Homework 2.2 Math 344

Chris Rytting

September 17, 2015

2.8

Assume to the contrary that L_1L_2 is invertible. Then $(L_1L_2)^{-1} = (L_2)^{-1}(L_1)^{-1}$ which is a contradiction since L_1 is not invertible and therefore L_1^{-1} does not exist.

2.9 (i)

Our inductive hypothesis is that $\mathscr{N}(A^{n-1}) = \mathscr{N}(A^k)$. We want to show that $\mathscr{N}(A^n) = \mathscr{N}(A^k)$. To do so we need to show that $\mathscr{N}(A^n) \subset \mathscr{N}(A^k)$ and that $\mathscr{N}(A^k) \subset \mathscr{N}(A^n)$. The second hypothesis we have by exercise 2.7. As for the first, Let $x \in \mathscr{N}(A^n)$, which implies that

$$\begin{aligned}
&A^n(x) = 0 \\
&\implies A^{n-1}(A(x)) = 0 \\
&\implies A(x) \in \mathscr{N}(A^{n-1}) \\
&\implies A(x) \in \mathscr{N}(A^k) \text{ (by the inductive hypothesis)} \\
&\implies A^k(A(x)) = 0 \\
&\implies A^{k+1}(A(x)) = 0 \\
&\implies x \in \mathscr{N}(A^{k+1}) = \mathscr{N}(A^k) \\
&\implies x \in \mathscr{N}A^k \quad x \in \mathscr{N}A^n \\
&\implies \mathscr{N}(A^n) \subset \mathscr{N}(A^k) \\
&\implies \mathscr{N}(A^n) = \mathscr{N}(A^k)
\end{aligned}$$

2.9 (ii)

We know that $\mathscr{R}(A^{k+1}) = \mathscr{R}(A^k)$, and we want to show that $\mathscr{R}(A^n) = \mathscr{R}(A^k)$ by showing that $\mathscr{R}(A^n) \subset \mathscr{R}(A^k)$ and that $\mathscr{R}(A^k) \subset \mathscr{R}(A^n)$. The first we've already proved in exercise 2.7. As for the second, Our inductive hypothesis is that $\mathscr{R}(A^{k-1}) = \mathscr{R}(A^k)$. Now let $x \in \mathscr{R}(A^k) = \mathscr{R}(A^{n-1}) \implies A(x) \in A^n(x) \implies A(x) \in \mathscr{R}A^n$ We also have that $x \in \mathscr{R}(A^k) \implies x = A^k(z) \implies A(x) = (A^{k+1}(z)) \implies A(x) \in$

$\mathscr{R}(A^{k+1})$

$$\implies A(x) \in \mathscr{R}(A^k) \quad A(x) \in \mathscr{R}(A^n) \quad \forall x \in \mathscr{R}(A^k) \implies \mathscr{R}(A^k) = \mathscr{R}(A^k)$$

2.9 (iii)

□IMG_1297.JPG

2.10

Let $A' : \mathscr{R}(B) \to W$ By the Rank-Nullity theorem, we have that $\dim(\mathscr{R}(B)) = \text{rank}(A') + \text{nullity}(A') \implies \text{rank}(B) = \text{rank}(A') + \text{nullity}(A')$. Now we have to show that

$$\mathscr{R}(A') = \mathscr{R}(AB) \quad \text{and that}$$

$$\mathscr{N}(A') = \mathscr{N}(A) \cap \mathscr{R}(B)$$

For the first equality, take an arbitrary $x \in \mathscr{R}(A')$. x is mapped from $\mathscr{R}(B)$ to W. An arbitrary $y \in \mathscr{R}(AB)$ is mapped from $U \to \mathscr{R}(B) \to W \implies \mathscr{R}(A') = \mathscr{R}(AB)$ For the second equality, take an arbitrary $x \in \mathscr{N}(A'), y \in \mathscr{N}(A) \cap \mathscr{R}(B)$. The $\mathscr{N}A'$ is everything that is in the range of B but mapped to zero. The second term is every y s.t. $B(y) = 0 y \in V$ that is mapped to zero but is also in V. Therefore, it is clear that

$$\mathscr{R}(A') = \mathscr{R}(AB)$$

$$\mathscr{N}(A') = \mathscr{N}(A) \cap \mathscr{R}(B)$$

2.11 (i)

$$\text{Need to show rank}(AB) \leq \text{rank} A \text{and} rank(AB) \leq \text{rank}(B).$$

We know that $\text{rank}(AB) = \text{rank}B - \dim(\mathscr{N}(A) \cap \mathscr{R}(B))$. $\dim(\mathscr{N}(A) \cap \mathscr{R}(B))$ cannot be negative, so we know that $\text{rank}(AB) \leq \text{rank}B$. As for $\text{rank}(A)$, we know that

$$\text{rank}(AB) = \text{rank}(B) - \dim(\mathscr{N}(A) \cap \mathscr{R}(B)) \leq \text{rank}(A)$$

$$\implies \text{rank}(B) \leq \dim(\mathscr{N}(A) \cap \mathscr{R}(B)) + \text{rank}(A)$$

Case one: $\dim \mathscr{N}(A) > \dim \mathscr{R}(B) : \dim(\mathscr{N}(A) \cap \mathscr{R}(B)) \leq \dim \mathscr{R}(B) = \text{rank}(B)$ so we have that

$$\text{rank}(B) \leq \text{rank}(A) + \text{rank}(B) \text{ if and only if } 0 \leq \text{rank}(A)$$

Case two: $\dim(\mathscr{N}(A) \cap \mathscr{R}(B)) \leq \dim \mathscr{N}(A)$, so

$$\text{rank}(B) \leq \text{rank}(A) + \text{nullity}(A) = \dim(A) \text{by rank nullity}$$

$$\mathscr{R}(B) \subset V \implies \text{rank}B \leq \dim V \implies \text{rank}(AB) \leq \text{rank}(A)$$

2.11 (ii)

We want to show that

$$\text{rank}(A) + \text{rank}(B) - \text{dim}V \leq \text{rank}(AB)$$

Note that

$$\begin{aligned}
\text{rank}(A) + \text{rank}(B) - \text{nullity}A - \text{rank}A = \text{rank}B - \text{dim}\mathscr{N}(A) &\leq \text{rank}(AB) \\
\implies \text{rank}B - \text{dim}\mathscr{N}(A) &\leq \text{rank}B - \text{dim}(\mathscr{N}(A) \cap \mathscr{R}(B)) \\
\implies -\text{dim}\mathscr{N}(A) &\leq -\text{dim}(\mathscr{N}(A) \cap \mathscr{R}(B)) \\
\implies \text{dim}\mathscr{N}(A) &\leq \text{dim}(\mathscr{N}(A) \cap \mathscr{R}(B)) \\
\text{which is true} \quad \implies \text{rank}(A) + \text{rank}(B) - \text{dim}V &\leq \text{rank}(AB)
\end{aligned}$$

2.12

For the mapping L, and $a, b \in \mathbb{R},\ x, y \in V$ we have the following:

$$\begin{aligned}
L(ax+by) &= \Big((ax+by) + W_1, (ax+by) + W_2, \ldots, (ax+by) + W_n\Big) \\
aL(x) + bL(y) &= a\Big(x + W_1, x + W_2, \ldots, x + W_n\Big) + b\Big(y + W_1, y + W_2, \ldots, y + W_n\Big) \\
&= \Big(ax + W_1, ax + W_2, \ldots, ax + W_n\Big) + \Big(by + W_1, by + W_2, \ldots, by + W_n\Big) \\
&= \Big((ax+by) + W_1, (ax+by) + W_2, \ldots, (ax+by) + W_n\Big)
\end{aligned}$$

$\implies L$ is a linear transformation.

To show that $\mathscr{N}(L) = \cap_{i=1}^{n} W_i$, we notice that:

$$\begin{aligned}
\mathscr{N}(L) &= \{x \in V \mid x + W_i = 0 + W_i \ \forall\ i\} = \{x \in V \mid x \in W_i \ \forall\ i\} \\
&= W_1 \cap W_2 \cap \cdots \cap W_n = \cap_{i=1}^{n} W_i
\end{aligned}$$

2.13

If V is a vector space and $S \subset T \subset V$, and $L : V/S \to V/T$ such that $L(x+S) = (x+T)$.

We know that the mapping L is well defined because $S \subset T$ and for any $(x+T) \in V/T$

the mapping is given by $(x+T) \in V/S$. This allows us to say that the mapping L is surjective.

We now want to show that our mapping L is linear:

$$\begin{aligned}
\mathscr{N}(L) &= \{(x+S) \in V/S \mid L(x+S) = (x+T)\} \\
&= \{(x+S) \in V/S \mid x \in T\} \\
&= \{x+S \in T/S\}
\end{aligned}$$

$\implies$ L is linear and by the FIT we have $\frac{V/S}{T/S}$ is isomorphic to V/T.