Imaging Geometry

Computer Vision

Modelling from 3D to 2D world

Perspective Matters!!

Taj Mahal


Paris town hall
“**Anamorphosis**”

Cricket

Cameras, Multiple Views, and Motion

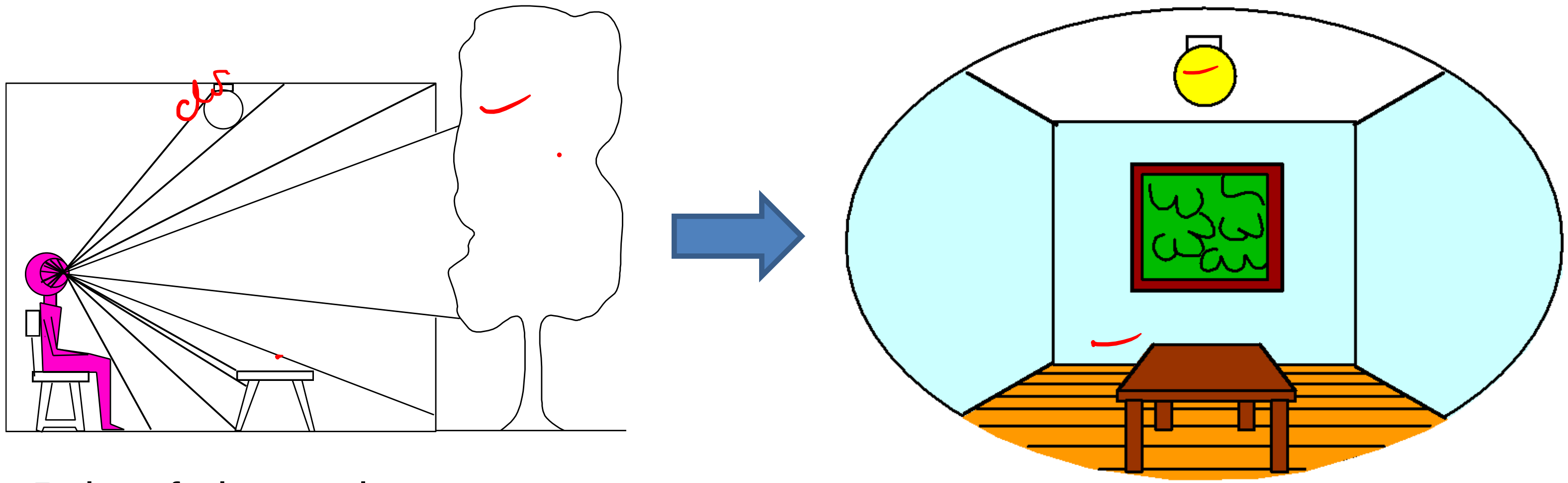
- Imaging Geometry
  - Image transforms like scaling, rotation etc
- Perspective Transformation
- Projective Transformation
- Camera Model
- Camera Calibration

Camera

Dimensionality Reduction Machine (3D to 2D)

3D world → *2D image*

↓ depth



Point of observation

How to recover knowledge about 3D from 2D…??

Figures © Stephen E. Palmer, 2002

Common transformations

Original

Transformed

Translation

Rotation

Scaling

Affine

Perspective

Slide credit (next few slides):
A. Efros and/or S. Seitz

Parametric (global) transformations

i/p

(x,y)

100

$\mathbf{p} = (x,y)$ 100

100 x 100

$\longrightarrow \boxed{T} \longrightarrow$

scaling? Rotation? aff-?

o/p

(x'y')

80

120

$\mathbf{p'} = (x',y')$ (Scaled version)

80 x 120

Transformation T is a coordinate-changing machine:

$$p' = T(p)$$

$(x'y') = T(x,y)$

What does it mean that T is global?

– T is the same for any point p

T can be described by just a few numbers (parameters)

(1) For linear transformations, we can represent T as a matrix

$$\boxed{p' = \mathbf{T}p}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{T} \begin{bmatrix} x \\ y \end{bmatrix}$$

Common transformations

Original

Transformed

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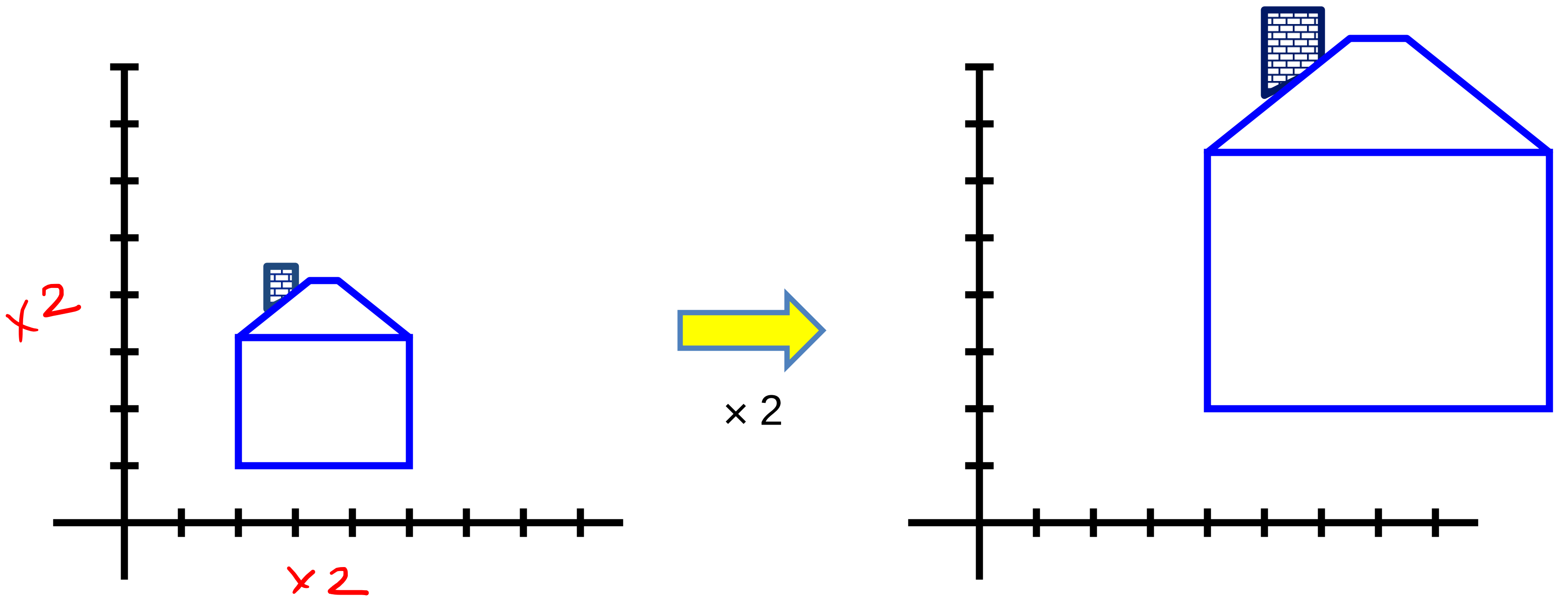
Scaling

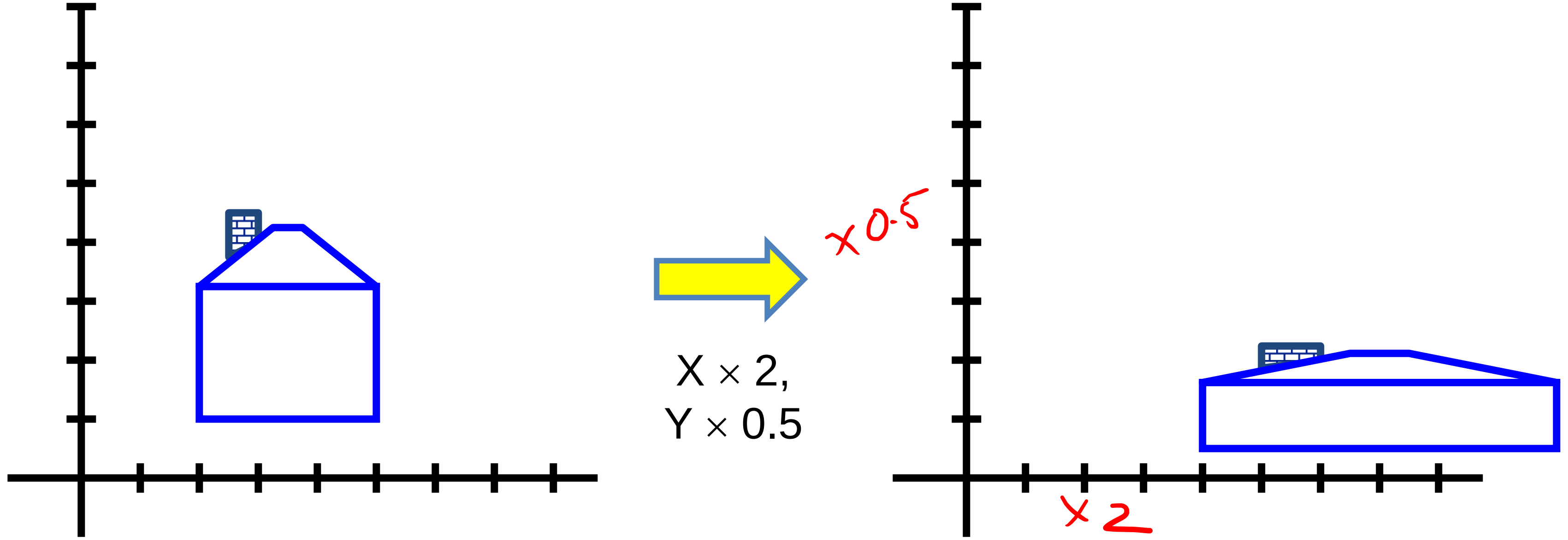
1

- *Scaling* a coordinate means multiplying each of its components by a scalar
- *Uniform scaling* means this scalar is the same for all components:

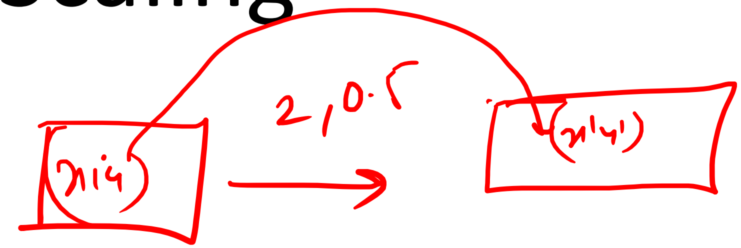


Scaling

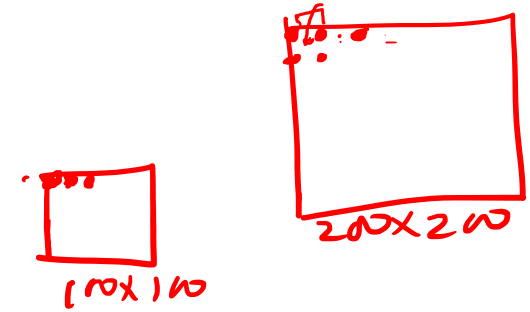
- *Non-uniform scaling*: different scalars per component:



Scaling



$x' = ax$
$y' = by$



- Scaling operation: $$x' = ax$$ $$y' = by$$

$I(x,y) = I(x',y')$

how to compute (x',y')

- Or, in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{\text{scaling matrix } S} \begin{bmatrix} x \\ y \end{bmatrix}$$

(x,y) $\qquad$ (x',y')

$(1,2) \longrightarrow (1.5, 2.3)$

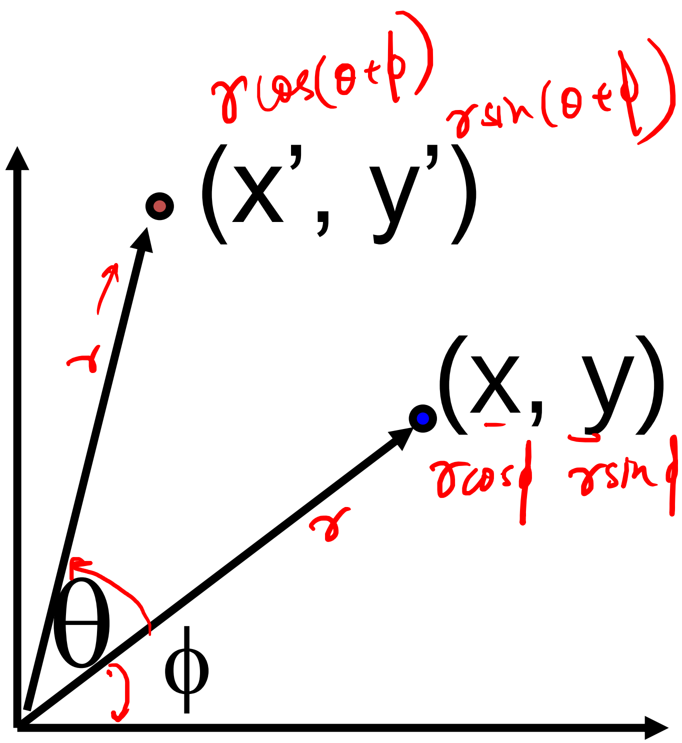
$\downarrow$

$(2,2)$

$x' = ax + 0y$
$y' = 0x + by$

$x' = ax$
$y' = by$

2-D Rotation



Polar coordinates…

x = r cos (φ)

y = r sin (φ)

x' = r cos (φ + θ)

y' = r sin (φ + θ)

Trig Identity…

x' = $\overset{x}{\text{r cos(φ)}}$ cos(θ) – $\overset{y}{\text{r sin(φ)}}$ sin(θ)

y' = $\underset{y}{\text{r sin(φ)}}$ cos(θ) + $\underset{x}{\text{r cos(φ)}}$ sin(θ)

Substitute…

x' = x **cos**(θ) - y **sin**(θ)

y' = x **sin**(θ) + y **cos**(θ)

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2-D Rotation

This is easy to capture in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} x \\ y \end{bmatrix}$$

Even though sin(θ) and cos(θ) are nonlinear functions of θ,

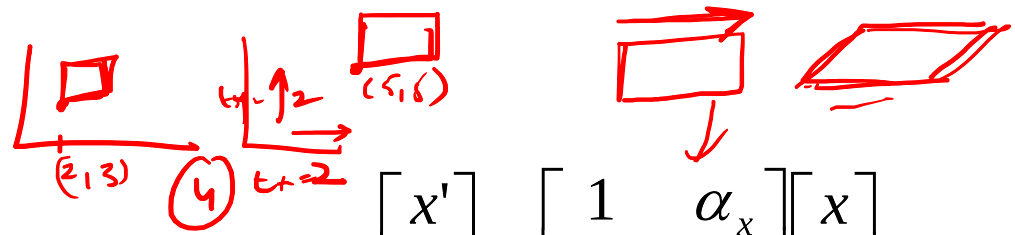
- ***x' is a linear combination of x and y***
- ***y' is a linear combination of x and y***

What is the inverse transformation?

- Rotation by –θ
- For rotation matrices $\mathbf{R}^{-1} = \mathbf{R}^{T}$

Basic 2D transformations

$x' = x + \alpha_x y$
$y' = \alpha_y x + y$



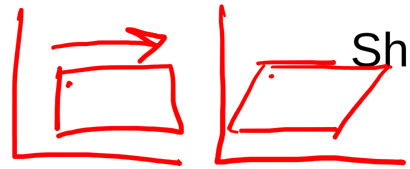
1

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & \alpha_x \\ \alpha_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Shear



2

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Rotate

3

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate

$x' = x + t_x$
$y' = y + t_y$

4

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Affine

Affine is any combination of translation, scale, rotation, and shear

Affine Transformations

1 Affine transformations are combinations of

- Linear transformations, and (S, T, R, Sh)
- Translations

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

or

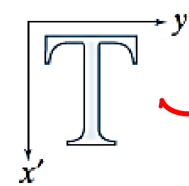
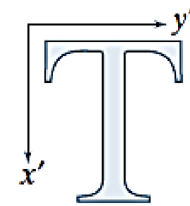
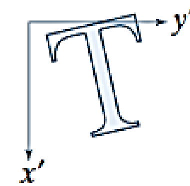
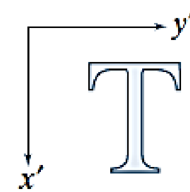
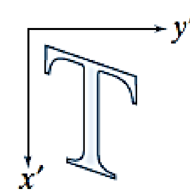
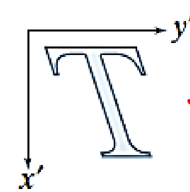
3x3

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

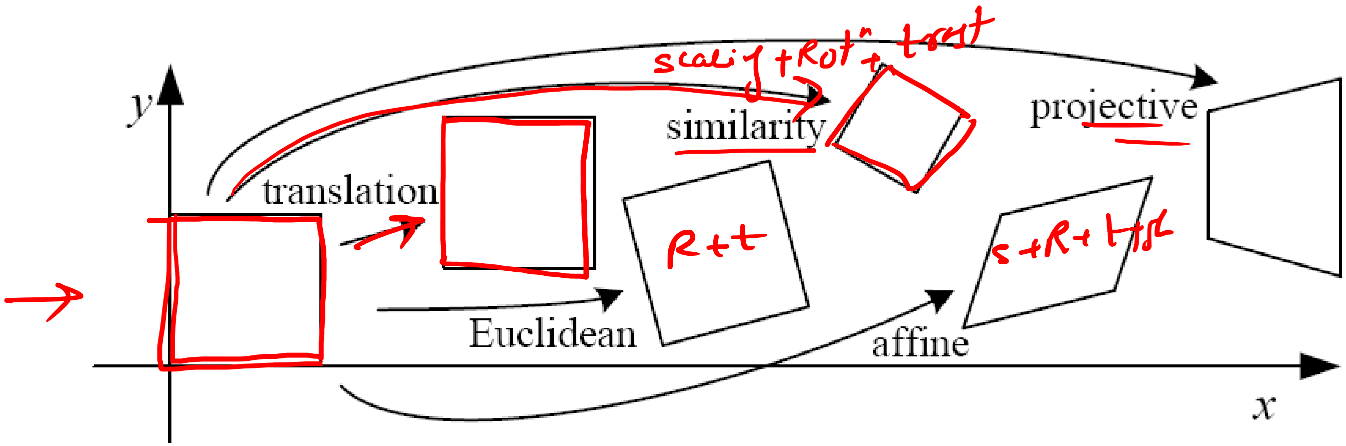
Properties of affine transformations:

- Lines map to lines
- Parallel lines remain parallel
- Ratios are preserved
- Closed under composition

basic image transfor

Transformation Name	Affine Matrix, A	Coordinate Equations	Example
Identity	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x' = x$ $y' = y$	
Scaling/Reflection (For reflection, set one scaling factor to –1 and the other to 0)	$\begin{bmatrix} c_x & 0 & 0 \\ 0 & c_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x' = c_x x$ $y' = c_y y$	
Rotation (about the origin)	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x' = x\cos\theta - y\sin\theta$ $y' = x\sin\theta + y\cos\theta$	
Translation	$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$x' = x + t_x$ $y' = y + t_y$	
Shear (vertical)	$\begin{bmatrix} 1 & s_v & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x' = x + s_v y$ $y' = y$	
Shear (horizontal)	$\begin{bmatrix} 1 & 0 & 0 \\ s_h & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x' = x$ $y' = s_h x + y$	

2D image transformations (reference table)



Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} I \mid t \end{bmatrix}_{2\times 3}$	2	orientation	
rigid (Euclidean)	$\begin{bmatrix} R \mid t \end{bmatrix}_{2\times 3}$	3	lengths	
similarity	$\begin{bmatrix} sR \mid t \end{bmatrix}_{2\times 3}$	4	angles	
affine	$\begin{bmatrix} A \end{bmatrix}_{2\times 3}$	6	parallelism	
projective	$\begin{bmatrix} \tilde{H} \end{bmatrix}_{3\times 3}$	8	straight lines	

'Homography'

Table 2.1 Hierarchy of 2D coordinate transformations. Each transformation also preserves the properties listed in the rows below it, i.e., similarity preserves not only angles but also parallelism and straight lines. The 2×3 matrices are extended with a third $[\mathbf{0}^T\ 1]$ row to form a full 3×3 matrix for homogeneous coordinate transformations.

Szeliski 2.1

Projective Transformations

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}}_{3\times 3} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

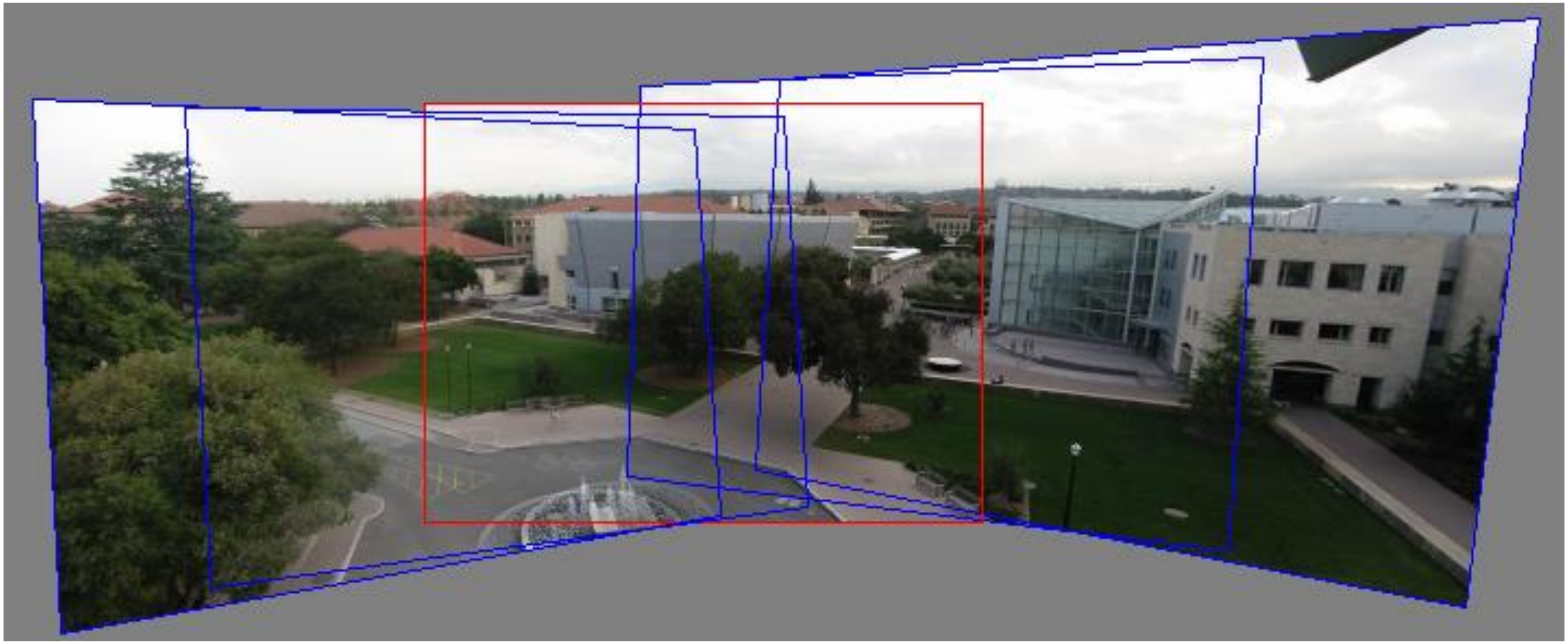
Projective transformations are combos of

- (1) Affine transformations, and (S+R+T+Sh)
- (2) Projective warps

Properties of projective transformations:

- Lines map to lines
- Parallel lines do not necessarily remain parallel
- Ratios are not preserved
- Models change of basis
- Projective matrix is defined up to a scale (8 DOF)

we use projective transforms to create a 360 panorama



- In order to figure this out, we need to learn what a **camera** is

The Geometry of Image Formation

Szeliski 2.1, parts of 2.2

To understand projective transfor

Mapping between image and world coordinates

- Pinhole camera model ✓
- Projective geometry ✓
  - Vanishing points and lines
- Projection matrix ✓

Slides from James Hays, Derek Hoiem, Alexei Efros, Steve Seitz, and David Forsyth

Image Formation: Orthographic Projection

- Means of representing 3-dimensional objects in 2-Dimensions.
- It is a form of parallel projection, in which all the projection lines are orthogonal to the projection plane

cap → 3D
small → 2D

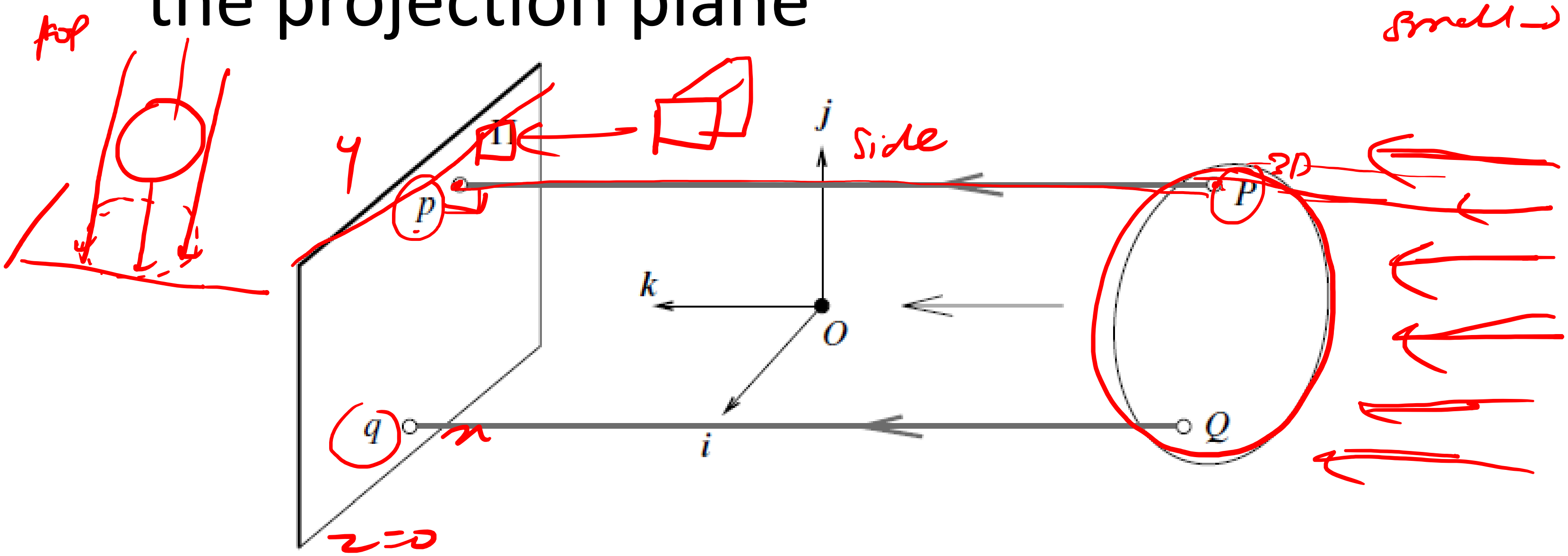


FIGURE 1.6: Orthographic projection. Unlike other geometric models of the image formation process, orthographic projection does not involve a reversal of image features.

Orthographic Projections

"H.W"
Homogeneous Coord

- A simple orthographic projection onto the plane z = 0 can be defined by the following matrix:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$2D = (v_x', y_y')$

$V_x \cdot V_y \; V_z$

- For each point $v = (v_x, v_y, v_z)$, the transformed point Pv would be

3x3

$$Pv = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix}$$

$Pv_x = Vx + 0 + 0$
$Pv_y = 0 + v_y + 0$
$Pv_z = 0 + 0 + 0$

$Pv = \begin{bmatrix} Pv_x = x \\ Pv_y = y \end{bmatrix}$

- Often, it is more useful to use homogeneous coordinates. The transformation in homogeneous coordinates

one extra [illegible]

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4x4

$(v_x, v_y, v_z) \longrightarrow (v_x, v_y, v_z, 1)$

- For each homogeneous vector $\boxed{v = (v_x, v_y, v_z, 1)}$, the transformed vector Pv would be

$$Pv = \underset{4\times4}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \underset{4\times1}{\begin{bmatrix} v_x \\ v_y \\ v_z \\ 1 \end{bmatrix}} = \underset{4\times1}{\begin{bmatrix} v_x \\ v_y \\ 0 \\ 1 \end{bmatrix}}$$

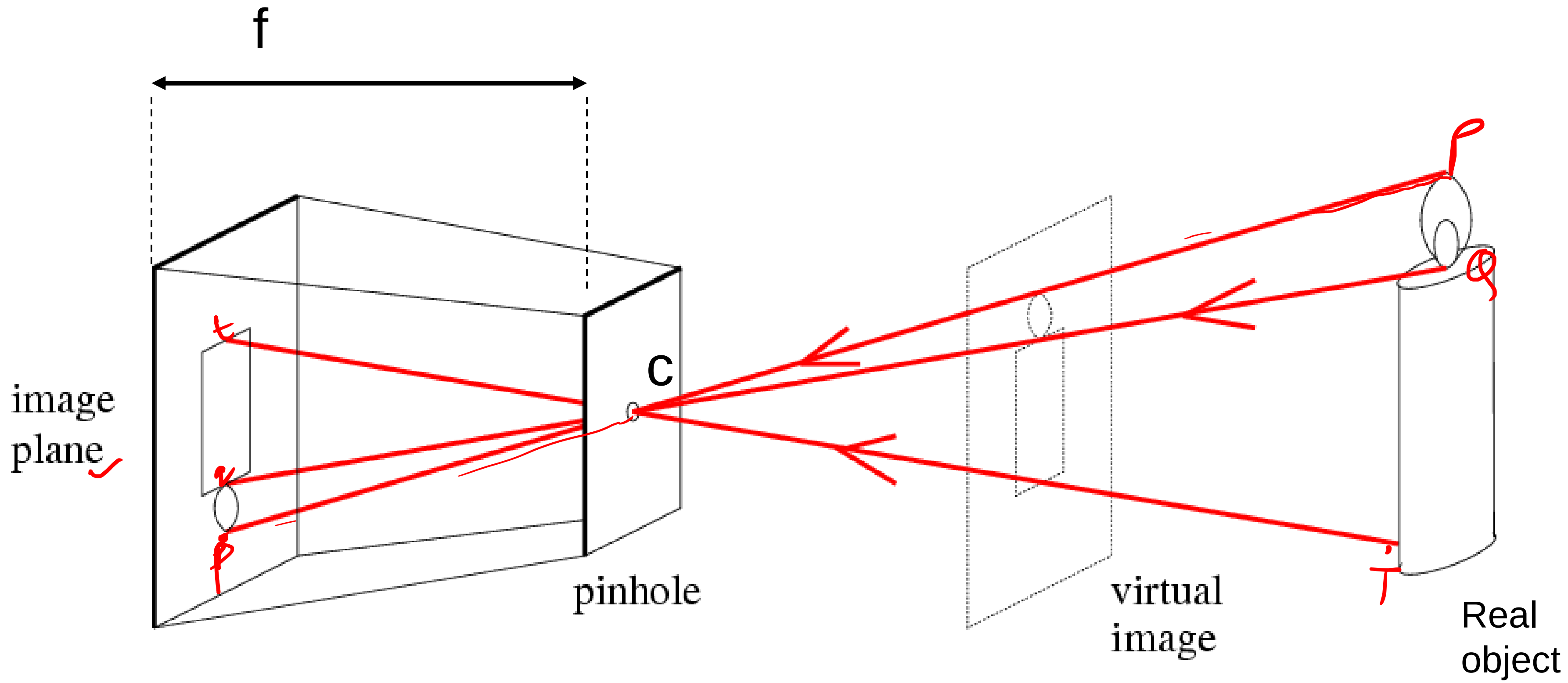
$v_x \to V_x'$
$v_y \to v_y'$
$1 \to 1$

$(v_x', v_y', 1)$

(v_x', v_y')

https://en.wikipedia.org/wiki/Orthographic_projection

Pinhole camera model



f = Focal length

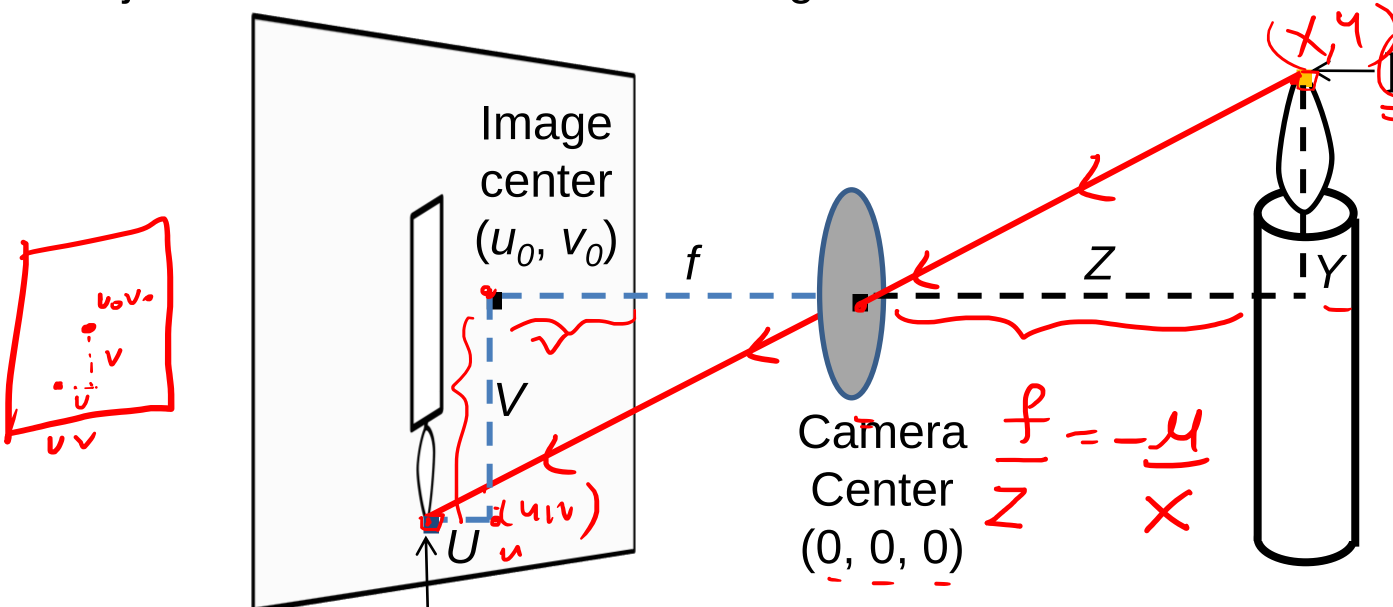
c = Optical center of the camera

Figure from Forsyth

Perspective Projection

3D 2D

Projection: world coordinates→image coordinates



$$\mathbf{P} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$ 3D

$\frac{f}{Z} = -\frac{u}{X}$

$\frac{f}{Z} = -\frac{v}{Y}$ $\therefore v = -\frac{fy}{z}$

$\therefore U = -\frac{X \cdot f}{Z}$

$$\mathbf{p} = \begin{bmatrix} U \\ V \end{bmatrix}$$

2D

p = distance from image center

x com of 3D point

Y of the 3D pt

$$U = -X * \frac{f}{Z}$$ 3D

x comp of the img 2D

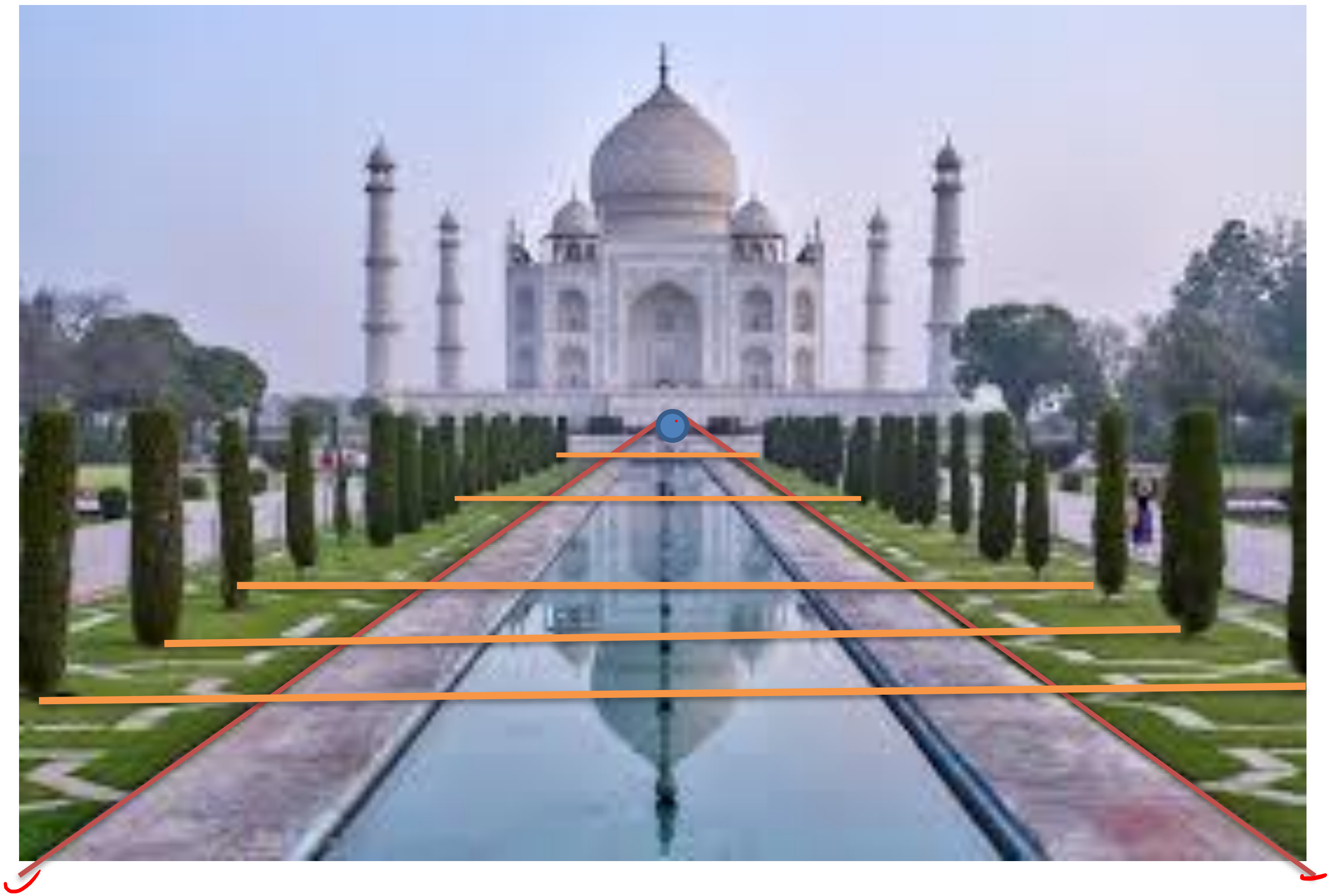
$$V = -Y * \frac{f}{Z}$$ 3D

y com of 2D

What is the effect if f and Z are equal?

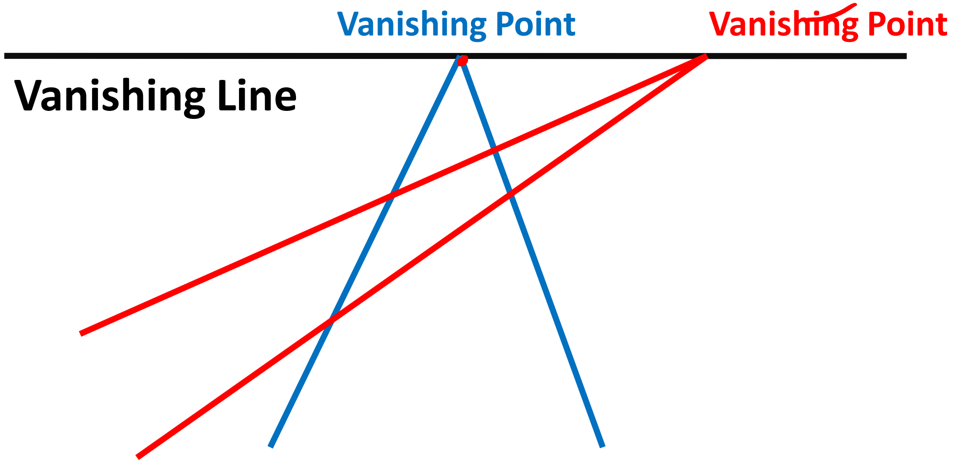
Taj Mahal

Taj Mahal



Perspective Transforms

- Parallel lines in the world intersect in the projected image at a "vanishing point".
- Parallel lines on the same plane in the world converge to vanishing points on a "vanishing line".



Perspective Projection:

- (https://www.youtube.com/watch?v=17kqhGRDHc8)

Basic

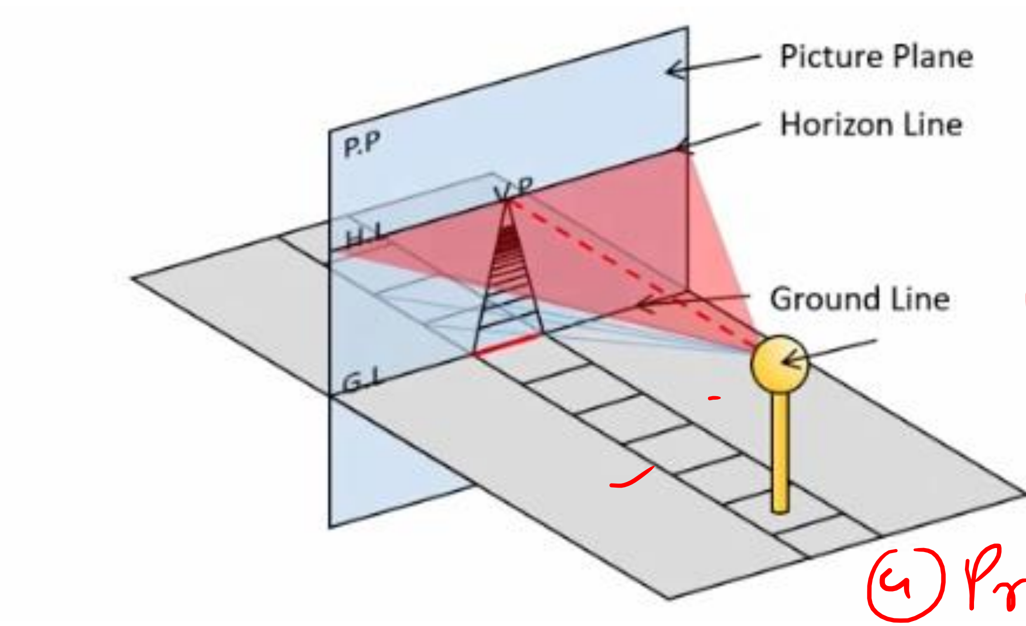
Scali → 2D

Rot → 2D

Shear → 2D

Tran

(1) Affine = S+R+Sh+T



(4) Projective Traf.

3D → 2D.

Ortho |||

Persp

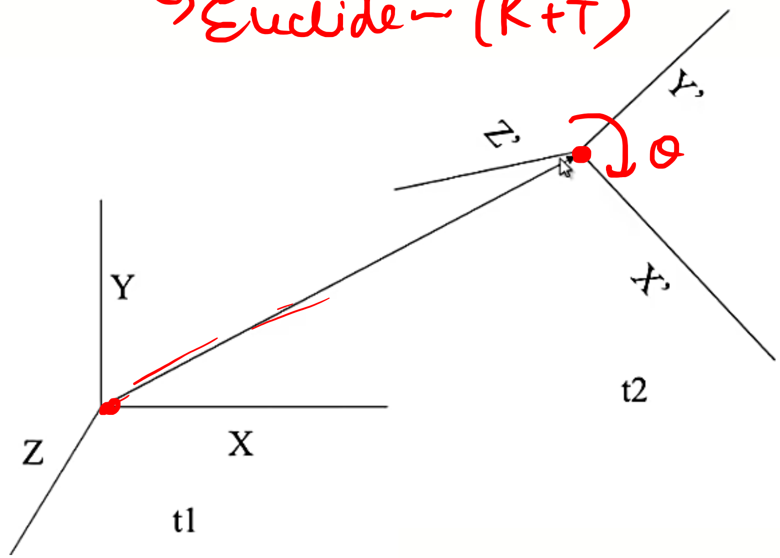
(2) Euclidean ⇒ R+T

(3) Similarity → R+T+S

3D Rigid Body Transform

→ Euclidean (R+T)

3D object → R. T.



Transform

Object in the 3D?
How the 3D world point of the object changes when the object R & T.

3x3

$$\begin{bmatrix}X'\\Y'\\Z'\end{bmatrix} = R\begin{bmatrix}X\\Y\\Z\end{bmatrix} + T = \underbrace{\begin{bmatrix}r_{11} & r_{12} & r_{13}\\ r_{21} & r_{22} & r_{23}\\ r_{31} & r_{32} & r_{33}\end{bmatrix}}\begin{bmatrix}X\\Y\\Z\end{bmatrix} + \begin{bmatrix}T_X\\T_Y\\T_Z\end{bmatrix}$$

New 3D point

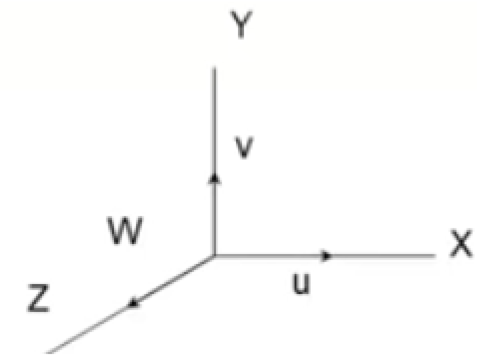
old 3D p.

Rotation matrix (9 unknowns)

Translation (3 unknowns)

Rotation

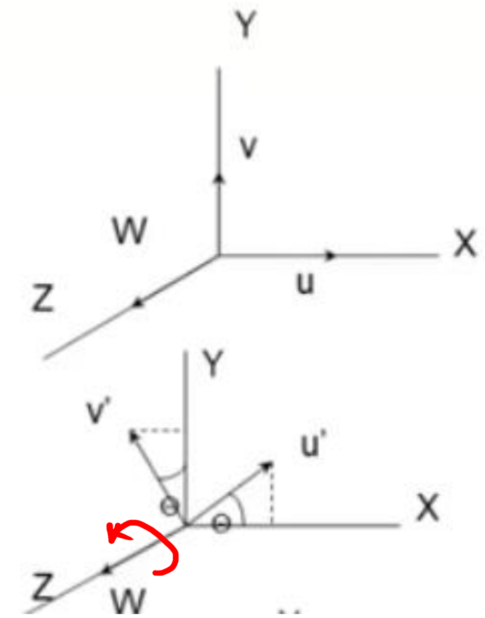
$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Rotation

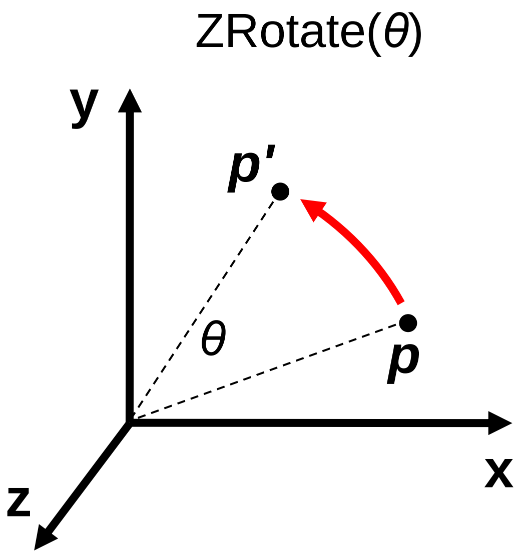
$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Rotation

- About z axis



Homogen

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Rotation

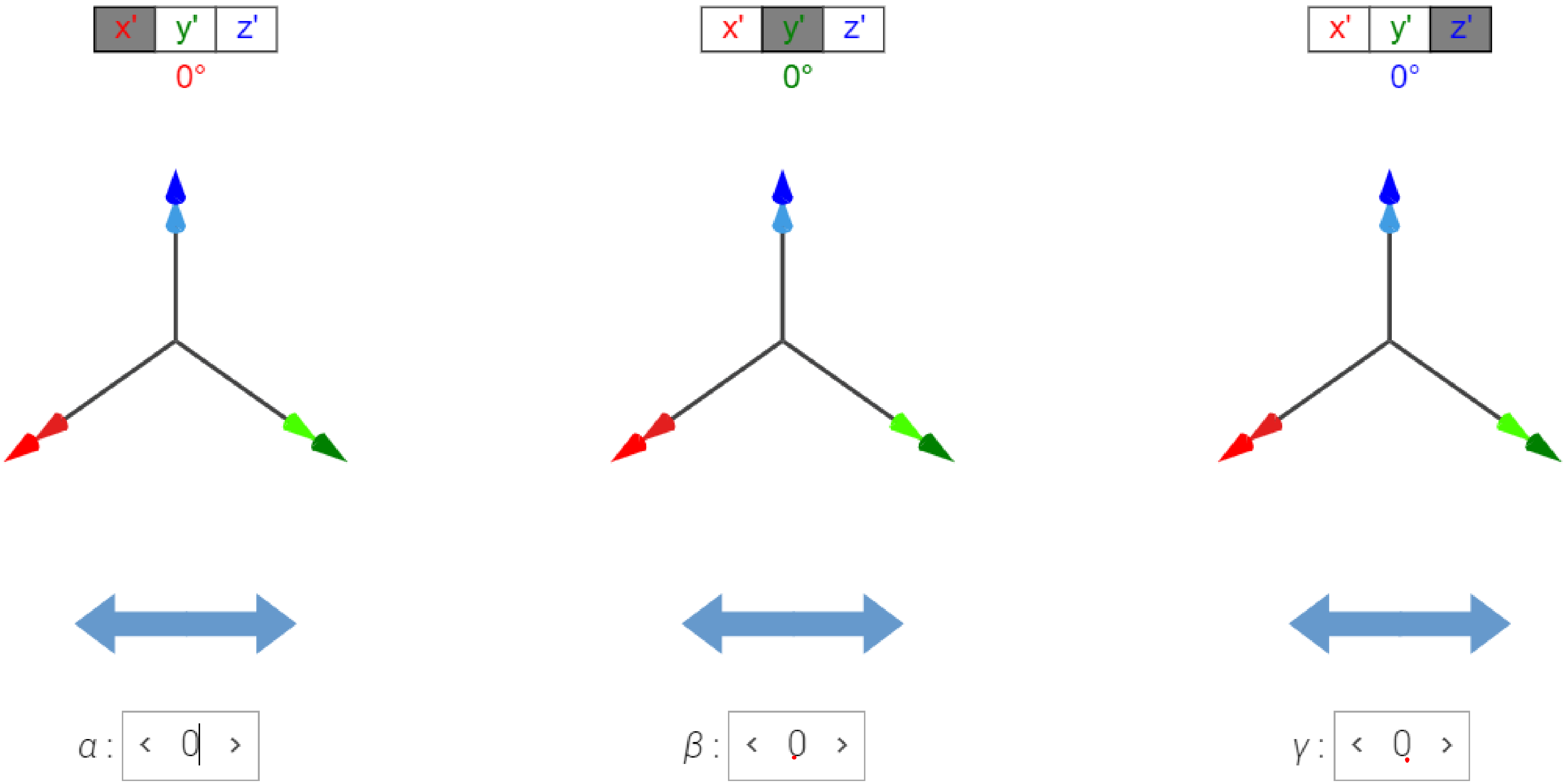
- About x axis:

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

- About y axis:

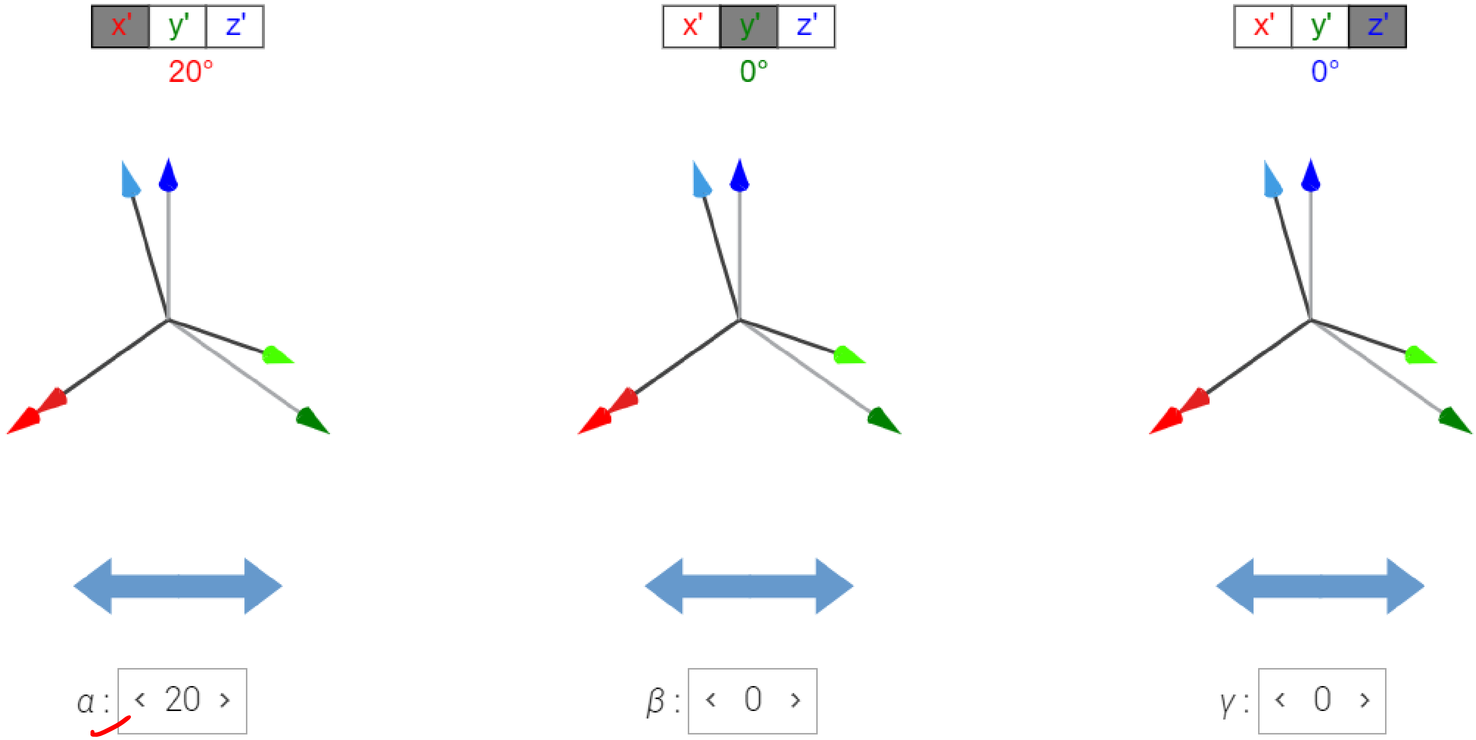
$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 1 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Euler Angles



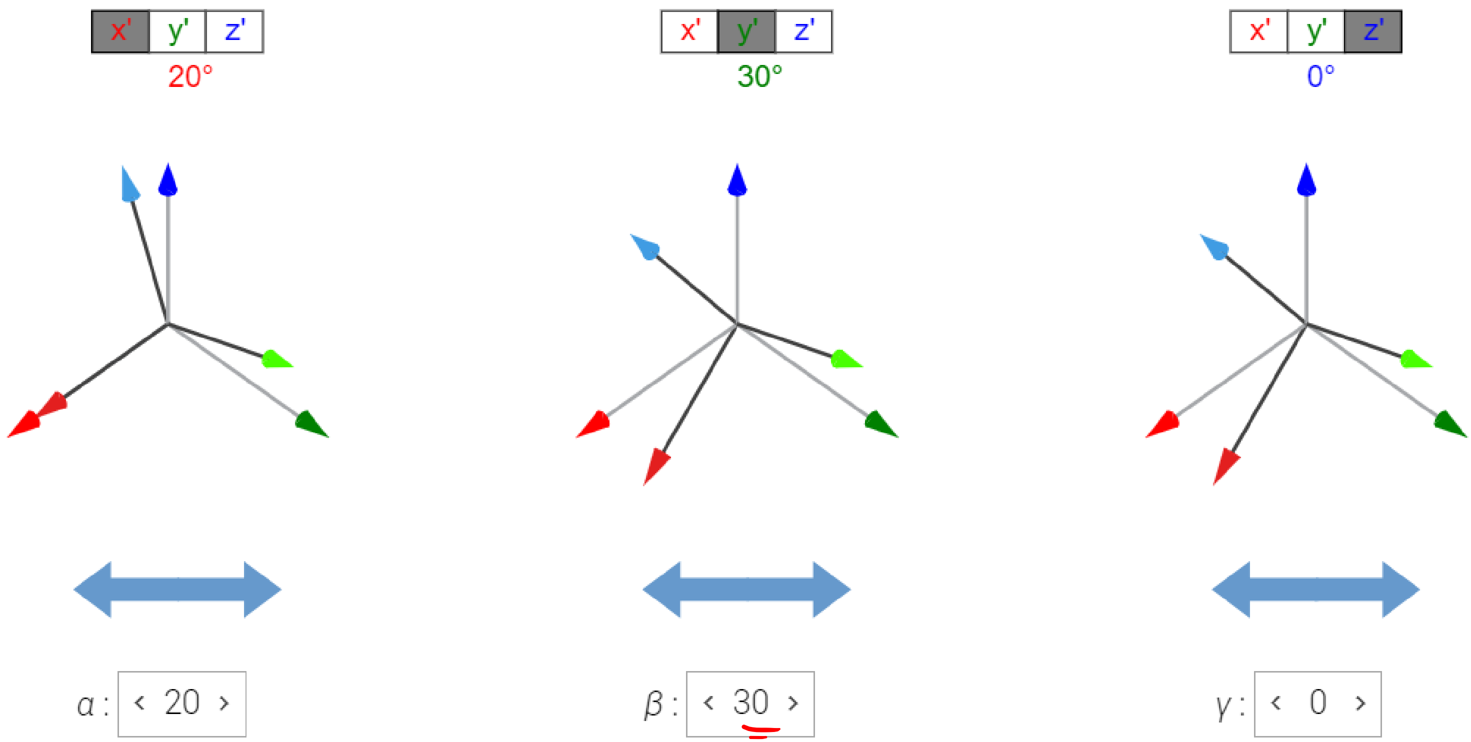
$$\mathbf{R} = \mathbf{R}_x(0°)\ \mathbf{R}_y(0°)\ \mathbf{R}_z(0°) = \begin{bmatrix} 1.000 & 0.000 & 0.000 \\ 0.000 & 1.000 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix}$$

Euler Angles



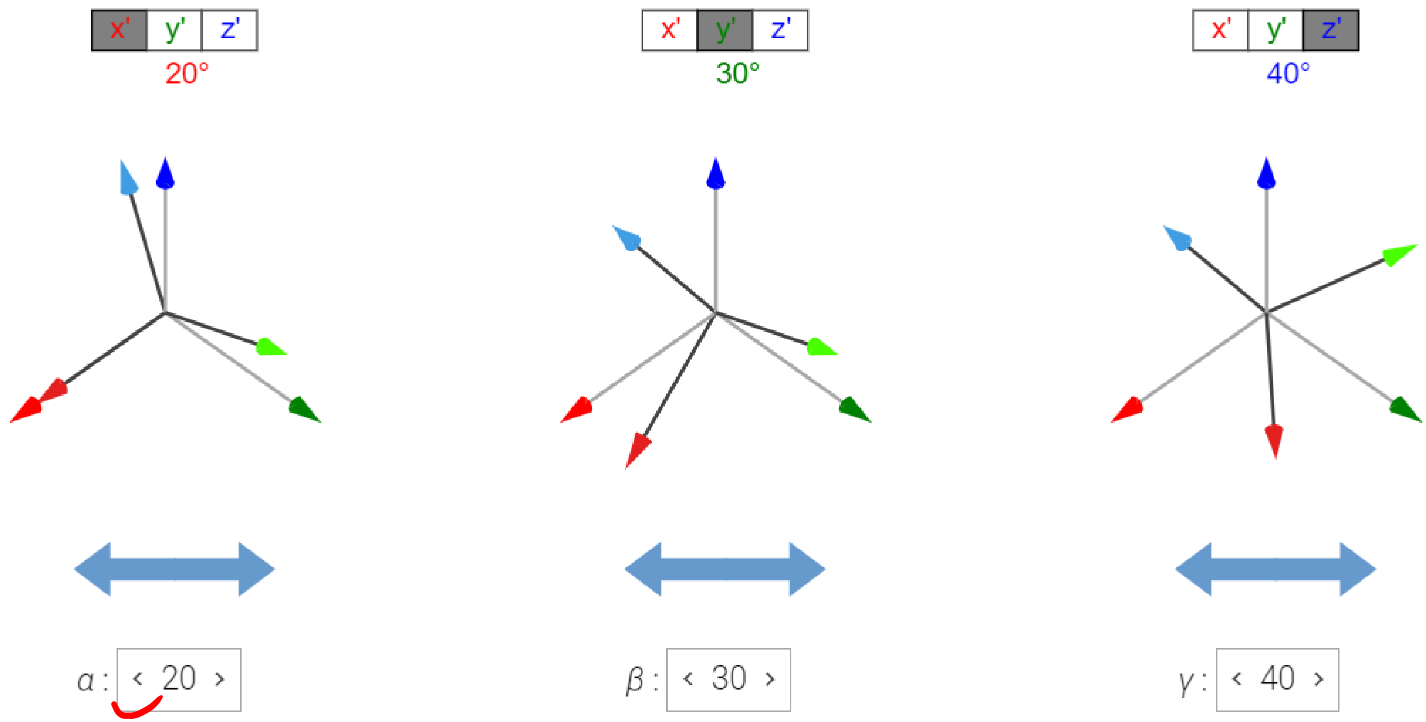
$$\mathbf{R} = \mathbf{R}_x(20°)\,\mathbf{R}_y(0°)\,\mathbf{R}_z(0°) = \begin{bmatrix} 1.000 & 0.000 & 0.000 \\ 0.000 & 0.940 & -0.342 \\ 0.000 & 0.342 & 0.940 \end{bmatrix}$$

Euler Angles



$$\mathbf{R} = \mathbf{R}_x(20°)\ \mathbf{R}_y(30°)\ \mathbf{R}_z(0°) = \begin{bmatrix} 0.866 & 0.000 & 0.500 \\ 0.171 & 0.940 & -0.296 \\ -0.470 & 0.342 & 0.814 \end{bmatrix}$$

Euler Angles



$$\mathbf{R} = \mathbf{R}_x(20°)\ \mathbf{R}_y(30°)\ \mathbf{R}_z(40°) = \begin{bmatrix} 0.663 & -0.557 & 0.500 \\ 0.735 & 0.610 & -0.296 \\ -0.140 & 0.564 & 0.814 \end{bmatrix}$$

Euler Angles

$R_z(\alpha)$ $\quad$ R_y^{β} $\quad$ R_x^{γ}

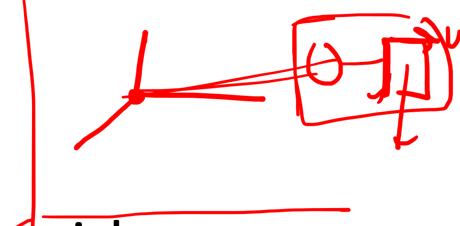
$$R = R_Z^{\alpha} R_Y^{\beta} R_X^{\gamma} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

$$R = R_Z^{\alpha} R_Y^{\beta} R_X^{\gamma} = \begin{bmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{bmatrix}$$

if angles are small $\cos\Theta \approx 1$ $\quad \sin\Theta \approx \Theta$

$$\boxed{R = \begin{bmatrix} 1 & -\alpha & \beta \\ \alpha & 1 & -\gamma \\ -\beta & \gamma & 1 \end{bmatrix}}$$

Important Definitions



- **Frame of reference:** a measurements are made with respect to a particular coordinate system called the frame of reference.
- **World Frame:** a fixed coordinate system for representing objects (points, lines, surfaces, etc.) in the world.
- **Camera Frame:** coordinate system that uses the camera center as its origin (and the optic axis as the Z-axis)
- **Image or retinal plane:** plane on which the image is formed, note that the image plane is measured in camera frame coordinates (mm)
- **Image Frame:** coordinate system that measures pixel locations in the image plane.
- ***Intrinsic Parameters:*** Camera parameters that are internal and fixed to a particular camera/digitization setup
- ***Extrinsic Parameters:*** Camera parameters that are external to the camera and may change with respect to the world frame. ?

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} C \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$ f[c]

3D point [ ] cm $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ = 2D $\begin{pmatrix} x \\ y \end{pmatrix}$