Math 344 Homework 2.4

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2.21

We observe that D is given by:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

While

$$D^2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is the desired result.

2.22

We know that

$$\begin{aligned} \operatorname{tr}(AB) &= \sum_{i=1}^{n} a_{1i}b_{i1} + \sum_{i=1}^{n} a_{2i}b_{i2} + \cdots + \sum_{i=1}^{n} a_{ni}b_{in} = \sum_{k=1}^{n}\Big(\sum_{i=1}^{n} a_{ki}b_{ik}\Big) \\ \operatorname{tr}(BA) &= \sum_{i=1}^{n} b_{1i}a_{i1} + \sum_{i=1}^{n} b_{2i}a_{i2} + \cdots + \sum_{i=1}^{n} b_{ni}a_{in} = \sum_{k=1}^{n}\Big(\sum_{i=1}^{n} b_{ki}a_{ik}\Big) \end{aligned}$$

as each of these individual summations yields an entry of the diagonal of AB and BA, respectively. We know, however, that

$$\sum_{i=1}^{n}\Big(\sum_{k=1}^{n} a_{ik}b_{ki}\Big) = \sum_{i=1}^{n}\Big(\sum_{k=1}^{n} b_{ik}a_{ki}\Big)$$

Therefore, we know that

$$\begin{aligned}
\operatorname{tr}(AB) &= \sum_{i=1}^{n} a_{1i}b_{i1} + \sum_{i=1}^{n} a_{2i}b_{i2} + \cdots + \sum_{i=1}^{n} a_{ni}b_{in} \\
&= \sum_{k=1}^{n}\Big(\sum_{i=1}^{n} a_{ki}b_{ik}\Big) = \sum_{k=1}^{n}\Big(\sum_{i=1}^{n} b_{ki}a_{ik}\Big) \\
&= \sum_{i=1}^{n} b_{1i}a_{i1} + \sum_{i=1}^{n} b_{2i}a_{i2} + \cdots + \sum_{i=1}^{n} b_{ni}a_{in} = \operatorname{tr}(AB)
\end{aligned}$$

2.22 (ii)

We know that

$$\begin{aligned}
A &= P^{-1}BP \\
\operatorname{tr}(A) &= \operatorname{tr}(P^{-1}BP) \\
\operatorname{tr}(A) &= \operatorname{tr}((P^{-1}B)P) \\
&\text{And by part (i),} \\
\operatorname{tr}(A) &= \operatorname{tr}(P(P^{-1}B)) \\
\operatorname{tr}(A) &= \operatorname{tr}(PP^{-1}B) \\
\operatorname{tr}(A) &= \operatorname{tr}(IB) \\
\operatorname{tr}(A) &= \operatorname{tr}(B)
\end{aligned}$$

Which is the desired result.

2.22 (iii)

We know that

$$\operatorname{tr}(AB) = \sum_{i=1}^{n} a_{1i}b_{i1} + \sum_{i=1}^{n} a_{2i}b_{i2} + \cdots + \sum_{i=1}^{n} a_{ni}b_{in} = \sum_{k=1}^{n}\Big(\sum_{i=1}^{n} a_{ki}b_{ik}\Big)$$

where the rows of A are multiplied by the columns of B. If we transpose B, though, we will be multiplying the rows of A by the rows of B, yielding

$$\operatorname{tr}(AB^T) = \sum_{i=1}^{n} a_{1i}b_{1i} + \sum_{i=1}^{n} a_{2i}b_{2i} + \cdots + \sum_{i=1}^{n} a_{ni}b_{ni} = \sum_{k=1}^{n}\Big(\sum_{i=1}^{n} a_{ki}b_{ki}\Big)$$

which is equivalent to showing the desired result, just with a k index instead of a j index.

2.23

We know that $A = P^{-1}BP$, and that

$$
\begin{aligned}
p(A) &= a_n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A^1 + a_0 A^0 \\
p(A) &= a_n (P^{-1}BP)^n + a_{n-1}(P^{-1}BP)^{n-1} + \cdots + a_1 (P^{-1}BP)^1 + a_0 (P^{-1}BP)^0
\end{aligned}
$$

Notice, however, that

$$
\begin{aligned}
A^n &= (P^{-1}BP)^n \\
&= (P^{-1}BP)(P^{-1}BP)\dots(P^{-1}BP) \\
&= (P^{-1}BPP^{-1}BP\dots P^{-1}BP) \\
&= (P^{-1}BIBI\dots IBP) \\
&= (P^{-1}B^n P)
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
p(A) &= a_n P^{-1}(B)^n P + a_{n-1}P^{-1}(B)^{n-1}P + \cdots + a_1 P^{-1}(B)^1 P + a_0 P^{-1}(B)^0 P \\
&= P^{-1}(a_n(B)^n + a_{n-1}(B)^{n-1} + \cdots + a_1(B)^1 + a_0(B)^0)P \\
&= P^{-1}p(B)P
\end{aligned}
$$

2.24

Reflexivity: Note that

$$A = I^{-1}AI = A$$

So A is similar to itself.

Symmetrical: Let A be similar to B, then

$$A = P^{-1}BP \implies B = PBP^{-1}$$

So B is also similar to A.

Transitive: Let A be similar to B and let B be similar to C, then

$$
\begin{aligned}
&A = P^{-1}BP \\
&B = Q^{-1}CQ \\
\implies &A = P^{-1}Q^{-1}CQP \\
\implies &A = S^{-1}CS \quad \text{where } S = QP
\end{aligned}
$$

Now, since we know that the product of two nonsingular matrices is nonsingular, we have that A is similar to C.

The set of all 1×1 matrices

2.25

Let A, B be similar matrices, with A being invertible. Then we have that

$$A = P^{-1}BP \quad B = PAP^{-1}$$

Now, since we know that P is invertible by the definition of similar matrices, and that A is invertible by assumption, we know that B is invertible as it is the product of three nonsingular matrices, which we know results in a nonsingular matrix from linear algebra.

2.26

Consider two similar matrices A, B. Let

$$S = \{b_1, b_2, \ldots, b_n\}$$

be a basis be a basis for $\mathscr{N}(B)$. Therfore,

$$\begin{aligned} b_i \in S, Bb = 0 \\ \implies P^{-1}APb = 0 \implies APb = P0 \implies APb = 0. \implies \\ \{Pb_1, Pb_2, \ldots, Pb_n\} \end{aligned}$$

Which is linearly independent. Now let $x \in \mathscr{N}(A)$. So $PBP^{-1}x = 0 = Ax \implies BP^{-1} = 0$. So $P^{-1}x \in \mathscr{N}(B)$. So $P^{-1}x \in \mathscr{N}(B) \implies P^{-1}x = a_1b_1 + a_2b_2 + \cdots + a_nb_n$ because of our assumption about $\mathscr{N}(B)$.

$$\begin{aligned} \implies x = a_1Pb_1 + a_2Pb_2 + \cdots + a_nPb_n \\ \implies \{Pb_1, Pb_2, \ldots, Pb_n\} \text{ spans } \mathscr{N}(A) \text{ and forms a basis for } \mathscr{N}(A). \end{aligned}$$

So we have that $\dim(\mathscr{N}B) = \dim(\mathscr{N}A) = \text{ Nullity } (A) = \text{ Nullity } (B)$. By rank-nullity, we have

$$\text{Rank } (A) = \mathscr{N}(A) - \text{ nullity } (A) = \mathscr{N}(B) - \text{ nullity } (B) = \text{ rank } (B)$$