Math 344 HW 3.6

Chris Rytting

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3.31

By Lemma 3.6.1, to find the minimum let $x = 1$. From the proof for Young's inequality, let $x = 1 = \frac{a^{p-1}}{b}$,

$$ab = (\frac{a^p}{p} + \frac{b^q}{q})$$

Which will hold iff $b = a^{p-1}$. Raising both sides to the q, we have the following:

$$b^q = a^{pq-q} = a^p$$

Thus, equality holds iff $b^p = a^p$.

3.32

Let $\epsilon \geq 1$. By Young's inequality, we have the following:

$$\begin{aligned} ab &\leq \frac{a^2}{2} + \frac{b^2}{2} \\ &\leq \frac{\epsilon^2}{\epsilon}(\frac{a^2}{2} + \frac{b^2}{2}) \\ &\leq \frac{a^2 + \epsilon^2 b^2}{2\epsilon} \\ ab &\leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2} \end{aligned}$$

Now, suppose $\epsilon < 1$. Thus, by Young's inequality,

$$\begin{aligned} ab &\leq \frac{a^2}{2} + \frac{b^2}{2} \\ &\leq \frac{1}{\epsilon}(\frac{a^2+b^2}{2}) \\ &\leq \frac{a^2}{2\epsilon} + \frac{b^2}{2\epsilon} \\ ab &\leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2} \end{aligned}$$

In both of these cases, $ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$

3.33

Note that if $a = b$, then

$$a^\theta b^{1-\theta} = a^\theta a^{1-\theta} = a = \theta a + (1-\theta)a = \theta a + (1-\theta)b$$

On the other hand, if $a \neq b$

$$\begin{aligned} a^\theta b^{1-\theta} &\leq \theta a + (1-\theta)b \\ \ln(a^\theta b^{1-\theta}) &\leq \ln(\theta a + (1-\theta)b) \end{aligned}$$

Due to the convexity of ln, $\theta \ln(a) < \ln(\theta a)$. Therefore,

$$\begin{aligned} \ln(a^\theta b^{1-\theta}) &= \theta \ln(a) + (1-\theta)\ln(b) \\ &< \ln(\theta a) + \ln((1-\theta)b) \\ &< \ln(\theta a + (1-\theta)b) \end{aligned}$$

Thus, equality holds if and only if $a = b$.

3.34

If we let $\theta = \frac{1}{2}$, then

$$\begin{aligned} &a^{\frac{1}{2}} b^{\frac{1}{2}} \leq \frac{1}{2}(a+b) \\ &(ab)^{\frac{1}{2}} \leq \frac{1}{2}(a+b) \\ &(\mathrm{A})^{\frac{1}{2}} \leq \frac{1}{4}\mathrm{P} \\ &P \geq 4\sqrt{A} \end{aligned}$$

Where A is the area and P is the perimeter. The minimum of the area is where $P = 4\sqrt{A}$ where

$$2(a+b) = 4\sqrt{ab} \Rightarrow 4(a+b)^2 = 16(ab) \Rightarrow 4(a-b)^2 = 0 \Rightarrow a = b$$

3.35

Using the Arithmetic Geometric Mean inequality,

$$(x_1 \cdot ... \cdot x_n)^{\frac{1}{n}} \leq \frac{x_1 + \cdots + x_n}{n}$$

Where the two quantities are equal iff each $x_i = x_j \quad \forall i, j$.
The n-dimensional cube must have n vertices, and 2^{n-1} edges by theorem, so the total length of all vertices is going to be $2^{n-1}(x_1 + x_2 + \cdots + x_n)$, where each x is a length of a vertex. Volume is given by $2^n(x_1 \cdot \cdots \cdot x_n)^{\frac{1}{n}}$, Thus, the inequality yields

$$2^n(x_1 \cdot ... \cdot x_n)^{\frac{1}{n}} \leq 2^{n-1}\sum_{i=1}^{n} x_i$$
$$(x_1 \cdot ... \cdot x_n)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} x_i$$
$$\text{Area}^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} x_i$$

The minimum value is yielded when when these are equal.
Note, if $x_i = y$ for all i, then

$$y^n = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n}(n(y^n)) = y^n$$

Otherwise, we know that these will not be equal by our definition of the Arithmetic Geometric Mean inequality.
Thus, the n dimensional rectangle with the least perimeter and a fixed area will be the square.

3.36

Lemma: For $p > 1, a \geq 0$ we have $\|\mathbf{x}\|_p \geq \|\mathbf{x}\|_{p+a}$

Pf:

$$\sum_{k=1}^{n} \|\mathbf{x}\|^p = \sum_{k=1}^{n} \big((\|\mathbf{x}\|^{p+a})^{\frac{p}{p+a}}\big)$$
$$\geq \big(\sum_{k=1}^{n}(\|\mathbf{x}\|^{p+a})\big)^{\frac{p}{p+a}}$$

By Jensen's inequality, $\frac{p}{p+a} < 1 \implies$ convexity, implying

$$\big(\sum_{k=1}^{n} \|\mathbf{x}\|^p\big)^{\frac{1}{p}} \geq \big(\sum_{k=1}^{n}(\|\mathbf{x}\|^{p+a})\big)^{\frac{1}{p+a}}$$
$$\|\mathbf{x}_p\| \geq \|\mathbf{x}\|_{p+a}$$

Pf for 3.36: Note, $\frac{1}{\frac{p}{r}} + \frac{1}{\frac{q}{r}} = 1$

$$\big(\sum_{k=1}^{n} \|\mathbf{x}_k\mathbf{y}_k\|^r\big)^{1/r} \leq \big(\big(\sum_{k=1}^{n} \|\mathbf{x}_k\mathbf{y}_k\|\big)^r\big)^{\frac{1}{r}}$$

By Jensen's inequality, as $r > 1$

$$= \sum_{k=1}^{n} \|\mathbf{x}_k\mathbf{y}_k\|$$

By Holder's inequality we have

$$= \|\mathbf{x}\|_{\frac{p}{r}}\|\mathbf{y}\|_{\frac{q}{r}}$$

Note that $\frac{p}{r} < p$ and $\frac{q}{r} < q$. Therefore, by Lemma:

$$\|\mathbf{x}\|_{\frac{p}{r}}\|\mathbf{y}\|_{\frac{q}{r}} \leq \|\mathbf{x}\|_p\|\mathbf{y}\|_q$$