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Stability and Numerical Simulation of Prey-predator System with Holling Type-II Functional Responses for Adult Prey

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Abstract. In this paper, we analyze the local stability of the prey-predator model. This model was constructed from two prey involving stage-structure and one predator. The population of prey is divided into two subpopulations, adult prey and immature. The ratio of intrinsic growth of predator population density divided by adult prey conversion factors into a predator. Reduction of predator populations has a reciprocal relationship with the availability of favorite foods determined by the environmental resources. We also consider that Holling type II the functional response for the predator. We have investigated the local stability and the solution of the system. We have studied the existence and feasibility of various equilibrium points. The system has three positive equilibria, namely the original, the extinction of the predator and the interior point. We explore the stabilities of all nonnegative equilibrium point by the real parts of the eigenvalues of the Jacobian matrix at each equilibrium point must be negative. We also use the Routh-Hurwitz criterion to investigate the stability at the interior equilibrium point. This study aims to detect the limit cycle with its phase portrait and its time-series graphs of the prey-predator system. Numerical simulated are represented as phase portraits to draw the stability of the equilibrium point. In our Numerical outcomes confirmed the analytical results and the local behavior prey-predator model which start from their positive initial condition. The parameter value is taken mainly from the literature and assumption. By a computational Python using the fourth-order Runge-Kutta method, we solved the prey-predator system and showed some new phase portraits such as the existence of the stable or unstable equilibrium point under a suitable value of the parameter. The next research, we will explore the global stability analysis behavior of the interior equilibrium point.

1.   Introduction

The dynamical behaviour of prey-predator model has investigated the stability of the positive equilibria and the existence of all feasible nonnegative equilibria. In this paper, we discuss the dynamical behaviour of stage-structured prey predator interactions with Holling type II response function for adult prey [1]. Model structure stages in the dynamics of interacting species have long been known [2]. This model has a much simpler way to simulate diversity than existing models and presents new phenomena in the real world.

In a dynamic system, if there is limit cycles, then the interior equilibrium point is center. In these references the limit cycles were all generated by Hopf Bifurcation [3]. Bifurcation is the stability change of a system that occurs due to changes in parameter values [4].

This paper is organized as follows. First, it considers the stability of the equilibria in detail with the conditions for existence. Second, it illustrates the local stability of equilibria and the third, it shows the

dynamical behaviour the stability of predator-prey model depending on the parameters by performed numerical simulations. In the last study, several numerical simulations were completed for illustrating the results.

2.   Mathematical Model

Huenchucona, Falconi and Vidal [5] proposed interaction of a two prey and one predator with the same predation process using the Beddington De-Angelis functional responses. The response function combines the predator rate of the adult prey population to the young prey populations with density depends on the young prey population.

$$\frac{dx_1}{dt} = rx_1\left(1 - \frac{x_1}{K}\right) - \beta x_1 - \frac{\beta_1 x_1 y}{1 + m_1 x_1{}^2 + n_1 x_2}$$
$$\frac{dx_2}{dt} = \beta x_1 - \frac{\beta_2 x_2 y}{1 + m_2 x_1 + n_2 x_2} - \mu_1 x_2 \quad (1)$$
$$\frac{dy}{dt} = \frac{\alpha_1 \beta_1 x_1 y}{1 + m_1 x_1{}^2 + n_1 x_2} + \frac{\alpha_2 \beta_2 x_2 y}{1 + m_2 x_1 + n_2 x_2} - \mu_2 y$$

Savitri, Abadi[6] studied the dynamical behavior of prey-predator model with stage-structure for prey. They used different predation rate, Holling type I the functional responses for immature prey and Holling type II functional response for mature prey.

$$\frac{dx_1}{dt} = rx_1\left(1 - \frac{x_1}{K}\right) - \beta x_1 - \alpha x_1 y$$
$$\frac{dx_2}{dt} = \beta x_1 - \frac{\varepsilon x_2 y}{1 + m x_2} - \mu_1 x_2 \tag{2}$$
$$\frac{dy}{dt} = \frac{\gamma \varepsilon x_2 y}{1 + m x_2} - \mu_2 y$$

Castellanos and Chan-Lopez [7], analyse a three level trophic chain model. They consider that Holling type I the functional response for the predator in middle and Holling type II for the predator at the top. The presence of the top predator is determinant in the stability of the chain model. The chain model is represented by the following system:

$$\frac{dx}{dt} = \rho x\left(1 - \frac{x}{k}\right) - a_1 xy$$
$$\frac{dy}{dt} = ca_1 xy - dy - \frac{a_2 yz}{y + b_2} \tag{3}$$
$$\frac{dz}{dt} = \alpha z^2 - \frac{\beta z^2}{y + b_2}$$

Before we introduce the mathematical model, let us describe the basic assumptions that we made to formulate it. In this work, we construction from the system (1), (2) and system (3). Motivated by these, in this paper we have presented the dynamical behaviour of three populations. Juvenile-adult stage structure is added to both prey and predator in model.

2.1.   Contruction of Mathematical Model

The first equation, juvenile prey grows with intrinsic growth rate r cause increasing the juvenile prey population growth rate, as follow

$$\frac{dx}{dt} = rx \tag{4}$$

Second, predatory predation rate for both of the prey population are different. It is assumed that predator can prey upon either juvenile prey or the adult prey in the structured model. Predatory predation rate used Holling type II for adult prey species, and Holling type I for juvenile prey. The rate of predation is given in the form of Holling type I functional response:

$$f(x, y) = \alpha x z \tag{5}$$

The rate of predation is given in the original form of Holling type II functional response:

$$f(x,y) = \frac{\eta yz}{y+m} \tag{6}$$

where η and m are the maximum value of the per capita reduction rate of y due to z and the coefficient of environmental protection for the adult prey, respectively. Therefore, the adult prey population growth rate is reduced due to predation, as follow

$$\frac{dx}{dt} = -\frac{\eta yz}{y+m}. \tag{7}$$

The population densities of juvenile prey, adult prey and predator, denoting by $x(t), y(t)$ and $z(t)$, respectively. The mathematical model a two prey one predator interactions are described by the following system of ODEs.

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \beta x - \alpha xz$$
$$\frac{dy}{dt} = \beta x - \frac{\eta yz}{y+m} - \mu y \tag{8}$$
$$\frac{dz}{dt} = \alpha_1 xz + \rho z^2 - \frac{\eta_1 yz}{y+m}$$

Changes in population densities of juvenile prey by intrinsic growth rates are reduced by the rate at which juvenile prey is consumed by predators. Juvenile prey populations grow with logistical equations that are limited by carrying capacity (k) to maintain their populations. The predation rate for juvenile prey is proportional to the rate at which predators and juvenile prey find each other or the so-called Lotka-Volterra equation. With parameter $r, \beta, \alpha, \alpha, \rho, m, \mu, \eta, \alpha_1, \eta_1$ and k are positive.

Changes in population density of adult prey, represent the growth of adult prey due to the conversion factor of juvenile prey to adult prey. The η parameter in the predation process causes a decrease in adult prey population which is interpreted as the rate that has an encounter with adult prey per unit density of adult prey. Without predation, the adult prey experiences a natural process of death. The rate of predator predation follows the Holling Type II functional response. The $\frac{m}{\eta}$ ratio is the average time in food processing for predator. The parameter ρ is the rate of population growth of predator population.

The rate of change in predator population is assumed to decrease in predator population following the Leslie-Gower model, with m representing loss of predator residue due to lack of their favorite food. Parameter η_1 is the ratio of intrinsic growth divided by the conversion factor of predation to adult prey. This affects the availability of favorite foods per capita, which is determined by the ability of environmental resources and is proportional to the abundance of their favorite foods [8].

2.2. Equilibrium point

In this section, we have investigated the local stability of all possible equilibrium point of the system (8). By the definition of equilibria of a system (8) are satisfied when

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0$$

We have studied the existence and feasibility of various equilibrium points. The system has three positive equilibria,

1. The original equilibrium point $E_1 = (0,0,0)$
2. The equilibrium of the predator extinction $E_2 = \left(\frac{k(r-\beta)}{r}, \frac{\beta k(r-\beta)}{\mu r}, 0\right)$
3. The interior equilibrium point $E_3 = (x^*, y^*, z^*)$

2.3. Local Stability

The local asymptotic stability of each equilibrium point is analyzed by calculating the Jacobian matrix and finding the eigenvalues assessed at each equilibrium point. The real parts of the eigenvalues of the Jacobian matrix must be negative for the stability of the equilibrium points.

The system (8) is a nonlinear autonomous. The stability of the nonlinear autonomous can be obtained by linearising the system (8) with the Jacobian matrix is given by:

$$J(x^*,y^*,z^*) = \begin{bmatrix} r - \dfrac{2rx}{k} - \beta - \alpha z & 0 & -\alpha x \\[2mm] \beta & -\dfrac{\eta z}{y+m} + \dfrac{\eta yz}{(y+m)^2} - \mu & -\dfrac{\eta y}{y+m} \\[2mm] \alpha_1 z & \dfrac{\eta_1 z^2}{(y+m)^2} & 2pz - \dfrac{2\eta_1 z}{y+m} + \alpha_1 x \end{bmatrix}$$

The characteristic equation is given by $\det(J(x^*,y^*,z^*) - \lambda I) = 0$ where (x^*,y^*,z^*) is the equilibrium point of (8). We explore the stabilities of all nonnegative equilibrium point by the real parts of the eigenvalues of the Jacobian matrix at each equilibrium point.

The characteristic equation of equilibrium point $E_1 = (0,0,0)$ is

$$\begin{vmatrix} (-1+r) - \lambda & 0 & 0 \\ \beta & -\mu - \lambda & 0 \\ 0 & 0 & 0-\lambda \end{vmatrix} = 0$$

The eigenvalues of the matrix $J(E_1)$ are $((-1+r) - \lambda)(-\mu - \lambda)(0 - \lambda) = 0$

It has always one negative eigenvalue $\lambda_1 = -\mu$, the other eigenvalue negative if $r < 1$, and $\lambda_3 = 0$. Therefore the equilibrium point E_1 is unstable.

The characteristic equation of equilibrium point $E_2 = \left(\dfrac{k(r-\beta)}{r}, \dfrac{\beta k(r-\beta)}{\mu r}, 0\right)$ is

$$\det\left(J\left(\dfrac{k(r-\beta)}{r}, \dfrac{\beta k(r-\beta)}{\mu r}, 0\right) - \lambda I\right) = 0$$

$$J(E_2) = \begin{vmatrix} (\beta - r) - \lambda & 0 & \dfrac{\alpha k(\beta - r)}{r} \\[3mm] \beta & -\mu - \lambda & \dfrac{\eta \beta k(\beta - r)}{r\mu\left(\dfrac{\beta k(r-\beta)}{\mu r} + m\right)} \\[5mm] 0 & 0 & -\dfrac{\alpha_1 k(\beta - r)}{r} \end{vmatrix} = 0$$

The eigenvalue of matrix $J(E_2)$ are $\lambda_1 = \beta - r$, $\lambda_2 = -\mu$ and $\lambda_3 = -\dfrac{\alpha_1 k(\beta-r)}{r}$. The equilibrium point $E_2 = \left(\dfrac{k(r-\beta)}{r}, \dfrac{\beta k(r-\beta)}{\mu r}, 0\right)$ is stable under condition if $r > \beta$.

The characteristic polynomial corresponding to the Jacobian matrix, equilibrium point $E_3 = (x^*, y^*, z^*)$

$$J(x^*,y^*,z^*) = \begin{bmatrix} r - \dfrac{2rx^*}{k} - \beta - \alpha z^* & 0 & -\alpha x^* \\[2mm] \beta & -\dfrac{\eta z^*}{y^*+m} + \dfrac{\eta yz^*}{(y^*+m)^2} - \mu & -\dfrac{\eta y^*}{y^*+m} \\[2mm] \alpha_1 z^* & \dfrac{\eta_1 z^{*2}}{(y^*+m)^2} & 2pz^* - \dfrac{2\eta_1 z^*}{y^*+m} + \alpha_1 x^* \end{bmatrix}$$

The characteristic equation of system (8) at the coexistence equilibrium point $E_3 = (x^*, y^*, z^*)$ is

$\lambda^2 - (\text{trace}J(x^*,y^*,z^*))\lambda + \det J(x^*,y^*,z^*) = 0$.

it is obvious that $\text{trace}(J(x^*,y^*,z^*)) < 0$ and $\det(J(x^*,y^*,z^*)) > 0$.

$$\lambda^3 + \gamma_1\lambda^2 + \gamma_2\lambda + \gamma_3 = 0$$

We use the Routh–Hurwitz criterion to investigate the stability at the interior equilibrium point, which has a negative real part if and only if $\gamma_1 > 0$, $\gamma_3 > 0$, and $\gamma_1\gamma_3 - \gamma_3 > 0$, the coexistence steady-state E_3 is locally asymptotically stable. The findings local stability condition of all equilibria described in table 1.

Table 1. Stability condition of all equilibrium point

	Type of stability	Local stability condition
$E_1 = (0,0,0)$	unstable	
$E_2 = \left(\dfrac{k(r-\beta)}{r}, \dfrac{\beta k(r-\beta)}{\mu r}, 0\right)$	stable	$r > \beta$
$E_3 = (x^*, y^*, z^*)$	Asymptotically stable	

3.   Numerical Simulation

We observed the dynamic behavior of the system (8) by confirming the results of analysis and simulation using Python to show the dynamics of the predator model around the positive interior equilibrium point.

We show that the results of interactions in the system can be influenced by the predation process by phase-portrait analysis and numerical simulations.
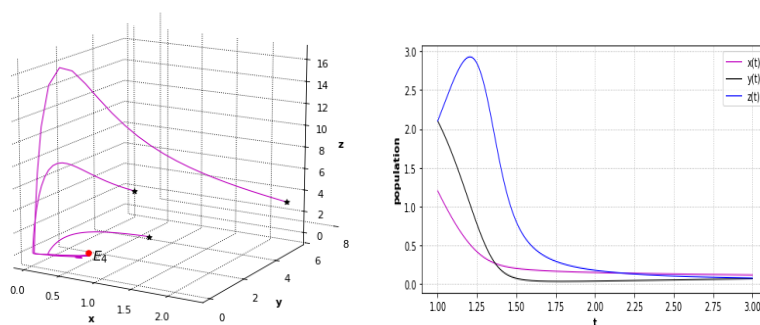
3.1.   Parameter value

Table 2. Parameter value

	r	β	α	α_1	η	η_1	K	ρ	m	μ
S1	0.82	0.87	1.56	1.12	2.41	1.83	12	1.38	0.13	0.11
S2	1.32	0.87	1.16	0.72	1.6	0.41	2.8	0.78	0.23	0.11
S3	1.32	0.87	0.76	0.72	0.6	0.41	2.8	0.78	0.23	0.11
S4	1.32	0.87	1.16	0.72	1.2	0.41	2.8	0.78	0.23	0.11
S5	1.32	0.87	1.16	0.72	0.3095	0.41	2.8	0.78	0.23	0.11

3.1.1.   The Dynamic Behaviour the first simulation stable the interior equilibrium point

The first simulation, we set the parameter value $r = 0.82, \beta = 0.87, \alpha = 1.56, \eta = 2.41, \rho = 1.38$, and $K = 12$. Base on analysis, the model of system (8) only has one equilibrium point is the locally asymptotically stable. The numerical simulation shows that all solutions are going to the interior equilibrium point with different initial value. Initial value 1: [3.01. 5.05. 4.28], initial value 2: [4.6,5.9,3.1] and initial value 3: [12.2,22.1,21.1]. The results of numerical simulation indicate that all populations can survive.
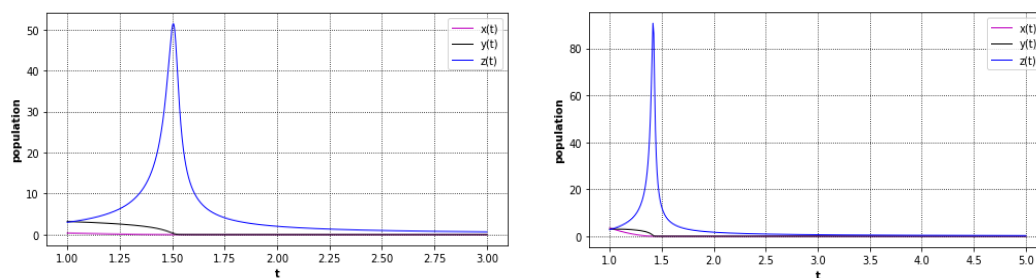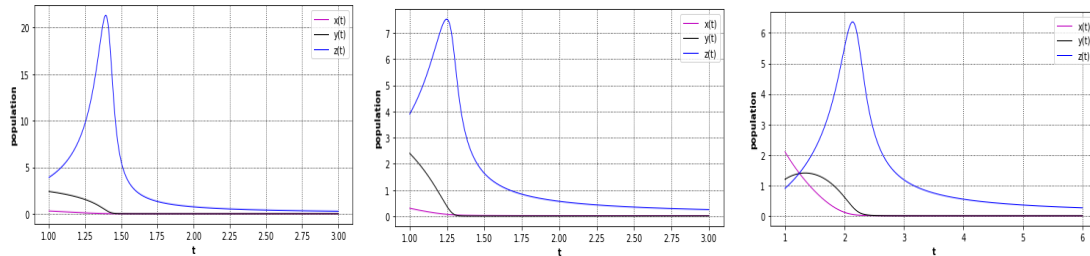

Figure 1 Time series of the solution system in the Interior Equilibrium Point if $K = 12$
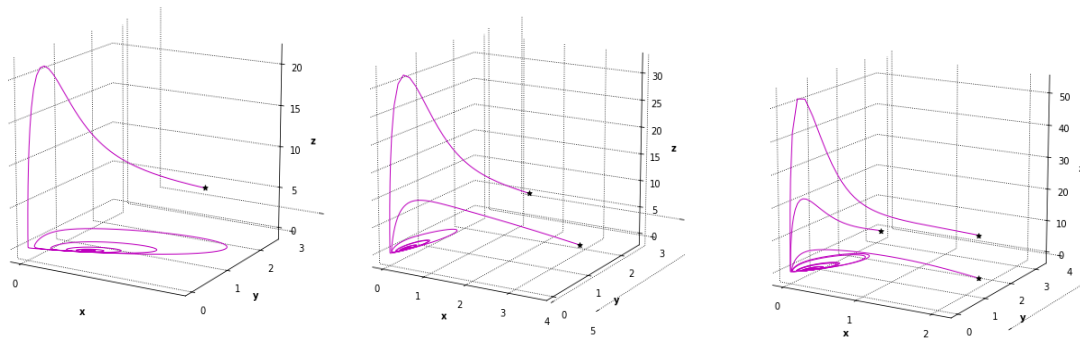

Figure 2 The Dynamic behaviours of the solution system in the Interior Equilibrium Point

3.1.2.   The Dynamic Behaviour $\eta > \beta$ and $\eta > \alpha$

The second simulation, we set the parameter value $r = 1.32, \beta = 0.87, \alpha = 1.56, \eta = 2.41, \rho = 1.38,$ and $K = 2,8$. Base on analysis, the model of system (8) only has one equilibrium point is the locally asymptotically stable. The numerical simulation shows that all solutions are going to the interior equilibrium point with different initial value. Initial value 1: [0.3. 2.4. 3.9], initial value 2: [0.6,2.4,4.1] and initial value 3: [2.1,1.2,1.1]

Figure 3 Time series of the solution system in the Interior Equilibrium Point if $K = 2.8$

Figure 4 The Dynamic behaviours of the solution system in the Interior Equilibrium Point with $K = 2.8$

3.1.3.   The Dynamic Behaviour if alpha < beta

The next simulation $\alpha < \beta$, we set the parameter value $\beta = 0.87, \alpha = 0.76, \eta = 0.6, \rho = 0.78,$ and $K = 2,8$. The numerical simulation shows that solution are going to the interior equilibrium point with N1: [0.3. 2.4. 3.9]

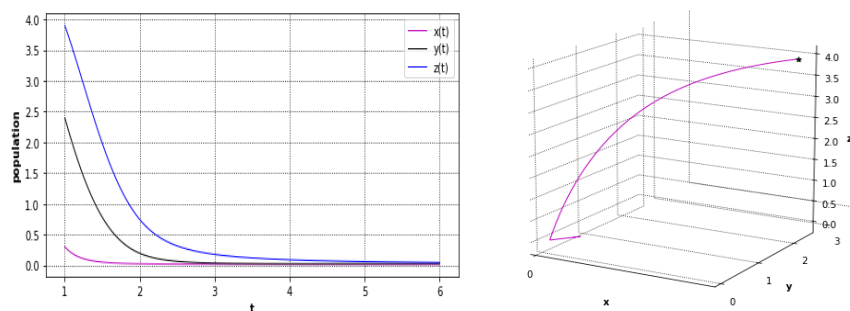
Figure 5 The Dynamic behaviours of the solution system with $\eta = 0.6$ *and* $\alpha < \beta$.

3.1.4.   The Dynamics Behaviour $\eta > \beta$ dan $\eta > \alpha$

The first simulation, we set the parameter value $r = 0.82, \beta = 0.87, \alpha = 0.76, \eta = 1.2, \rho = 1.38,$ and $K = 2.8$. The numerical simulation shows that all solution are going to the interior equilibrium point with initial value N1: [1.2, 2.1, 2.4]
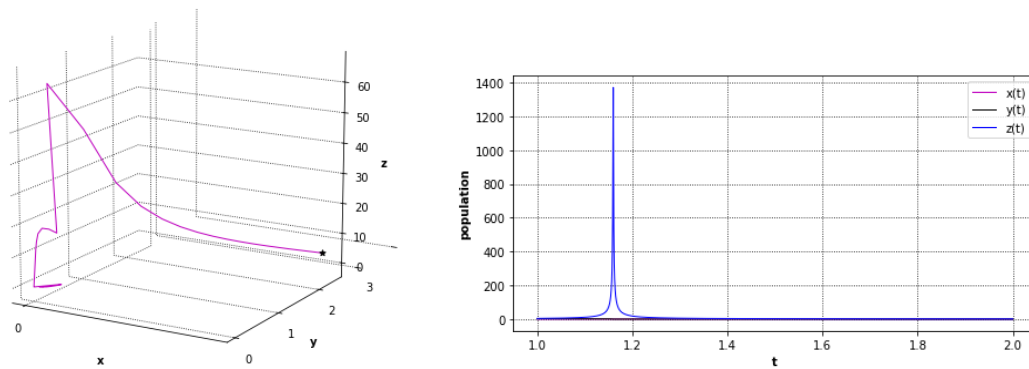
Figure 6 The Dynamic behaviours of the solution system with $\eta = 1.2$

3.1.5. *The Dynamics Behaviour* $\eta = \alpha$

The next simulation, we set the parameter value $r = 0.82, \beta = 1.3, \alpha = 1.2, \eta = 1.2, \rho = 1.38$, and $K = 2.8$. The numerical simulation shows that all solution are going to the interior equilibrium point with N1: [1.2, 2.1. 4.28],
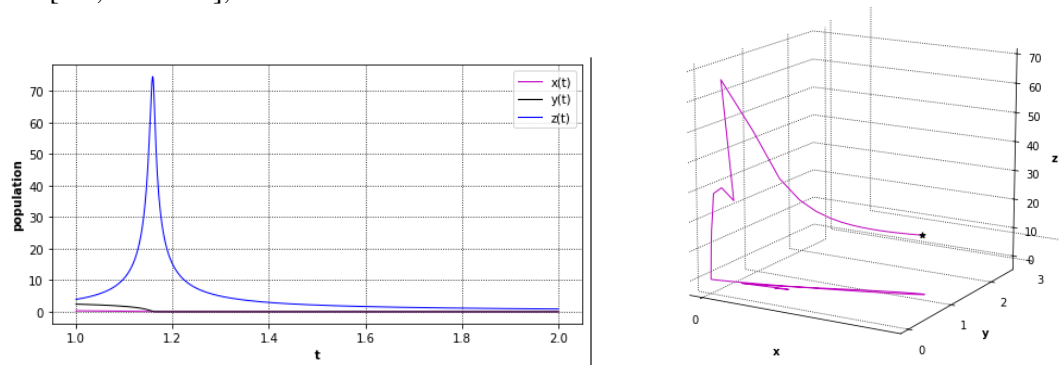

Figure 7 The Dynamic behaviours of the solution system with $\eta = \alpha = 1.2 < \beta = 1.3$.

3.2. *Continuation Numeric*

Then a numerical continuation is carried out on the predation parameter $\eta = 0.6$ which results in one equilibrium point E_3 which is stable. All of parameter are $r = 1.32, \beta = 0.87, \alpha = 1.16, \eta_1 = 0.41, \rho = 0.78, m = 0.23, \mu = 0.11$ and $K = 2.8$. Eigenvalues consist of two λ complexes with a real negative portion, and one negative eigenvalue. $\lambda_1 = -0.0519514704367125 + 0.351175046207591\, I, \lambda_2 = -0.0519514704367125 - 0.351175046207591\, I$ and $\lambda_3 = -0.445962288036575$

Based on these parameter values, an equilibrium point is obtained

$E_1 = (0; 0; 0)$
$E_2 = (0.9545454547; 7.549586778; 0)$
$E_3 = (0.1194522936; 0.1667424272; 0.3393851516)$

The results of numerical continuity when the value of $\eta = 0.30951031$ causes a change at the equilibrium point E_3 from the stable spiral to center. If you pass the predation parameter value at $\eta = 0.3$, a change at the equilibrium point E_3 becomes an unstable spiral. Based on these results, it can be seen that the higher the value of the parameter η, the equilibrium point E_3 will experience changes that are stable. The stability of the equilibrium point changes when Hopf bifurcation occurs when $\eta = 0.30951031$. when the parameter $\eta = 0.30951031$ the stability of the equilibrium point E_3 changes from a stable spiral to a center.
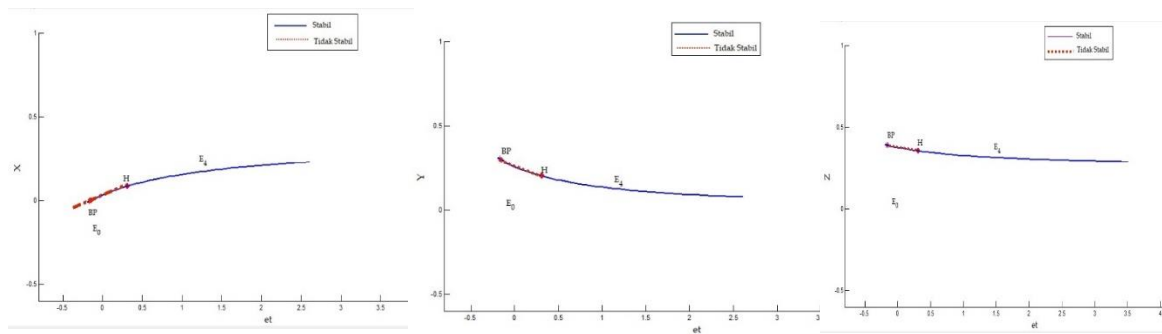
Figure 8. Bifurcation diagram of the juvenile-adult prey and predator population with respect to η
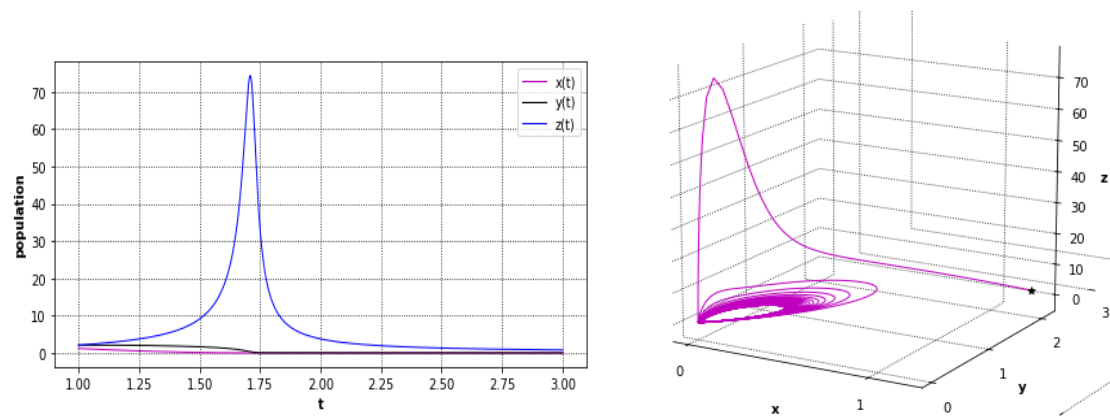

Figure 9 The Dynamic behaviours of the solution system with initial value $N1 = [0.3, 2.4, 3.9]$
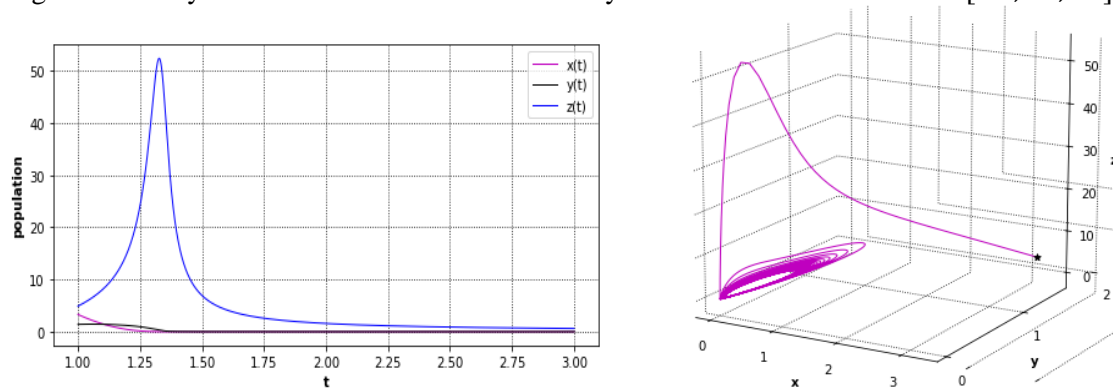

Figure 10 The Dynamic behaviours of the solution system with initial value $N2 = [4.1, 2.2, 5.1]$
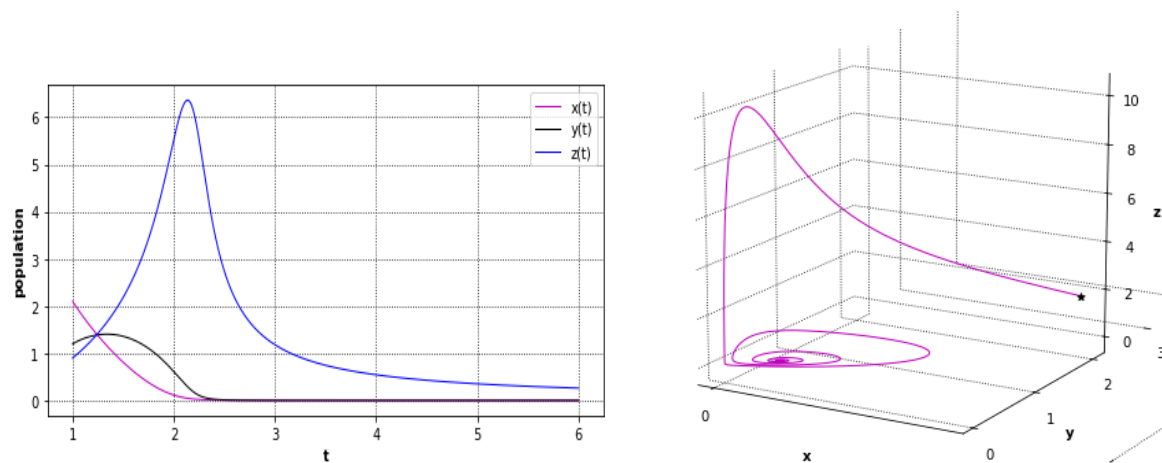

Figure 11 The Dynamic behaviours of the solution system with initial value $N3 = [2.1, 1.2, 1.1]$
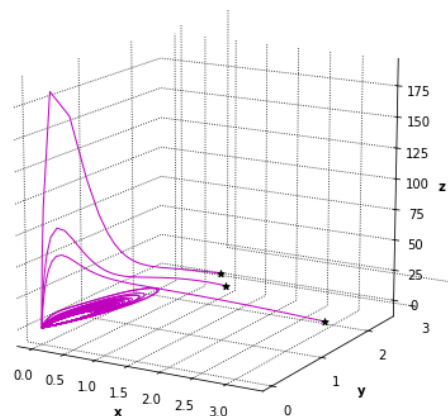
Figure 12 The Dynamic behaviours of the solution system with $\eta = 0.30951031$

It is shown numerically that there exists a stable limit cycle. In this case, the population of juvenile prey, adult prey and the predators will exis and the existence of Hopf bifurcation we will studied further

4.   Conclution

In this research, we are investigated with the local stability in predator-prey model with stage-structure on both of the prey population. The model has three e positive equilibria, namely the original, the extinction of the predator and the interior point. The Interior equilibrium point is local asymptotically stable with certain conditions. This study aims to detect the limit cycle with its phase portrait and its time-series graphs of the prey-predator system. Numerical simulated are represented as phase portraits to shows the stability of the equilibrium point. In our Numerical outcomes confirmed the analytical results and the local behaviour juvenile-adult prey and predator model which start from their positive initial condition. The parameter value is taken mainly from the literature and assumption. It is shown numerically that there exists a stable limit cycle. The next research, we will studied further the existence of Hopf bifurcation analysis of the interior equilibrium point.

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