Math 344 Homework 6.3

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6.12

i

$$\begin{aligned}
Df(x) &= DAx \\
&= D\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
&= D\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}
\end{aligned}$$

Since the Jacobian is the matrix representation of the linear map $Df(x)$, we have

$$\begin{aligned}
D\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} &= \begin{bmatrix} D_1f_1(x) & D_2f_1(x) & \cdots & D_nf_1(x) \\ D_1f_2(x) & D_2f_2(x) & \cdots & D_nf_2(x) \\ \vdots & \vdots & \ddots & \vdots \\ D_1f_m(x) & D_2f_m(x) & \cdots & D_nf_m(x) \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \\
&= A
\end{aligned}$$

ii

$$
\begin{aligned}
Df(x) &= Dx^T A \\
&= D\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \\
&= D\begin{bmatrix} a_{11}x_1 + \cdots + a_{m1}x_n & a_{12}x_1 + \cdots + a_{m2}x_n & \cdots & a_{1n}x_1 + \cdots + a_{mn}x_n \end{bmatrix} \\
&= D\begin{bmatrix} f_1 & f_2 & \cdots & f_n \end{bmatrix}
\end{aligned}
$$

by similar logic as **i**, we have that

$$
\begin{aligned}
D\begin{bmatrix} f_1 & f_2 & \cdots & f_n \end{bmatrix} &= \begin{bmatrix} D_1 f_1(x) & D_2 f_1(x) & \cdots & D_n f_1(x) \\ D_1 f_2(x) & D_2 f_2(x) & \cdots & D_n f_2(x) \\ \vdots & \vdots & \ddots & \vdots \\ D_1 f_m(x) & D_2 f_m(x) & \cdots & D_n f_m(x) \end{bmatrix}^T \\
&= A^T
\end{aligned}
$$

6.13 (i)

Let $f(x) = u(x)^T v(x)$. It suffices to show that

$$
\lim_{h \to 0} \frac{\|u(x+h)^T v(x+h) - u(x)v(x) - u(x)^T Dv(x)h - v(x)^T Du(x)h\|}{\|h\|} = 0
$$

By Proposition 6.2.17, u, v are locally Lipschitz, therefore we have

$$
\|u(x+h) - u(x)\| \leq L\|h\| \quad \|v(x+h) - v(x)\| \leq L\|h\|
$$

implying

$$
\begin{aligned}
\|u(x+h) - u(x)^T - Du(x)\| &\leq \frac{\epsilon\|h\|}{3(\|v(x)\| + 1)} \\
\|v(x+h) - v(x)^T - Dv(x)\| &\leq \frac{\epsilon\|h\|}{3(\|u(x)\| + 1)}
\end{aligned}
$$

$\|h\| < \delta_x$ implies that both of these derivatives exist.

Given $\epsilon > 0$, let $\delta = \min\left\{\delta_x, \frac{\epsilon}{3L\|Dv(x)\|}\right\}$
then $\|h\| < \delta$ implies

$$
\begin{aligned}
&\|u(x+h)^T v(x+h) - u(x)v(x) - u(x)^T Dv(x)h - v(x)^T Du(x)h\| \\
&\leq \|u(x+h)^T\|\|v(x+h) - v(x) - Dv(x)h\| + \|v(x)\|\|u(x+h) - u(x)^t - Du(x)h\| \\
&\qquad + \|u(x+h)^T - u(x)^T\|\|Dv(x)h\|\|h\| \\
&\leq \|u(x)^T + L\|\|v(x+h) - v(x) - Dv(x)h\| + \|v(x)\|\|u(x+h)^T - u(x)^T - Du(x)h\| \\
&\qquad + \delta L\|Dv(x)\|\|h\| \\
&< \epsilon\|h\|
\end{aligned}
$$

6.13 (ii)

Given $f(x) = x^T \quad g(x) = Ax$, we have

$$D(fg) = x^T A + x^T A^T = x^T(A + A^T)$$

by (i).

6.13 (iii)

Let $f(x) = Bw$. Then

$$\begin{aligned} Df(x) &= B(x)Dw(x) + w^T DB(x)^T \\ &= B(x)Dw(x) + \begin{bmatrix} w^T Db_1^T(x) \\ \vdots \\ w^T(x)Db_k^T(x) \end{bmatrix} \end{aligned}$$

by (i).

6.14

We want to show that the inner product between the two of these is zero. Keep in mind that our field is the reals, so a hermitian is analogous to a transpose. Also, the derivative of a constant is 0. Then given the standard inner product, we have

$$\begin{aligned} \int \overline{DF(\gamma(t))^T}\gamma'(t)dt &= \int DF(\gamma(t))\gamma'(t)dt \\ &= \int DF(\gamma(t))D\gamma(t)\gamma'(t)dt \\ &= \int DC \cdot D\gamma(t)\gamma'(t)dt \\ &= \int 0 \cdot D\gamma(t)\gamma'(t)dt \\ &= 0 \end{aligned}$$

showing orthogonality, the desired result.

6.15

Note,

$$D(g \circ f) = D(g(f(\mathbf{x})))D(f(\mathbf{x}))$$

$$D(g(x)) = \begin{bmatrix} y & x \\ 2x & 2y \end{bmatrix}$$

$$D(f(x)) = \begin{bmatrix} -1 & cos(y) \\ e^x & -1 \end{bmatrix}$$

We also know that

$$f(0,0) = (0,1)$$

yielding

$$Dg(f(0,0)) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$$

6.16

We have that

$$\begin{aligned} \langle f, g \rangle &= xy(\sin(y) - x) + (e^x - y)(x^2 + y^2) \\ &= xy\sin(y) - x^2y + x^2e^x + y^2e^x - yx^2 - y^3 \end{aligned}$$

Differentiating, we have

$$D(\langle f, g \rangle) = \begin{bmatrix} y\sin(y) - 2xy + 2xe^x + x^2e^x + y^2e^x - 2xy \\ x\sin(y) + xy\cos(y) - x^2 + 2ye^x - x^2 - 3y^2 \end{bmatrix}$$

Evaluating at $(0,0)$ yields

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

6.17

By the chain rule,

$$Df(\mathbf{x}_0) = 2\|A\mathbf{x}_0 - \mathbf{b}\|_2 \cdot \nabla\|A\mathbf{x}_0 - \mathbf{b}\|_2 \cdot A$$