Imaging Geometry

Computer Vision

Modelling from 3D to 2D world

Perspective Matters!!

Taj Mahal

Paris town hall
"**Anamorphosis**"

Lake Sørvágsvatn in Faroe Islands

100 metres above sea level

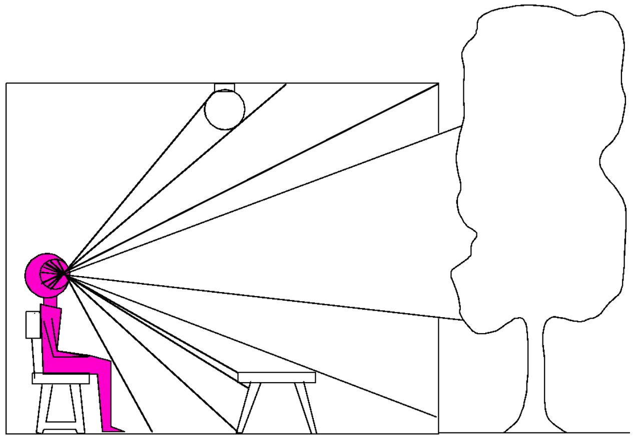
Cricket

Cameras, Multiple Views, and Motion

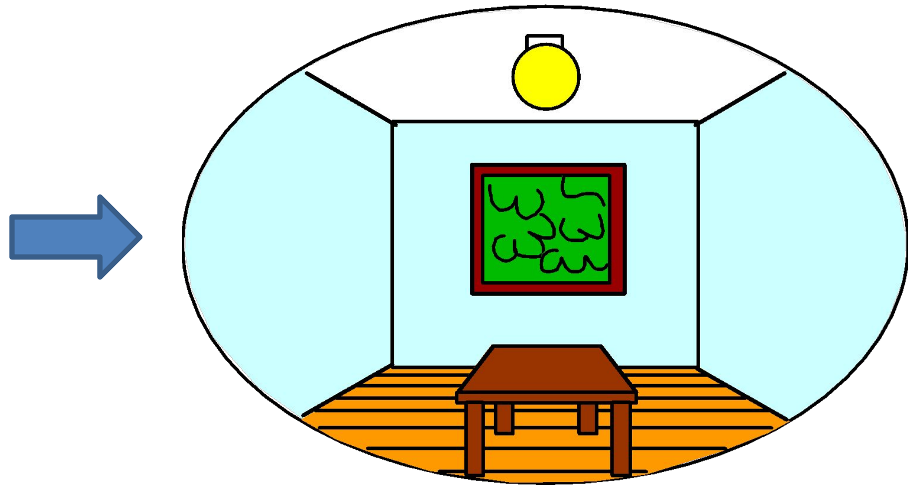
- Imaging Geometry
  - Image transforms like scaling, rotation etc
- Perspective Transformation
- Projective Transformation
- Camera Model
- Camera Calibration

Dimensionality Reduction Machine (3D to 2D)

3D world *2D image*



Point of observation



How to recover knowledge about 3D from 2D…??

Common transformations

Original

Transformed

Translation

Rotation

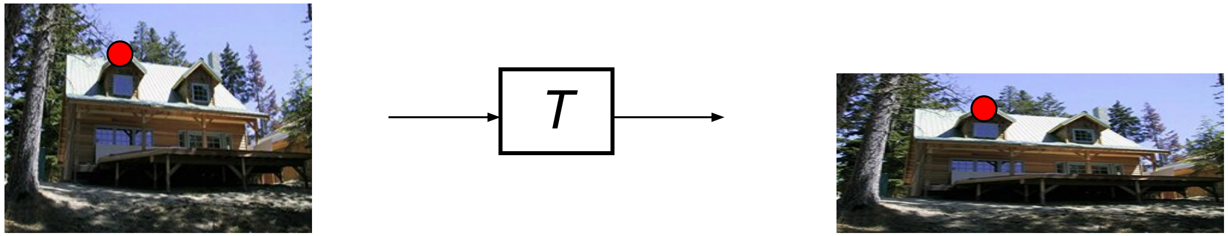
Scaling

Affine

Perspective

Slide credit (next few slides):
A. Efros and/or S. Seitz

Parametric (global) transformations



p = (x,y) **p’** = (x’,y’)

Transformation T is a coordinate-changing machine:

p’ = *T*(p)

What does it mean that *T* is global?

– *T* is the same for any point p

T can be described by just a few numbers (parameters)

For linear transformations, we can represent T as a matrix

p’ = **T**p

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{T} \begin{bmatrix} x \\ y \end{bmatrix}$$

Common transformations

Original

Transformed

Translation

Rotation

Scaling

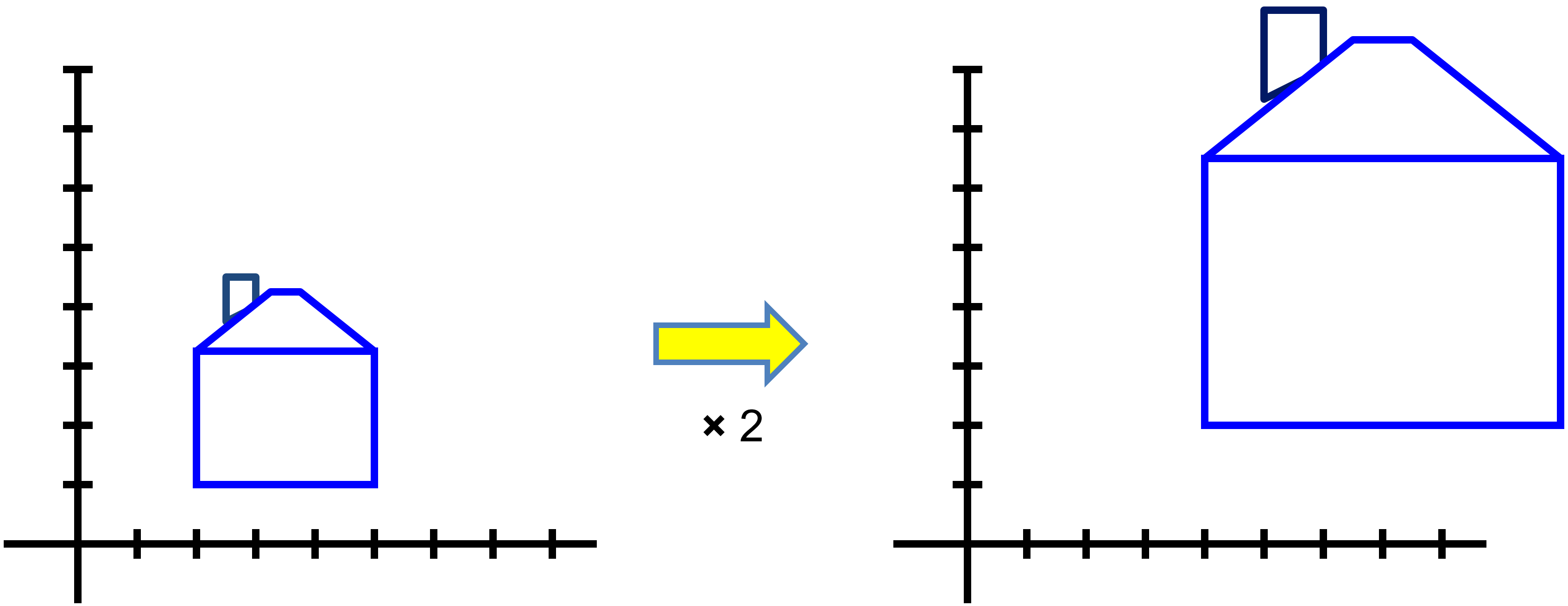
Affine

Perspective

Slide credit (next few slides):
A. Efros and/or S. Seitz

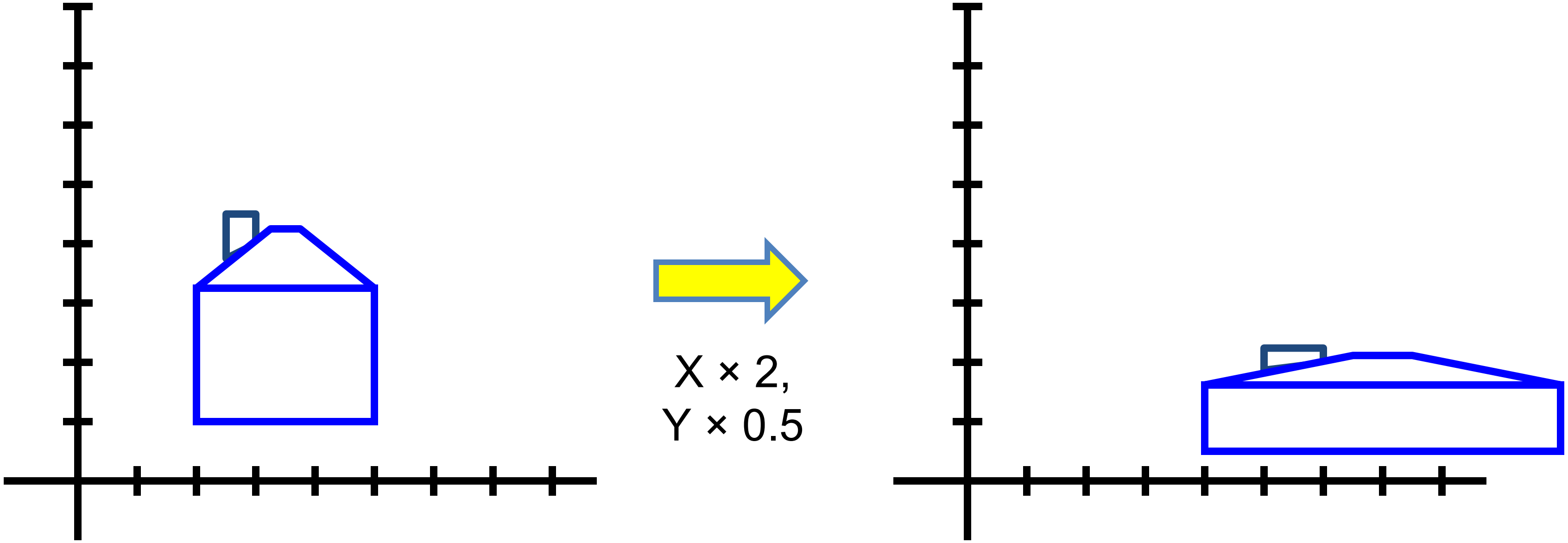
Scaling

- *Scaling* a coordinate means multiplying each of its components by a scalar
- *Uniform scaling* means this scalar is the same for all components:



Scaling

- *Non-uniform scaling*: different scalars per component:



Scaling

- Scaling operation:

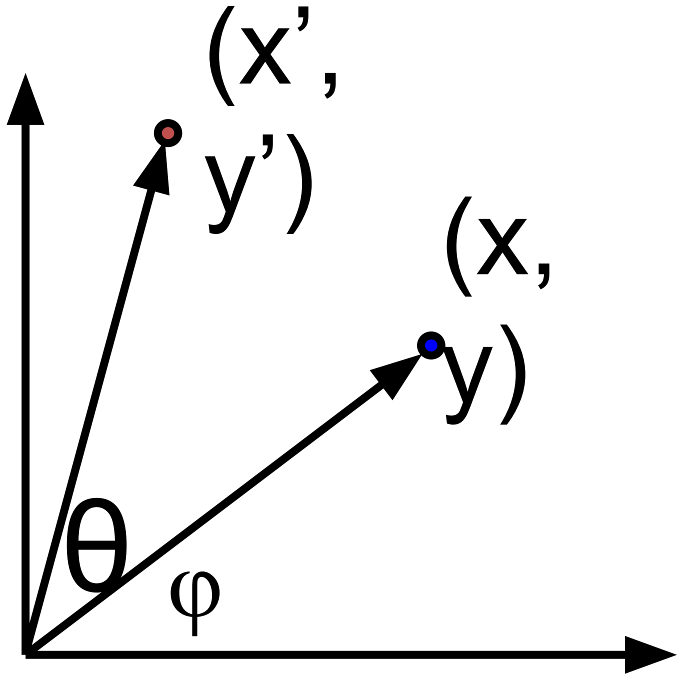
$$x' = ax$$

$$y' = by$$

- Or, in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{\textit{scaling matrix } S} \begin{bmatrix} x \\ y \end{bmatrix}$$

2-D Rotation



Polar coordinates…

$x = r \cos(\varphi)$
$y = r \sin(\varphi)$
$x' = r \cos(\varphi + \theta)$
$y' = r \sin(\varphi + \theta)$

Trig Identity…

$x' = r \cos(\varphi) \cos(\theta) - r \sin(\varphi) \sin(\theta)$
$y' = r \sin(\varphi) \cos(\theta) + r \cos(\varphi)$
$\sin(\theta)$

Substitute…

$x' = x\ \mathbf{cos}(\theta) - y\ \mathbf{sin}(\theta)$
$y' = x\ \mathbf{sin}(\theta) + y\ \mathbf{cos}(\theta)$

2-D Rotation

This is easy to capture in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} x \\ y \end{bmatrix}$$

Even though sin(θ) and cos(θ) are nonlinear functions of θ,

- ***x' is a linear combination of x and y***
- ***y' is a linear combination of x and y***

What is the inverse transformation?

- Rotation by –θ
- For rotation matrices $\mathbf{R}^{-1} = \mathbf{R}^{T}$

Basic 2D transformations

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & \alpha_x \\ \alpha_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Shear

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Rotate

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Affine

Affine is any combination of translation, scale, rotation, and shear

Affine Transformations

Affine transformations are combinations of

- Linear transformations, and
- Translations

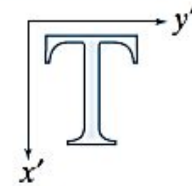
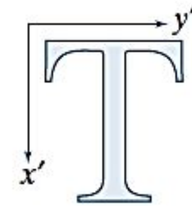
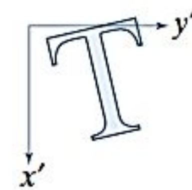
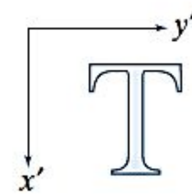
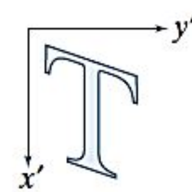
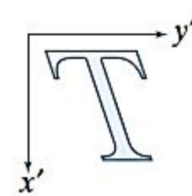
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

or

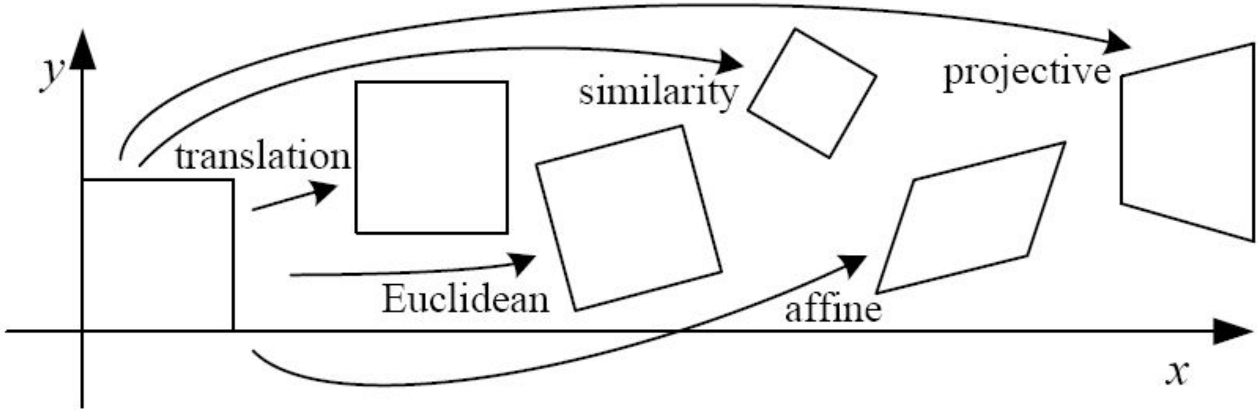
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Properties of affine transformations:

- Lines map to lines
- Parallel lines remain parallel
- Ratios are preserved
- Closed under composition

Transformation Name	Affine Matrix, A	Coordinate Equations	Example
Identity	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x' = x$ $y' = y$	
Scaling/Reflection (For reflection, set one scaling factor to –1 and the other to 0)	$\begin{bmatrix} c_x & 0 & 0 \\ 0 & c_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x' = c_x x$ $y' = c_y y$	
Rotation (about the origin)	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x' = x\cos\theta - y\sin\theta$ $y' = x\sin\theta + y\cos\theta$	
Translation	$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$x' = x + t_x$ $y' = y + t_y$	
Shear (vertical)	$\begin{bmatrix} 1 & s_v & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x' = x + s_v y$ $y' = y$	
Shear (horizontal)	$\begin{bmatrix} 1 & 0 & 0 \\ s_h & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x' = x$ $y' = s_h x + y$	

2D image transformations (reference table)



Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} I \mid t \end{bmatrix}_{2\times 3}$	2	orientation	
rigid (Euclidean)	$\begin{bmatrix} R \mid t \end{bmatrix}_{2\times 3}$	3	lengths	
similarity	$\begin{bmatrix} sR \mid t \end{bmatrix}_{2\times 3}$	4	angles	
affine	$\begin{bmatrix} A \end{bmatrix}_{2\times 3}$	6	parallelism	
projective	$\begin{bmatrix} \tilde{H} \end{bmatrix}_{3\times 3}$	8	straight lines	

'Homography'

Table 2.1 Hierarchy of 2D coordinate transformations. Each transformation also preserves the properties listed in the rows below it, i.e., similarity preserves not only angles but also parallelism and straight lines. The 2×3 matrices are extended with a third $[\mathbf{0}^T \; 1]$ row to form a full 3×3 matrix for homogeneous coordinate transformations.

Szeliski 2.1

Projective Transformations

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

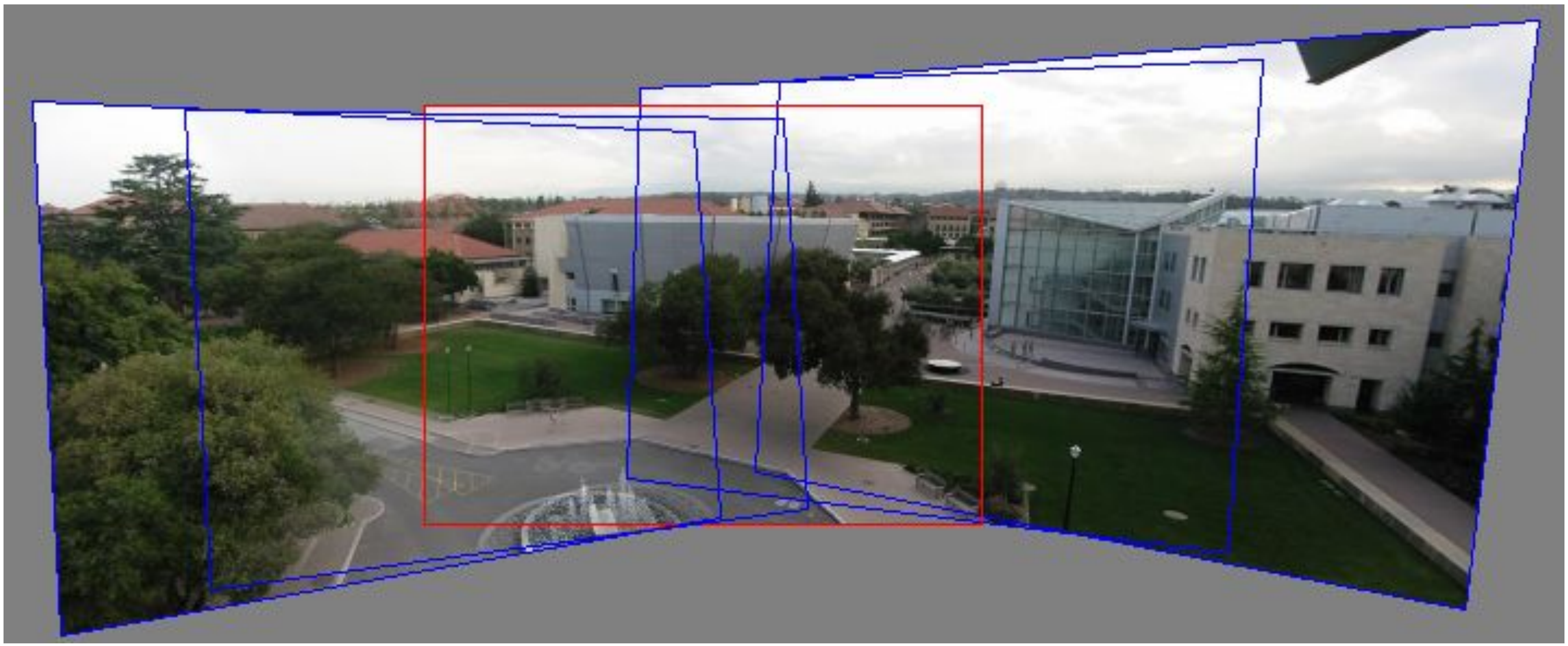
Projective transformations are combos of

- Affine transformations, and
- Projective warps

Properties of projective transformations:

- Lines map to lines
- Parallel lines do not necessarily remain parallel
- Ratios are not preserved
- Models change of basis
- Projective matrix is defined up to a scale (8 DOF)

we use projective transforms to create a 360 panorama



- In order to figure this out, we need to learn what a **camera** is

The Geometry of Image Formation

Szeliski 2.1, parts of 2.2

Mapping between image and world coordinates

- Pinhole camera model
- Projective geometry
  - Vanishing points and lines
- Projection matrix

Slides from James Hays, Derek Hoiem, Alexei Efros, Steve Seitz, and David Forsyth

Image Formation: Orthographic Projection

- Means of representing 3-dimensional objects in 2-Dimensions.
- It is a form of parallel projection, in which all the projection lines are orthogonal to the projection plane

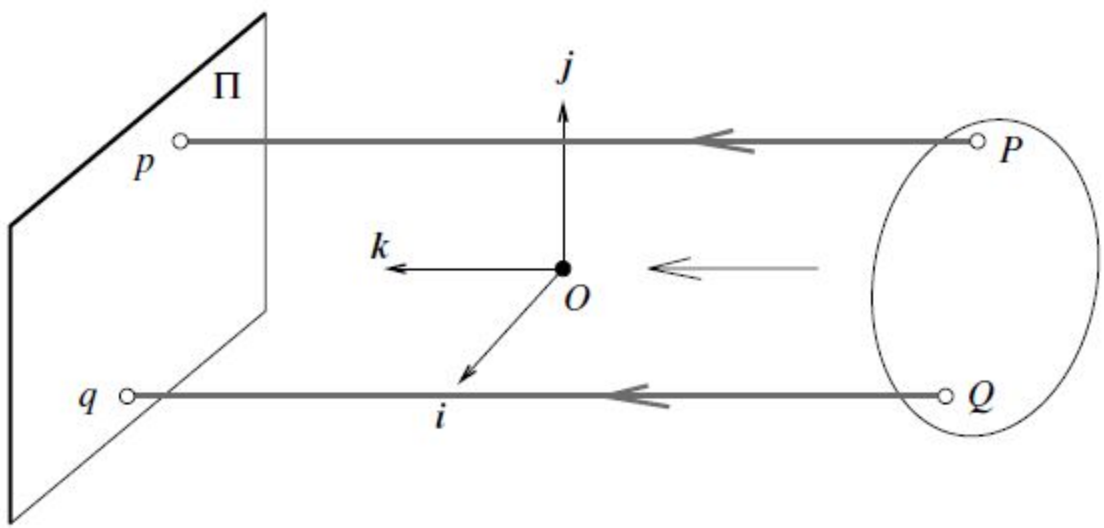


FIGURE 1.6: Orthographic projection. Unlike other geometric models of the image formation process, orthographic projection does not involve a reversal of image features.

Orthographic Projections

- A simple orthographic projection onto the plane z = 0 can be defined by the following matrix:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- For each point $v = (v_x, v_y, v_z)$, the transformed point Pv would be

$$Pv = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix}$$

- Often, it is more useful to use homogeneous coordinates. The transformation in homogeneous coordinates

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- For each homogeneous vector $v = (v_x, v_y, v_z, 1)$, the transformed vector Pv would be

$$Pv = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \\ 1 \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ 0 \\ 1 \end{bmatrix}$$

https://en.wikipedia.org/wiki/Orthographic_projection

Orthographic Projections

- In computer graphics, one of the most common matrices used for orthographic projection can be defined by a 6-tuple, (left, right, bottom, top, near, far), which defines the clipping planes.
- These planes form a box with the minimum corner at (left, bottom, -near) and the maximum corner at (right, top, -far).
- The box is translated so that its center is at the origin, then it is scaled to the unit cube which is defined by having a minimum corner at $(-1,-1,-1)$ and a maximum corner at $(1,1,1)$.

Orthographic Projections

The orthographic transform can be given by the following matrix:

$$P = \begin{bmatrix} \frac{2}{right-left} & 0 & 0 & -\frac{right+left}{right-left} \\ 0 & \frac{2}{top-bottom} & 0 & -\frac{top+bottom}{top-bottom} \\ 0 & 0 & \frac{-2}{far-near} & -\frac{far+near}{far-near} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

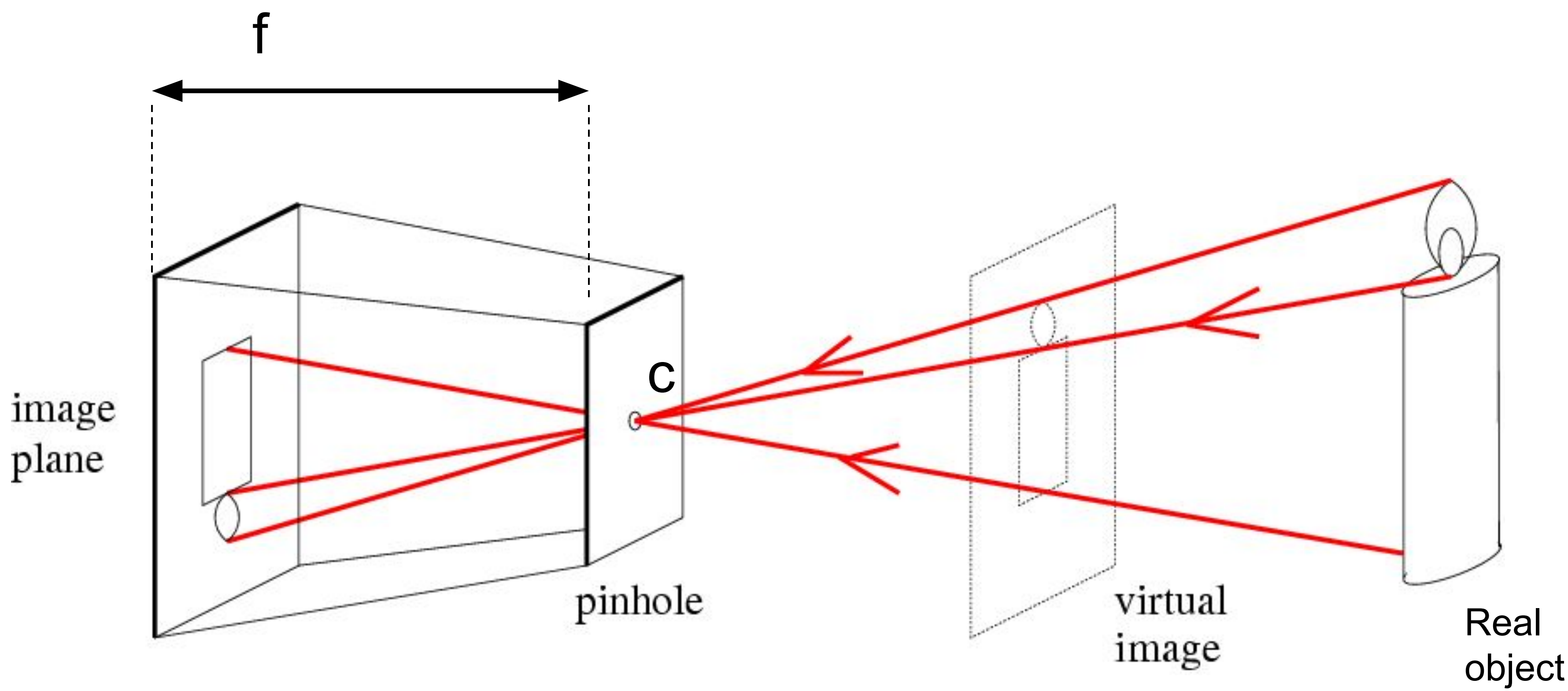
which can be given as a scaling S followed by a translation T of the form

$$P = ST = \begin{bmatrix} \frac{2}{right-left} & 0 & 0 & 0 \\ 0 & \frac{2}{top-bottom} & 0 & 0 \\ 0 & 0 & \frac{2}{far-near} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -\frac{left+right}{2} \\ 0 & 1 & 0 & -\frac{top+bottom}{2} \\ 0 & 0 & -1 & -\frac{far+near}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The inversion of the projection matrix P^{-1}, which can be used as the unprojection matrix is defined:

$$P^{-1} = \begin{bmatrix} \frac{right-left}{2} & 0 & 0 & \frac{left+right}{2} \\ 0 & \frac{top-bottom}{2} & 0 & \frac{top+bottom}{2} \\ 0 & 0 & \frac{far-near}{-2} & -\frac{far+near}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Pinhole camera model



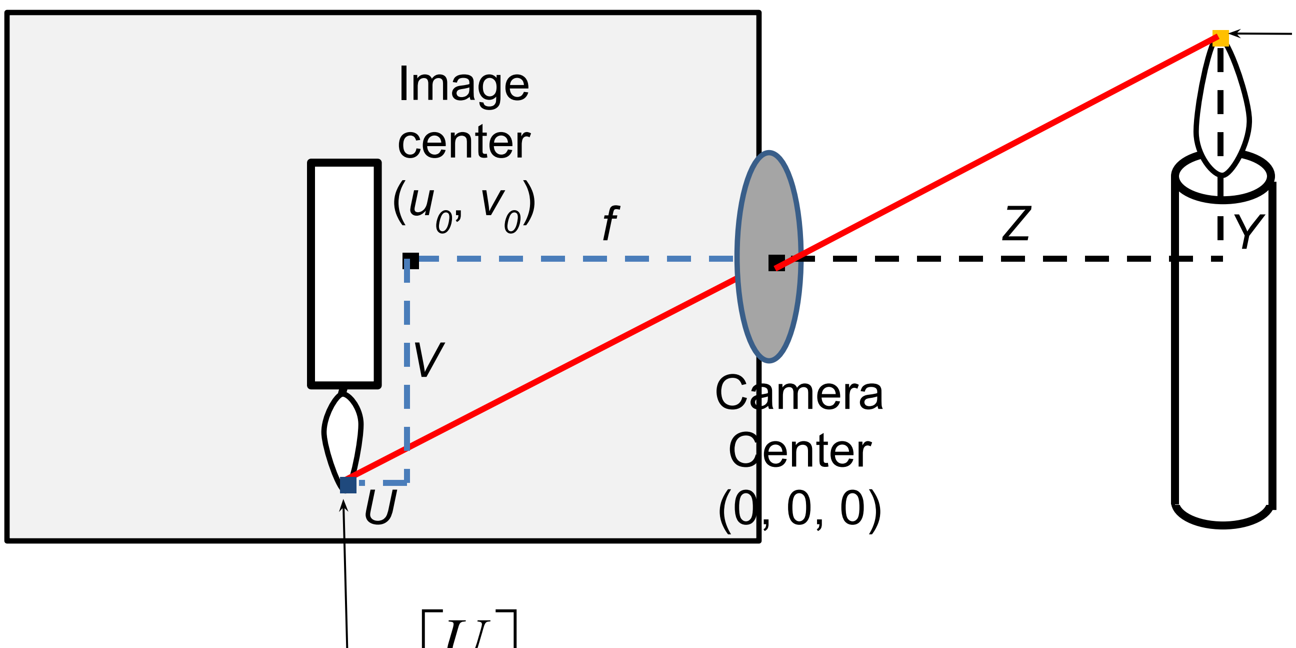
f = Focal length
c = Optical center of the camera

Figure from

Perspective Projection

Projection: world coordinates□image coordinates

$$\mathbf{P} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$



$$\mathbf{p} = \begin{bmatrix} U \\ V \end{bmatrix}$$

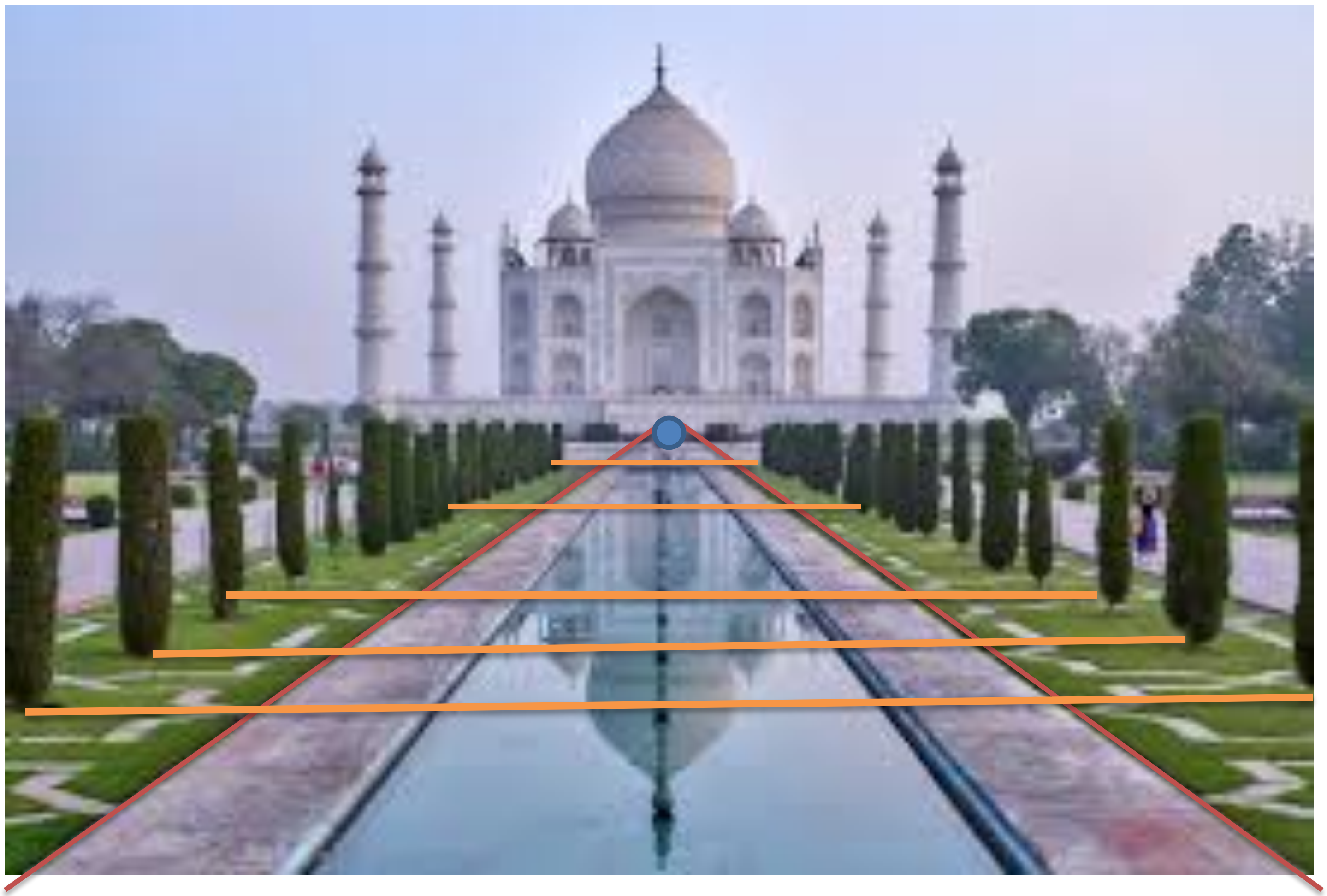
p = distance from image center

$$U = -X * \frac{f}{Z} \qquad V = -Y * \frac{f}{Z}$$

What is the effect if f and Z are equal?

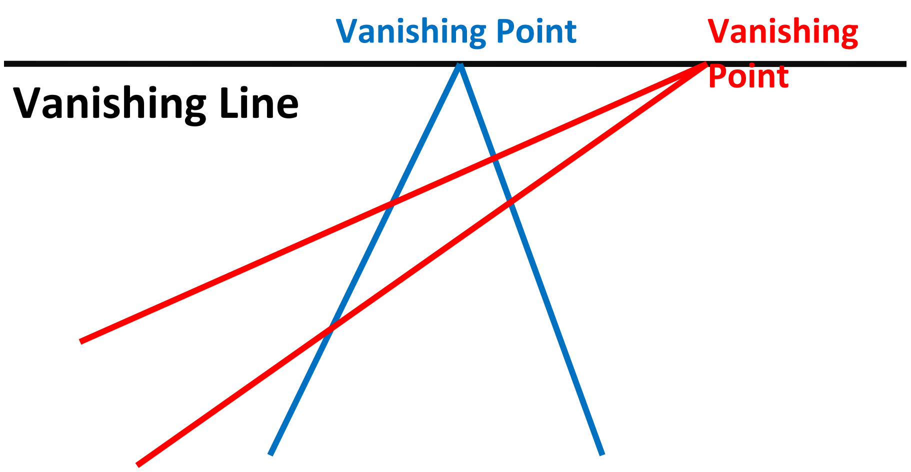
Taj Mahal

Taj Mahal



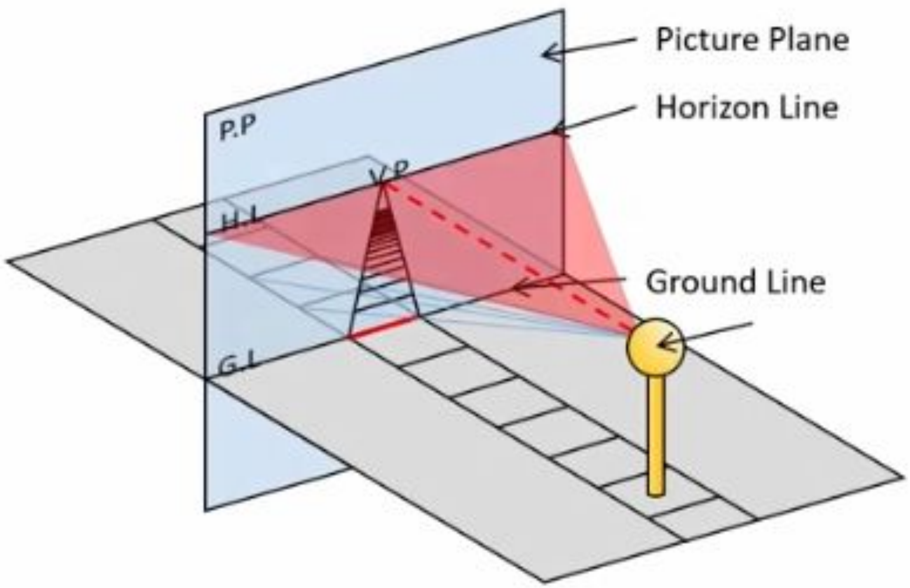
Perspective Transforms

- Parallel lines in the world intersect in the projected image at a "vanishing point".
- Parallel lines on the same plane in the world converge to vanishing points on a "vanishing line".

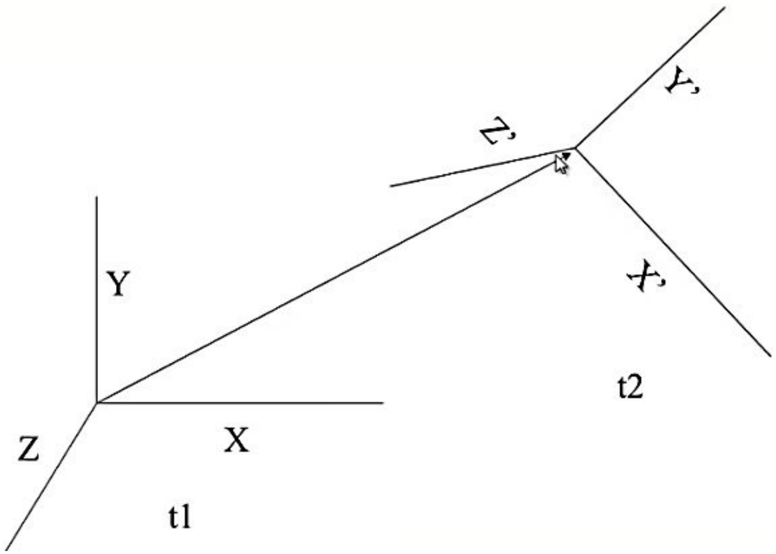


Perspective Projection:

- https://www.youtube.com/watch?v=17kqhGRDHc8



3D Rigid Body Transform

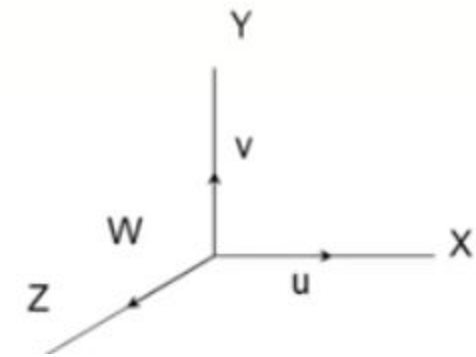


$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = R\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + T = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

Rotation matrix (9 unknowns) Translation (3 unknowns)

Rotation

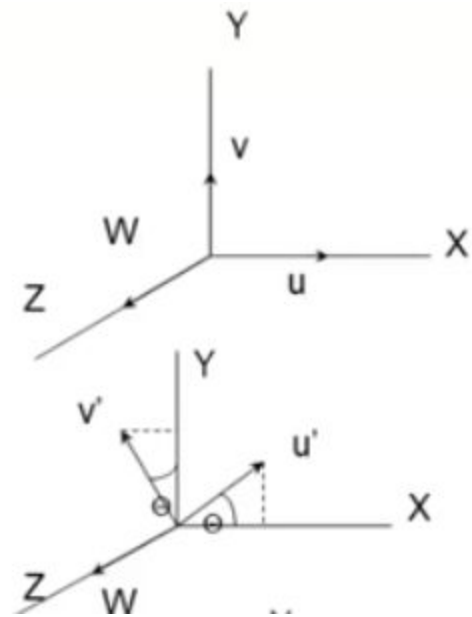
$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Rotation

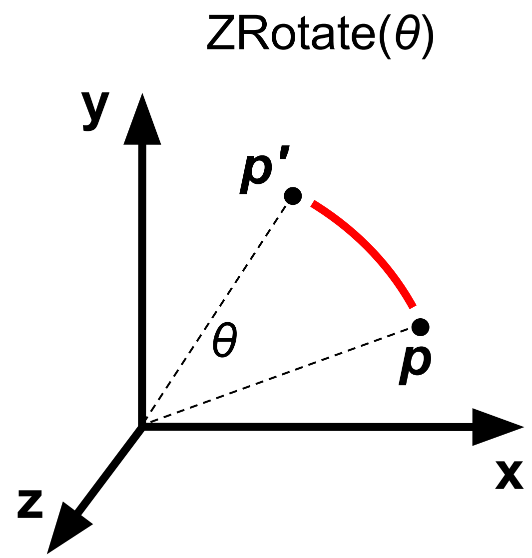
$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Rotation

- About z axis



$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Rotation

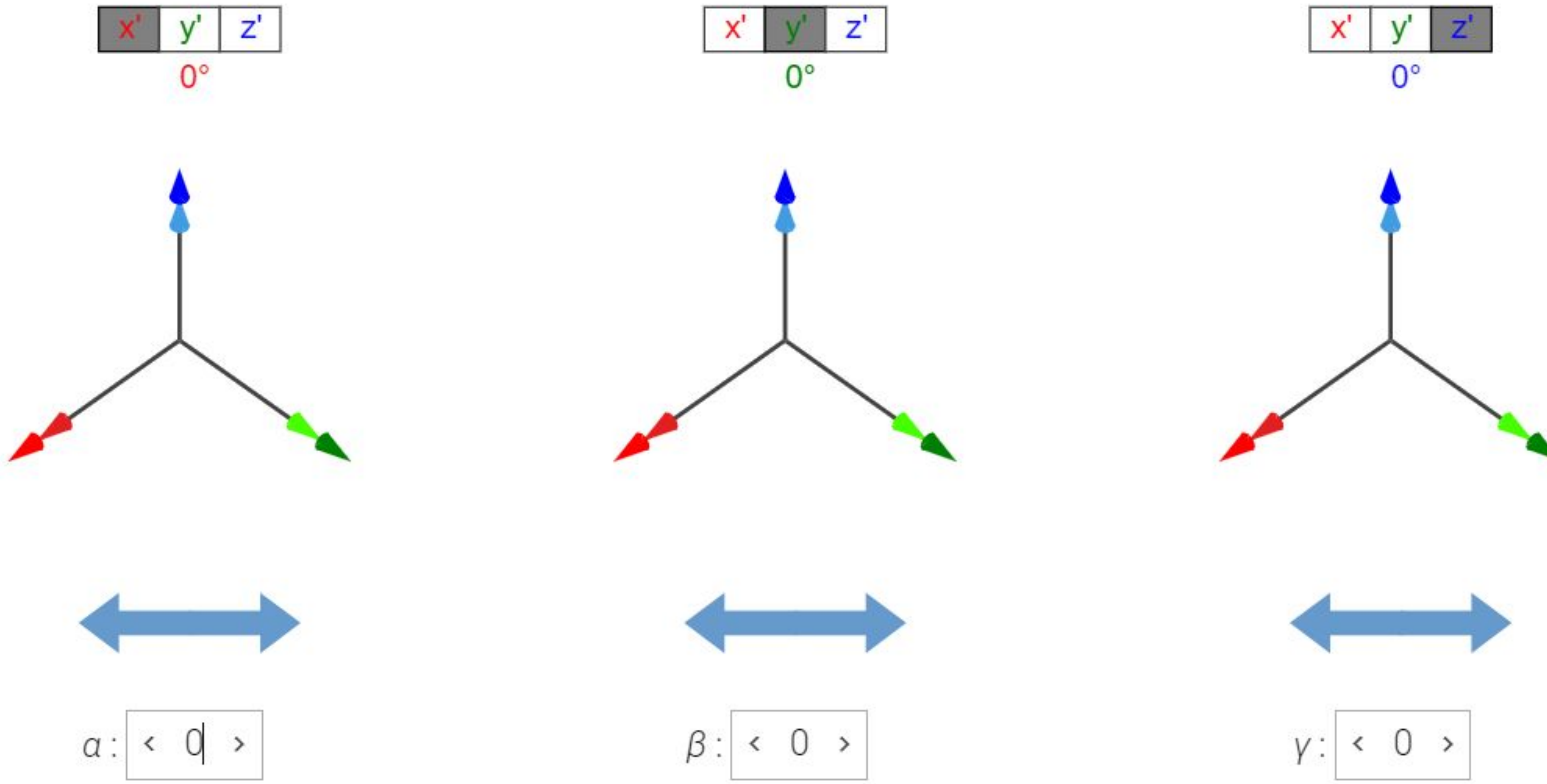
- About x axis:

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

- About y axis:

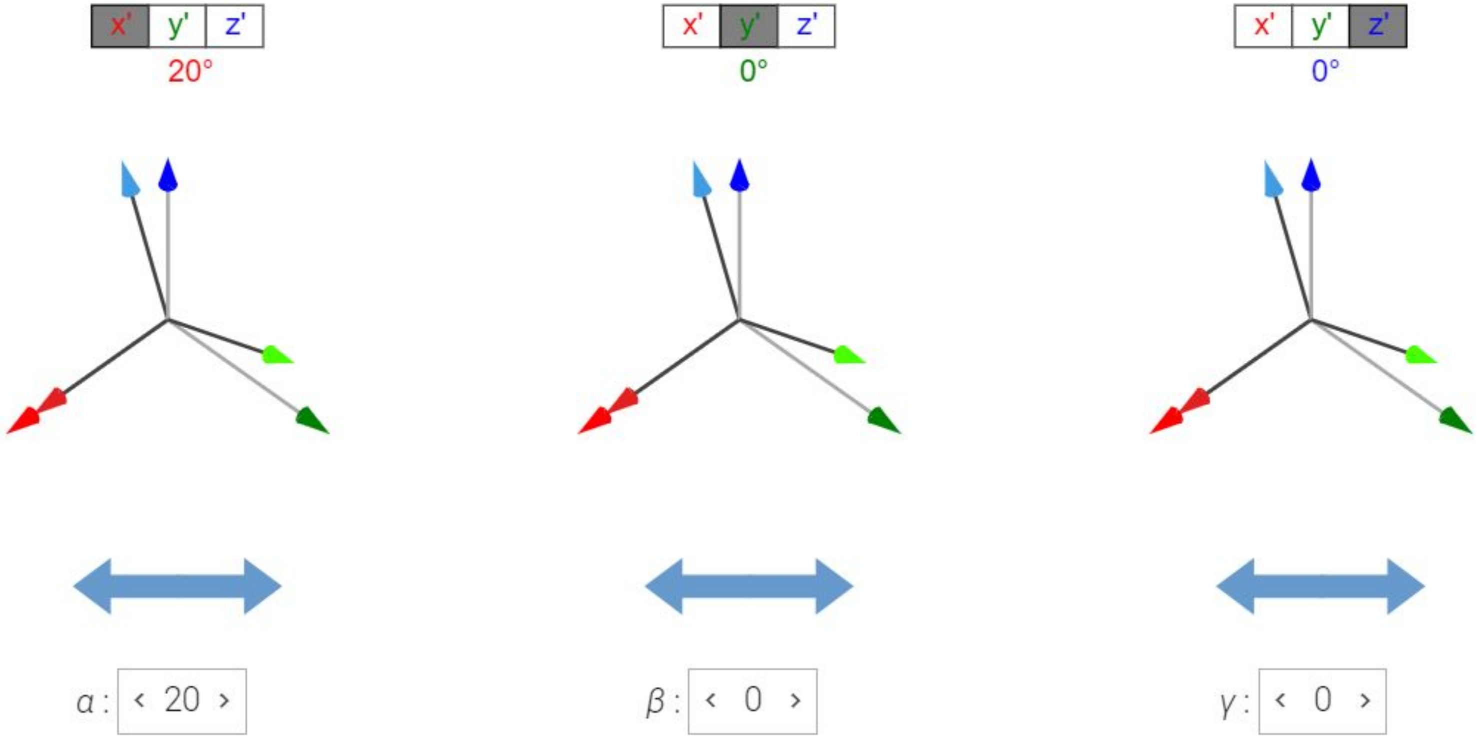
$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 1 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Euler Angles



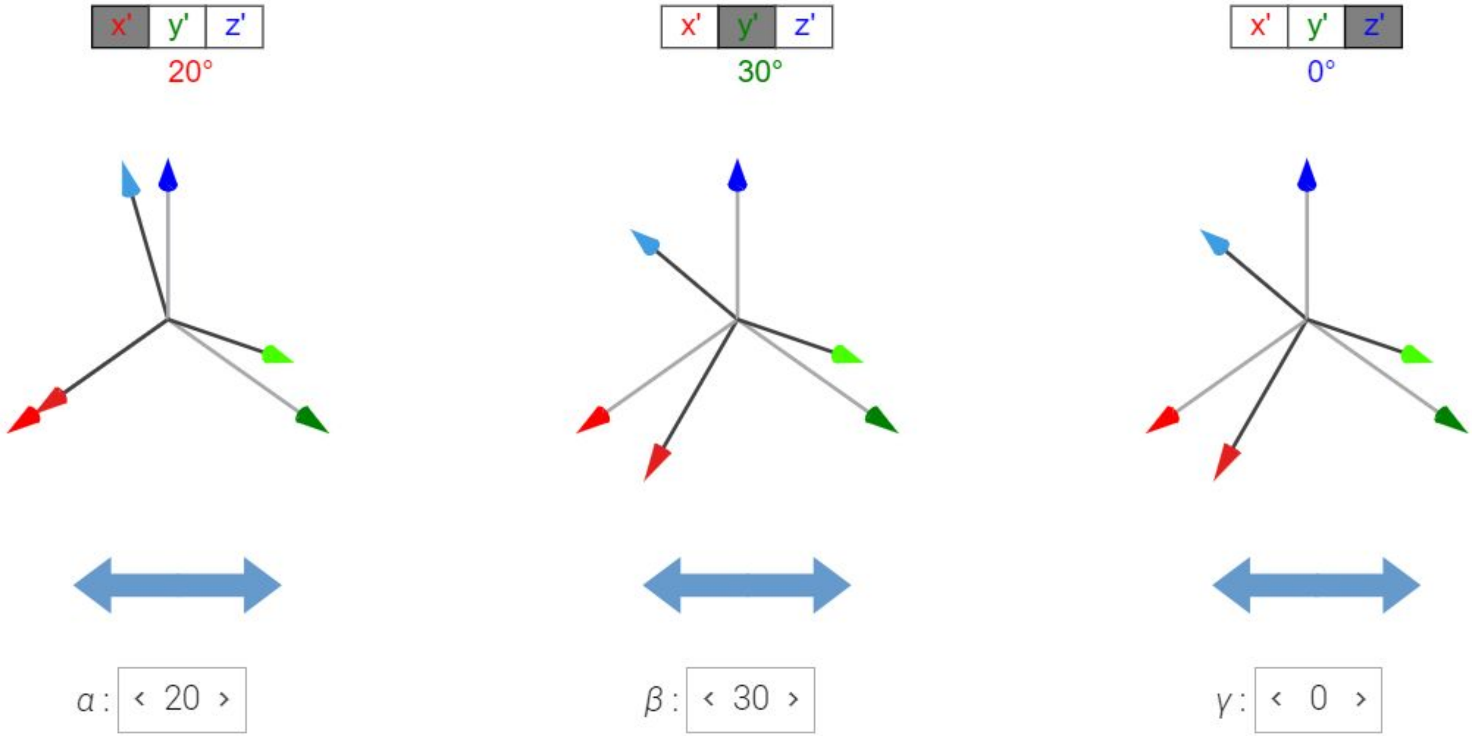
$$\mathbf{R} = \mathbf{R}_x(0°)\ \mathbf{R}_y(0°)\ \mathbf{R}_z(0°) = \begin{bmatrix} 1.000 & 0.000 & 0.000 \\ 0.000 & 1.000 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix}$$

Euler Angles



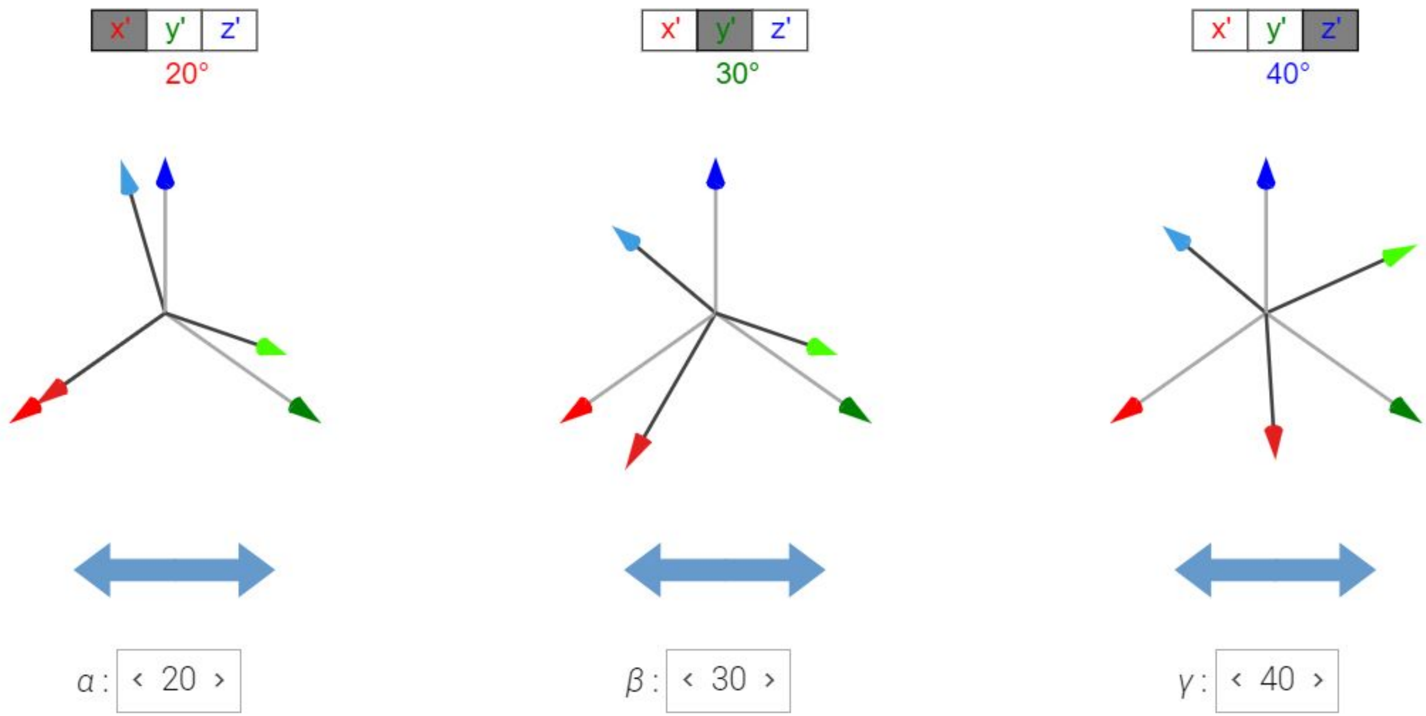
$$\mathbf{R} = \mathbf{R}_x(20°)\ \mathbf{R}_y(0°)\ \mathbf{R}_z(0°) = \begin{bmatrix} 1.000 & 0.000 & 0.000 \\ 0.000 & 0.940 & -0.342 \\ 0.000 & 0.342 & 0.940 \end{bmatrix}$$

Euler Angles



$$\mathbf{R} = \mathbf{R}_x(20°)\ \mathbf{R}_y(30°)\ \mathbf{R}_z(0°) = \begin{bmatrix} 0.866 & 0.000 & 0.500 \\ 0.171 & 0.940 & -0.296 \\ -0.470 & 0.342 & 0.814 \end{bmatrix}$$

Euler Angles



$$\mathbf{R} = \mathbf{R}_x(20°)\,\mathbf{R}_y(30°)\,\mathbf{R}_z(40°) = \begin{bmatrix} 0.663 & -0.557 & 0.500 \\ 0.735 & 0.610 & -0.296 \\ -0.140 & 0.564 & 0.814 \end{bmatrix}$$

Euler Angles

$$R = R_Z^\alpha R_Y^\beta R_X^\gamma = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

$$R = R_Z^\alpha R_Y^\beta R_X^\gamma = \begin{bmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{bmatrix}$$

$\Downarrow$ if angles are small $\cos\Theta \approx 1 \quad \sin\Theta \approx \Theta$

$$R = \begin{bmatrix} 1 & -\alpha & \beta \\ \alpha & 1 & -\gamma \\ -\beta & \gamma & 1 \end{bmatrix}$$

Important Definitions

- **Frame of reference:** a measurements are made with respect to a particular coordinate system called the frame of reference.
- **World Frame:** a fixed coordinate system for representing objects (points, lines, surfaces, etc.) in the world.
- **Camera Frame:** coordinate system that uses the camera center as its origin (and the optic axis as the Z-axis)
- **Image or retinal plane:** plane on which the image is formed, note that the image plane is measured in camera frame coordinates (mm)
- **Image Frame:** coordinate system that measures pixel locations in the image plane.
- ***Intrinsic Parameters:*** Camera parameters that are internal and fixed to a particular camera/digitization setup
- ***Extrinsic Parameters:*** Camera parameters that are external to the camera and may change with respect to the world frame.

HW Questions

- Read and define a few parmeters with respect to camera that are:

1. Intrinsic
2. Extrinsic

Camera Model & Calibration

Due to extensive Mathematical Computations please study the links:

Video Lecture Links

. Web link:**Projective geometry, camera models and calibration, IIT Delhi:**

http://www.cse.iitd.ernet.in/~suban/vision/geometry/index.html

Next Class

- Please Read the content of the lectures to understand the mathematics of forming
- Camera Matrix
- Ping me the points which you are not able to understand….
- So that I can send you solutions to those points.