LINEAR ALGEBRA TUTORIAL

VECOTRS:

In linear algebra, a vector is a mathematical object that represents both magnitude and direction. Vectors are commonly used in machine learning for representing data points, features, or model parameters. They can be one-dimensional (column vectors) or multi-dimensional (row vectors or column vectors with multiple components).

Here are some basic operations of vectors:

1. Vector Addition:

Vector addition is performed by adding corresponding elements of two vectors. If we have two vectors u and v of the same dimension, the sum is obtained by adding their corresponding elements:

u = [u1, u2, ..., un]

v = [v1, v2, ..., vn]

u + v = [u1 + v1, u2 + v2, ..., un + vn]

For example, if u = [1, 2, 3] and v = [4, 5, 6], then u + v = [1 + 4, 2 + 5, 3 + 6] = [5, 7, 9].

2. Scalar Multiplication:

Scalar multiplication is performed by multiplying each element of a vector by a scalar value. If we have a vector u and a scalar c, the scalar multiplication is given by:

u = [u1, u2, ..., un]

c * u = [c * u1, c * u2, ..., c * un]

For example, if u = [1, 2, 3] and c = 2, then c * u = [2 * 1, 2 * 2, 2 * 3] = [2, 4, 6].

3. Dot Product:

The dot product (also known as the inner product or scalar product) is an operation that takes two vectors and produces a scalar value. If we have two vectors u and v of the same dimension, the dot product is calculated as the sum of the products of their corresponding elements:

u = [u1, u2, ..., un]

v = [v1, v2, ..., vn]

u · v = u1 * v1 + u2 * v2 + ... + un * vn

For example, if u = [1, 2, 3] and v = [4, 5, 6], then u · v = 1 * 4 + 2 * 5 + 3 * 6 = 4 + 10 + 18 = 32.

4. Cross Product:

The cross product is an operation that is only defined for vectors in three-dimensional space. It produces a new vector that is perpendicular (orthogonal) to the two input vectors. The cross product of vectors u and v is denoted as u x v.

For example, if u = [1, 2, 3] and v = [4, 5, 6], then the cross product u x v is calculated as:

u x v = [(2 * 6) - (3 * 5), (3 * 4) - (1 * 6), (1 * 5) - (2 * 4)] = [-3, 6, -3].

These basic vector operations are essential in various machine learning algorithms and applications. They help manipulate and combine vectors to perform computations and solve problems in the field of machine learning.

5. Angel between Vectors:

The cosine formula for finding the angle between vectors A and B is given by: $\cos(\theta) = (A \cdot B) / (|A|\ |B|)$.

Example:

Consider two vectors A = (1, 2, 3) and B = (-2, 0, 1).

Step 1: Calculate the dot product: $A \cdot B = (1 * -2) + (2 * 0) + (3 * 1) = -2 + 0 + 3 = 1$.

Step 2: Calculate the magnitudes: $|A| = \sqrt{(1^2 + 2^2 + 3^2)} = \sqrt{14}$, and $|B| = \sqrt{((-2)^2 + 0^2 + 1^2)} = \sqrt{5}$.

Step 3: Substitute the values into the formula: $\cos(\theta) = 1 / (\sqrt{14} * \sqrt{5})$.

Step 4: Calculate the angle: $\theta = \arccos(1 / (\sqrt{14} * \sqrt{5})) \approx 52.5$ degrees.

So, the angle between vectors A and B is approximately 52.5 degrees.

Remember that the angle between vectors is always positive and ranges from 0 to 180 degrees.

Type of Vectors:

Here are some types of vectors in linear algebra along with examples:

1. Zero Vector: The zero vector, denoted by 0, is a vector with all components equal to zero.

Example: Vector 0 = [0, 0, 0].

2. Unit Vector: A unit vector is a vector with a magnitude of 1.

Example: Unit vector u = $[1/\sqrt{2}, 1/\sqrt{2}]$.

3. Column Vector: A column vector is a vector arranged in a single column.

Example: Column vector v = [2, 4, -1]^T.

4. Row Vector: A row vector is a vector arranged in a single row.

Example: Row vector w = [3, -2, 1].

5. Position Vector: A position vector represents the position of a point in space relative to a reference point or origin.

Example: Position vector p = [2, -1, 3].

6. Displacement Vector: A displacement vector represents the change in position between two points in space.

Example: Displacement vector d = [4, -2, 1].

7. Normal Vector: A normal vector is a vector that is perpendicular (orthogonal) to a surface or plane.

Example: Normal vector n = [0, 1, 0].

8. Eigenvector: An eigenvector is a non-zero vector that, when multiplied by a square matrix, results in a scalar multiple of itself.

Example: Consider the matrix A = [2, 1; 4, 3]. The eigenvector for A is v = [1, 2].

9. Orthogonal Vectors: Orthogonal vectors are vectors that are perpendicular to each other in n-dimensional space. Mathematically, two vectors u and v are orthogonal if their dot product is zero

Example of Orthogonal Vectors:

Consider the vectors u = [2, 1] and v = [-1, 2].

To check if they are orthogonal, we calculate their dot product:

$u \cdot v = (2 * -1) + (1 * 2) = -2 + 2 = 0$.

Since the dot product is zero, vectors u and v are orthogonal.

10. Orthonormal Vectors: Orthonormal vectors are unit vectors that are orthogonal to each other

Example of Orthonormal Vectors:

Consider the vectors i = [1, 0] and j = [0, 1].

To check if they are orthonormal, we calculate their dot product and magnitudes:

$i \cdot j = (1 * 0) + (0 * 1) = 0$,

$|i| = \sqrt{(1^2 + 0^2)} = 1$, $|j| = \sqrt{(0^2 + 1^2)} = 1$

Since the dot product is zero and the magnitudes are 1, vectors i and j are orthonormal.

These examples illustrate different types of vectors in linear algebra. Each type of vector has its own characteristics and applications in various mathematical and scientific contexts.

Vector Norms:

The three common norms used in mathematics: L1 norm, L2 norm, and L-infinity norm. These norms are used to measure the size or magnitude of a mathematical object, such as a vector or a matrix. Here's a brief definition of each norm and a few examples with solutions:

1. L1 Norm (Manhattan norm or Taxicab norm):

The L1 norm is defined as the sum of the absolute values of the elements in a vector or the sum of the absolute differences between corresponding elements in two vectors. It is denoted as $||x||1$.

Example 1:

Let's consider a vector x = [2, -5, 3]. The L1 norm of x can be calculated as:

$||x||1 = |2| + |-5| + |3| = 2 + 5 + 3 = 10$.

Example 2:

Consider two vectors x = [1, 2, 3] and y = [4, 5, 6]. The L1 norm of the difference between these vectors can be calculated as:

$||x - y||1 = |1 - 4| + |2 - 5| + |3 - 6| = 3 + 3 + 3 = 9$.

2. L2 Norm (Euclidean norm):

The L2 norm is defined as the square root of the sum of the squares of the elements in a vector or the square root of the sum of the squared differences between corresponding elements in two vectors. It is denoted as $||x||2$.

Example 1:

Consider a vector x = [2, -5, 3]. The L2 norm of x can be calculated as:

||x||2 = sqrt(2^2 + (-5)^2 + 3^2) = sqrt(4 + 25 + 9) = sqrt(38) ≈ 6.16.

Example 2:

Consider two vectors x = [1, 2, 3] and y = [4, 5, 6]. The L2 norm of the difference between these vectors can be calculated as:

||x - y||2 = sqrt((1 - 4)^2 + (2 - 5)^2 + (3 - 6)^2) = sqrt(9 + 9 + 9) = sqrt(27) ≈ 5.20.

3. L-infinity Norm (Maximum norm or Chebyshev norm):

The L-infinity norm is defined as the maximum absolute value of the elements in a vector or the maximum absolute difference between corresponding elements in two vectors. It is denoted as $||x||_\infty$.

Example 1:

Consider a vector x = [2, -5, 3]. The L-infinity norm of x can be calculated as:

$||x||_\infty = \max(|2|, |-5|, |3|) = \max(2, 5, 3) = 5$.

Example 2:

Consider two vectors x = [1, 2, 3] and y = [4, 5, 6]. The L-infinity norm of the difference between these vectors can be calculated as:

$||x - y||_\infty = \max(|1 - 4|, |2 - 5|, |3 - 6|) = \max(3, 3, 3) = 3$.

These are basic examples demonstrating the calculation of the norms for vectors. The norms can also be applied to matrices and have various applications in fields like optimization, signal processing, machine learning, and more.

Matrix:

In linear algebra, a matrix is a rectangular array of numbers arranged in rows and columns. Matrices are fundamental mathematical objects used in various fields, including machine learning, to represent and manipulate data. Here is an overview of matrices and the basic operations that can be performed on them, along with examples:

1. Matrix Addition:

Matrix addition involves adding corresponding elements of two matrices to produce a new matrix. For example, consider two matrices A and B:

A = [[1, 2], [3, 4]]

B = [[5, 6], [7, 8]]

The sum of A and B (A + B) is calculated by adding corresponding elements:

A + B = [[1 + 5, 2 + 6], [3 + 7, 4 + 8]] = [[6, 8], [10, 12]]

2. Scalar Multiplication:

Scalar multiplication involves multiplying every element of a matrix by a scalar value. For example, let's consider a matrix A:

A = [[1, 2], [3, 4]]

If we multiply A by a scalar c = 2:

c * A = [[2 * 1, 2 * 2], [2 * 3, 2 * 4]] = [[2, 4], [6, 8]]

3. Matrix Multiplication:

Matrix multiplication is an operation that combines the rows of the first matrix with the columns of the second matrix to produce a new matrix. For example, let's consider two matrices A and B:

A = [[1, 2, 3], [4, 5, 6]]

B = [[7, 8], [9, 10], [11, 12]]

The product of A and B (A * B) is calculated as follows:

A * B = [[(1 * 7) + (2 * 9) + (3 * 11), (1 * 8) + (2 * 10) + (3 * 12)],

[(4 * 7) + (5 * 9) + (6 * 11), (4 * 8) + (5 * 10) + (6 * 12)]]

= [[58, 64], [139, 154]]

4. Transpose:

The transpose of a matrix involves interchanging its rows with columns. For example, consider a matrix A:

A = [[1, 2, 3], [4, 5, 6]]

The transpose of A (A^T) is obtained by swapping the rows and columns:

A^T = [[1, 4], [2, 5], [3, 6]]

5. Determinant:

The determinant is a scalar value associated with a square matrix. It represents certain properties of the matrix. For example, consider a matrix A:

A = [[1, 2], [3, 4]]

The determinant of A (|A|) is calculated as follows:

|A| = (1 * 4) - (2 * 3) = -2

6. Trace of a Matrix:

The trace of a square matrix is the sum of the elements on its main diagonal, which runs from the top left to the bottom right. It is denoted as "tr(A)" or "Tr(A)" for a matrix A.

The trace of matrix A is calculated as follows:

Consider the matrix A = [2 4; -1 3].

Tr(A) = 2 + 3 = 5. Therefore, the trace of matrix A is 5

We can also calculate frobenius norms using trace of matrix ||A||f = Tr(A)^1/2

7. Inverse:

To calculate the inverse of a 3x3 matrix, you can use the following steps:

1. Check if the matrix is invertible:

   - The matrix must be square, meaning it has the same number of rows and columns.

   - The determinant of the matrix should not be zero. If the determinant is zero, the matrix is singular and does not have an inverse.

2. Find the adjugate of the matrix:

   - The adjugate of a matrix is obtained by taking the transpose of the cofactor matrix.

   - To calculate the cofactor matrix, you need to calculate the determinant of each minor matrix formed by removing a row and a column from the original matrix.

   - The cofactor matrix is obtained by alternating the signs of the determinants of the minor matrices.

3. Calculate the determinant of the original matrix.

4. Divide the adjugate of the matrix by the determinant:

   - Each element of the adjugate matrix is divided by the determinant to obtain the inverse of the matrix.

Here is an example to illustrate the process:

Consider a 3x3 matrix A:

A = [[1, 2, 3], [4, 5, 6], [7, 8, 9]]

1. Check if the matrix is invertible:

   - The matrix is square, so it meets the first requirement.

   - Calculate the determinant of A:

|A| = (1 * (5 * 9 - 6 * 8)) - (2 * (4 * 9 - 6 * 7)) + (3 * (4 * 8 - 5 * 7))

= (1 * -3) - (2 * -6) + (3 * 3)

= -3 + 12 + 9

= 18

- Since the determinant is not zero, the matrix is invertible.

2. Find the adjugate of the matrix:

- Calculate the cofactor matrix:

C = [[-3, 6, -3], [6, -12, 6], [-3, 6, -3]] (cofactors are obtained by swapping signs of elements and taking determinants of minors)

- Transpose the cofactor matrix to get the adjugate matrix:

adj(A) = [[-3, 6, -3], [6, -12, 6], [-3, 6, -3]]

3. Calculate the determinant of the original matrix:

|A| = 18 (already calculated)

4. Divide the adjugate of the matrix by the determinant:

- Divide each element of the adjugate matrix by the determinant:

A^-1 = (1 / 18) * [[-3, 6, -3], [6, -12, 6], [-3, 6, -3]] = [[-1/6, 1/3, -1/6], [1/3, -2/3, 1/3], [-1/6, 1/3, -1/6]]

Therefore, the inverse of the matrix A is:

A^-1 = [[-1/6, 1/3, -1/6], [1/3, -2/3, 1/3], [-1/6, 1/3, -1/6]]

You can verify the result by multiplying A and its inverse to obtain the identity matrix.

These are some of the basic operations on matrices. They are foundational to linear algebra and extensively used in various mathematical calculations, data analysis, and machine learning algorithms.

Type of Matrix:

There are several types of matrices commonly used in linear algebra. Here are some of the most frequently encountered types:

1. Square Matrix: A square matrix has an equal number of rows and columns. For example, a matrix with dimensions 3x3 or 4x4 is a square matrix.

2. Rectangular Matrix: A rectangular matrix has a different number of rows and columns. For example, a matrix with dimensions 2x3 or 4x2 is a rectangular matrix.

3. Diagonal Matrix: A diagonal matrix is a square matrix where all the elements outside the main diagonal (the diagonal from the top left to the bottom right) are zero. The non-zero elements can be any value. Example:

[3 0 0]

[0 5 0]

[0 0 2]

4. Identity Matrix: An identity matrix is a square matrix where all the elements on the main diagonal are 1, and all other elements are 0. It is denoted by the symbol "I" or "I_n", where "n" represents the dimensions of the matrix. Example:

[1 0 0]

[0 1 0]

[0 0 1]

5. Zero Matrix: A zero matrix is a matrix where all the elements are zero. It is denoted by the symbol "O" or "Omn", where "m" represents the number of rows and "n" represents the number of columns. Example:

[0 0 0]

[0 0 0]

[0 0 0]

6. Symmetric Matrix: A symmetric matrix is a square matrix where the element at row "i" and column "j" is equal to the element at row "j" and column "i" or A=A^T. Example:

[2 5 7]

[5 1 -3]

[7 -3 4]

7. Skew-Symmetric Matrix: A skew-symmetric matrix is a square matrix where the element at row "i" and column "j" is equal to the negative of the element at row "j" and column "i" or A=-A^T.. Example:

[0 3 -4]

[-3 0 5]

[4 -5 0]

8. Orthogonal Matrix: An orthogonal matrix is a square matrix where the columns (or rows) are orthogonal unit vectors. The dot product of any two distinct columns (or rows) of an orthogonal matrix is zero, and the norm (magnitude) of each column (or row) is 1. In other words, if A is an orthogonal matrix, then A^T * A = I, where A^T is the transpose of A and I is the identity matrix.

Example of an orthogonal matrix:

Consider the following matrix A:

[0.6 0.8]

[-0.8 0.6]

To check if A is orthogonal, we calculate A^T * A and check if it equals the identity matrix:

A^T * A =

[0.6 -0.8] * [0.6 0.8]   = [1 0]

[0.8 0.6]   [-0.8 0.6]   = [0 1]

Since A^T * A equals the identity matrix, matrix A is orthogonal.

9. Orthonormal Matrix: An orthonormal matrix is a special case of an orthogonal matrix where the columns (or rows) are orthogonal unit vectors. In addition to satisfying the properties of an orthogonal matrix, an orthonormal matrix has the property that the dot product of any two distinct columns (or rows) is zero, and the norm of each column (or row) is 1.

Example of an orthonormal matrix:

Consider the following matrix B:

[1/√2 -1/√2]

[1/√2 1/√2]

To check if B is orthonormal, we calculate B^T * B and check if it equals the identity matrix:

B^T * B =

[1/√2 1/√2] * [1/√2 -1/√2] = [1 0]

[-1/√2 1/√2]   [1/√2 1/√2] = [0 1]

Since B^T * B equals the identity matrix, matrix B is orthonormal.

Both orthogonal and orthonormal matrices are important in linear algebra and have applications in various areas such as rotation transformations, coordinate systems, and solving systems of linear equations.

These are some of the commonly encountered types of matrices in linear algebra. Each type has its own properties and characteristics, and they are used for various mathematical operations and applications.

The Frobenius norm (Matrix Norms):

The Frobenius norm, also known as the Euclidean norm or the L2 norm of a matrix, measures the "size" or "magnitude" of a matrix. It is defined as the square root of the sum of the squares of all the elements in the matrix. The Frobenius norm of a matrix A, denoted as ||A||F, is calculated as:

||A||F = sqrt(∑∑ |A[i][j]|^2)

where i iterates over the rows and j iterates over the columns of the matrix A.

Here are a few examples of calculating the Frobenius norm:

Example 1:

Consider the matrix A:

A = [[1, 2], [3, 4], [5, 6]]

To calculate the Frobenius norm of matrix A, we square each element, sum the squares, and take the square root of the result:

||A||F = sqrt(|1|^2 + |2|^2 + |3|^2 + |4|^2 + |5|^2 + |6|^2)

= sqrt(1 + 4 + 9 + 16 + 25 + 36)

= sqrt(91)

≈ 9.54

Example 2:

Let's consider another matrix B:

B = [[-1, 0, 1], [2, -2, 2]]

The Frobenius norm of matrix B can be calculated as:

||B||F = sqrt(|-1|^2 + |0|^2 + |1|^2 + |2|^2 + |-2|^2 + |2|^2)

= sqrt(1 + 0 + 1 + 4 + 4 + 4)

= sqrt(14)

≈ 3.74

The Frobenius norm is commonly used in various applications, such as matrix analysis, machine learning, and optimization. It provides a measure of the overall "size" or "magnitude" of a matrix and can be used to compare matrices or assess the error between two matrices.

Method of Solving Simultaneous equations:

1. Solving using matrix inversion:

To solve simultaneous equations using matrix inversion in linear algebra, you can follow these steps:

Step 1: Write the system of equations in matrix form.

Let A be the coefficient matrix, X be the column vector of variables, and B be the column vector of constant terms. The system of equations can be written as AX = B.

Step 2: Calculate the inverse of the coefficient matrix A.

If the inverse of A exists, you can find it by calculating A^-1.

Step 3: Multiply both sides of the equation AX = B by A^-1.

This gives you X = A^-1B.

Step 4: Calculate the product of A^-1 and B.

Multiply the inverse matrix A^-1 by the column vector B to obtain the solution vector X.

Let's consider an example to illustrate the process:

Example:

Solve the following system of linear equations:

2x + 3y = 8

4x - 2y = 2

Step 1: Write the system of equations in matrix form:

[2   3] * [x] = [8]

[4 -2] * [y] = [2]

Step 2: Calculate the inverse of the coefficient matrix A.

A = [2   3]

[4 -2]

To calculate A^-1, we need to find the determinant of A and the adjugate of A.

The determinant of A (|A|) = (2 * -2) - (3 * 4) = -16

The adjugate of A (adj(A)) = [-2   -3]

[-4   2]

Finally, A^-1 = (1 / |A|) * adj(A) = (1 / -16) * [-2   -3] = [1/8   3/16]

[-4   2] = [1/4 -1/8]

Step 3: Multiply both sides by A^-1:

[x] = [1/8   3/16] * [8]

[y] = [1/4   -1/8] * [2]

Step 4: Calculate the product of A^-1 and B:

[x]   [(1/8)*8 + (3/16)*2]

[y] = [(1/4)*8 - (1/8)*2]

[x]   [11/8]

[y] = [7/4]

So, the solution to the system of equations is x = 11/8 and y = 7/4 by using matrix inversion.

Using matrix inversion in linear algebra allows you to find the exact solution to the system of simultaneous equations by finding the inverse of the coefficient matrix and performing matrix operations.

Method of Matrix Decomposition:

1. Eigendecomposition:

Eigendecomposition is a process in linear algebra that decomposes a square matrix into a set of eigenvectors and eigenvalues. It allows us to express the original matrix as a product of three matrices: V, D, and V^(-1), where V is a matrix whose columns are the eigenvectors(v), D is a diagonal matrix with the eigenvalues(λ) on the diagonal, and V^(-1) is the inverse of matrix V.

Decomposition of A is represented as: **A = VDV^-1**

Example: Let's consider an example to illustrate eigendecomposition:

Consider the matrix A:

A = [[4, 2],

[1, 3]]

To perform eigendecomposition, we follow these steps:

1. Calculate the eigenvalues: The eigenvalues of A can be found by solving the characteristic equation |A - λI| = 0, where I is the identity matrix.

The characteristic equation is: |A - λI| = Det. ([4-λ, 2]

[1, 3-λ])

Expanding the determinant, we get: (4 - λ)(3 - λ) - (2)(1) = 0

Simplifying further, we have: λ^2 - 7λ + 10 = 0

Factoring the equation, we obtain: (λ - 5)(λ - 2) = 0

So, the eigenvalues are λ1 = 5 and λ2 = 2.

2. Calculate the eigenvectors: For each eigenvalue, we find the corresponding eigenvector by solving the equation (A - λI)x = 0.

For eigenvalue λ1 = 5: (A - λ1I)x = 0

(A - 5I)x = 0

|4-5, 2 | |x1| = |0|

|1, 3-5| |x2| |0|

Simplifying, we have:

|-1, 2| |x1| = |0|

|1, -2| |x2| |0|

This leads to the equation -x1 + 2x2 = 0.

Solving for x1 in terms of x2, we get: x1 = 2x2

So, the eigenvector corresponding to λ1 = 5 is [2, 1].

For eigenvalue λ2 = 2: (A - λ2I)x = 0

(A - 2I)x = 0

|4-2, 2 | |x1| = |0|

|1, 3-2| |x2| |0|

Simplifying, we have:

|2, 2| |x1| = |0|

|1, 1| |x2| |0|

This leads to the equation 2x1 + 2x2 = 0.

Solving for x1 in terms of x2, we get: x1 = -x2

So, the eigenvector corresponding to λ2 = 2 is [-1, 1].

3. Construct the eigenvector matrix V: The matrix V is formed by arranging the eigenvectors as columns.

V = [[2, -1],

[1, 1]]

Also calculate V^-1 (Inverse of V):

Det. |v|= (2*1 – 1*(-1)) = 3

V^-1 = 1/3 * [[1 , 1],

[-1, 2]]

V^-1= [[1/3, 1/3]

[-1/3, 2/3]]

4. Construct the eigenvalue matrix D: The eigenvalue matrix D is a diagonal matrix with the eigenvalues on the diagonal.

D = [[5, 0],

[0, 2]]

Finally, we can write the eigendecomposition of matrix A as:

A = VDV^(-1)

A = [[4, 2],

[1, 3]]

V = [[2, -1],

[1, 1]]

D = [[5, 0],

[0, 2]]

V^-1= [[1/3, 1/3]

[-1/3, 2/3]]

Verify the eigendecomposition:

PDP^(-1) = [[2, -1], * [[5, 0], * [[1/3, 1/3],

[1, 1]] [0, 2]] [-1/3, 2/3]]

Calculating the product, we obtain the original matrix A, confirming the eigendecomposition.

Eigendecomposition allows us to analyze the behavior and properties of the original matrix A in terms of its eigenvalues and eigenvectors.

2. Singular Value Decomposition:

Singular Value Decomposition (SVD) is a technique in linear algebra that decomposes a matrix into three separate matrices: U, Σ, and V^T. It is a powerful tool for understanding the properties and structure of a matrix.

Decomposition of A is: A = UΣV^T where U, Σ, and V^T satisfy the properties of orthogonal/orthonormal matrices. The matrix U and V^T are both square matrices, while Σ is a rectangular diagonal matrix (with possible zero entries for rectangular matrices).

U: The matrix U is called the left singular matrix. It is a square matrix whose columns are the left singular vectors of A. These vectors are orthogonal to each other and form an orthonormal basis for the column space of A.

Σ: The matrix Σ is called the singular value matrix. It is a diagonal matrix that contains the singular values of A. The singular values are non-negative and are arranged in descending order on the diagonal of Σ.

V^T: The matrix V^T (transpose of V) is called the right singular matrix. It is a square matrix whose columns are the right singular vectors of A. These vectors are orthogonal to each other and form an orthonormal basis for the row space of A.

Example:

Let's consider an example to illustrate Singular Value Decomposition:

Consider the matrix A:

A = [[2, 1],

[1, 2]]

To perform Singular Value Decomposition, we follow these steps:

1. Calculate the eigenvalues and eigenvectors of A * A^T:

Compute A * A^T:

A*A^T = [[2, 1], [1, 2]] * [[2, 1], [1, 2]] = [[5, 4], [4, 5]].

Find the eigenvalues of A*A^T by solving the characteristic equation:

|A * A^T - λI| = 0.

(5 - λ)(5 - λ) - 4 * 4 = 0.

Simplifying the equation, we have λ^2 - 10λ + 9 = 0.

Solving the quadratic equation, we find two eigenvalues: λ1 = 1 and λ2 = 9.

For each eigenvalue, find the corresponding eigenvector.

For $\lambda1 = 1$, solve the equation $(A * A^T - \lambda 1I) * x = 0$:

$[[5, 4], [4, 5]] * x = 0$.

We obtain one linearly independent eigenvector $x1 = [-1, 1]$.

For $\lambda2 = 9$, solve the equation $(A * A^T - \lambda 2I) * x = 0$:

$[[5, 4], [4, 5]] * x = 0$.

We obtain another linearly independent eigenvector $x2 = [1, 1]$.

Normalize the eigenvectors:

Normalize each eigenvector by dividing it by its norm to obtain unit vectors.

Normalize x1: $||x1|| = \sqrt{(1^2 + (-1)^2)} = \sqrt{2}$.

$v1\_normalized = [-1/\sqrt{2}, 1/\sqrt{2}]$.

Normalize x2: $||x2|| = \sqrt{(1^2 + 1^2)} = \sqrt{2}$.

$v2\_normalized = [1/\sqrt{2}, 1/\sqrt{2}]$.

Calculate the singular values:

The singular values σi are the square roots of the eigenvalues of $A * A^T$ or $A^T * A$.

$\sigma 1 = \sqrt{\lambda 1} = 1$, $\sigma 2 = \sqrt{\lambda 2} = 3$.

2. Calculate the singular value matrix Σ: The singular value matrix Σ is a diagonal matrix where the singular values are arranged in descending order.

$\Sigma = [[\sqrt{\lambda 2}, 0],$ = $[[3, 0],$

$[0, \sqrt{\lambda 1}]]$ $[0, 1]]$

3. Calculate the left singular matrix U: The left singular matrix U is composed of the eigenvectors of $A * A^T$, normalized to unit length.

Taken v1 & v2 from step 2 & Normalize v1 and v2 to unit length:

$v1\_ normalized = [1/\sqrt{2}, 1/\sqrt{2}]$ and $v2\_ normalized = [-1/\sqrt{2}, 1/\sqrt{2}]$

The left singular matrix U is formed by arranging the normalized eigenvectors as columns:

$U = [[1/\sqrt{2}, -1/\sqrt{2}],$

$[1/\sqrt{2}, 1/\sqrt{2}]]$

4. Calculate the right singular matrix V: The right singular matrix V is composed of the eigenvectors of A^T * A, normalized to unit length.

The eigenvectors of A^T * A are the same as the eigenvectors of A * A^T, so V will be the same as U:

V = [[1/√2, -1/√2],
  [1/√2, 1/√2]]

Therefore, the Singular Value Decomposition of matrix A is:

A = UΣV^T

Substituting the calculated values:

A = [[1/√2, -1/√2], * [[3, 0], * [[1/√2, 1/√2], = [[2, 1],
  [1/√2, 1/√2]] [0, 1]], * [-1/√2, 1/√2]] [1, 2]]

SVD allows us to understand the structure and properties of the matrix A, and it is useful in various application such as data compression, image processing, and linear regression.

PCA:

Principal Component Analysis (PCA) is a statistical technique used to reduce the dimensionality of high-dimensional data while preserving its key patterns and structures. In linear algebra, PCA involves finding the orthogonal vectors, called principal components, that best represent the data's variance.

Here's the definition of PCA with an example:

Consider a dataset X with n data points, each described by d-dimensional feature vectors. To perform PCA:

1. Standardize the data: Subtract the mean from each feature and divide by the standard deviation to ensure features are on a similar scale.

2. Compute the covariance matrix: Form the d x d covariance matrix C of the standardized data, which captures the relationships between the features.

3. Calculate eigenvectors and eigenvalues: Find the eigenvectors and corresponding eigenvalues of C. Eigenvectors represent the principal components, and eigenvalues indicate the amount of variance explained by each principal component.

4. Select principal components: Sort the eigenvectors based on their eigenvalues in descending order. Choose the top k eigenvectors, which correspond to the highest eigenvalues, to form the principal components matrix P.

5. Transform the data: Project the standardized data onto the principal components by multiplying X with P to obtain the transformed data Y, which has reduced dimensionality.

Example:

Let's assume we have a dataset X with two features (2D data) represented by the following points:

X = [[1, 3], [2, 4], [3, 5], [4, 6], [5, 7]]

1. Standardize the data: Subtract the mean of each feature and divide by their standard deviations.

2. Compute the covariance matrix: Calculate the 2x2 covariance matrix C.

3. Calculate eigenvectors and eigenvalues: Find the eigenvectors and eigenvalues of C. Suppose we obtain the eigenvectors [0.707, 0.707] and [-0.707, 0.707], with eigenvalues [9.856, 0.144].

4. Select principal components: Sort the eigenvectors based on their eigenvalues. We choose the eigenvector corresponding to the highest eigenvalue, [0.707, 0.707], as the first principal component

5. Transform the data: Project the standardized data onto the principal component. Multiply X with the selected principal component to obtain the transformed data Y.

PCA allows for dimensionality reduction while preserving the most important information. It is widely used in various applications such as data visualization, feature extraction, and noise reduction.