Math 344 Homework 5.3

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November 13, 2015

5.16

Suppose that $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ be Cauchy. Then

$$d(x, x_n) < \frac{\epsilon}{2}$$

and

$$d(y, y_n) < \frac{\epsilon}{2}$$

for some N. Note, by triangle inequality we have

$$|d(x_k, y_k) - d(x_j, y_j)| \leq |d(x_k, x_j) + d(y_k, y_j)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which is the desired result.

5.17

Suppose $\{x_i\}_{i=0}^{\infty}$ has a closer point $x \in X$. By Remark 5.2.33, there is a subsequence convergent to x.

Furthermore, by Proposition 5.3.8, the whole Cauchy sequence converges to the unique cluster point referred to in Prop 5.2.30.

5.18 (i)

x^3 is uniformly continuous on the interval $(0, 1)$, but not on the latter.

5.18 (ii)

$\frac{\sin(x)}{x}$ is uniformly and Lipshitz continuous on both intervals.

5.18 (iii)

$x\log(x)$ is uniformly continuous on the inverval $(0,1)$, but not on the latter. Let $\epsilon > 0$ be arbitrary, and let $\delta = \frac{\epsilon}{3}$. Then when $|x-y| < \delta$, we have

$$|x^3 - x^3| = |x-y|(x^2+xy+y^2) < 3|x-y| < \frac{3\epsilon}{3} = \epsilon \implies \text{ uniformly continuous}$$

On the other hand, for the other interval, note that for any fixed difference in $|x-y|$, there exists an x such that,

$$|x^2 - y^2| = x^2|1 - \left(\frac{y}{x}\right)^2| \not< \epsilon$$

5.19

Let $f : X \to Y$ be a bounded linear transformation. Note that for all unit vectors,

$$\sup \|f\mathbf{x}\| = \|f\| < M, \quad M \in \mathbb{N}$$

Let $\delta = \frac{\epsilon}{\|f\|}$. Then

$$\|f\mathbf{x}\| \leq \|f\|\|\mathbf{x}\|$$

$$\|f\mathbf{x} - f\mathbf{y}\| = \|f(\mathbf{x}-\mathbf{y})\| \leq \|f\|\|x-y\| < \epsilon$$

which is the desired result.

5.20

By way of contradiction, assume that there exists a Cauchy sequence in Z that does not converge to something in Z. Then, since Y is dense in Z, every $z \in Z$ is either a limit point of Y or found in Y.

Now, let $\{z_i\}_{i=0}^{\infty}$ be this sequence that does not converge in Z. We can see that $\exists j, n$ such that

$$d(z_j, z_n) = d(d(y_{j,i}, y_{j,k}, d(y_{n,m}, y_{n,p}) < \epsilon$$

By Exercise 5.16, distance is a real number, but is arbitrarily small in this case, and a limit point of Y, therefore we know that it is in Z, and the sequence converges to it in Z.

We have a contradiction and the desired result.

5.21 (i)

Suppose $f(B)$ is not bounded, then there exists an

$$\mathbf{x} \in B0 \text{ such that } f(x) > M \forall M \in \mathbb{N}$$

Then, given $d(x,y) < \delta$,

$$|f(x) - f(y)| \not< \epsilon \forall \epsilon > 0$$

Then by contrapositive, we have that if f is uniformly continuous, $f(B)$ is bounded.

5.21 (ii)

$f(\mathbf{x}) = \sqrt{\mathbf{x}}$ is continuous, but unbounded.

5.22

No. Let X be $\mathbb{R}$, and the metric on X be

$$d(x, y) = \frac{1}{2}, x \neq y, 0 \text{ otherwise}$$

Consider as a counterexample, the funciton $f(x) = x$ which is uniformly continious on this metric yet unbounded.