Math 320 Homework 4.3

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4.15

By the definition of the characteristic function of the random variable X, we have that

$$\Phi_X(t) = E[e^{itX}] = \int_{-\infty}^{\infty} p(x)e^{itx}dx$$

Now, by the definition of the fourier transform of p to be

$$\hat{p}(x) = \int_{-\infty}^{\infty} e^{-itx}p(x)dx \implies \hat{p}(-x) = \int_{-\infty}^{\infty} e^{itx}p(x)dx$$

and we have the desired result, that

$$\Phi_X(t) = E[e^{itX}] = \int_{-\infty}^{\infty} p(x)e^{itx}dx = \hat{p}(-x)$$

4.16 (i)

$$\phi_X(0) = E[e^{i0X}] = E[e^0] = E[1] = 1$$

4.16(ii)

$$\begin{aligned}
\phi_{-X}(T) &= E[e^{it(-X)}] \\
&= \int_{\infty}^{-\infty} f_X(x)e^{it(-X)}dx \\
&= \int_{\infty}^{-\infty} f_X(x)e^{-itX}dx \\
&= \overline{\int_{\infty}^{-\infty} f_X(x)e^{itX}dx} \\
&= \overline{\phi_X(t)}
\end{aligned}$$

4.16(iii)

$$
\begin{aligned}
\phi_Z(t) &= \phi_{X+Y}(t) \\
&= E[e^{it(X+Y)}] \\
&= E[e^{itX+itY}] \\
&= E[e^{itX}e^{itY}] \\
&= E[e^{itX}]E[e^{itY}] \\
&= \phi_X(t)\phi_Y(t)
\end{aligned}
$$

4.16(iv)

$$
\begin{aligned}
\phi_{aX}(t) &= E[e^{itaX}] \\
&= E[e^{iatX}] \\
&= \phi_X(at)
\end{aligned}
$$

4.17

Note,

$$
\begin{aligned}
\hat{f}_Z(t) &= \phi_Z(-t) \\
&= \phi_X(-t)\phi_Y(-t) \\
&= \hat{f}_X(t)\hat{f}_Y(t) \\
&= \hat{f}(f_X(t) * f_Y(t))
\end{aligned}
$$

Now, as the convolution is invertible, this can be expressed as

$$f_Z(t) = f_X(t) * f_Y(t)$$

4.18

By the stretch theorem and example 4.3.3, and letting $t = x - \mu$, we have that

$$
\begin{aligned}
\phi_N(\frac{x-\mu}{\sigma}) &= \hat{f}_N(\frac{-(x-\mu)}{\sigma}) \\
&= e^{\frac{-(x-\mu)^2}{2\sigma^2}}
\end{aligned}
$$

Note that, given the random variable $Y = \sum \frac{X_i - \mu}{\sqrt{n}\sigma^2}$, let $\bar{x} = x - \mu$. The variance will be the same, but we will have that $\mu_{\bar{x}} = 0$. Therefore, we see that

$$\begin{aligned}
\phi_Y(t) &= \phi_{\sum \frac{\bar{x}}{\sqrt{n}\sigma^2}}(t) \\
&= \phi_{\bar{x}}(\frac{t}{\sqrt{n}\sigma^2})^n \\
&= \int_{-\infty}^{\infty} e^{\frac{-it\bar{x}}{\sqrt{n}\sigma^2}} f_{\bar{x}}(\bar{x}) d\bar{x} \\
&= \int_{-\infty}^{\infty} (1 - \frac{it\bar{x}}{\sqrt{n}\sigma^2} + \frac{(it\bar{x})^2}{2!n\sigma^4} - \frac{it\bar{x}}{3!n^{\frac{3}{2}}\sigma^6} + \dots) f_{\bar{x}}(\bar{x}) d\bar{x} \\
&= \int_{-\infty}^{\infty} f_{\bar{x}} d\bar{x} - \frac{i\bar{x}}{\sqrt{n}\sigma^2} \int_{-\infty}^{\infty} \bar{x} f_{\bar{x}} d\bar{x} - \frac{\bar{x}^2}{2n\sigma^4} \int_{-\infty}^{\infty} \bar{x}^2 f_{\bar{x}} d\bar{x} + \frac{i\bar{x}^3}{2n^{\frac{3}{2}}\sigma^6} \int_{-\infty}^{\infty} f_{\bar{x}}(\bar{x}^3 \dots) d\bar{x} \\
&= 1 - \frac{i\bar{x}}{\sqrt{n}\sigma^2}\mu_{\bar{x}} - \frac{\bar{x}^2\sigma^2}{2n\sigma^4} + h(n,t) \quad \text{where we know } h \to 0 \text{ and } \mu_{\bar{x}} = 0 \\
&= \left(\phi_{\bar{x}}\left(\frac{\bar{x}}{\sqrt{n}\sigma^2}\right)\right)^n \to e^{\frac{-(\bar{x})}{2\sigma^2}}
\end{aligned}$$

Now, by the stretch theorem,

$$\hat{f}_Y(-\bar{x}) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\bar{x}^2}{2\sigma^2}}$$

and substituting in for $\bar{x}$,

$$\hat{f}_Y(-(x-\mu)) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

and

$$P(Y < S) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{S} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

4.19

Calculating fourier of PDF, we have

$$\begin{aligned}
g(\xi) &= \frac{\hat{f}}{\int_{-\infty}^{\infty} \hat{f} d\xi} \\
&= \frac{\int_{-\infty}^{\infty} \sqrt{2\pi} e^{-i\xi t} e^{-\frac{t^2}{2\sigma^2}} dt}{\sqrt{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\xi t} e^{-\frac{t^2}{2\sigma^2}} dt d\xi} \\
&= \frac{\int_{-\infty}^{\infty} e^{-i\xi t} e^{-\frac{t^2}{2\sigma^2}} dt}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\xi t} e^{-\frac{t^2}{2\sigma^2}} dt d\xi}
\end{aligned}$$

Now,

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-i\xi t} e^{-\frac{t^2}{2\sigma^2}} dt &= \int_{-\infty}^{\infty} e^{-\frac{t^2+2\sigma^2 i\xi+(\sigma^2 i\xi)^2-(\sigma^2 i\xi)^2}{2\sigma^2}} dt \\
&= \int_{-\infty}^{\infty} e^{-\frac{(t+2\sigma^2 i\xi)^2}{2\sigma^2}} e^{\frac{(\sigma^2 i\xi)^2}{2\sigma^2}} dt \\
&= e^{\frac{(\sigma^2 i\xi)^2}{2\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{(t+2\sigma^2 i\xi)^2}{2\sigma^2}} dt \\
\text{making an 'x-substitution'} \\
&= e^{\frac{(\sigma^2 i\xi)^2}{2\sigma^2}} \sigma \int_{-\infty}^{\infty} e^{-\frac{x}{2}} dx \\
&= e^{\frac{(\sigma^2 i\xi)^2}{2\sigma^2}} \sigma\sqrt{2\pi}
\end{aligned}$$

giving us

$$\begin{aligned}
\frac{\int_{-\infty}^{\infty} e^{-i\xi t} e^{-\frac{t^2}{2\sigma^2}} dt}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\xi t} e^{-\frac{t^2}{2\sigma^2}} dt d\xi} &= \frac{e^{\frac{(\sigma i\xi)^2}{2}} \sigma\sqrt{2\pi}}{\int_{-\infty}^{\infty} e^{\frac{(\sigma i\xi)^2}{2}} \sigma\sqrt{2\pi} d\xi} \\
&\text{making a u-substitution} \\
&= \frac{e^{\frac{(\sigma i\xi)^2}{2}}}{\frac{1}{\sigma}\int_{-\infty}^{\infty} e^{\frac{-(u)^2}{2}} du} \\
&= \frac{\sigma e^{\frac{(\sigma i\xi)^2}{2}}}{\sqrt{2\pi}}
\end{aligned}$$

Knowing that the normal pdf is

$$\frac{e^{-\frac{x}{2\sigma_x^2}}}{\sigma_x\sqrt{2\pi}}$$

We can make the substitution,

$$\sigma_g = \frac{1}{\sigma}$$

and our pdf is a normal distribution with variance

$$\frac{1}{\sigma} \implies \frac{1}{\sigma_g} \cdot \sigma = 1$$

This is very similar to the uncertainty principle because if you have a low variance, and thereore high certaintiy of one state, you have a high variance and low certainity of the other, because the σ values are so interrelated.

It is like the uncertainty principle because given a low variance we have high certainty, and given a high variance we have low certainty.