n+1):
        total += (2./k)*np.sin(k*x)
    f = np.pi - x
    total -= f
    return total
```

```
print g_n(5, 100)
x = np.linspace(0,2*np.pi, 100)
plt.plot(g_n(x,1))
plt.plot(g_n(x,2))
plt.plot(g_n(x,3))
plt.plot(g_n(x,10))
plt.plot(g_n(x,100))
plt.plot(g_n(x,1000))
plt.plot(g_n(x,10000))
plt.show()
```

4.5 (ii)

$$g_n(x) = (\pi - x) - \sum_{k=1}^{n} 2\text{sin}(x)$$
$$\implies g_n'(x) = -1 + \sum_{k=1}^{n} 2\text{cos}(x)$$

Now we want to show that

$$\frac{\sin\left((2n+1)\frac{x}{2}\right)\frac{x}{2}}{\sin(\frac{x}{2})} = -1 + \sum_{k=1}^{n} 2\text{cos(x)}$$

Note that this is true iff we have

$$\sum_{k=1}^{n} 2\text{cos}(x) = \frac{\sin\left((2n+1)\right)\frac{x}{2}}{\sin(\frac{x}{2})} + 1$$
$$\implies \sum_{k=1}^{n} 2\text{cos}(x)\text{sin}(\frac{x}{2}) = \sin((2n+1)\frac{x}{2}) + \sin(\frac{x}{2})$$

Now, we know by Brigg's Idenity that we have

$$\implies \sum_{k=1}^{n} \sin((k+\frac{1}{2})x) - \sin((k-\frac{1}{2})x) = \sin((2n+1)\frac{x}{2}) + \sin(\frac{x}{2})$$

Which telescopes to

$$-\text{sin}(\frac{-1}{2}x) + \sin((2n+1)\frac{x}{2}) = \sin((2n+1)\frac{x}{2}) + \sin(\frac{x}{2})$$
$$\implies \sin((2n+1)\frac{x}{2}) + \sin(\frac{x}{2}) = \sin((2n+1)\frac{x}{2}) + \sin(\frac{x}{2})$$

And we have that

$$\frac{\sin\left((2n+1)\frac{x}{2}\right)}{\sin(\frac{x}{2})} = -1 + \sum_{k=1}^{n} 2\text{cos}(x)$$

which is the desired result.

4.5(iii)

Note, the critical point will be when $G_n'(x) = 0$. Now, by (ii),

$$\frac{\sin\left((2n+1)\frac{x}{2}\right)}{\sin(\frac{x}{2})} = 0$$

$$\sin\left((2n+1)\frac{x}{2}\right) = 0 \quad \text{where } \sin(x) \neq 0$$

The first instance in which this will happen will be

$$(2n+1)\frac{x}{2} = \pi$$

$$\implies x = \frac{2\pi}{(2n+1)}$$

4.6

By FTFC, we have

$$g_n(\theta_n) - g_n(0) = \int_0^{\theta_n} \frac{\sin\left((2n+1)\frac{x}{2}\right)}{\sin(\frac{x}{2})} dx$$

Now,

$$g_n(0) = f(0) - \sum_{k=1}^{\infty} \frac{2}{k} \sin(0) = \pi$$

Thus,

$$g_n(\theta_n) = \int_0^{\theta_n} \frac{\sin\left((2n+1)\frac{x}{2}\right)}{\sin(\frac{x}{2})} dx - \pi$$

Finally, by integrating numerically in python, we have that the last result holds.