Math 320 Homework 3.5

Chris Rytting

October 22, 2015

3.27

$$
\begin{aligned}
E[X] &= \int_a^b x \frac{1}{b-a} dx \\
&= \frac{x^2}{2(b-a)}\Big|_a^b \\
&= \frac{b^2-a^2}{2(b-a)} \\
&= \frac{(a+b)(b-a)}{2(b-a)} \\
&= \frac{a+b}{2}
\end{aligned}
$$

$$
\begin{aligned}
V(X) &= E[X^2] - E[X]^2 \\
&= \int_a^b \frac{x^2}{b-a} - \left(\frac{b+a}{2}\right)^2 \\
&= \frac{x^3}{3(b-a)}\Big|_a^b - \left(\frac{b+a}{2}\right)^2 \\
&= \frac{b^3-a^3}{3(b-a)} - \left(\frac{b+a}{2}\right)^2 \\
&= \frac{(b-a)(a^2+ab+b_2)}{3(b-a)} - \left(\frac{b+a}{2}\right)^2 \\
&= \frac{(a^2+ab+b_2)}{3} - \frac{b^2+2ab+a^2}{4} \\
&= \frac{(4a^2+4ab+4b_2)}{3} - \frac{3b^2+6ab+3a^2}{4} \\
&= \frac{(4a^2+4ab+4b_2)-3b^2-6ab-3a^2}{12} \\
&= \frac{(a^2-2ab+b_2)}{12} \\
&= \frac{(b-a)^2}{12}
\end{aligned}
$$

3.28 (i)

$$\int_0^5 \frac{2}{15} e^{\frac{-2x}{15}} dx = .48658$$

3.28 (ii)

$$1 - \int_0^{15} \frac{2}{15} e^{\frac{-2x}{15}} dx = .135335$$

3.28 (iii)

$$\int_0^{15} \frac{(\frac{2}{15})^3 x^2 e^{\frac{-2x}{15}}}{\Gamma(3)} = .32332$$

3.29 (i)

We will maximize the p.d.f. to do so:

$$\frac{\partial}{\partial x}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)=\frac{1}{\sqrt{2\pi\sigma^2}}\left(-\frac{2(x-\mu)}{2\sigma^2}\right)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Which equals 0 at $x=\mu$, meaning that the p.d.f. is maximized at this point.

3.29 (ii)

Maximizing p.d.f., we have

$$\frac{\partial f_X(x)}{\partial x}=\frac{b^a(a-1)x^{a-2}e^{-xb}-b^ax^{a-1}be^{-xb}}{\Gamma(a)}=0$$

yields

$$b^a(a-1)x^{a-2}e^{-xb}=b^ax^{a-1}be^{-xb}$$

yielding

$$x=\frac{a-1}{b}<\frac{a}{b}=\mu$$

So the mode is less than the mean, as desired.

3.29 (iii)

$$\frac{\partial f_X(x)}{\partial x}=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\left((a-1)x^{a-2}(1-x)^{b-1}-x^{a-1}(b-1)(1-x)^{b-2}\right)$$

yields

$$(a-1)x^{a-2}(1-x)^{b-1}=x^{a-1}(b-1)(1-x)^{b-2}$$

yielding

$$x=\frac{a-1}{a+b-2}$$

which is less than μ when $a=1,b=2$ and greater than μ when $a=500,b=1$.

3.30 (i)

We have that the normal distribution is given by

$$\int_{\mu-k\sigma}^{\mu+k\sigma}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Now, making the substitution $u=\frac{x-\mu}{\sigma}$, and finding our new bounds given by

$$\frac{\mu+k\sigma-\mu}{\sigma}=k \text{ and } \frac{\mu-k\sigma-\mu}{\sigma}=-k$$

And our new integral is

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-k}^{k}\sigma e^{-\frac{u^2}{2}} = \frac{1}{\sqrt{2\pi}}\int_{-k}^{k}e^{-\frac{u^2}{2}}$$

Which is the standard normal distribution's p.d.f., the desired result.

3.30 (ii)

Probabilities that X lies in between $-k$ and k for

$$k = 1 : 0.68268949213708585$$

$$k = 2 : 0.95449973610364158$$

$$k = 3 : 0.99730020393673979$$

$$k = 4 : 0.99993665751633376$$

$$k = 5 : 0.99999942669685615$$

$$k = 6 : 0.9999999980268246$$

3.31

Using the beta distribution, we have that this probability is given by

$$\frac{\Gamma(6)}{\Gamma(3)\Gamma(3)}\int_{0}^{1/3} x^2 - 2x^3 + x^4 dx \approx .2099$$

3.32

Letting $y = xb$, we have that

$$\int \frac{b^a x^{a-1} e^{-xb} dx}{\Gamma(a)} = \int \frac{y^{a-1}e^{-y}}{\Gamma(a)} dy = \frac{1}{\Gamma(a)}\int y^{a-1}e^{-y} dy = \frac{\Gamma(a)}{\Gamma(a)} = 1$$

which is the desired result.