Math 344 Homework 5.6

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5.36

Let (x, d) be a metric space, suppose

$$f : [0, \infty) \to [0, \infty) \quad f(0) = 0, f'(x) > 0, f(a + b) \leq f(a) + f(b)$$

We know that this is positive semidefinite, and that $\delta(x, y) = \delta(y, x)$, therefore $f(\delta(x, y)) = f(\delta(y, x))$, and the triangle inequality holds by definition.

Now let $\varepsilon > 0$. Consider the open ball $B_\delta(x, \varepsilon)$. Note, the open ball $B_\rho(x, f(\varepsilon/2)) \subset B_\delta(x, \varepsilon)$. Consider that for B_ρ, $\exists \gamma \in [0, \infty)$ such that $f(\gamma) < \delta$, and the open ball $B_\delta(x, \gamma) \subset B_\rho(x, \varepsilon)$.

Thus, they are topologically equivalent.

5.37

We know that this is positive as $\|f^{-1}(\mathbf{y})\| \geq 0$. Thus it fulfills positivity.

Note that $\|a\mathbf{y}\|_f = |a|\|\mathbf{y}\|_f$ and since $\exists x \in Y$ such that $f(x) = \mathbf{y}$. Since isomorphisms are linear,

$$\|a\mathbf{y}\|_f = \|af^{-1}(\mathbf{y}\|_X = |a|\|f^{-1}(\mathbf{y})\|_X = |a|\|\mathbf{y}\|_F$$

and thus it fulfills scalar preservation.

Note that

$$\|\mathbf{x} + \mathbf{y}\|_f = \|f^{-1}(\mathbf{x}) + f^{-1}(\mathbf{y})\| \leq \|f^{-1}(\mathbf{x})\| + \|f^{-1}(\mathbf{y})\| = \|\mathbf{x}\|_f + \|\mathbf{y}\|_f$$

And thus it fulfills the triangle inequality.

5.38

$(\to)$ Suppose that two matrices, δ and ρ are topologically equivalent.

The identity mapping $I : (x, \delta) \to (x, \rho)$ is clearly bijective.

As for continuity, since these are topologically equivalent, we have

$$B_\delta(x, \epsilon) \subset B_\rho(x, \gamma) \quad \epsilon, \gamma > 0$$

$$\implies I(B_\delta(x, \epsilon)) \subset I(B_\rho(x, \gamma))$$

This holds for I^{-1}, therefore I is a homeomorphism.
($\leftarrow$) Now suppose I is a homeomorphism. Then it is bijective, continous, and its inverse is continuous. Thanks to continuity,

$$i(B_\delta(x, \epsilon)) = B + \rho(x, \gamma)$$

for some γ.

$$\implies B_\rho(x, \gamma/2) \subset B_\delta(x, \epsilon)$$
$$\implies I(B_\rho(x, \epsilon)) = B_\delta(x, r)$$
$$\implies B_\delta(x, \gamma/2) \subset B_\rho(x, \epsilon)$$

implying topological equivalence.

5.39

Let

$$f : S^1 \to \mathbb{R}$$

Now, S^1 can be uniquely expressed as $[0, 2\pi]$, so let $f : [0, 2\pi] \to \mathbb{R}$ where $f(0) = f(2\pi)$, where f is continuous.

Splitting f into two distinct functions by splitting the domain, letting

$$f_1(x) = f(x) \quad x \in [0, \pi]$$

$$f_2(x) = f(x + \pi, \quad x \in [0, \pi]$$

Note each is continuous and has the same beginning and end points.

Now let $g(x) = f_1(x) - f_2(x)$. g maps from $(f(0) - f(\pi), -(f(0) - f(\pi)))$. Therefore, by the IVT,

$$\exists c \quad \text{such that } g(c) = 0 \implies f_1(x) = f_2(x)$$

and there exists two equivalent anti-podal points.

5.40

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a homeomorphism. Then there exists a homeomorphism

$$\mathbb{R} \setminus 0 \to \mathbb{R}^2 \setminus f(0)$$

but this would suggest that $\mathbb{R}^2\ f(0)$ is connected. However, this is a contradiction of Theorem 5.6.25, and we have that it is not.

5.41

No. Consider the sets of invertible matrices which are continuous and open such that

$$S_1 = \{A \in M_2(\mathbb{R}) |\det(A) > 0\}$$

$$S_2 = \{A \in M_2(\mathbb{R}) |\det(A) < 0\}$$

Note that $S_1 \bigcup S_2 = M_2(\mathbb{R})$ and they are disjoint.