Accelerated Life Testing — Step-Stress Models and Data Analyses

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Key Words—**Accelerated life testing, Step-stressing, Cumulative exposure models, Maximum likelihood.**

Reader Aids—
Purpose: Advance the theory and practice
Special math needed for explanations: Statistics
Special math needed for results: Same
Results useful to: Reliability practitioners

Abstract—**This paper presents statistical models and methods for analyzing accelerated life-test data from step-stress tests. Maximum likelihood methods provide estimates of the parameters of such models, the life distribution under constant stress, and other information. While the methods are applied to the Weibull distribution and inverse power law, they apply to many other accelerated life test models. These methods are illustrated with step-stress data on time to breakdown of an electrical insulation.**

1. INTRODUCTION

Accelerated life testing of a product or material is used to get information quickly on its life distribution. Test units are run under severe conditions and fail sooner than under usual conditions. A model is fitted to the accelerated failure times and then extrapolated to estimate the life distribution under usual conditions. This is quicker and cheaper than testing at usual conditions, which is usually impractical because life is so long.

Accelerated test conditions involve higher than usual temperature, voltage, pressure, vibration, cycling rate, load, etc., or some combination of them. The use of such accelerating variables for a specific product or material is usually established by engineering practice. For example, for tests with accelerated voltage, the model of Section 3.1 is often used. Constant stress is used in the test and the model, because 1) stress is constant in most applications and 2) constant-stress models are available and verified by experience. Refs [8, 14] and [6, chap 9] survey constant-stress models and statistical methods for planning and analyzing tests.

Tests at constant, high stresses can run too long because there is usually great scatter in failure times. Step-stress testing is intended to reduce test time and to assure that failures occur quickly enough. A step-stress test runs through a pattern of specified stresses, each for a specified time, until the test specimen fails. The test stresses and their hold times are chosen to assure that failure occurs quickly enough. Often, different step-stress patterns are used on different specimens. Such tests can be further shortened by stopping the test with specimens unfailed. Running times on unfailed specimens are called censored data.

This paper presents maximum likelihood (ML) methods for estimating a model for life as a function of constant stress from step-stress test data, which can be censored. This paper documents what appears to be the first application of valid statistical methods for analyzing such data.

In this paper:

- Section 2 describes the illustrative data
- Section 3 describes the model for step-stress data
- Section 4 presents results for the illustrative data
- Section 5 introduces ML theory for estimating the model parameters

2. THE DATA

The data in Table 2.1 illustrate the data analysis methods. They are from a step-stress test of cable insulation. The test was run to estimate life at a design stress of 400 volts/mil. Also, this insulation was to be compared with another insulation that was tested.

Table 2.1 Step-Stress Test Data on Cable I

Voltage Pattern		Specimen Data			
Step*	Kilovolts	Hold (min)	Final Step	Total Time to Failure (min)	Thickness (mils)
1	26.0	15	5	102	27
		15	5	113	27
2	28.5	15	5	113	27
3	31.0	60	6	370+	29.5
		60	6	345	29.5
4	33.4	60	6	345+	28
5	36.0	240	6	1249	29
		240	6	1333	29
6	38.5	240	6	1333+	29
		240	5	1096.9	29
7	41.0	240	6	1250.8	30
		240	5	1097.9	29
8	43.5				
		960	3	2460.9	30
9	46.0	960	3	2460.9+	30
		960	3	2700.4	30
10	48.5	960	4	2923.9	30
		960	2	1160.0	30
		960	3	1962.9	30
		960	1	363.9+	30
		960	1	898.4+	30
		960	5	4142.1	30

* Before step 1, each specimen was held 10 min. each at 5, 10, 15, and 20 kV.
\+ denotes a running time without failure.

Each specimen was first held for 10 minutes each at 5kV, 10kV, 15kV, and 20kV before it went into step 1 at 26kV. In steps 1 through 10, a specimen has the same hold

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time at each voltage (15 minutes, 1 hour, 4 hours, or 16 hours). Table 2.1 shows the step number and the total time on test when a specimen broke down and its insulation thickness (used to calculate its stress as the voltage divided by the thickness).

3. THE MODEL FOR STEP-STRESS TESTING

The model for step-stress data consists of three parts: 1) the model for the distribution of life as a function of constant stress, 2) the model for the effect on life of the 'size' of a unit, and 3) the model for the cumulative effect of exposure in a step-stress test. In applications, one should verify that all three parts of the model are valid.

3.1 Constant-Stress Model

For many products and materials tested at constant stress, the following model adequately describes specimen life as a function of the stress. Its assumptions are:

1) For any constant stress V (which must be positive), the life distribution is Weibull.

2) The Weibull shape parameter β is constant.

3) The Weibull scale parameter α is —

$$\alpha(V) = (V_o/V)^p. \tag{3.1}$$

Here β, V_o, p are positive parameters characteristic of the material and the test method. Eq. (3.1) is called the inverse power law.

These assumptions imply that the population fraction $F(t; V)$ of units failing by time t under constant stress V is

$$F(t; V) = 1 - \exp[-\{t(V/V_o)^p\}^\beta], \qquad t > 0. \tag{3.2}$$

The model and applications of it are given by Endicott and others [2-4]. For $\beta = 1$, the Weibull distribution is an exponential distribution. The Weibull and exponential distributions and their properties are described, for example, by Hahn & Shapiro [5].

The F-th fractile of the life distribution for a stress V is

$$t_F(V) = \exp[p \ln(V_o/V) + (1/\beta)\, u(F)]; \tag{3.3}$$

$u(F) \equiv \ln\{-\ln(1 - F)\}$ is the F-th standard extreme value fractile.

3.2 Effect of Size on Life

Usually, test specimens are smaller than actual units. For example, a cable is longer than a specimen and so is more failure prone. A model for the effect of size follows. It uses the size, S^*, of a unit and the size, S, of the specimen. A unit is regarded as a series system made up of (S^*/S) specimens with s-independent life times; that is, the unit fails when the first specimen fails. For example, if cable specimens have length S and a cable in an application has length S^*, then the cable is regarded as a series system of (S^*/S) specimens. For cable, some engineers use dielectric volume as size, and others use "exposed area".

The series system assumption and (3.2) imply that the fraction of units of size S^* failing by time t under constant stress V is

$$F(t; V, S^*) = 1 - \exp[-(S^*/S)\{t(V/V_o)^p\}^\beta]. \tag{3.4}$$

This reduces to (3.2) if $S^* = S$.

Eq. (3.4) can underestimate life of a large unit. Adjoining specimen-size pieces of cable or other product can have positively correlated lifetimes, instead of s-independent ones. That is, if a piece has a short (long) life-time, pieces adjoining it tend to have short (long) lifetimes. Positive correlation in the model yields longer lifetimes for large units. As an extreme, if pieces in a unit were perfectly correlated, all pieces would have the exact same lifetime, and the unit would have the lifetime of a specimen. Thus, the life distribution (3.2) of specimens is an upper bound on the life distribution of larger units. For some applications, knowledge of such lower (3.4) and upper (3.2) bounds for the distribution might suffice.

Nelson [9, 10] derives and presents in detail the models above.

3.3 Cumulative Effect of Exposure

For a step-stress pattern, there is a Cdf for time to failure under test. Data from this Cdf are observed in the test. Not interested in life under a step-stress pattern, one usually wants the life distribution under constant stress, which units see in use. To analyze data, one needs a model that relates the distribution (or cumulative exposure) under step-stressing to the distribution (or exposure) under constant stress. The following describes one such model.

The model assumes that the remaining life of specimens depends only on the current cumulative fraction failed and current stress — regardless how the fraction accumulated. Moreover, if held at the current stress, survivors will fail according to the Cdf for that stress but starting at the previously accumulated fraction failed.

Figure 3.1 depicts the model. Part A depicts a step-stress pattern with four steps. The figure shows the failure and censoring times of test specimens. Part B depicts the four Cdf's for the constant stresses (V_1, V_2, V_3, V_4). The arrows show that the specimens first follow the Cdf for V_1 up to the first hold time t_1. When the stress increases from V_1 to V_2, the unfailed specimens continue along the Cdf for V_2, starting at the accumulated fraction failed. Similarly, when the stress increases from V_2 to V_3, from V_3 to V_4, etc., the unfailed specimens continue along the next Cdf, starting at the accumulated fraction failed. The Cdf for life under the step-stress pattern appears in part C and consists of the segments of the Cdf's for the constant stresses.

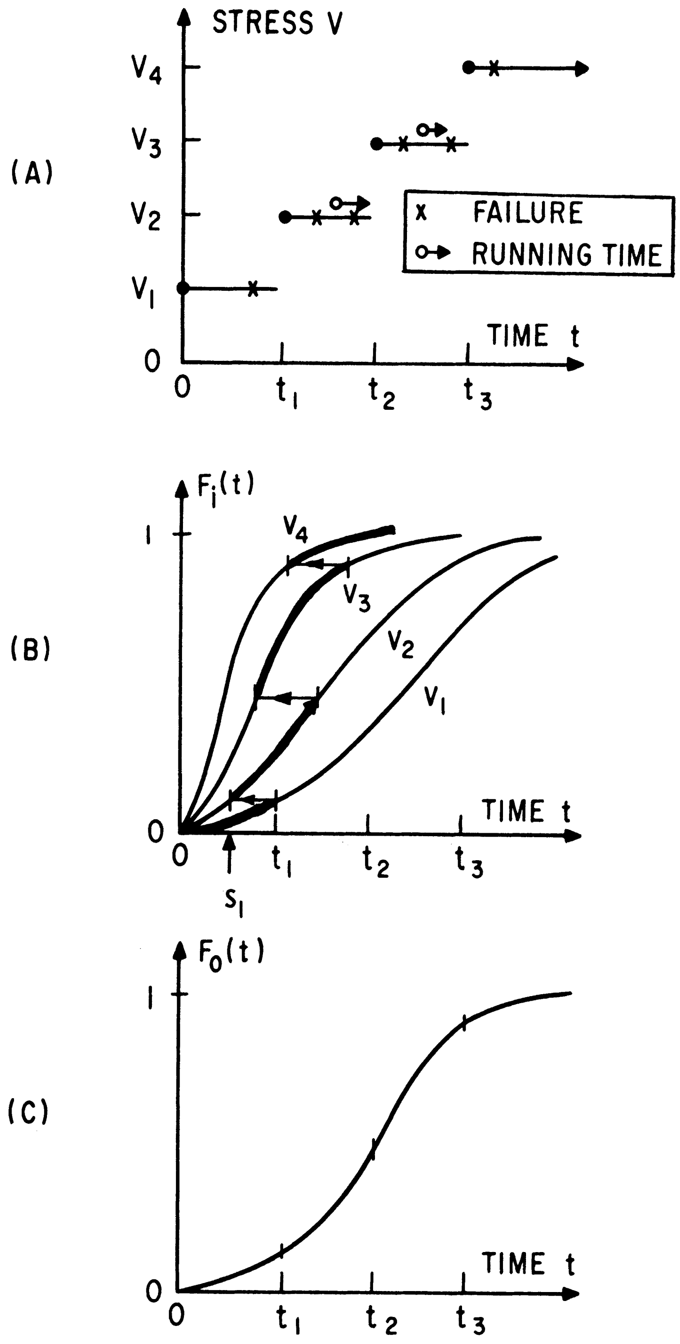

Fig. 3.1. Relationship between Constant- and Step-Stress Distributions

This model is mathematically expressed as follows to obtain the Cdf $F_o(t)$ of time to failure under a particular step-stress pattern. Less mathematically inclined readers can skip to Section 4. Suppose that, for a particular pattern, step i runs at stress V_i, starts at time t_{i-1}, and runs to time $t_i (t_o = 0)$. The Cdf of time to failure for units run at a constant stress V_i is denoted by $F_i(t)$.

The population cumulative fraction of specimens failing in step 1 is

$$F_o(t) = F_1(t), \qquad 0 \leqslant t \leqslant t_1. \tag{3.5}$$

Step 2 has an equivalent start time s_1 which would have produced the same population cumulative fraction failing (as depicted in Figure 3.1B); that is, s_1 is the solution of

$$F_2(s_1) = F_1(t_1). \tag{3.6}$$

The population cumulative fraction of specimens failing in step 2 by total time t is

$$F_o(t) = F_2[(t - t_1) + s_1], \qquad t_1 \leqslant t \leqslant t_2. \tag{3.7}$$

Similarly, step 3 has the equivalent start time s_2 that is the solution of

$$F_3(s_2) = F_2(t_2 - t_1 + s_1). \tag{3.8}$$

Similarly, for step 3,

$$F_o(t) = F_3[(t - t_2) + s_2], \qquad t_2 \leqslant t \leqslant t_3. \tag{3.9}$$

In general, step i has the equivalent start time s_{i-1} that is the solution of

$$F_i(s_{i-1}) = F_{i-1}(t_{i-1} - t_{i-2} + s_{i-2}), \tag{3.10}$$

and

$$F_o(t) = F_i[(t - t_{i-1}) + s_{i-1}], \qquad t_{i-1} \leqslant t \leqslant t_i. \tag{3.11}$$

Thus, $F_o(t)$ for the step-stress pattern is made up of segments of the Cdf's $F_1(\ )$, $F_2(\ )$, etc., as shown in part C of Figure 3.1. A different step-stress pattern would have different $F_o(t)$ distribution.

Various models for cumulative exposure under step-stress testing have been proposed [1, 10, 14]. For example, Endicott and others [2-4] successfully used such models for voltage-accelerated life testing of capacitors subject to dielectric breakdown. For step-stress tests, the validity of such models (including the one above) is an open question.

The preceding specializes to the inverse power law model as follows. By (3.2), the Cdf for the fraction of specimens failing by time t for the constant stress V_i is

$$F_i(t) = 1 - \exp[-\{t(V_i/V_o)^p\}^\beta]; \tag{3.12}$$

Then, for step 1, (3.5) becomes

$$F_o(t) = 1 - \exp[-\{t(V_1/V_o)^p\}^\beta], \qquad 0 \leqslant t \leqslant t_1. \tag{3.5'}$$

The equivalent time s_1 at V_2 is given by (3.6) as

$$s_1 = t_1(V_1/V_2)^p. \tag{3.6'}$$

For step 2,

$$F_o(t) = 1 - \exp[-\{(t - t_1 + s_1)(V_2/V_o)^p\}^\beta], \quad \cdot\ t_1 \leqslant t \leqslant t_2. \tag{3.7'}$$

Similarly, for step 3,

$$s_2 = (t_2 - t_1 + s_1)(V_2/V_3)^p, \tag{3.8'}$$

$$F_o(t) = 1 - \exp[-\{(t - t_2 + s_2)(V_3/V_o)^p\}^\beta], \quad \cdot\ t_2 \leqslant t \leqslant t_3. \tag{3.9'}$$

In general, for step i,

$$s_{i-1} = (t_{i-1} - t_{i-2} + s_{i-2})(V_{i-1}/V_i)^p, \tag{3.10'}$$

$$F_o(t) = 1 - \exp[-\{(t - t_{i-1} + s_{i-1})(V_i/V_o)^p\}^\beta], \quad \cdot\ t_{i-1} \leqslant t \leqslant t_i. \tag{3.11'}$$

Thus, $F_o(t)$ consists of segments of Weibull distributions.

Once one has estimates of the model parameters β, p, V_o, one can then use the procedure above to estimate $F_o(t)$ under any varying stress pattern that might occur in actual use.

4. RESULTS ON THE CABLE EXAMPLE

This section presents:

1) results of the ML fitting of the model in Section 3 to the cable data,

2) ML estimates of the model parameters (β, V_o, p) and fractiles $t_F(V)$,

3) a comparison of the model fitted to the data with the model fitted to data on another type of cable.

Section 5 presents ML theory.

4.1 Model Fitting Using Maximum Likelihood Estimates

The ML fitted model for the fraction of cables that fail by age t in minutes is —

$$F(t;\ V,\ S^*) = 1 - \exp[-\ (S^*/9.425)\ \{t(V/1619.4)^{19.937}\}^{0.75597}];$$

here S^* is the area of the dielectric on a cable (the area of a test specimen is 9.425 sq. in.), $\hat\beta = 0.75597$ is the Weibull shape parameter, $\hat p = 19.937$ is the power in the inverse power law model (3.2) for life and voltage stress, and V is the stress in volts per mil. The values are not known as accurately as the number of significant figures might imply.

4.2 Estimates and Confidence Intervals

The ML estimates of V_o, p, β come from a small number of specimens. So the estimates contain considerable uncertainty, shown by the approximate 95% *s*-confidence intervals in Tables 4.1 and 4.2. The tables also show the ML estimate of the first percentile of the specimen life distribution at the design stress of 400 volts per mil and approximate 95 percent *s*-confidence limits. Figure 4.1 depicts the first percentile versus stress. The method for estimating the model parameters and percentiles and for obtaining the *s*-confidence intervals is indicated in Section 5. Other information in the table is explained later. The intervals tend to be too narrow (unconservative) for small samples.

Table 4.1 ML Results for Cable I

Parameter	Estimate	95% *s*-Conf. Limits Lower	Upper
V_o	1616.4	1291.0	1941.8
p	19.937	6.2	33.7
β	0.75597	0.18	1.33
1% point (min) at 400 V/mil	2.81×10^9	2.65×10^4	2.98×10^{14}

Asymptotic Covariance Matrix of Maximum Likelihood Estimates

	V_o	p	β
V_o	27566.		symmetric
p	−1145.7	49.004	
β	41.572	−1.7561	0.086575

Maximum Log Likelihood = −103.53

Table 4.2 ML Results for Cable II

Parameter	Estimate	95% *s*-Conf. Limits Lower	Upper
V_o	3056.3	2177.6	3934.9
p	9.6015	5.6	13.6
β	0.96910	0.54	1.40
1% point (min) at 400 V/mil	2.62×10^6	2.96×10^4	2.32×10^8

Asymptotic Covariance Matrix of Maximum Likelihood Estimates

	V_o	p	β
V_o	200,957		symmetric
p	−901.11	4.1599	
β	61.017	−0.27439	0.047608

Maximum Log Likelihood = −141.66

4.3 The Fitting Calculations

ML fitting was done with STATPAC [11, 13], a general program for data analysis and for fitting statistical models to data. One need not understand the mathematical details of ML, since STATPAC calculates estimates and approximate *s*-confidence intervals for model parameters and other quantities of interest, such as percentiles.

4.4 Validity of the Model and Data

The ML fitting assumes 1) the model is valid and 2) the data are valid. Some checks [7, 10] of these assumptions can be made by means of graphical analyses of residuals, calculated from the fitted model. Overall, the model and data appear to be adequate.

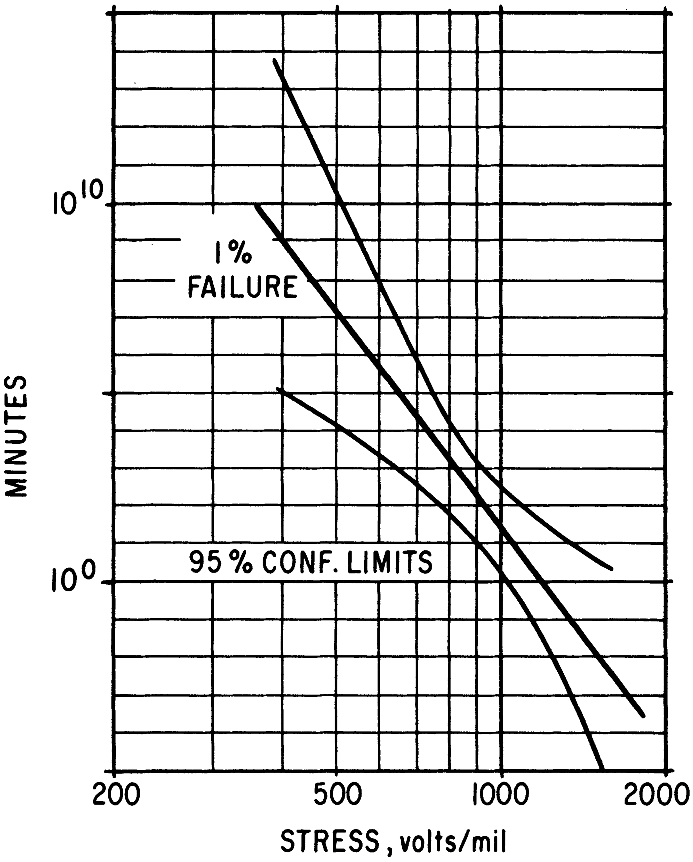

Fig. 4.1. 1% Point and 95% s-Confidence Limits vs. Stress—Cable I.

4.5 Comparisons

The step-stress test was also carried out on another type of cable. The results of the ML analyses of the data appear in Table 4.2. A purpose of the testing is to compare the two types of cable to determine whether corresponding parameter estimates differ by a convincing amount relative to their uncertainties. The two types of cable are compared below with respect to:

1) The shape parameter, β,
2) The power parameter, p,
3) The entire model (β, p, and V_o simultaneously),
4) The 1 percent points of the life distribution at 400 volts per mil.

Detailed statistical theory appears in [10].

Shape parameters. The *s*-confidence intervals for a β in Tables 4.1 and 4.2 overlaps the estimate of the other β. This indicates that the difference between the two β estimates is not *s*-significant. A more formal comparison, described below for the power parameter estimates, could be used.

Power parameters. The observed difference of the two power parameter estimates is 19.937 − 9.601 = 10.336. Because the estimates are *s*-independent, the variance of their difference is the sum of their variances, 49.004 + 4.1599 = 53.1639. Approximate 95% *s*-confidence limits for the true difference is the observed difference ± 1.96 times the square root of the variance (1.96 is the standard *s*-normal 97.5 percentile). These limits are 10.336 ± 1.96 $\sqrt{53.1639}$ or − 3.955 and 24.627. This interval encloses a difference of zero. So the observed difference in the power parameters is not *s*-significantly different from zero.

Models. One can compare entire models for equality, simultaneously comparing corresponding parameter estimates. For the two cables, the sum of their maximum log likelihoods is (− 103.53) + (141.66) = − 245.19. For the same model fitted to the pooled data on the two cables, the maximum log likelihood is −265.15. Twice the difference 2{(−245.19) − (− 265.15)} = 39.92 is the test statistic. If the two models are the same, this statistic is approximately chi-square distributed with three degrees of freedom. An unusually large statistic indicates that some corresponding model parameters differ *s*-significantly. A 'large' statistic is one above a high percentile of the chi-square distribution; for example, $\chi^2(0.999;3) = 16.3$. Since 39.93 > 16.3, the parameter estimates of the models for the two types of cable differ highly *s*-significantly, undoubtedly due largely to the difference between the V_o estimates, which can be compared as the power parameters were. Ref. [10] gives the theoretical basis for this simultaneous comparison.

First percentiles. The fitted model for each type of cable was used to estimate the first percentile at 400 volts per mil. The *s*-confidence interval for each first percentile in Tables 4.1 and 4.2 overlaps the estimate of the other first percentile. This indicates that the two estimates do not differ *s*-significantly. The estimates could be compared more formally with the method described above for the power parameters.

5. MAXIMUM LIKELIHOOD THEORY

Standard ML theory applies to the step-stress models and data described above. Then the corresponding Cdf for each test specimen is the $F_o(t)$ given in the previous section and the specimen's pdf is the time derivative of its $F_o(t)$. The $F_o(t)$ differs for specimens with different step-stress patterns. The sample likelihood is the product of such pdf's evaluated at failure times and of such survival functions evaluated at censoring times. This sample likelihood is used to obtain ML estimates, approximate *s*-confidence limits, and likelihood ratio tests, as described in detail in [10, 11].

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Dr. Wayne Nelson: For biography, see vol R-25, 1976 Apr, p 24.

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Metaphysical Electronics Ltd.

INTERNAL MEMORANDUM

TO: Pluman Hatt, Chief Meditator

SUBJECT: Legal Review of Specification for Tactical Psychic Detector

There is some ambiguity in the reliability requirements of the subject specification which must be resolved before we can bid intelligently. Paragraph 3.2 cites the requirement as "Operation without failure 24 hours a day, 365 days a year". This can be interpreted as requiring the system never to have a failure. If this is the intent, we can reduce our bid significantly by firing the entire Reliability Department, without fear of liability.

However, please note that the requirement makes no note of leap years. Hence, it could be interpreted as allowing 24 hours of downtime every four years or an availability of 0.99932. This is difficult but plausible. If this is the intent, we will have to pursue a serious Reliability effort and must consider this in our pricing.

Request you query the customer for clarification. If possible, try to steer him towards the first interpretation.

It is likewise not clear how the transitions to and from daylight savings time will affect the requirement. One transition day has only 23 hours and the other has 25 hours. Suggest we ignore this until it turns to our advantage.

COURTLY MANNORS
Chief, Legal Staff