

# Problem Set 1

## Chapter 5

### 5.1

The following condition characterizes the optimal amount of cake to eat in period 1:

$$u(w_1 - \psi(w_1)) = u(w_1)$$

For period 2

$$w_{t+1} = \psi(w_1) = 0 \quad \forall w_1$$

### 5.2

Optimal amount of cake to leave for period 3 from period 2:

$$w_3 = \psi_2(w_2) = 0$$

$$\beta u'(w_2) = u'(w_1 - w_2)$$

### 5.3

If the individual lives for three periods  $T = 3$  what are the conditions that characterize the optimal amount of cake to leave for the next period in each period  $\{W_2, W_3, W_4\}$ ? Now assume that the initial cake size is  $W_1 = 1$ , the discount factor is  $\beta = 0.9$ , and the period utility function is  $\ln(c_t)$ . Show how  $\{c_t\}_{t=1}^3$  and  $\{W_t\}_{t=1}^4$  evolve over the three periods.

$$\beta u'(w_3) = u'(w_2 - w_3)$$

$$\beta u'(w_2 - \psi_2(w_2)) = u'(w_1 - w_2)$$

$$w_1 = 1$$

$$w_2 = .631$$

$$w_3 = .299$$

$$w_4 = 0$$

$$c_1 = .369$$

$$c_2 = .332$$

$$c_3 = .299$$

## 5.4

$$\beta v'_T(w_T) = u'(W_{T-1} - W_T)$$

$$W_T = \psi(W_{T-1}) : u(W_{T-1} - \psi(W_{T-1})) + \beta V_T(\psi_{T-1}(W_{T-1})) = 0$$

## 5.5

$$V_T = \ln(\bar{W}) \neq V_{T-1} = (\ln \bar{W} - \psi_{T-1}(\bar{W})) + \beta \ln(\psi_{T-1} \text{bar} W)$$

$$W_{T+1} = \psi_T(\bar{W}) = 0$$

$$W_T = \psi_{T-1}(\bar{W}) = \ln(\bar{W} - \psi_{T-1}(\bar{W})) + \beta \ln(\psi_{T-1}(\bar{W})) \neq 0$$

$$\implies \psi_{T-1}(\bar{W}) \neq \psi_T(\bar{W})$$

## 5.6

Solution for the period  $T - 2$  policy function for how much cake to save for the next period  $W_{T-1}$ :

$$W_{T-1} = \beta \frac{(1 + \beta)}{(1 + \beta + \beta^2)W} T_{T-2}$$

Analytical Solution for  $V_{T-2}$ :

$$V_{T-2} = \ln\left(\frac{1}{1 + \beta + \beta^2} W_{T-2}\right) + \beta \ln\left(\frac{\beta}{1 + \beta + \beta^2} W_{T-2}\right) + \beta^2 \ln\left(\frac{\beta^2}{1 + \beta + \beta^2} W_{T-2}\right)$$

## 5.7

Analytical solution for  $\psi_{T-s}(W_{T-s})$ :

$$\psi_{T-s}(W_{T-s}) = \frac{\sum_{i=0}^s \beta^i - 1}{\sum_{i=0}^s \beta^i}$$

Analytical solution for  $V_{T-s}(W_{T-s})$ :

$$V_{T-s}(W_{T-s}) = \sum_{i=0}^s \beta^i \ln\left(\frac{\beta^i}{\sum_{j=0}^s \beta^j} W_{T-s}\right)$$

Proofs:

$$\lim_{s \rightarrow \infty} \psi_{T-s}(W_{T-s}) = \lim_{s \rightarrow \infty} \frac{\frac{1}{1-\beta} - 1}{\frac{1}{1-\beta}} W_{T-s} = \lim_{s \rightarrow \infty} \frac{\frac{\beta}{1-\beta}}{\frac{1}{1-\beta}} W_{T-s} = \beta(W_{T-s}) = \psi(W_{T-s})$$

And as  $s \rightarrow \infty$ :

$$\begin{aligned}
V_{T-s}(W_{T-s}) &= \sum_{i=0}^s \beta^i \ln\left(\frac{\beta^i}{\sum_{j=0}^s \beta^j} W_{T-s}\right) = \sum_{i=0}^{\infty} \beta^i \ln\left(\frac{\beta^i}{\frac{1}{1-\beta}} W_{T-s}\right) - \sum_{i=0}^{\infty} \beta^i \ln(\beta^i (1-\beta) W_{T-s}) \\
&= \lim_{s \rightarrow \infty} \sum_{i=0}^s i \beta^i \ln(\beta) + \frac{\ln((1-\beta) W_{T-s})}{1-\beta} = \ln(\beta) \lim_{s \rightarrow \infty} \sum_{i=0}^s i \beta^i + \frac{\ln((1-\beta) W_{T-s})}{1-\beta} \\
&\implies V_{T-s}(W_{T-s}) \text{ converges to a function } V(W_{T-s}), \text{ which is independent of time.}
\end{aligned}$$

## 5.8

Bellman equation with infinite horizon:

$$V(W) = \max_{W' \in [0, W]} u(W - W') + \beta V(W')$$

## 5.11

$$\delta_T = 709.1159$$

## 5.12

$$\delta_{T-1} = 855.9830$$

$\delta_{T-1}$  is higher than  $\delta_T$

## 5.13

$$\delta_{T-2} = 838.5298$$

$\delta_{T-2}$  is higher than  $\delta_T$ , but lower than  $\delta_{T-1}$ .

## 5.18

$$\delta_T = 7,006,262.4138$$

## 5.19

$$\delta_{T-1} = 5,684,014.4287$$

$\delta_{T-1}$  is lower than  $\delta_T$

## 5.20

$$\delta_{T-2} = 4,609,092.4524$$

$\delta_{T-2}$  is lower than both  $\delta_{T-1}$  and  $\delta_T$

## 5.25

$$\delta_T = 24,819.0573$$

## 5.26

$$\delta_{T-1} = 85,327.4341$$

$\delta_{T-1}$  is higher than  $\delta_T$

## 5.27

$$\delta_{T-2} = 71,280.8333$$

$\delta_{T-2}$  is lower than  $\delta_{T-1}$  but not  $\delta_T$

# Chapter 4

## 4.1

$$\bar{k}2 = .028$$

$$\bar{k}3 = .091$$

$$\bar{c}1 = .214$$

$$\bar{c}2 = .223$$

$$\bar{c}3 = .232$$

$$\bar{w} = .242$$

$$\bar{r} = 2.192$$

## 4.2

$$\bar{k}2 = .041$$

$$\bar{k}3 = .117$$

$$\bar{c}1 = .226$$

$$\bar{c}2 = .240$$

$$\bar{c}3 = .255$$

$$\bar{w} = .268$$

$$\bar{r} = 1.818$$

Consumption increases,  $\bar{k}$  increases. This is due to the increase in patience. People will save more and therefore in the long-run will be able to consume more. The only variable which decreases is the interest rate, which also makes sense because more people are saving more.

## 4.4

$T = 7$  before within .0001 of steady state.